Fixed-point subalgebra of quiver Hecke algebras for a quiver automorphism

and application to the Hecke algebra of $G(r, p, n)$

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Motivations

Let $n, e, p \in \mathbb{N}^*$ with $e \geq 2$. Let q, ζ be some elements of a field F of respective order e, p. Let $\mathbf{\Lambda} = (\Lambda_i)_i$ be a $\mathbb{Z}/e\mathbb{Z}$-tuple of non-negative integers and set $r := p \sum_i \Lambda_i$.

The *Ariki–Koike algebra* $\mathrm{H}_n^{\mathbf{\Lambda}}(q, \zeta)$ is a Hecke algebra of the complex reflection group $G(r, 1, n)$. It is a F-algebra generated by $S, T_1, \dots, T_{n-1}$, the “cyclotomic relation” being:

$$\prod_{i \in \mathbb{Z}/e\mathbb{Z}} \prod_{j \in \mathbb{Z}/p\mathbb{Z}} \left(S - \zeta^j q^i\right)^{\Lambda_i} = 0.$$

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There is an automorphism σ^{H} of $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q, \zeta)$ of order p given by:

$$\sigma^{\mathrm{H}}(S) := \zeta S,$$
$$\forall a, \sigma^{\mathrm{H}}(T_a) := T_a.$$

The subalgebra $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q, \zeta)^{\sigma^{\mathrm{H}}}$ of fixed points is a Hecke algebra of $G(r, p, n)$.

Cyclotomic quiver Hecke algebra

Let Γ be a quiver ($=$ oriented graph) with vertex set K. The *quiver Hecke algebra* $\mathrm{R}_n(\Gamma)$ is generated over F by:

$$e(\boldsymbol{k}) \text{ for } \boldsymbol{k} \in K^n,$$

$$y_1, \dots, y_n,$$

$$\psi_1, \dots, \psi_{n-1},$$

together with some relations.

Exemple of relation

For $\boldsymbol{k} \in K^n$ such that $k_a \xrightarrow{\Gamma} k_{a+1}$ then $\psi_a^2 e(\boldsymbol{k}) = (y_{a+1} - y_a)e(\boldsymbol{k})$.

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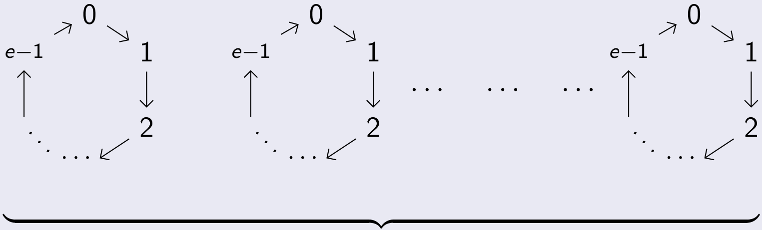
For $\boldsymbol{\Lambda} = (\Lambda_k)_{k \in K} \in \mathbb{N}^K$, the *cyclotomic* quiver Hecke algebra $\mathrm{R}_n^{\boldsymbol{\Lambda}}(\Gamma)$ is the quotient of $\mathrm{R}_n(\Gamma)$ by the following relations:

$$\forall \boldsymbol{k} \in K^n, y_1^{\Lambda_{k_1}} e(\boldsymbol{k}) = 0.$$

Graded isomorphism theorem

Theorem (Brundan–Kleshchev, Rouquier)

The Ariki–Koike algebra $\mathrm{H}_n^{\Lambda}(q,\zeta)$ *is isomorphic over* F *to the cyclotomic quiver Hecke algebra* $\mathrm{R}_n^{\Lambda}(\Gamma_{e,p})$*, where* $\Gamma_{e,p}$ *is given by:*



$\underbrace{\qquad\qquad}_{p' := \frac{p}{\gcd(p,e)} \text{ copies}}$

Remark

The integer p' is the smallest integer $m \geq 1$ such that $\zeta^m \in \langle q \rangle$.

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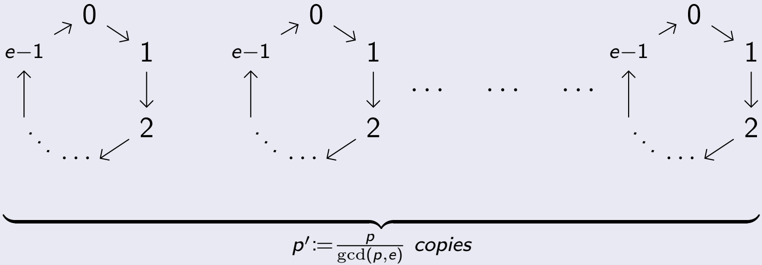

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Our aim is to find an isomorphism $\Phi : \mathrm{H}_n^{\Lambda}(q,\zeta) \to \mathrm{R}_n^{\Lambda}(\Gamma_{e,p})$ such that we get a "nice" automorphism $\Phi \circ \sigma^{\mathrm{H}} \circ \Phi^{-1}$ of $\mathrm{R}_n^{\Lambda}(\Gamma_{e,p})$.

Fixed-point quiver Hecke subalgebra

Let $\sigma : K \to K$ a bijection of finite order p such that:

$$\forall k, k' \in K, k \to k' \implies \sigma(k) \to \sigma(k').$$

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Theorem

The map σ induces a well-defined automorphism of $\mathrm{R}_n(\Gamma)$ by:

$$\begin{aligned} \forall \boldsymbol{k} \in K^n, &\quad \sigma(e(\boldsymbol{k})) := e(\sigma(\boldsymbol{k})), \\ \forall a \in \{1, \dots, n\}, &\quad \sigma(y_a) := y_a, \\ \forall a \in \{1, \dots, n-1\}, &\quad \sigma(\psi_a) := \psi_a. \end{aligned}$$

Definition

We set:

$$\mathrm{R}_n(\Gamma)^\sigma := \{h \in \mathrm{R}_n(\Gamma) : \sigma(h) = h\}.$$

Fixed-point cyclotomic quiver Hecke subalgebra

Theorem

We can give a presentation of $\mathrm{R}_n(\Gamma)^\sigma$ in terms of the following generators:

$$e(\gamma) := e(\boldsymbol{k}) + e(\sigma(\boldsymbol{k})) + \cdots + e(\sigma^{p-1}(\boldsymbol{k})) \text{ for } \gamma = [\boldsymbol{k}] \in K^n/\langle\sigma\rangle,$$

$$y_1, \ldots, y_n,$$

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Exemple of relation

If $\gamma \in K^n/\langle\sigma\rangle$ verifies "$\gamma_a \to \gamma_{a+1}$" then $\psi_a^2 e(\gamma) = (y_{a+1} - y_a)e(\gamma)$.

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We now assume that $\Lambda_k = \Lambda_{\sigma(k)}$ for all $k \in K$.

Theorem

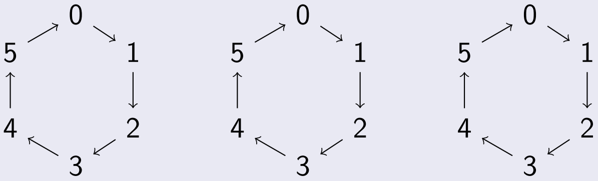
The automorphism σ *induces an automorphism of* $\mathrm{R}_n^{\boldsymbol{\Lambda}}(\Gamma)$*. Moreover:*

$$\mathrm{R}_n^{\boldsymbol{\Lambda}}(\Gamma)^\sigma \simeq \mathrm{R}_n(\Gamma)^\sigma \Big/ \Big\langle y_1^{\Lambda_{\gamma_1}} e(\gamma) = 0 : \gamma \in K^n/\langle\sigma\rangle \Big\rangle .$$

Application to the Hecke algebra of $G(r, p, n)$

Theorem

We can choose Φ *such that the automorphism* $\Phi \circ \sigma^{\mathrm{H}} \circ \Phi^{-1}$ *of* $\mathrm{R}_n^{\Lambda}(\Gamma)$ *comes from the following bijection of the vertices (case* $e := 6, p := 9$ *and* $p' = \frac{9}{\gcd(9,6)} = 3$*):*



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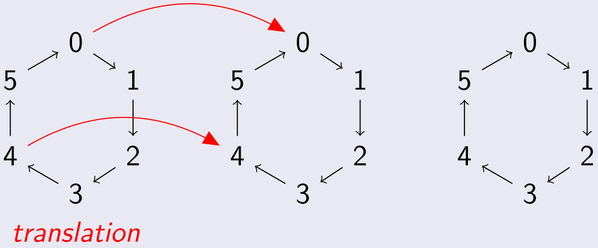

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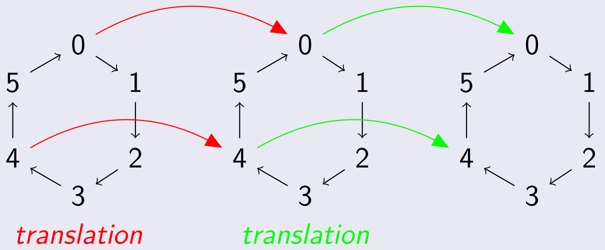

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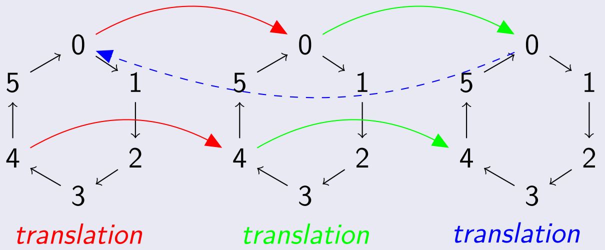

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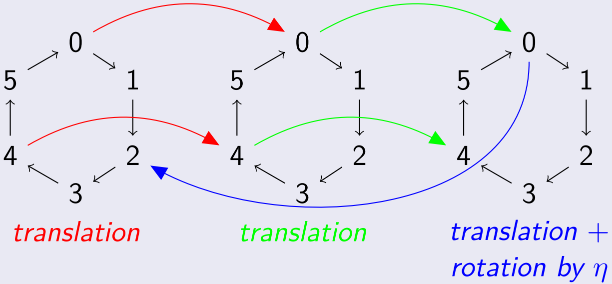

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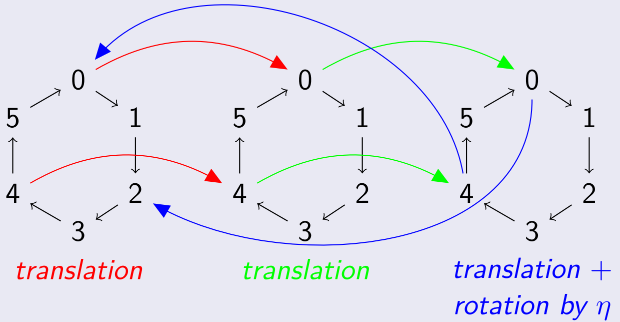

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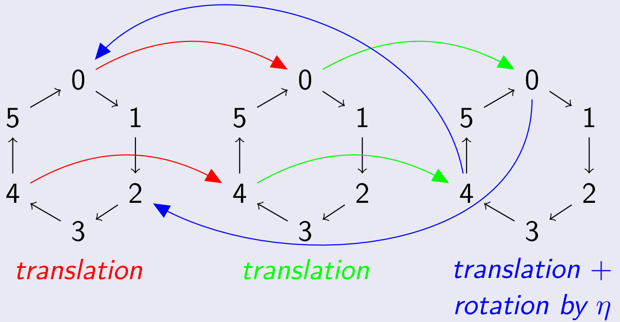

where $\eta \in \mathbb{Z}/e\mathbb{Z}$ *is determined by the equality* $\zeta^{p'} = q^{\eta}$*.*