Combinatorial representation theory of the symmetric group

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Let $G \coloneqq \mathfrak{S}_n$.

Definition. A **representation** of degree n of G is the data of a k-vector space V of dimension n together with a group homomorphism $\rho : G \to \mathrm{GL}(V)$. It is associated to a linear action $G \curvearrowright V$ (*i.e.* compatible with the action of the scalars), in other words V has a structure of a $k[G]$-module.

Definition. Let (V, ρ) be a representation of G.

— A subrepresentation is a subvector space W that is stable under $\rho(G)$.

— A representation is **irreducible** if it has no nontrivial proper subrepresentation.

Example. — A representation of degree 1 is irreducible.

— Let k be a field and let $n \geq 2$. The action of $\mathfrak{S}_n$ on $\{1, \dots, n\}$ endows k^n with a structure of $\mathfrak{S}_n$-module. The associated representation is reducible since $k(1, \dots, 1)^\top$ and $\{(x_1, \dots, x_n) \in k^n : \sum_{i=1}^n x_i = 0\}$ are nontrivial proper subrepresentations.

Definition. We define $\mathrm{Irr}(G)$ to be the set of irreducible representations, up to isomorphism.

Proposition. *Let $n \geq 2$ and assume that $\mathrm{char}(k) \neq 2$. The symmetric group $\mathfrak{S}_n$ has exactly two irreducible representations of degree 1, namely, the trivial representation and the sign representation.*

Démonstration. We look for the group homomorphisms $\mathfrak{S}_n \to k^\times$. If ρ is such a nontrivial homorphism then it has to take a nontrivial value $x \in k^\times$ on a transposition, with $x^2 = 1$ thus $x = \pm 1$. Now all the transpositions are conjugated in $\mathfrak{S}_n$ thus $\rho(\tau) = x$ for every transposition $\tau \in \mathfrak{S}_n$. The transpositions generate $\mathfrak{S}_n$ thus $x \neq 1$ since ρ is nontrivial thus $x = -1$. Hence ρ is the sign morphism. $\square$

1 Complex representations

Proposition. *The number of (classes of isomorphism of) irreducible complex representations is the number of conjugacy classes in G, moreover :*

$$|G| = \sum_{V \in \mathrm{Irr}(G)} (\dim V)^2.$$

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Remark. In particular, there is a natural probability measure on the set of irreducible representations of G, where $\mathbb{P}(V) := \frac{(\dim V)^2}{|G|}$.

Recall that any permutation $\sigma \in \mathfrak{S}_n$ factors uniquely (up to a reordering of the factors) as a product of cycles $c_1, \dots, c_{h(\sigma)}$ with pairwise disjoint supports. If c_i has length λ_i^σ, we can assume that $\lambda^\sigma = (\lambda_1^\sigma, \dots, \lambda_{h(\sigma)}^\sigma)$ is non-decreasing.

Proposition. *Two permutation $\sigma, \rho \in \mathfrak{S}_n$ are conjugated if and only if $\lambda^\sigma = \lambda^\rho$.*

Démonstration. Follows from the fact that for any permutation $w \in \mathfrak{S}_n$ we have :

$$w(a_1, \dots, a_k)w^{-1} = \Big(w(a_1), \dots, w(a_k)\Big).$$

□

Definition. A non-decreasing sequence of positive integers $\lambda = (\lambda_1 \geq \cdots \geq \lambda_h > 0)$ with sum $|\lambda| := \sum_{i=1}^h \lambda_i = n$ is a **partition** of n.

If $\mathcal{P}_n$ denotes the set of partitions of n, we can thus write $(S^\lambda)_{\lambda \in \mathcal{P}_n}$ for the set of irreducible complex representations of $\mathfrak{S}_n$. We can thus study the following problems :

1. Can we construct S^λ (combinatorially) from the partition λ ?
2. Can we deduce some information (as the dimension as $\mathbb{C}$-vector space) on S^λ from the partition λ ?
3. Any complex representation writes as a sum of irreducible ones. Can we determine the irreducible summands for $S^\lambda \uparrow_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}}$, $S^\lambda \downarrow_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}}$, $S^\lambda \otimes S^\mu$,. . . ?

These questions are of **algebraic combinatorics** nature. Here are some tracks for the aswers.

1. We can, using the notion of **tabloïd** of shape λ : these are the orbits for the action of $\mathfrak{S}_\lambda := \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_h}$ on the set of **tableaux** of shape λ (which are bijection between $\{1, \dots, n\}$ and the Young diagram of λ), the action of $\sigma_k \in \mathfrak{S}_{\lambda_k}$ being given by its action on the k-th row of the tableau.
2. We can construct a $\mathbb{C}$-basis indexed by the **standard tableaux** of shape λ, which are the tableaux with increasing rows (from left to right) and columns (from top to bottom). There is a closed formula for the number of such tableaux : the hooklength formula, where the hooklength of a box is the number of boxes directly below and directly to the right (including the box itself, in the English convention). The formula then reads :
$$\dim S^\lambda = \frac{n!}{\prod_{\gamma \in \mathcal{Y}(\lambda)} h_\gamma}.$$
For instance, for the partition $(4, 1)$ there are 4 standard tableaux (any number except 1 can be on the second row, while then the first row has only one choice), and indeed $\frac{5!}{5 \cdot 3 \cdot 2 \cdot 1 \cdot 1} = 4$. Finally, note that the equality $n! = \sum_{\lambda \in \mathcal{P}_n} \#\mathrm{Std}(\lambda)^2$ can be obtained via the **Robinson–Schensted algorithm** provides, which is an explicit bijection between $\mathfrak{S}_n$ and $\prod_{\lambda \in \mathcal{P}_n} \mathrm{Std}(\lambda)^2$.
3. The induction (resp. restriction) is given by the sum of all the S^μ where μ is a partition obtained by adding (resp. removing) a box to λ. The tensor product is given by the **Littlewood–Richardson rule**, the coefficient $c_{\lambda,\mu}^\nu$ of S^ν (where $|\nu| = |\lambda| + |\mu|$) is given by the the number of skew semi-standard tableau of shape ν/λ of weight μ.

2 Modular representations

We now look at the represention of $G = \mathfrak{S}_n$ over a field of characteristic p. If p is large enough then there is not much difference with the complex case, however for small p the main difference is that some representation may not be written as the sum of irreducible ones.

However, given a $\mathbb{F}_p[G]$-module V, we may look at a **Jordan–Hölder** series of V (i.e. maximal composition series with simple quotients) and look at the simple quotients that occurs. In particular, it is still interesting to look at the irreducible modules.

2.1 Irreducible representations

Proposition (Brauer)**.** *The number of irreducible representations of G over $\overline{\mathbb{F}_p}$ is the number of p-**regular** conjugacy classes, that is, the conjugacy classes whose order is coprime to p.*

For $\mathfrak{S}_n$, the algebraic closure can be forgotten (cf. Specht modules are defined over $\mathbb{Z}$). Moreover, the cardinality of the conjugacy class associated with a partition $\lambda = (\lambda_i)$ is $\mathrm{lcm}(\lambda_i)$, thus the p-regular conjugacy classes are given by the partition with no parts divisible by p. We will be interested by another indexing set.

Definition. A partition $\lambda = (\lambda_1 \geq \ldots \lambda_h > 0)$ is p-**regular** if no part repeats p times or more, that is, if $\lambda_i \neq \lambda_{i+p-1}$ for all $i \leq h - p + 1$.

The next result generalises the fact that there are as many partitions of n into odd parts as partitions of n with distincts part (which is the below result for $p = 2$).

Proposition. *The above two sets of partitions are in bijection.*

Now the constructions of the irreducible modules is a bit more delicat than is characteristic zero. First, we can realise the **complex** representations over $\mathbb{Z}$, that is, the representation S^λ can be constructed via a morphism $\mathfrak{S}_n \to \mathrm{GL}_{d_\lambda}(\mathbb{Z})$. Hence, we can reduce it modulo p and obtain a representation $\overline{S}^\lambda$.

Theorem (James 1976)**.** *One can contsruct a family $\{D^\lambda\}_{\lambda \in \mathcal{P}_n}$ of $\mathbb{F}_p$-representations, where D^λ is a certain quotient of $\overline{S}^\lambda$, such that :*

- *the representation D^λ is non-zero if and only if λ is p-regular ;*
- *the set of these D^λ form a complete family of pairwise non-isomorphic irreducible $\mathbb{F}_p$-representations.*

Now all that is known about the irreducible complex representations is essentially unknown for the irreducible modular representations. . .

2.2 Regularisation

Definition (James 1976)**.** Let λ be a partition. The p-**regularisation** of λ is the partition $\mathrm{reg}_p(\lambda)$ obtained from the Young diagram of λ by moving as much as possible each box in the $(1, p-1)$ direction (1 right step and $p - 1$ up steps).

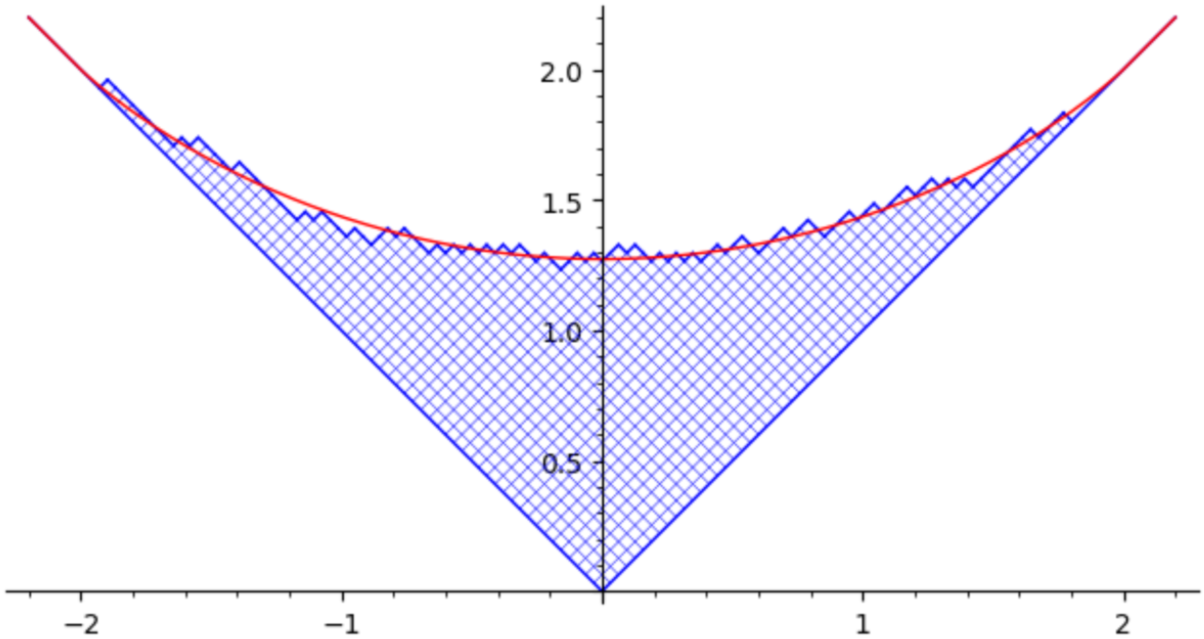

FIGURE 1 – Limit shape for a Young diagram under Pl_{1000}

Note that it is not immediately clear that $\mathrm{reg}_p(\lambda)$ is a partition.

Example. The 3-regularisation of $(2,2,2,1,1,1)$ is $(3,3,2,1)$.

Theorem (James 1976)**.** *Let λ be a partition. The irreducible modular representation $D^{\mathrm{reg}_p(\lambda)}$ appears exactly once is the series of quotients of a Jordan–Hölder series of $\overline{S}^{\lambda}$.*

3 Asymptotics for the Plancherel measure

Recall that the Plancherel measure on the set $\mathcal{P}_n$ of partitions of n is given by $\mathrm{Pl}_n(\lambda) = \frac{\#\mathrm{Std}(\lambda)^2}{n!}$.

3.1 For partitions

Theorem (Kerov–Vershik, Logan–Shepp, 1977)**.** *The upper rim of the rescaled Young diagramm, tilted by $\frac{3\pi}{4}$, of a partition taken under the Plancherel measure, gets closer and closer as $n \to +\infty$ to the curve of the map $\Omega : \mathbb{R} \to \mathbb{R}$ given by :*

$$\Omega(s) := \begin{cases} \frac{2}{\pi}\left(s \arcsin\left(\frac{s}{2}\right) + \sqrt{4-s^2}\right), & \textit{if } |s| \leq 2, \\ |s|, & \textit{otherwise.} \end{cases}$$

In particular, the length of the first row and the first column of λ has magnitude $2\sqrt{n}$.

Remark. The limit shape Ω is the antiderivative on $(-2,2)$ of $s \mapsto \frac{2}{\pi}\arcsin(\frac{s}{2})$.

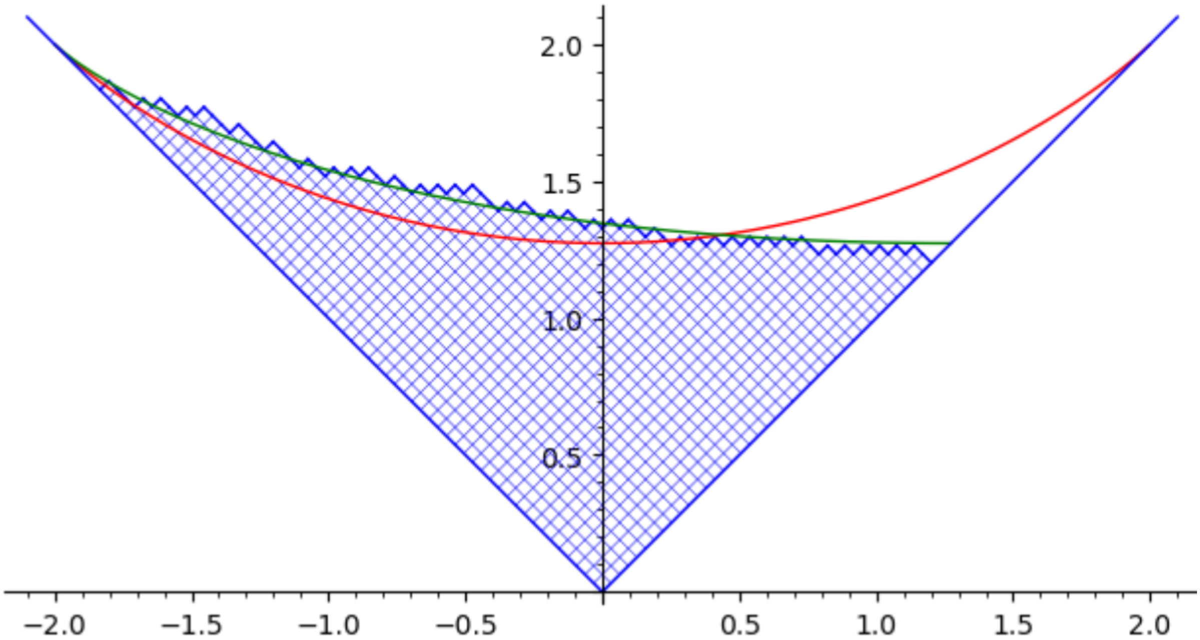

FIGURE 2 – Limit shape (in green) for the 2-regularisation of a partition under Pl_{1000}, with in red the limit shape Ω.

Remark. The problem of determining the lenth of the first row of a partition under the Plancherel measure is known as the **Ulam problem** : via the Robinson–Schensted correspondance, this corresponds to the length of a maximal increasing subsequence of permutation in $\mathfrak{S}_n$ taken uniformly.

This is the **Russian convention** for Young diagrams. We illustrate the above theorem in Figure 1. The result was originally proved by minimising an integral (the "hook integral"), and can also be proved using **central characters** of $\mathfrak{S}_n$.

3.2 For regularisations

Theorem (R. 2023)**.** *The upper rim of the rescaled tilted Young diagram, of the* p-***regularisation*** *of a partition taken under the Plancherel measure, gets closer and closer as* $n \to +\infty$ *to a curve* Ω_e*, which is defined implicitly from* Ω*. For intance, for* $p = 2$ *one has :*

$$\begin{aligned} \Omega_2\Big(2s + \Omega(s)\Big) &= \Omega(s), && \textit{for } -2 < s < \Omega(0) = \frac{4}{\pi}, \\ \Omega_2(s) &= |s|, && \textit{otherwise.} \end{aligned}$$

The idea behind the proof is to move the move the limit Young diagram along the same direction as the p-regularisation (this is a **shaking** process).

Despite the implicit characterisation of Ω_e, we are still able to recover some information on the length of the first row and column.

Corollary (R. 2023)**.** *If* λ *is taken under* Pl_n *then :*

— *the length of the first row of* $\mathcal{Y}\Big(\mathrm{reg}_e(\lambda)\Big)$ *has magnitude* $2\sqrt{n}$ *;*

— *the length of the first row of* $\mathcal{Y}\Big(\mathrm{reg}_e(\lambda)\Big)$ *has magnitude* $\frac{2e\sqrt{n}}{\pi}\sin\frac{\pi}{e}$.

4 Generalised regularisations

Diego Millan Berdasco studied a generalisation of the p-regularisation.

Definition (Millan Berdasco 2021)**.** Let $i \in \{1, \ldots, p-1\}$. Let λ be a partition. The (p, i)**-regularisation** of λ is the unique partition $\mathrm{reg}_{p,i}(\lambda)$ obtained from the Young diagram of λ by moving each box in the $(i, p-i)$ direction (i right steps and $p-i$ up steps) so that it **dominates** every other partitions obtained via this process.

Note that $\mathrm{reg}_{p,1} = \mathrm{reg}_p$. The dominance order between partitions is obtained by comparing the partial sums : we have $\lambda \trianglelefteq \mu$ if $\sum_{i=1}^k \lambda_i \leq \sum_{i=1}^k \mu_i$ for all k (taking zero parts if k is beyond the number of parts).

The representation theoretic results are not established yet (namely, an (p, i)-regular partition, that is, a partition stable under $\mathrm{reg}_{p,i}$ is not necessarily p-regular !), and neither are the asymptotic ones ! For the asymptotics, the difficulty is that $\mathrm{reg}_{p,i}(\lambda)$ is not obtained via pushing boxes as much as possible ; it seems that the limit shape is random.