Stuttering multipartitions

and blocks of Ariki–Koike algebras

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Motivations

Let $\mathcal{H}_n^{\mathrm{X}}$ be a Hecke algebra of type $\mathrm{X} \in \{\mathrm{B}, \mathrm{D}\}$.

- If $\mathcal{H}_n^{\mathrm{B}}$ is semisimple, its irreducible representations are indexed by the *bipartitions* $\{(\lambda, \mu)\}$ of n.

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- In this case, by Clifford theory the irreducible $\mathcal{H}_n^{\mathrm{D}}$-modules are exactly the irreducible summands in the restrictions $\mathcal{D}^{\lambda,\mu}\big\downarrow_{\mathcal{H}_n^{\mathrm{D}}}^{\mathcal{H}_n^{\mathrm{B}}}$. The number of these irreducible summands entirely depends whether $\lambda = \mu$ or $\lambda \neq \mu$.

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The above problem appears when studying the cellularity of $\mathcal{H}_n^{\mathrm{D}}$.

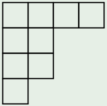
Bipartitions

Definition

A *partition* is a finite non-increasing sequence of positive integers.

We can picture a partition with its *Young diagram*.

Example

The sequence $(4, 2, 2, 1)$ is a partition and its Young diagram is  .

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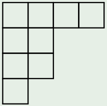

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Definition

A *bipartition* is a pair of partitions.

Example

The pair $\big((5, 1), (2)\big)$ is a bipartition, constructed with the partitions $(5, 1)$ and (2).

Multiset of residues

Let η be a positive integer and set $e := 2\eta$.

Definition

The *multiset of residues* of the bipartition (λ, μ) is the part of

$$\begin{array}{|c|c|c|c} \hline 0 & 1 & 2 & \dots \\ \hline -1 & 0 & 1 & \dots \\ \hline -2 & -1 & 0 & \dots \\ \hline \vdots & \vdots & \vdots & \ddots \end{array} \qquad \begin{array}{|c|c|c|c} \hline \eta & \eta+1 & \eta+2 & \dots \\ \hline \eta-1 & \eta & \eta+1 & \dots \\ \hline \eta-2 & \eta-1 & \eta & \dots \\ \hline \vdots & \vdots & \vdots & \ddots \end{array} \pmod{e},$$

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Example

The multiset of residues of the bipartition $\big((5,1),(2)\big)$ is given for $e = 4$ by

$$\begin{array}{|c|c|c|c|c|}\hline 0 & 1 & 2 & 3 & 0 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline \end{array}.$$

Residues multiplicity and shift

Let $e = 2\eta \in 2\mathbb{N}^*$. If (λ, μ) is a bipartition, write $\alpha(\lambda, \mu) \in \mathbb{N}^e$ for the e-tuple of multiplicities of the multiset of residues.

Example

The multiset of residues of the bipartition $\big((4,2),(1)\big)$ for $e = 6$ is $\begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 3 \\ \hline 5 & 0 \\ \cline{1-2}\end{array}\ \begin{array}{|c|}\hline 3 \\ \hline\end{array}$, thus $\alpha\big((4,2),(1)\big) = (2,1,1,2,0,1)$.

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Definition (Shift)

For $\alpha = (\alpha_i) \in \mathbb{N}^e$, we define $\sigma \cdot \alpha \in \mathbb{N}^e$ by $(\sigma \cdot \alpha)_i := \alpha_{\eta+i}$.

We have $\sigma \cdot \alpha = (\alpha_\eta, \alpha_{\eta+1}, \dots, \alpha_{e-1}, \alpha_0, \alpha_1, \dots, \alpha_{\eta-1})$.

Stutterness

Proposition

We have $\alpha(\mu, \lambda) = \sigma \cdot \alpha(\lambda, \mu)$. *In particular, if* $\alpha := \alpha(\lambda, \lambda)$ *then* $\sigma \cdot \alpha = \alpha$.

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Theorem (R.)

Let (λ, μ) *be a bipartition and let* $\alpha := \alpha(\lambda, \mu) \in \mathbb{N}^{\mathsf{e}}$*. If* $\sigma \cdot \alpha = \alpha$ *then there exists a partition* ν *such that* $\alpha = \alpha(\nu, \nu)$*.*

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Example

Take $e = 6$. The multisets

$$\begin{array}{|c|}\hline 0\\\hline 5\\\hline\end{array}\quad \begin{array}{|c|c|c|}\hline 3&4&5\\\hline 2&3\\\cline{1-2} 1&2\\\cline{1-2} 0\\\cline{1-1}\end{array}, \qquad \text{and} \qquad \begin{array}{|c|c|c|c|}\hline 0&1&2&3\\\hline 5\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|c|}\hline 3&4&5&0\\\hline 2\\\cline{1-1}\end{array},$$

coincide (and $\alpha = (2,1,2,2,1,2)$).

Proof by example

We have $\alpha(\ydiagram{1,1}, \ydiagram{3,2,2,1}) = (2, 1, 2, 2, 1, 2)$.

0	1	2
5	0	
4	5	
3		

3	4	5
2	3	
1	2	
0		

$$\alpha = ({\color{green}3}, 2, 3, {\color{green}3}, 2, 3)$$

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0	1	2
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3		

3	4	5
2	3	
1	2	
0		

$$\alpha = (3, 2, 3, 3, 2, 3)$$

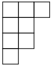
$$\downarrow$$

0	1	2
5	0	
4	**5**	

3	4	5
2	3	
1	**2**	

$$\alpha = (2, 2, 3, 2, 2, 3)$$

Proof by example

We have $\alpha($, $) = (2, 1, 2, 2, 1, 2)$.

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0		

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$\downarrow$ $\downarrow$

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4		

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$$\downarrow \qquad\qquad \downarrow$$

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Failure of the proof by example

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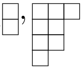
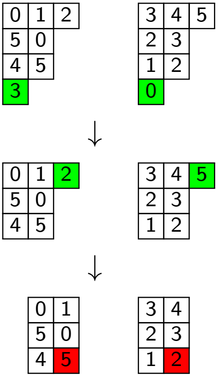
We have $\alpha($$) = (2, 1, 2, 2, 1, 2)$.



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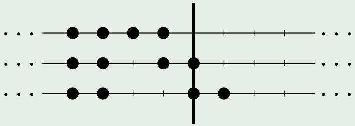
Abaci and cores

To a partition $\lambda = (\lambda_1, \dots, \lambda_h)$, we associate an abacus with e runners such that for each $a \in \mathbb{N}^*$,

> there are exactly λ_a gaps above and on the left of the bead a.

Example

The 3 and 4-abaci associated with the partition $(6, 4, 4, 2, 2)$ are



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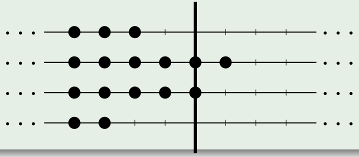

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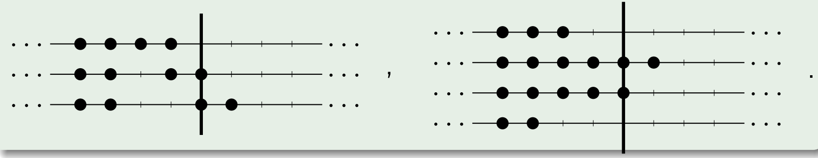


Definition

If no runner of the e-abacus of a partition λ has a gap between its beads, we say that λ is an *e-core*.

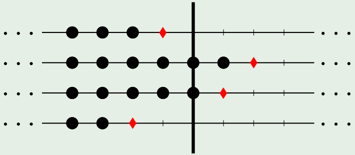
The partition of the above example is not a 3-core but a 4-core.

Parametrisation

To the e-abacus of an e-core λ, we associate the coordinates $x(\lambda) \in \mathbb{Z}^e$ of the first gaps.

Example

For the 4-core $(6, 4, 4, 2, 2)$ we have



,

where each $\color{red}\blacklozenge$ denote a first gap, hence $x = (-1, 2, 1, -2)$.

Using the parametrisation

Proposition

Let λ be an e-core, let $\alpha := \alpha(\lambda) \in \mathbb{N}^e$ be the e-tuple of multiplicities of the multiset of residues and $x := x(\lambda) \in \mathbb{Z}^e$ the parameter of the e-abacus. We have:

$$x_0 + \cdots + x_{e-1} = 0,$$

$$\frac{1}{2}\|x\|^2 = \alpha_0,$$

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Corollary

If $x = x(\lambda)$ and $y = x(\mu)$ then $\alpha_0(\lambda, \mu) = q(x, y)$, where

$$q : \left| \begin{array}{ccc} \mathbb{Q}^e \times \mathbb{Q}^e & \longrightarrow & \mathbb{Q} \\ (x, y) & \longmapsto & \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - y_0 - \cdots - y_{\eta-1} \end{array} \right. .$$

Key lemma

Let (λ, μ) be an e-bicore, define $x := x(\lambda)$ and $y := x(\mu) \in \mathbb{Z}^e$. We assume that $\alpha := \alpha(\lambda, \mu)$ satisfies $\sigma \cdot \alpha = \alpha$ and we want to prove that there exists a partition ν such that $\alpha(\nu, \nu) = \alpha$.

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Lemma

It suffices to find an element $z \in \mathbb{Z}^e$ such that:

$$\begin{cases} q(z, z) \leq q(x, y), \\ z_0 + \cdots + z_{e-1} = 0, \\ z_i + z_{i+\eta} = x_i + y_{i+\eta}, & \text{for all } i. \end{cases} \tag{E}$$

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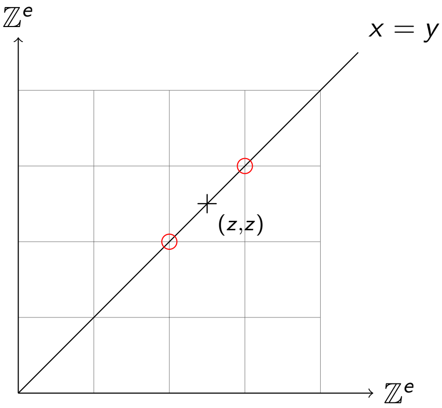
Thanks to the convexity of q, the element $z := \frac{x+y}{2}$ satisfies (E). However, we may have $z \notin \mathbb{Z}^e$: in general $z \in \frac{1}{2}\mathbb{Z}^e$.

First try



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- the constraints are still satisfied
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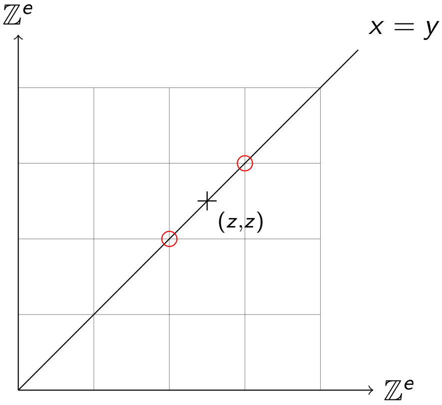

We want to prove that we can choose a red point such that:

- the constraints are still satisfied $\rightarrow$ binary matrices
- estimate the error made $\rightarrow$ strong convexity

End

a	t	t	e	n	t	i	o	n
T	h	a	n	k				
y	o	u	r					
y	o	u						
f	o	r						
!								