Partition combinatorics seasoned with asymptotics

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January 2023
ANR CORTIPOM Workshop, IECL Nancy

Abstract

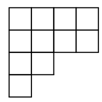
Partitions, or Young diagrams, index the representation of symmetric groups. There are plenty of combinatorial objects, constructed from partitions, that allow a better understanding of these representations. For instance: hooks, with the "hook formula" giving the dimensions of the representations in characteristic zero, and the notion of core, used in the theory of modular representations. The Plancherel measure on partitions has a natural meaning in representation theory, and since a theorem of Kerov–Vershik and Logan–Shepp we know that large partitions, chosen under the Plancherel measure, have a prescribed limit shape. One can then study the following problem: given a (combinatorial) quantity defined on partitions, study its behaviour as the size of the partition goes to infinity. For instance, Kerov–Vershik have studied the largest dimension of the representations, and recently we have studied the size of the cores. We will give other examples of quantities, defined on partitions, which should be interesting to study.

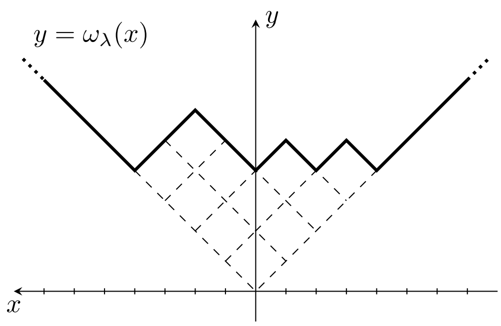
1 Partitions combinatorics

Definition. A *partition* is a finite sequence of decreasing integers $\lambda = (\lambda_1 \geq \cdots \geq \lambda_h > 0)$. We write $|\lambda| \coloneqq \lambda_1 + \cdots + \lambda_h$.

We denote by $\mathcal{P}_n$ the set of partitions of n, that is, partitions λ with $|\lambda| = n$. The *Young diagram* of $\lambda = (\lambda_1 \geq \ldots \lambda_h > 0)$ is the following subset of $\mathbb{Z}^2_{\geq 1}$:

$$\mathcal{Y}(\lambda) \coloneqq \big\{(a, b) \in \mathbb{Z}^2 : 1 \leq a \leq h \text{ and } 1 \leq b \leq \lambda_a\big\}.$$

For instance, the Young diagram of the partition $(4, 4, 2, 1)$ is . Titling the Young diagram by $\frac{3\pi}{4}$ gives the Russian convention (the former being the English one):



The set $\mathcal{P}_n$ of partitions of n parametrises the conjugacy classes of the symmetric group $\mathfrak{S}_n$ on n letters (via the cycle decomposition). In particular, it parametrises the set $\{S^\lambda\}_{\lambda \vdash n}$ of *irreducible complex representations*. This parametrisation can be made such that S^λ has a basis indexed by the set $\mathrm{Std}(\lambda)$ of *standard Young tableaux* of shape λ, that is, bijective maps $\mathfrak{t} : \{1, \dots, n\} \to \mathcal{Y}(\lambda)$ that increase along the rows and down the columns of $\mathcal{Y}(\lambda)$. Then by standard representation theory we obtain that:

$$n! = \sum_{\lambda \vdash n} \#\mathrm{Std}(\lambda)^2.$$

Remark.

- This equality can also be proved via a direct computation on the Young graph, or via the Robinson–Schensted correspondence, which is a bijection between $\mathfrak{S}_n$ and pairs of standard tableaux of same shape .
- The quantity $\#\mathrm{Std}(\lambda)^2$ is *not* the number of partition with conjugacy class λ.

Definition. The *Plancherel* measure Pl_n on $\mathcal{P}_n$ is defined by $\mathrm{Pl}_n(\lambda) := \frac{\#\mathrm{Std}(\lambda)^2}{n!}$.

Example. The measure Pl_4 on $\mathcal{P}_4$ is given by the following table.

λ	Pl_4
(4)	$\frac{1}{24} \approx 0.04$
$(3,1)$	$\frac{9}{24} = 0.375$
$(2,2)$	$\frac{4}{24} \approx 0.17$
$(2,1,1)$	$\frac{9}{24} = 0.375$
$(1,1,1,1)$	$\frac{1}{24} \approx 0.04$

In comparison, the uniform measure on $\mathcal{P}_4$ has value $\frac{1}{5} = 0.2$ on each partition.

2 Asymptotics related to the partition itself

We rescale on the two axes the upper border in the Russian convention ω_λ into $\widetilde{\omega}_\lambda$ such that each box has area $\frac{2}{|\lambda|}$.

Definition. Let $\Omega : \mathbb{R} \to \mathbb{R}$ be defined by $\Omega(s) := |s|$ if $|s| \geq 2$ and by:

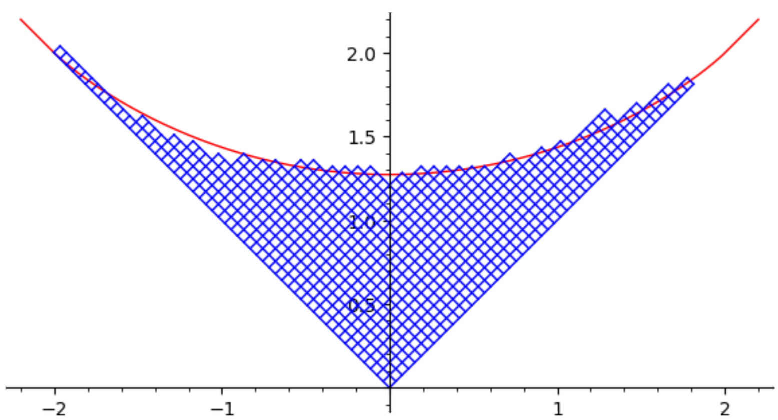
$$\Omega(s) := \frac{2}{\pi}\left(s \arcsin\left(\tfrac{s}{2}\right) + \sqrt{4 - s^2}\right),$$

otherwise.

Note that if $s \in (-2, 2)$ then $\Omega'(s) = 1 - \frac{2}{\pi} \arccos \frac{s}{2}$.

Theorem (Kerov–Vershik [KV77], Logan–Shepp [LoSh], 1977)**.** *Under Pl_n, the function $\widetilde{\omega}_\lambda$ converges uniformly in probability to Ω as $n \to +\infty$, that is, for all $\epsilon > 0$ we have:*

$$\mathrm{Pl}_n\left(\sup_{\mathbb{R}} \left|\widetilde{\omega}_\lambda - \Omega\right| > \epsilon\right) \xrightarrow{n \to +\infty} 0.$$



Remark. Kerov–Vershik [KV85] in fact also studied $M_n := \max_{\lambda \in \mathcal{P}_n} \mathrm{Pl}_n(\lambda)$, proving that M_n decreases to 0 faster than any polynomial, more precisely: there exist positive constants $c_0, c_1 > 0$ such that $e^{-c_0\sqrt{n}} \leq M_n \leq e^{-c_1\sqrt{n}}$. They have also proved a similar result for the asymptotics of the typical dimension: there exist positive constants $c'_0, c'_1 > 0$ such that the set $\mathcal{TP}_n$ of partitions $\lambda \in \mathcal{P}_n$ that satisfy $e^{-c'_0\sqrt{n}} \leq \mathrm{Pl}_n(\lambda) \leq e^{-c'_1\sqrt{n}}$ satisfy $\mathrm{Pl}_n(\mathcal{TP}_n) \xrightarrow{n\to+\infty} 1$. Finally, these typical diagrams also converge to the limit shape Ω, more precisely, $\max\big\{\|\widetilde{\omega}_\lambda - \Omega\|_\infty : \lambda \in \mathcal{TP}_n\big\} = o\big(n^{-1/6}\big)$.

We also have convergence of the supports, in particular:

$$\mathrm{Pl}_n\left(\left|\frac{\lambda_1}{2\sqrt{n}} - 1\right| > \epsilon\right) \xrightarrow{n\to+\infty} 0,$$

which answers the Ulam's problem about the length of a longest increasing subsequence of a uniformly random chosen permutation in $\mathfrak{S}_n$ (by the Robinson–Schensted correspondence). The proofs of the above results mainly consists in optimising an integral functional (the "hook integral"), obtained from an explicit formula for $\#\mathrm{Std}(\lambda)$ (the "hook length formula").

The story for higher order asymptotics is a bit more involved. Baik-Deift-Johansson [BDJ99] proved that the random variable $n^{-1/6}\big(\lambda_1 - 2\sqrt{n}\big)$ converges in probability in distribution to the *Tracy–Widom* distribution, that is:

$$\mathrm{Pl}_n\left(n^{-1/6}\big(\lambda_1 - 2\sqrt{n}\big) \leq t\right) \xrightarrow{n\to+\infty} F(t),$$

for all $t \in \mathbb{R}$, where F is the Tracy–Widom c.d.f.. A key point in the proof is the resolution of a Riemann–Hilbert problem. Note also that the Tracy–Widom distribution first arose in the random matrix theory, as the limiting distribution of the rescaled largest eigenvalue of a random GUE matrix. In fact the same hold for λ_i and the i-th largest eigenvalue: Baik-Deift-Johansson [BDJ00] proved the case $i = 2$ and the general case was proved independently by Borodin–Okounkov-Olshanski [BOO] (via a determinantal point processus — we will go back on this later), Johansson [Joh] (via discrete orthogonal polynomial ensembles) and Okounkov [Ok] (via ramified covering of the sphere).

3 Cores of partitions

3.1 Descent set

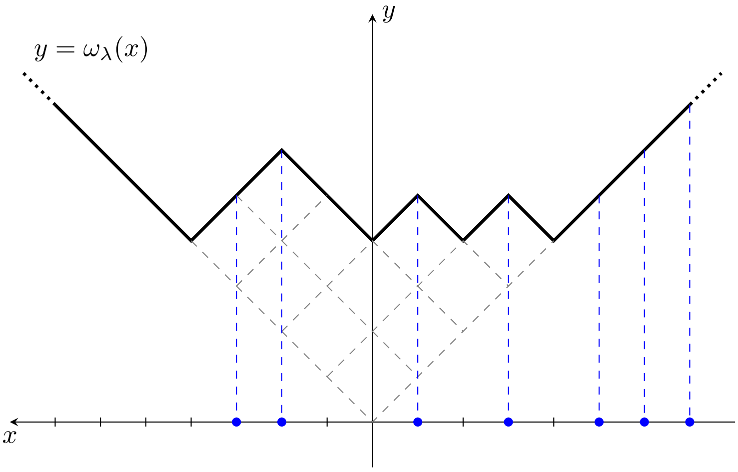
The proof of [BOO] that we mentioned uses the notion of *descent set* for partitions.

Definition. The *descent set* of the partition $\lambda = (\lambda_i)_{i\geq 1}$ is:

$$\mathcal{D}(\lambda) := \{\lambda_i - i : i \geq 1\} \subseteq \mathbb{Z}.$$

In other words, we have $h \in \mathcal{D}(\lambda)$ if and only if there is a descent in the Young diagram of λ in the Russian convention between the x-coordinates h and $h+1$.

For instance, for $\lambda = (4,4,2,1)$ we have $\mathcal{D}(\lambda) = \{3, 2, -1, -3, -5, -6, -7, \dots\}$:



We will now state the result of BOO; we first need two more definitions. Let $t \in \mathbb{R}_{>0}$.

- The *discrete Bessel kernel* $\mathcal{J}^t$ is:

$$\mathcal{J}^t(x,y) := \sqrt{t}\frac{J_x J_{y+1} - J_{x+1} J_y}{x-y}(2\sqrt{t}),$$

  for $x, y \in \mathbb{R}$, where J_z is the Bessel function of the first kind of order $z \in \mathbb{R}$.

- The *Poissonised* Plancherel measure pl_t on the set of *all* partitions $\mathcal{P} := \sqcup_{n\geq 0}\mathcal{P}_n$ is given by first choosing n by a Poisson random variable with parameter t and then choosing a partition inside $\mathcal{P}_n$ via Pl_n, in other words:

$$\mathrm{pl}_t(\lambda) = e^{-t}\sum_{n\geq 0}\frac{t^n}{n!}\mathrm{Pl}_n(\lambda) = e^{-t}t^{|\lambda|}\left(\frac{\#\mathrm{Std}(\lambda)}{|\lambda|!}\right)^2.$$

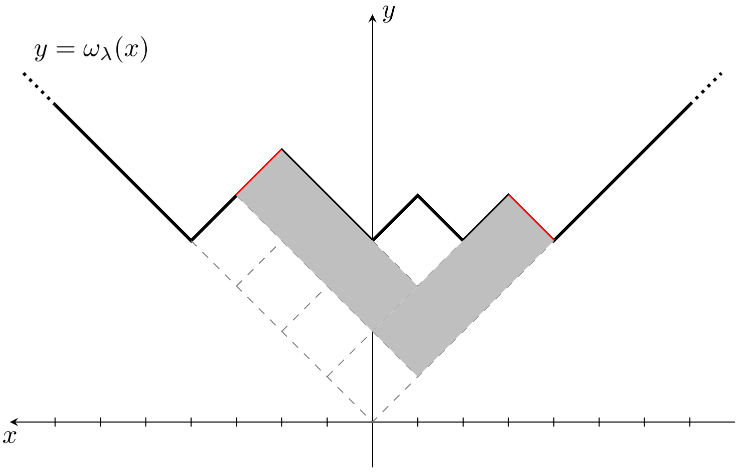
Theorem (Borodin-Okounkov-Olshanski 2000)**.** *Let $x_1, \dots, x_s \in \mathbb{Z}$ be distinct. The associated correlation function for the descent set are a determinantal point process with kernel $\mathcal{J}^t$, that is:*

$$\mathrm{pl}_t\big(x_1, \dots, x_s \in \mathcal{D}(\lambda)\big) = \det\big[\mathcal{J}^t(x_a, x_b)\big]_{1\leq a,b\leq s}.$$

3.2 Core

The notion of *core* for a partition, more precisely, of e-core for $e \geq 2$, has a natural interpretation in terms of representation theory. For instance, if $e = p$ is prime, then the two representations S^λ and S^μ are in the same block of $\mathbb{F}_p\mathfrak{S}_n$ (indecomposable two-sided ideals; they are interesting to study since irreducible modules no longer suffice to study the representations) if and only if they share the same p-core.

We first define the notion of h-*hook* for a partition λ. In the Russian convention, a h-hook is just a the data of a box γ and all the boxes of the Young diagram that are on the "V"-shape where γ is the bottom of the "V", such that the corresponding V has exactly h boxes. For instance, here is the (6−)hook corresponding with the box standing on $(-1, 1)$:



Now we can remove this hook and let gravity acting by moving down the remaining boxes that are above the hook. It should be clear that the descent set remains almost the same, the only difference being that the direction of the red segments changes. More precisely, we have the following result.

Proposition. *The partition λ has an h-hook if and only if there exists $b \in \mathcal{D}(\lambda)$ with $b - e \notin \mathcal{D}(\lambda)$. Moreover, the partition μ that we obtain by removing the h-hook satisfies $\mathcal{D}(\mu) = \big(\mathcal{D}(\lambda) \setminus \{b\}\big) \sqcup \{b - e\}$.*

Thus, decreasing by e each possible element of $\mathcal{D}(\lambda)$ gives a uniquely defined descent set, which corresponds to a partition $\overline{\lambda}$. By the above proposition, this partition λ is obtained by successively removing all the e-hooks from λ.

Definition. We say that $\overline{\lambda}$ is the *e-core* of λ.

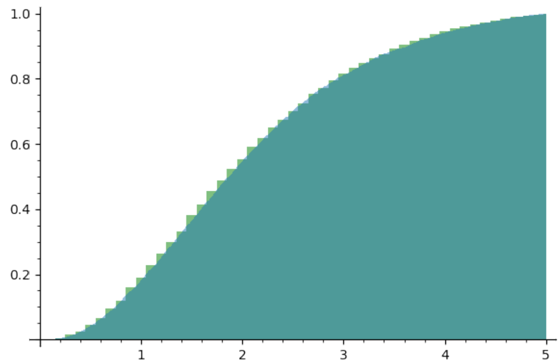
Example. We our previous example $\lambda = (4, 4, 2, 1)$, the 6-core of λ is $\overline{\lambda} = (4, 1)$ (obtained after removing only one 6-hook).

It should thus be possible to study $\overline{\lambda}$ from the knowledge of $\mathcal{D}(\lambda)$. Indeed, by this approach we can obtain the following result.

Theorem ([Ro])**.** *Let $e \geq 2$. Under the Poissonised Plancherel measure pl_t, the size of the e-core of λ converges in distribution as $t \to +\infty$ to a sum of independent Gamma distributions $\Gamma\big(\frac{1}{2}, \sin\frac{k\pi}{e}\big)$ for $k \in \{1, \dots, e-1\}$.*

Remark. Here $\Gamma\big(\frac{1}{2}, \sin\frac{k\pi}{e}\big)$ is obtained as the square of a centred Gaussian distribution of variance $\frac{1}{2}\sin\frac{k\pi}{e}$.

Here is an illustration for $e = 7$ and $n = 3000$:



4 What next?

There exist many other representation theoretic quantities defined for partitions. They should also be interesting to study when the size of the partition goes to infinity.

4.1 Regularisation map

For $e \geq 2$, consider the following labelling of $\mathbb{Z}^2_{\geq 1}$:

$$\begin{array}{|c|c|c|c|c|}
\hline
1 & e & 2e-1 & 3e-2 & \dots \\
\hline
2 & e+1 & 2e & \vdots & \ddots \\
\hline
3 & e+2 & 2e+1 & \vdots & \ddots \\
\hline
\end{array}.$$

In other words, we place the elements of $\mathbb{Z}_{\geq 1}$ in the first column, and then we add $e-1$ when we go right. For instance, for $e = 3$ we have the following (partial) labelling:

$$\begin{array}{|c|c|c|c|}
\hline
1 & 3 & 5 & 7 \\
\hline
2 & 4 & 6 & 8 \\
\hline
3 & 5 & 7 & 9 \\
\hline
\end{array}.$$

Definition.
- A box labelled by i belong to the *i-th ladder*.
- If λ is a partition, its *e-regularisation* is the partition $\mathrm{reg}_e(\lambda)$ obtained from the the Young diagram of λ by moving each box as high as possible in its ladder.

If λ is a partition then the partition $\mathrm{reg}_e(\lambda)$ is *e-regular*, that is, no e-parts of λ are equal. For instance, the partition $(2, 2, 2, 2, 1)$ is not 3-regular but $(2, 2, 1, 1)$ is. If $e = p$ is prime, then p-regular partitions of n parametrise the irreducible representations of $\mathbb{F}_p\mathbb{S}_n$ (*i.e.* irreducible representations of $\mathfrak{S}_n$ over $\mathbb{F}_p$) (with a similar statement for the Iwahori–Hecke algebra when e is not prime.

Remark. Where does regularisation come from? In characteristic p, one can still construct a set $\{S^\lambda\}_{\lambda\vdash n}$ of representations of $\mathbb{S}_n$ over $\mathbb{F}_p$, but now some of these representations are not irreducible. However, one can construct from this set a complete set of irreducible representations $\{D^\lambda\}_{\lambda\vdash_p n}$, pairwise non isomorphic, where $\vdash_p$ means "p-regular partitions". Looking at the Jordan–Hölder decomposition for S^λ (a sort of "decomposing into irreducible parts" that is valid whatever is the characteristic), James [Ja] proved that we will find the irreducible $\mathbb{F}_p\mathbb{S}_n$-module $D^{\mathrm{reg}_e(\lambda)}$ (and only once). And this is interesting since we don't know that much about the irreducible modules of $\mathbb{S}_n$ is positive characteristic!

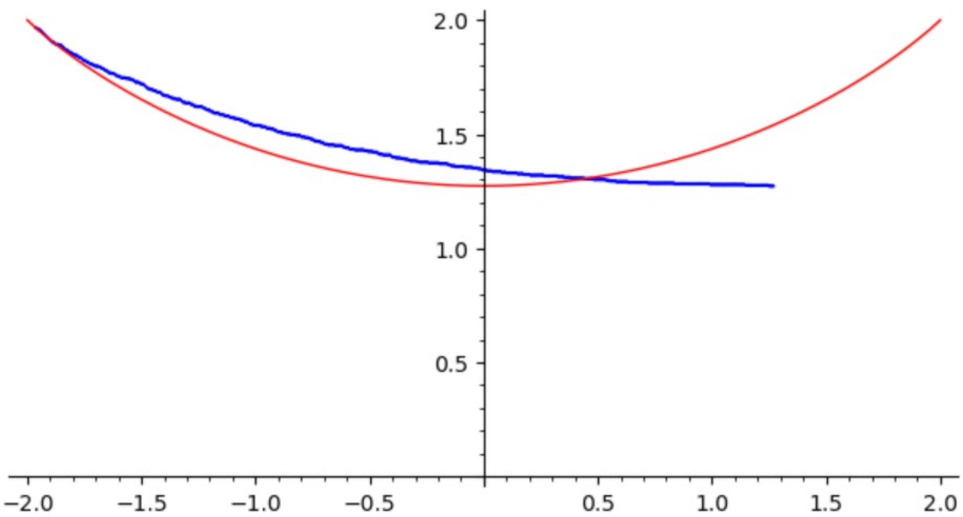
Example. We compute the 3-regularisation of $\lambda = (4, 3, 3, 3, 2, 1, 1, 1, 1)$. We first compute the ladders (in red, the ladder numbers corresponding to the partition):

$$\begin{array}{|c|c|c|c|c|}
\hline
\color{red}{1} & \color{red}{3} & \color{red}{5} & \color{red}{7} & 9 \\
\hline
\color{red}{2} & \color{red}{4} & \color{red}{6} & 8 \\
\cline{1-4}
\color{red}{3} & \color{red}{5} & \color{red}{7} & 9 \\
\cline{1-4}
\color{red}{4} & \color{red}{6} & \color{red}{8} \\
\cline{1-3}
\color{red}{5} & \color{red}{7} & 9 \\
\cline{1-3}
\color{red}{6} & 8 \\
\cline{1-2}
\color{red}{7} & 9 \\
\cline{1-2}
\color{red}{8} \\
\cline{1-1}
\color{red}{9} \\
\cline{1-1}
\end{array}$$

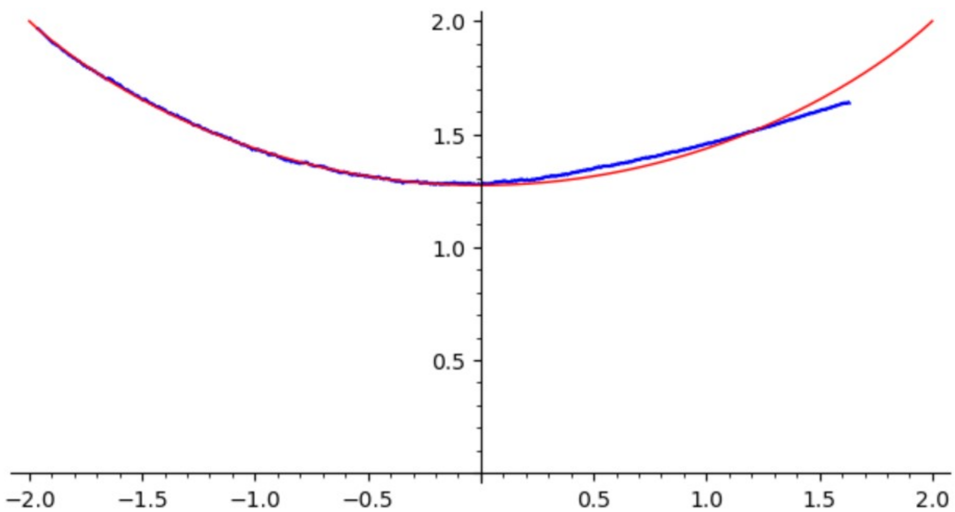
Only the ladders 8 and 9 are not upper-justified. We only have to move a box in the 8-th ladder and one in the 9-th to obtain $\mathrm{reg}_3(\lambda) = (5, 4, 3, 3, 2, 1, 1)$, which is 3-regular indeed (in blue the new boxes):

1	3	5	7	9
2	4	6	8	
3	5	7	9	
4	6	8		
5	7	9		
6	8			
7	9			
(8)				
(9)				

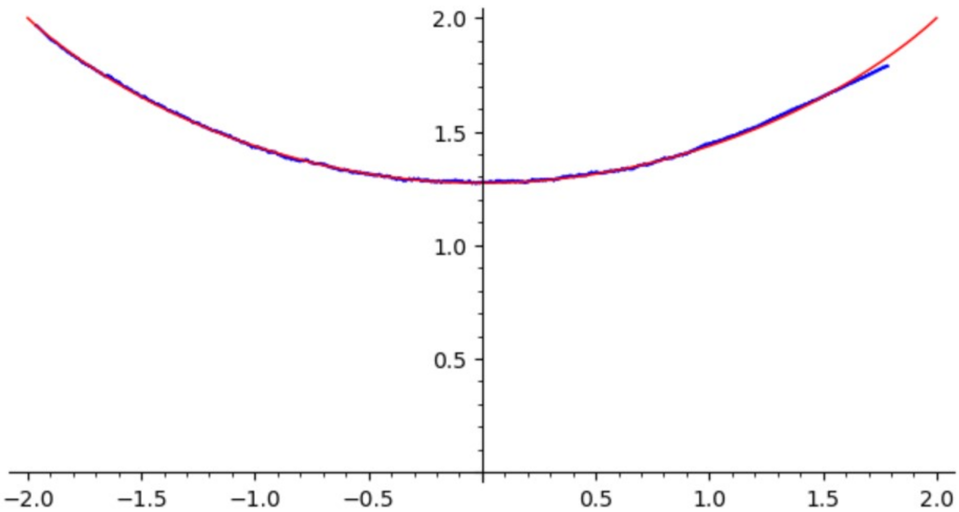
Now recall that we are interested in large partitions. Under the Plancherel measure, here is what the e-regularisation look like (in red the arcsin shape, in blue the regularisation of a large partition), for $e = 2$:



for $e = 3$:



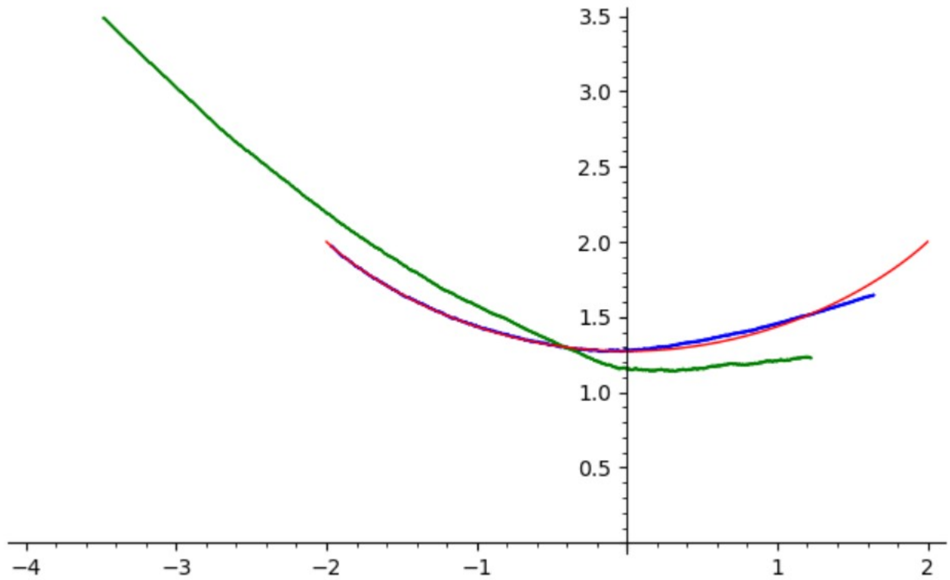
and for $e = 4$:



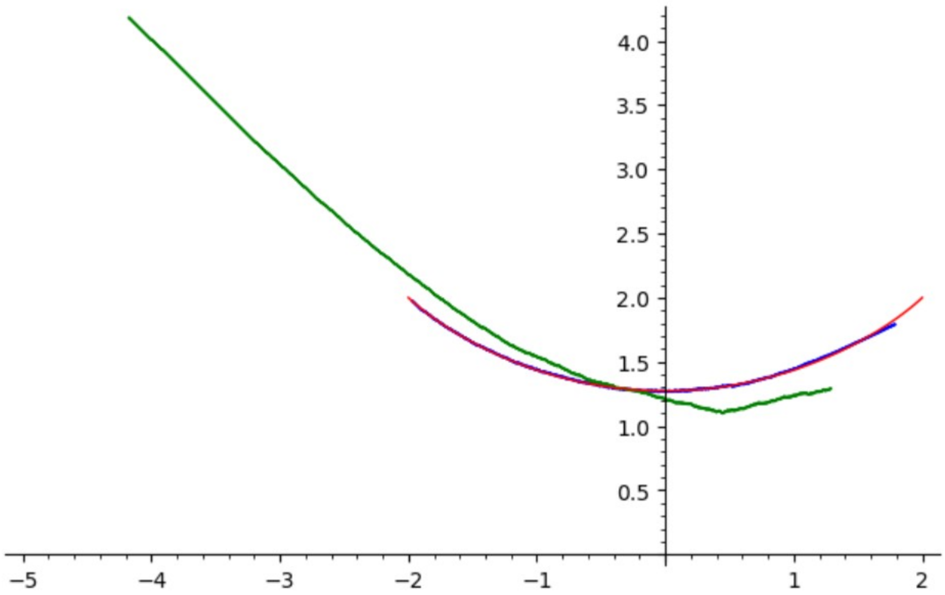
4.2 Mullineux involution

If $p \geq 3$ and $\{D^\lambda\}_{\lambda \vdash_p n}$ is the set of irreducible representations of $\mathfrak{S}_n$ over $\mathbb{F}_p$, indexed by p-regular partitions, we define the *Mullineux* image of $\lambda \vdash_p n$ to be the partition $\mathrm{m}_p(\lambda) \vdash_p n$ such that $D^{\mathrm{m}_p(\lambda)} \simeq D^\lambda \otimes \mathrm{sgn}$. Similarly, such an operation is defined for any $e \geq 3$. The Mullineux map has a combinatorial description, first conjectured by Mullineux [Mu] in 1979. It looks like removing p-hooks and putting them back. The Mullineux conjecture was first proved by Ford–Kleshchev in 1997. There are also many other descriptions of the Mullineux map, for instance by Kleshchev (crystal combinatorics), Fayers (idem), Bessenrodt–Olsson (crystal combinatorics and residue symbol), Brundan–Kujawa (representation theory of $\mathrm{GL}(m|n)$), Losev (wall-crossing functors in category $\mathcal{O}$).

Since by the last part we should have an asymptotic shape for e-regular partitions, it makes sense to look for an asymptotic shape for their Mullineux image (the Mullineux image of the regularisation of a large partition for the Plancherel measure). For partitions of 10^6 (!), we have (in red the arcsin shape, in blue its regularisation, in green its Mullineux image), for $e = 3$:



and for $e = 4$:



But now the limit shape looks a bit elaborated!

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