

# BMO-TEICHMÜLLER SPACES

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ABSTRACT. We show that the complex dilatation of the Douady-Earle extension of a strongly quasymmetric homeomorphism produces a Carleson measure. As an application, we study the BMO-Teichmüller theory compatible with a Fuchsian group.

## 1. INTRODUCTION

Let  $G$  be a Fuchsian group, i.e. a group of Möbius transformations acting properly discontinuously on the unit disk  $\mathbb{D}$ . For such a group we define  $M(G)$  as

$$M(G) = \{\mu \in L^\infty(\mathbb{D}) : \|\mu\|_\infty < 1 \text{ and } \forall g \in G, \mu = \frac{\bar{g}'}{g'} \mu \circ g\}.$$

If  $\mu \in M(G)$ , then there exists a unique quasiconformal self-mapping  $f^\mu$  of  $\mathbb{D}$  fixing  $1, -1, i$  and satisfying

$$\frac{\partial f^\mu}{\partial \bar{z}} = \mu \frac{\partial f}{\partial z}$$

in  $\mathbb{D}$ . Similarly there exists a unique quasiconformal homeomorphism of the plane  $f_\mu$  which is holomorphic outside  $\mathbb{D}$  with the normalization

$$f_\mu(z) = z + \frac{b_1}{z} + \dots$$

at  $\infty$  and such that in  $\mathbb{D}$  we have again

$$\frac{\partial f_\mu}{\partial \bar{z}} = \mu \frac{\partial f}{\partial z}.$$

These homeomorphisms conjugate  $G$  respectively to a new Fuchsian group and to a quasi-Fuchsian group, i.e. a Möbius group acting properly discontinuously on the quasideisk  $f_\mu(\mathbb{D})$ .

The mapping  $f^\mu$  has a geometric interpretation: If we denote by  $S$  the Riemann surface  $\mathbb{D}/G$  then  $f^\mu$  is the lift (to the universal covering) of a quasiconformal mapping from the Riemann surface  $S$  onto  $S' = \mathbb{D}/G'$  where

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$G' = f^\mu \circ G \circ (f^\mu)^{-1}$ . Conversely if  $F$  is a quasiconformal homeomorphism from  $S$  to a Riemann surface  $S'$  it has a lift to a quasiconformal homeomorphism  $f$  of  $\mathbb{D}$  and, replacing if necessary  $F$  by  $\theta \circ F$  where  $\theta : S' \rightarrow S''$  is a conformal isomorphism, we may assume that  $f = f^\mu$  for some  $\mu \in M(G)$ .

If  $\mu \in M(G)$ , then  $f^\mu$  has a well-defined boundary value which is a quasymmetric homeomorphism of the unit circle. We define an equivalence relation on  $M(G)$  by  $\mu \sim \nu$  if  $f^\mu|_{\partial\mathbb{D}} = f^\nu|_{\partial\mathbb{D}}$ . Again this equivalence relation has a geometric interpretation: if  $F, G$  represent the quasiconformal mappings on  $S$  whose lifts are precisely  $f^\mu, f^\nu$  then  $\mu \sim \nu$  is equivalent to saying that  $G \circ F^{-1}$  is homotopic to a conformal isomorphism between  $F(S)$  and  $G(S)$ , the homotopy being constant on the (possibly empty) boundary of  $F(S)$ .

The Teichmüller space  $T_S$  is the quotient space  $M(G)/\sim$ . We refer to [10] for details about this construction.

The preceding exposition imply that the mapping  $[\mu] \mapsto f^\mu$  is well defined and injective from  $T_S$  into  $\text{QS}(G)$ , the set of quasymmetric homeomorphisms  $h$  of the unit circle such that  $h \circ G \circ h^{-1}$  is a Möbius group (more precisely the trace on the unit circle of a Möbius group). A deep theorem of Tukia [12] asserts that this mapping is also onto, so that one may identify  $T_S$  with  $\text{QS}(G)$ .

There is a similar description of the Teichmüller space in terms of  $f_\mu$ . We call a quadratic differential for the group  $G$  a holomorphic mapping  $\varphi$  in  $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$  such that

$$\forall g \in G, \varphi = \varphi \circ g(g')^2.$$

If  $\mu \in M(G)$  it is easy to see that the Schwarzian derivative

$$S_{f_\mu}(z) = (\log f'_\mu)'' - \frac{1}{2}(\log f_\mu)'^2$$

is a quadratic differential for  $G$ . In [10] it is shown that the mapping  $[\mu] \mapsto S_{f_\mu}$  is well defined and injective on  $T_S$ . The image of this mapping is included in  $T(G)$ , the space of Schwarzian derivatives of injective holomorphic functions in  $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$  having a quasiconformal extension to  $\mathbb{C}$  which are quadratic differentials for  $G$ . A theorem due to Lehto and Tukia [10] asserts that this mapping is a bijection onto  $T(G)$ . This is the so called Bers imbedding; it allows to identify the Teichmüller space  $T_S$  with  $T(G)$ , a space of quadratic differential.

Both theorems (the identification of  $T_S$  with  $\text{QS}(G)$  and with  $T(G)$ ) have recieved a simplified proof with the use of the Douady-Earle extension Theorem.

The aim of this paper is to follow the same idea, i.e. the use of Douady-Earle extension Theorem, to prove analogs of the above statements in the setting of BMO-Teichmüller theory which was introduced by Astala and the second author [3]. Before stating these results we recall the basics of this non-standard Teichmüller theory.

## 2. BMO-TEICHMÜLLER THEORY.

A positive measure  $m$  in the unit disk is called a Carleson measure if

$$\sup_{I \subset \partial\mathbb{D} \text{ interval}} m(C(I))/|I| < +\infty,$$

where  $C(I) = \{rz : z \in I, (1 - |I|/(2\pi))r \leq r \leq 1\}$ . We will also need Carleson measures on  $\mathbb{C} - \overline{\mathbb{D}}$ ; the reader will easily guess their proper definition. We then define  $\text{CM}(\mathbb{D})$  the set of measurable functions  $\mu$  in the unit disk such that

$$\frac{|\mu|^2(z)}{1 - |z|} dx dy$$

is a Carleson measure.

An homeomorphism of the unit circle is called strongly quasimetric if it is absolutely continuous at every scale, i.e. if

$$\forall \epsilon > 0, \exists \delta > 0; \forall I \text{ interval}, \forall E \subset I \text{ Borel}, |E| \leq \delta|I| \Rightarrow |h(E)| \leq \epsilon|h(I)|.$$

We denote by SQS the set of strongly quasimetric homeomorphisms of the circle. SQS is a group, and more precisely it is the group of homeomorphisms  $h$  such that  $V_h : b \mapsto b \circ h$  is an isomorphism of the space  $\text{BMO}(\partial\mathbb{D})$  [6],[9]. We recall the definition of this space:

$$\text{BMO}(\partial\mathbb{D}) = \{b \in L^2(\partial\mathbb{D}); \sup_I V_I(b) < +\infty\}$$

where  $V_I(b)$  is the variance of  $b$  on the interval  $I$ .

Naturally a strongly quasimetric homeomorphism is quasimetric but the converse is far from being true since a quasimetry may be totally singular.

Let us denote by  $M(1)$ ,  $T(1)$  the spaces  $M(G)$ ,  $T(G)$  for  $G = \{I\}$ . The following theorem holds:

**Theorem 1.** *The following are equivalent:*

- (1)  $\mu \in M(1) \cap \text{CM}(\mathbb{D})$ ,
- (2)  $f^\mu \in \text{SQS}(\partial\mathbb{D})$ ,
- (3)  $S_{f^\mu} \in T(1)$  and  $|S_{f^\mu}|^2(|z| - 1)^3 dx dy$  is a Carleson measure in  $\mathbb{C} \setminus \overline{\mathbb{D}}$ .

The equivalence (1)  $\Leftrightarrow$  (2) is essentially due to Fefferman, Kenig and Pipher [8] while (1)  $\Leftrightarrow$  (3) is due to Astala and Zinsmeister [3]. The statement (2)  $\Rightarrow$  (1) must be read as: if  $h \in \text{SQS}$  then it has a quasiconformal extension to the unit disk whose complex dilatation satisfies (1). It should be noticed that a slight modification of Beurling-Ahlfors extension does the job [8]. Similarly the statement that (3)  $\Rightarrow$  (1) must be understood as follows: if  $f$  is holomorphic and injective in  $\mathbb{C} \setminus \overline{\mathbb{D}}$  with a qc extension to  $\mathbb{C}$  and such that  $|S_f|(z)^2(|z|-1)^3 dx dy$  is a Carleson measure in  $\mathbb{C} \setminus \overline{\mathbb{D}}$ , then it has a qc extension whose complex dilatation belongs to  $M(1) \cap \text{CM}(\mathbb{D})$ .

Let us now consider a Fuchsian group  $G$ . Define  $\mathcal{M}(G) = M(G) \cap \text{CM}(\mathbb{D})$ ,  $\text{SQS}(G) = \text{QS}(G) \cap \text{SQS}(\partial\mathbb{D})$ ,  $\mathcal{T}(G) = \{\varphi \in T(G); |\varphi|^2(z)(|z|-1)^3 dx dy \text{ is a Carleson measure on } \mathbb{C} \setminus \overline{\mathbb{D}}\}$ . The same equivalence relation as in the classical case may be defined on  $\mathcal{M}(G)$  and we denote by  $\mathcal{T}_S$  the quotient space ( $S = \mathbb{D}/G$ ). As a byproduct of the main result of this paper we will prove that

**Theorem 2.** *The mapping  $[\mu] \mapsto f^\mu$  is a bijection from  $\mathcal{T}_S$  onto  $\text{SQS}(G)$  while  $[\mu] \mapsto S_{f_\mu}$  is bijective from  $\mathcal{T}_S$  onto  $\mathcal{T}(G)$ .*

In the next paragraph we introduce Douady-Earle extension and use it to give a proof of this theorem.

We end this paragraph with two comments:

**2.1. Motivation for BMO-Teichmüller theory.** The whole Teichmüller theory as just described can be viewed geometrically as follows. In this section we take  $G = \{I\}$ , so that  $S = \mathbb{D}$ , and we put  $T = T_{\mathbb{D}}$ . If  $[\mu] \in T$  then  $f_\mu$  is a Riemann mapping (defined on  $\mathbb{C} \setminus \overline{\mathbb{D}}$ ) on a domain which is a quasiconformal image of the disk and  $f^\mu$  is then the conformal welding of the boundary of this domain, i.e.  $f^\mu = \psi^{-1} \circ f_\mu$  where  $\psi$  is a Riemann mapping from the disk. Very loosely speaking, the theory of the universal Teichmüller space is the theory dealing with quasiconformal geometry. The situation for the BMO-Teichmüller theory is not so clear but its starting point is the following theorem:

**Theorem 3.** *The following are equivalent for  $\mu \in M(1)$ :*

- (1)  $\exists \nu \in [\mu] \in \text{CM}(\mathbb{D})$  with a small norm,
- (2)  $\log(f^\mu)' \in \text{BMO}(\partial\mathbb{D})$  with a small norm,
- (3)  $(|\zeta|-1)^3 |S_{f_\mu}|^2 d\zeta d\bar{\zeta}$  is a Carleson measure with small norm.

These three conditions are equivalent to the fact that if  $f_\mu(\partial\mathbb{D})$  passes through  $\infty$  (which we may of course assume) it is the image of a line under a bilipschitz homeomorphism of the plane with constant close to 1. So at least in a neighborhood of the origin BMO-Teichmüller theory deals with bilipschitz geometry.

But this fact ceases to hold in general: in fact Bishop and Jones [2] have characterized domains arising in Theorem 3 and the corresponding Jordan curves need not be rectifiable. The following question is still open. Let  $\mu \in \mathcal{M}(1)$  be such that  $f_\mu(\partial\mathbb{D})$  is the bilipschitz image of a circle or a line. Is the same true for  $f_{t\mu}(\partial\mathbb{D})$ ,  $0 < t < 1$ ?

**2.2. Groups of convergence type.** Contrarily to the classical Teichmüller spaces  $\mathcal{T}_S$  can be trivial. More precisely the latter space is reduced to 0 if and only if Brownian motion is recurrent on  $S$ , which is equivalent to the fact that the Fuchsian group  $G$  is of divergence type:

$$\sum_{\gamma \in G} (1 - |\gamma(0)|) = +\infty.$$

The reason for this is the two-dimensional version of Mostow rigidity Theorem due to Agard and Pommerenke [1], [11]: if  $G$  is of divergence type and if  $h \in QS(G)$  then  $h$  must be singular. On the other hand it has been shown in [4] that  $\mathcal{T}_S$  is never trivial if  $G$  is of convergence type.

### 3. DOUADY-EARLE EXTENSION THEOREM

**Theorem 4.** [7] *There exists an application  $E$  mapping  $QS(\partial\mathbb{D})$  into the set of quasiconformal self-maps of the unit disk such that:*

- (1)  $\forall h \in QS(\partial\mathbb{D}), E(h)|_{\partial\mathbb{D}} = h,$
- (2)  $\forall h \in SQ(\partial\mathbb{D}), \forall \tau, \sigma \in Aut(\mathbb{D}), E(\sigma \circ h \circ \tau) = \sigma \circ E(h) \circ \tau.$

The main step in the construction of  $E(h)$  is the following fact that we mention for latter need: If  $h \in SQ(\partial\mathbb{D})$  we define the function  $F = F_h : \mathbb{D} \times \mathbb{D} \mapsto \mathbb{C}$  by

$$F(z, w) = \frac{1}{2\pi} \int_{\partial\mathbb{D}} \frac{h(\zeta) - w}{1 - \bar{w}h(\zeta)} \frac{1 - |z|^2}{|z - \zeta|^2} |d\zeta|.$$

Then for any  $z \in \mathbb{D}$  there exists a unique  $w \in \mathbb{D}$  such that  $F(z, w) = 0$ . We define  $E(h)(z) = w$ . Notice that if  $\int h = 0$  then  $E(h)(0) = 0$ .

Our main result is the following

**Theorem 5.** *If  $h \in SQS(\partial\mathbb{D})$  then, if  $\mu$  denotes the complex dilatation of the Douady-Earle extension  $E(h)$ , it holds that  $\mu \in CM(\mathbb{D})$ .*

The proof of this theorem will be given in the next paragraph. We end the present one by showing how it implies Theorem 2.

Let us first consider  $h \in SQS(G)$ : let  $\mu$  be the complex dilatation of  $E(h)$ . It suffices to prove that  $\mu \in M(G)$ : but if  $g \in G$  then since  $E(h \circ g) = E(h) \circ g$  and since  $h \in Q(G)$  there exists  $g_1$  Möbius such that  $h \circ g = g_1 \circ h$ . By a new application of Douady-Earle Theorem,  $E(g_1 \circ h) = g_1 \circ E(h)$ : it follows that

$E(h)$  and  $E(h) \circ g$  have the same complex dilatation, but this is equivalent to saying that  $\mu \in M(G)$ .

For the other part of the theorem we start with a univalent function  $f$  on  $\mathbb{C} \setminus \overline{\mathbb{D}}$  such that  $S_f \in T(G)$  and such that  $(|\zeta| - 1)^3 |S_f(\zeta)|^2 d\zeta d\bar{\zeta}$  is a Carleson measure. Let  $F$  be a Riemann mapping from  $\mathbb{D}$  onto  $\mathbb{C} \setminus \overline{f(\mathbb{D})}$  and  $h = F^{-1} \circ f$  the conformal welding. By Theorem 1, we have  $h \in SQS(\partial\mathbb{D})$ . Now the proof in [10], p.199 gives that  $h \in SQS(G)$ . Let  $F_1 = F \circ E(h)$ ; it is a quasiconformal extension of  $f$  whose dilatation is in  $\mathcal{M}(G)$  and the proof is complete.

#### 4. PROOF OF THEOREM 3.2

We adapt methods from [5]. Let  $h$  be an homeomorphism of the unit circle and  $H$  its harmonic extension to the unit disk. We assume that  $\int_{-\pi}^{\pi} h(e^{it}) dt = 0$ , which implies that  $f(0) = 0$  where  $f = E(h)$  is the Douady-Earle extension of  $h$ . We also assume that  $h$  is quasisymmetric and consider a quasiconformal extension  $g$  of  $h$ . Finally we denote by  $\nu$  the complex dilatation of  $g^{-1}$ .

**Proposition 6.** *For some universal constant  $C > 0$ ,*

$$\iint_{\mathbb{D}} |\bar{\partial}H|^2 dx dy \leq C \iint_{\mathbb{D}} \frac{|\nu|^2}{1 - |\nu|^2} dx dy.$$

**Proof.** We write  $H = H_1 + \overline{H_2}$  where  $H_1, H_2$  are holomorphic on  $\mathbb{D}$  and vanish at 0. Then  $\partial H = H_1'$ ,  $\bar{\partial}H = \overline{H_2'}$ ,  $|\nabla H|^2 = |\partial H|^2 + |\bar{\partial}H|^2$ ,  $J_H = |\partial H|^2 - |\bar{\partial}H|^2$ . The starting point is the inequality

$$\iint_{\mathbb{D}} |\nabla H|^2 dx dy \leq \iint_{\mathbb{D}} |\nabla g|^2 dx dy$$

due to the fact that  $H$  is harmonic and that  $H, g$  have the same boundary values. On the other hand, by Stokes formula (or by Choquet's Theorem asserting that  $H$  is a self-diffeomorphism of  $\mathbb{D}$ ), we also have

$$\iint_{\mathbb{D}} J_H dx dy = \iint_{\mathbb{D}} J_g dx dy = \pi.$$

Combining the two inequalities we get

$$\iint_{\mathbb{D}} |\bar{\partial}H|^2 dx dy \leq \iint_{\mathbb{D}} |\bar{\partial}g|^2 dx dy.$$

But

$$\iint_{\mathbb{D}} |\bar{\partial}g|^2 dx dy = \iint_{\mathbb{D}} \frac{|\bar{\partial}g|^2}{|\partial g|^2 - |\bar{\partial}g|^2} J_g dx dy = \iint_{\mathbb{D}} \frac{|\mu_g|^2}{1 - |\mu_g|^2} J_g dx dy.$$

Performing then the change of variable  $\zeta = g(z)$  we obtain that this integral is also equal to

$$\iint_{\mathbb{D}} \frac{|\mu_g \circ g^{-1}|^2}{1 - |\mu_g \circ g^{-1}|^2} dx dy = \iint_{\mathbb{D}} \frac{|\nu|^2}{1 - |\nu|^2} dx dy$$

since  $|\mu_g \circ g^{-1}| = |\mu_{g^{-1}}| = |\nu|$ .

**Proposition 7.** *There exists a constant  $C(K)$  ( $K$  is the constant of quasimetry of  $h$ ) such that*

$$\frac{|\mu_f(0)|^2}{1 - |\mu_f(0)|^2} \leq C \iint_{\mathbb{D}} |\bar{\partial}H|^2 dx dy.$$

**Proof.** We first recall [7] that  $f(z) = w$  is the unique solution of  $F(z, w) = 0$  where

$$F_z(z, w) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{h(e^{it}) - w}{1 - \bar{w}h(e^{it})} \frac{1 - |z|^2}{|z - e^{it}|^2} dt.$$

It is also shown in [7] that

$$F_z(0, 0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-it} h(e^{it}) dt = \hat{h}(1),$$

$$F_{\bar{z}}(0, 0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{it} h(e^{it}) dt = \hat{h}(-1),$$

$$F_w(0, 0) = -1, \text{ and}$$

$$F_{\bar{w}}(0, 0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(e^{it})^2 dt = \frac{1}{\pi} \int_{-\pi}^{\pi} H(e^{it}) \bar{H}_2(e^{it}) dt.$$

We next compute  $|\mu_f(0)|^2/(1 - |\mu_f(0)|^2)$  using implicit function Theorem and the formula  $F(z, f(z)) = 0$ . We get (writing  $F_z$  for  $F_z(0, 0)$  etc..) the system

$$F_{\bar{z}} + F_{\bar{w}} \bar{f}_{\bar{z}} + F_w f_{\bar{z}} = 0, \quad \bar{F}_{\bar{z}} + \bar{F}_w \bar{f}_{\bar{z}} + \bar{F}_{\bar{w}} \bar{f}_{\bar{z}} = 0,$$

whose solution is

$$f_{\bar{z}} = \frac{\bar{F}_{\bar{z}} F_{\bar{w}} - F_{\bar{z}} \bar{F}_{\bar{w}}}{|F_w|^2 - |F_{\bar{w}}|^2}, \quad f_z = \frac{\bar{F}_z F_{\bar{w}} - F_z \bar{F}_{\bar{w}}}{|F_w|^2 - |F_{\bar{w}}|^2},$$

and finally

$$\frac{|\mu_f(0)|^2}{1 - |\mu_f(0)|^2} = \frac{|\bar{F}_{\bar{z}} F_{\bar{w}} - F_{\bar{z}} \bar{F}_{\bar{w}}|^2}{(|F_z|^2 - |F_{\bar{z}}|^2) (|F_w|^2 - |F_{\bar{w}}|^2)}.$$

First of all it is proven in [7] that

$$|F_z|^2 - |F_{\bar{z}}|^2 = |\hat{h}(1)|^2 - |\hat{h}(-1)|^2 > 0,$$

$$|F_w|^2 - |F_{\bar{w}}|^2 = 1 - |h(e^{it})|^2 dt / (2\pi) > 0.$$

By compactness we deduce the existence of a constant  $C(K)$  such that if  $h$  is  $K$ -qs then

$$(|F_z|^2 - |F_{\bar{z}}|^2) (|F_w|^2 - |F_{\bar{w}}|^2) \geq C(K).$$

From all this we deduce

$$\frac{|\mu_f(0)|^2}{1 - |\mu_f(0)|^2} \leq C(K) \left| \overline{\hat{h}(1)} \frac{1}{\pi} \int_0^{2\pi} H(e^{it}) \bar{H}_2(e^{it}) dt + \hat{h}(-1) \right|^2.$$

But we have  $|\hat{h}(1)| \leq 1$ ,  $|\hat{h}(-1)| \leq \iint |\bar{\partial}H|^2$ , and

$$\left| \frac{1}{\pi} \int_0^{2\pi} H(e^{it}) \bar{H}_2(e^{it}) dt \right|^2 \leq \frac{1}{\pi} \int_0^{2\pi} |H_2(e^{it})|^2 dt \leq C \iint_{\mathbb{D}} |\bar{\partial}H|^2 dx dy$$

and the proposition is proven.

**Proposition 8.** *There exists a constant  $C(K)$  such that  $\forall z \in \mathbb{D}$ ,*

$$\frac{|\mu_{f^{-1}}(z)|^2}{1 - |\mu_{f^{-1}}(z)|^2} \leq C(K) \iint_{\mathbb{D}} \frac{|\mu_{g^{-1}}(w)|^2 (1 - |z|)^2}{1 - |\mu_{g^{-1}}(w)|^2 |1 - \bar{w}z|^4} dudv.$$

**Proof.** The case  $z = 0$  follows from propositions 6 and 7 and from the fact that  $|\mu_{f^{-1}}(0)| = |\mu_f(0)|$ . For the general case we use

$$M_1(\zeta) = \frac{\zeta + z}{1 + \bar{\zeta}z}, \quad M_2(\zeta) = \frac{\zeta - f(z)}{1 - \bar{f}(z)\zeta}$$

so that  $M_1(0) = z$ ,  $M_2 \circ f \circ M_1(0) = 0$ . Let  $F(\zeta) = M_2 \circ f \circ M_1(\zeta)$ ,  $G(\zeta) = M_2 \circ g \circ M_1(\zeta)$ . We have

$$|\mu_F(\zeta)| = |\mu_f(M_1(\zeta))|, \quad |\mu_{G^{-1}}(\zeta)| = |\mu_{g^{-1}}(M_2^{-1}(\zeta))|.$$

Applying then proposition 7 we obtain

$$\frac{|\mu_f(z)|^2}{1 - |\mu_f(z)|^2} \leq C \iint_{\mathbb{D}} \frac{|\mu_{g^{-1}}(M_2^{-1}(w))|^2}{1 - |\mu_{g^{-1}}(M_2^{-1}(w))|^2} dudv$$

and, recalling that  $\nu = \mu_{g^{-1}}$ ,

$$\leq C \iint_{\mathbb{D}} \frac{|\nu(\zeta)|^2}{1 - |\nu(\zeta)|^2} |M_2'(\zeta)|^2 d\zeta d\bar{\zeta} = C \iint_{\mathbb{D}} \frac{|\nu(\zeta)|^2}{1 - |\nu(\zeta)|^2} \frac{(1 - |f(z)|^2)^2}{|1 - \bar{f}(z)\zeta|^4} d\zeta d\bar{\zeta}$$

and the proposition follows by replacing  $z$  by  $f^{-1}(z)$ .

**Theorem 9.** *Let  $h \in SQS(\partial\mathbb{D})$  and  $f = E(h)$  its Douady-Earle extension. Then*

$$\frac{|\mu_f(z)|^2}{1 - |z|} dx dy$$

*is a Carleson measure in the unit disk.*

**Proof.** First of all there exists  $M \in \text{Aut}(\mathbb{D})$  such that  $M \circ f(0) = 0$ . As  $M \circ f = E(M \circ h)$  and  $\mu_{M \circ f} = \mu_f$  we may assume that  $f(0) = 0$ .

Next we consider an extension  $g$  of  $h$  such that

$$\frac{|\mu_g|^2}{1 - |z|} dx dy \in CM(\mathbb{D})$$

(for instance the modified Beurling-Ahlfors extension, see [8]).



**Lemma 10.** *If  $G$  is bilipschitz for the hyperbolic metric and if*

$$\frac{|\mu_g|^2}{1-|z|} dx dy \in CM(\mathbb{D}),$$

*then the same is true for  $g^{-1}$ .*

**Proof.** To simplify the notations we prove the analogous statement for the upper-half plane  $\mathbb{R}_+^2 = \{y > 0\}$ . Let  $I \subset \mathbb{R}$  be an interval and  $C_I = I \times [0, |I|]$  be the associated Carleson box. Then by an obvious change of variables we get

$$\mathcal{I} = \iint_{C_I} \frac{|\mu_{g^{-1}}(z)|^2}{\text{Im}(z)} dx dy = \iint_{g^{-1}(C_I)} \frac{|\mu_g(\zeta)|^2}{\text{Im}(\zeta)} \frac{\text{Im}(\zeta)}{\text{Im}(g(\zeta))} J_g(\zeta) d\zeta d\bar{\zeta}.$$

But there exists a constant  $\alpha = \alpha(K)$  such that

$$C_{\alpha J} \subset g^{-1}(C_I) \subset C_J, \quad J = h^{-1}(I)$$

where  $\alpha J$  is the interval with the same center as  $J$  but with length  $\alpha|J|$ .

On the other hand, by quasiconformality and from the fact that  $g$  is bilipschitz for the hyperbolic metric,

$$\frac{\text{Im}(\zeta)}{\text{Im}(g(\zeta))} J_g(\zeta) \sim \frac{|h(I_\zeta)|}{|I_\zeta|}$$

where  $I(\zeta)$  is the interval  $[a, b]$  such that the triangle  $(a, b, \zeta)$  is equilateral. Let then  $\omega = h'$ ,  $\varphi = \omega|_{2J}$ . By Carleson Theorem [?],

$$\mathcal{I} \leq \int_J \varphi^*(x) dx$$

where  $\varphi^*$  stands for the Hardy-Littlewood maximal function of  $\varphi$ .

By Muckenhoupt theory, there exists  $C, p > 1$  such that for any interval  $J$ ,

$$\frac{1}{|J|} \int_J \omega(x)^p dx \leq C \left( \frac{1}{|J|} \int_J \omega(x) dx \right)^p$$

We may then write

$$\mathcal{I} \leq |J|^{\frac{1}{p'}} \left( \int_J \varphi^{*p} \right)^{\frac{1}{p}} \leq C |J|^{\frac{1}{p'}} \left( \int_J \omega^p \right)^{\frac{1}{p}} \leq C \int_J \omega = C|I|,$$

from which Lemma 10 follows.

**Lemma 11.** *If  $A(z)dzd\bar{z}$  is a Carleson measure in  $\mathbb{D}$ , the same is true for  $B(z)dzd\bar{z}$  where*

$$B(z) = \iint_{\mathbb{D}} A(\omega) \frac{(1-|\omega|)(1-|z|)}{|1-\bar{\omega}z|^4} du dv$$

**Proof.** Here again we prove the statement for  $\mathbb{R}_+^2$ . In this case we write  $B = T(A)$  where

$$T(A)(x + iy) = \iint_{\mathbb{R}_+^2} A(w) \frac{vy}{|w - x + iy|^4} dudv.$$

By translation invariance it suffices to test the property on intervals  $I = [-b, b]$ . Furthermore if  $A(z)dxdy \in CM(\mathbb{R}_+^2)$  the same is true for  $\lambda A(\lambda z)$  with the same norm. Since  $T(\lambda A(\lambda)) = \lambda^{-1}B(\lambda^{-1}z)$ , we just have to show the property for  $b = 1/2$ . Let  $C = [-1/2, 1/2] \times [0, 1]$ , then

$$\iint_C B(x + iy)dxdy = \iint_{\mathbb{R}_+^2} vA(w) \left( \iint_C \frac{y}{|w - x + iy|^4} dxdy \right) dudv = I.$$

We put  $\bar{C} = [-1, 1] \times [0, 2]$  and write  $I = \mathcal{A} + \mathcal{B} = \iint_{\bar{C}} + \iint_{\mathbb{R}_+^2 \setminus \bar{C}}$ .

$$\mathcal{B} \leq C \iint_{\mathbb{R}_+^2 \setminus \bar{C}} \frac{vA(w)}{w^4} dudv \leq C \sum_{n \geq 1} \iint_{|w| \sim 2^n} \frac{2^n}{A}(w) 2^{4n} dudv \leq C \sum_{n \geq 1} 2^{-2n} \leq C.$$

To estimate  $\mathcal{A}$  it suffices to observe (simple computation) that

$$\iint_C \frac{y}{((u-x)^2 + (v+y)^2)^2} dxdy \leq \frac{C}{v}$$

The proof of the theorem is then completed by application of all the preceding propositions and lemmas.

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