Schubert polynomials and quasisymmetric operators

Philippe Nadeau (CNRS & ICJ, Univ Lyon 1)

Joint work with Hunter Spink & Vasu Tewari (Toronto)

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Schubert polynomials

The Schubert polynomials $\mathfrak{S}_w$ form a basis in $\mathbb{Z}[x_1, x_2, ...]$, indexed by permutations.

Ex $(w \in S_3)$

$$\mathfrak{S}_{123} = 1 \qquad \mathfrak{S}_{213} = x_1 \qquad \mathfrak{S}_{132} = x_1 + x_2 \qquad \mathfrak{S}_{321} = x_1^2 x_2 \qquad \mathfrak{S}_{312} = x_1^2 \qquad \mathfrak{S}_{231} = x_1 x_2$$

$\Rightarrow$ Positive coefficients, and rich combinatorics.

$\Rightarrow$ Contain the Schur polynomials $s_\lambda(x_1, \cdots, x_n)$ as special cases.

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Origin: $\mathfrak{S}_w$ encodes the (Chow/cohomology) class of the Schubert subvariety X_w (inside the full flag variety). (Lascoux-Schützenberger)

Consequence: write

$$\mathfrak{S}_u \mathfrak{S}_v = \sum_w c_{u,v}^w \mathfrak{S}_w$$

$c_{u,v}^w$ = triple intersection number of Schubert varieties.

$\Rightarrow c_{u,v}^w \geq 0$, but no known combinatorial proof.

Quasisymmetric polynomials

Fix $n \geq 1$, and let $f \in \mathrm{Pol}_n := \mathbb{Q}[x_1, \ldots, x_n]$.

f is symmetric $\Leftrightarrow f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = f(x_1, \ldots, x_n)$ for all $\sigma \in S_n$

$\Leftrightarrow$ For all $a_1, \ldots, a_k > 0$, Coeff of $x_1^{a_1} \cdots x_k^{a_k}$ = Coeff of $x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}$ in f whenever $i_1, \ldots, i_k$ are pairwise distinct.

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For $n = 2$, $f = x_1^2 x_2$.
For $n = 3$, $f = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3$.

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Motivation(s)

- Introduced in Stanley's thesis (1970), explicitly identified by Gessel (1984)
  They are the natural setting for certain generating functions for posets.
- Terminal object in a certain category of Hopf algebras.
- Active topic of research: create bases that refine symmetric bases, and expand (quasi)symmetric functions in these bases,...

Outline of the talk

1. Classical case (symmetric)

Space Sym_n of symmetric polynomials in $x_1, \dots, x_n$

$\Rightarrow$ Defined by the vanishing of divided difference operators ∂_i

$\Rightarrow$ Which in turn characterize by "duality" the family of Schubert polynomials $\mathfrak{S}_w$.

$$\mathrm{Sym}_n \longrightarrow \partial_i \longrightarrow \mathfrak{S}_w$$

Combinatorics: Permutations

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2. New case (quasisymmetric)

Space QSym_n of quasisymmetric polynomials in $x_1, \dots, x_n$.

$\Rightarrow$ Defined by the vanishing of trimming operators T_i.

$\Rightarrow$ Which in turn characterize by "duality" the family of Forest polynomials $\mathfrak{P}_F$.

$$\mathrm{QSym}_n \longrightarrow \mathrm{T}_i \longrightarrow \mathfrak{P}_F$$

Combinatorics: Plane binary forests

Further reading: arXiv:2406.01510

Table 1: Comparing the symmetric and m-quasisymmetric stories

§		${}^m\mathrm{QSym}_n$	Sym_n
2	**Divided differences**	T_i^m	∂_i
3	**Indexing combinatorics**	$F \in \mathsf{For}^m$	$w \in S_\infty$
		Fully supported forests For_n^m	S_n
		Forest code $\mathsf{c}(F)$	Lehmer code $\mathsf{lcode}(w)$
		Left terminal set $\mathsf{LTer}(F)$	Descent set $\mathrm{Des}(w)$
		F/i for $i \in \mathsf{LTer}(F)$	ws_i for $i \in \mathrm{Des}(w)$
		Trimming sequences $\mathrm{Trim}(F)$	Reduced words $\mathrm{Red}(w)$
		Zigzag forests $Z \in \mathsf{ZigZag}_n^m$	Grassmannian permutations λ
4	**Monoid**	m-Thompson monoid	nilCoxeter monoid
5	Pol-**basis**	Forest polynomials $\mathfrak{P}_F$	Schuberts $\mathfrak{S}_w$
	Composites	$\mathsf{T}_F^m = \mathsf{T}_{i_1}^m \cdots \mathsf{T}_{i_k}^m$ for $\mathbf{i} \in \mathrm{Trim}(F)$	$\partial_w = \partial_{i_1} \cdots \partial_{i_k}$ for $\mathbf{i} \in \mathrm{Red}(w)$
6	Pol_n-**basis**	$\{\mathfrak{P}_F \mid \mathsf{LTer}(F) \subset [n]\}$	$\{\mathfrak{S}_w \mid \mathrm{Des}(w) \subset [n]\}$
	Duality	$\mathrm{ev}_0\, \mathsf{T}_F^m \mathfrak{P}_G = \delta_{F,G}$	$\mathrm{ev}_0\, \partial_w \mathfrak{S}_{w'} = \delta_{w,w'}$
7	**Positive expansions**	$\mathfrak{P}_F \mathfrak{P}_H = \sum c_{F,H}^G \mathfrak{P}_G,\ c_{F,H}^G \geq 0$	$\mathfrak{S}_u \mathfrak{S}_w = \sum c_{u,w}^v \mathfrak{S}_v,\ c_{u,w}^v \geq 0$
8	**Invariant basis**	Fundamental m-qsyms $\mathfrak{P}_Z$	Schur polynomials s_λ
9	**Coinvariant basis**	$\{\mathfrak{P}_F \mid F \in \mathsf{For}_n^m\}$	$\{\mathfrak{S}_w \mid w \in S_n\}$
	Coinvariant action	$\mathsf{T}_i^m : {}^m\mathrm{QSCoinv}_n \to {}^m\mathrm{QSCoinv}_{n-m}$	$\partial_i : \mathrm{Coinv}_n \to \mathrm{Coinv}_n$
10	**Harmonic basis**	Forest volume polynomials	Degree polynomials

1. Classical case (symmetric)

Divided differences ∂_i

- Let $\mathrm{Pol}_n = \mathbb{Q}[x_1, \dots, x_n]$ and let $f \in \mathrm{Pol}_n$.

Recall $f \in \mathrm{Sym}_n \Leftrightarrow \sigma \cdot f := f(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = f(x_1, \dots, x_n)$ for all $\sigma \in S_n$.

Pick $\sigma = s_i = (i, i+1)$

Letting $\partial_i = \dfrac{\mathrm{id} - s_i}{x_i - x_{i+1}}$,

$$\mathrm{Sym}_n = \bigcap_{i=1}^{n-1} \ker \partial_i$$

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- Endomorphisms generated by the ∂_i ?

Relations $\partial_i^2 = 0$, $\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}$ and $\partial_i \partial_j = \partial_j \partial_i$ for $|i - j| \geq 2$.

“NilCoxeter monoid” $\simeq$ Permutations w in S_n with product

$$w \cdot w' = ww' \text{ if } \ell(w) + \ell(w') = \ell(ww'), \text{ and } 0 \text{ otherwise.}$$

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$$w \cdot w' = ww' \text{ if } \ell(w) + \ell(w') = \ell(ww'), \text{ and } 0 \text{ otherwise.}$$

- This gives

$w = w' \cdot s_i$ for some $w' \Leftrightarrow i \in \mathrm{Des}(w) = \{i : w(i) > w(i+1)\}$

$w = s_{i_1} \cdot s_{i_2} \cdots s_{i_k} \Leftrightarrow s_{i_1} s_{i_2} \cdots s_{i_k}$ is a reduced expression for w.

$\Rightarrow$ Define ∂_w as the composite $\partial_{i_1} \partial_{i_2} \cdots \partial_{i_k}$.

Schubert polynomials $\mathfrak{S}_w$

$\mathrm{Pol} = \lim_n \mathrm{Pol}_n = \mathbb{Q}[x_1, x_2, ...]$.

$S_\infty = \lim_n S_n$ = $\{$ Permutations w of $\{1, 2, ...\}$ such that $w(i) = i$ for i large enough$\}$.

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Definition-Theorem. The Schubert polynomials $\mathfrak{S}_w$ for $w \in S_\infty$, are the unique family of homogenous polynomials in Pol such that $\mathfrak{S}_{\mathrm{id}} = 1$ and

$$\partial_i \mathfrak{S}_w = \begin{cases} \mathfrak{S}_{ws_i} & \text{if } i \in \mathrm{Des}(w), \\ 0 & \text{otherwise.} \end{cases}$$

Proof Sketch: Pick n such that $w \in S_n$, define $\mathfrak{S}_w = \partial_{w^{-1}w_o^n}(x_1^{n-1}x_2^{n-2}\cdots x_{n-1}^1)$, and check that this does not depend on n. This proves existence, uniqueness is easier. □

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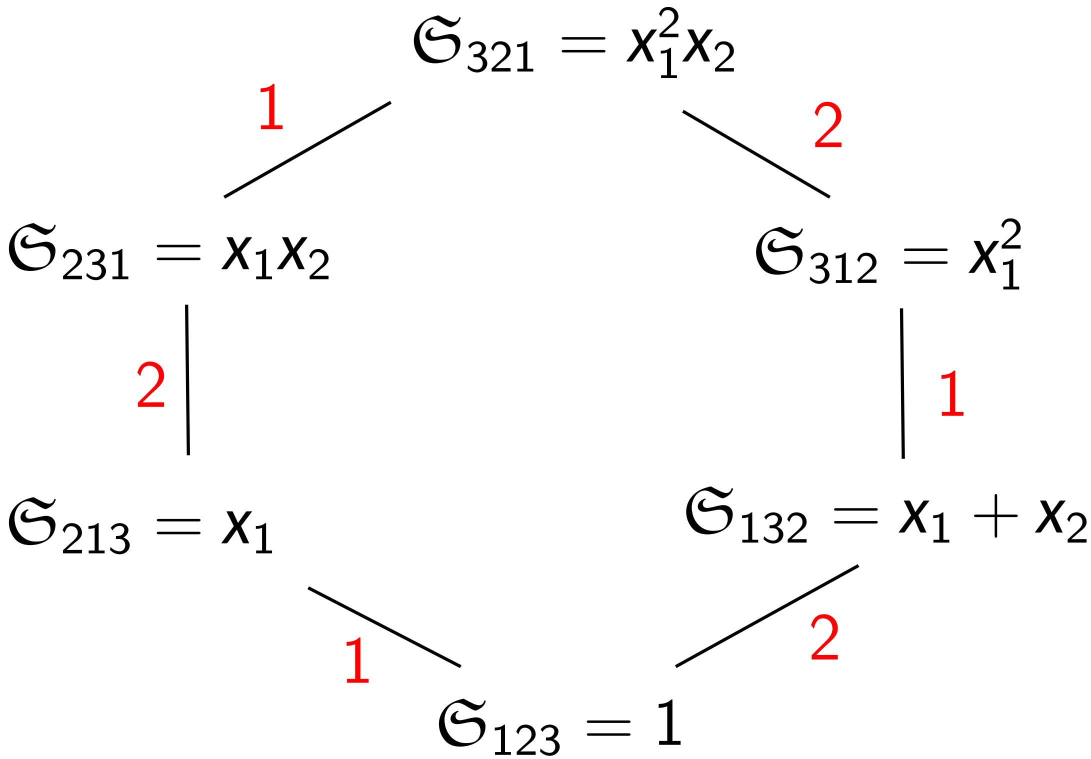
Iterating the equations above gives the following:

Corollary (Duality). For any $w, w' \in S_\infty$,

$$\text{Constant term of } \partial_w(\mathfrak{S}_{w'}) = \begin{cases} 1 & \text{if } w = w' \\ 0 & \text{otherwise.} \end{cases}$$

Back to example



Divisibility for the nilCoxeter monoid $=$ Weak order

What do we get ?

Nice bases of various spaces:

- $\mathfrak{S}_w$ is symmetric in $x_1, \dots, x_n$ if and only w has a unique descent at $i = n$.

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Positivity questions

- From their definition, not clear that they have positive coefficients. This requires some work $\Rightarrow$ Combinatorial interpretation as pipe dreams.
- This approach says very little about the positivity of the general structure coefficients c_{uv}^w.

2. New case (quasisymmetric)

Where are we ?

What we have just seen

$$\mathrm{Sym}_n \longrightarrow (\partial_i)_i \longrightarrow \langle \partial_i \rangle = (\partial_w)_{w \in S_\infty} \longrightarrow \mathfrak{S}_w$$

Combinatorics of permutations

Where we're going

$$\mathrm{QSym}_n \longrightarrow (\mathrm{T}_i) \longrightarrow \langle T_i \rangle = (T_F)_{F \in \mathrm{For}} \longrightarrow \mathfrak{P}_F$$

Combinatorics of plane binary forests

Trimming operators

- **Original approach:** (Hivert, 2000)

Define $\bar{s}_i$ on Pol_n by $\bar{s}_i(\cdots x_i^a x_{i+1}^b \cdots) = \begin{cases} \cdots x_i^a x_{i+1}^b \cdots & \text{if } a, b > 0 \\ s_i(\cdots x_i^a x_{i+1}^b \cdots) & \text{otherwise.} \end{cases}$

Proposition. Let $f \in \mathrm{Pol}_n$. Then $f \in \mathrm{QSym}_n \Leftrightarrow \bar{s}_i(f) = f$ for all $i < n$.

The $\bar{\partial}_i = id - \bar{s}_i$ vanish for $i < n$ on QSym_n.

Problem. The action of the $\bar{s}_i$ & the relations satisfied by the $\bar{\partial}_i$ are not very pleasant.

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- **New approach:** (N.-Spink-Tewari, '24+)

Definition. For $f \in \mathrm{Pol}_n$ and $i < n$, define

$$\mathrm{R}_i(f(x_1, \dots, x_n)) := f(x_1, \dots, x_{i-1}, 0, x_i, x_{i+1}, \dots, x_{n-1})$$

Lemma. $\mathrm{R}_i(f) = \mathrm{R}_{i+1}(f)$ if and only if $\bar{s}_i(f) = f$.

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$$\Rightarrow \quad \mathrm{QSym}_n = \cap_{i=1}^{n-1} \ker \mathrm{T}_i$$

Trimming operators

Explicitly,

$$\mathrm{T}_i(f) = \frac{f(x_1, \dots, x_{i-1}, x_i, 0, x_{i+1}, \dots, x_{n-1}) - f(x_1, \dots, x_{i-1}, 0, x_i, x_{i+1}, \dots, x_{n-1})}{x_i}$$

$$T_1(x_1^a x_2^b) = \begin{cases} 0 & \text{if } ab > 0 \text{ or } a = b = 0 \\ x_1^{a-1} & \text{if } a > 0 \text{ and } b = 0 \\ -x_1^{b-1} & \text{if } b > 0 \text{ and } a = 0. \end{cases}$$

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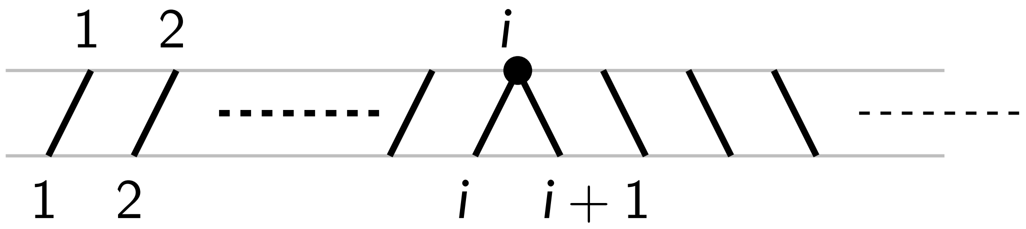
- We now let $n \to \infty$ and thus consider $\mathrm{T}_i : \mathrm{Pol} \to \mathrm{Pol}$.

The T_i satisfy the relations of the Thompson monoid

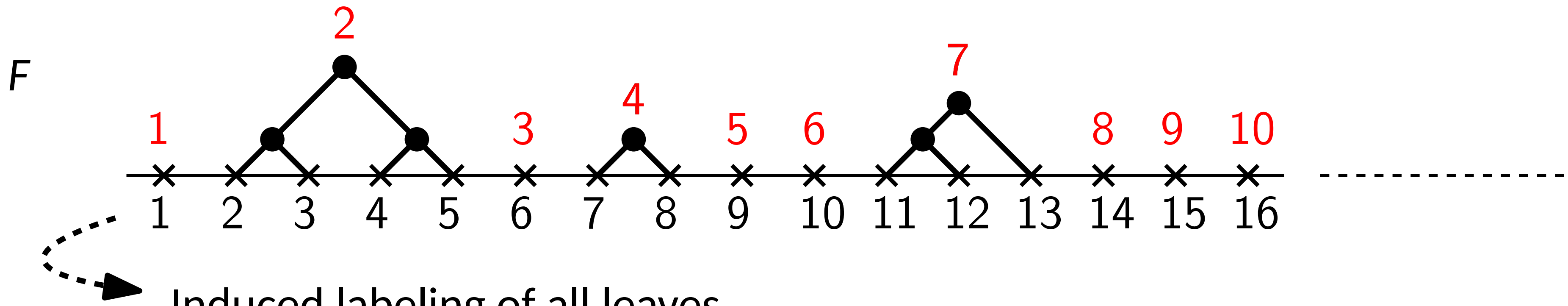
$$\boxed{\mathrm{T}_i \mathrm{T}_j = \mathrm{T}_j \mathrm{T}_{i+1} \text{ if } i > j.}$$

$T_i \Leftrightarrow$



Combinatorics

Definition. An indexed forest *F* is a sequence of plane binary trees, eventually trivial.



Induced labeling of all leaves.

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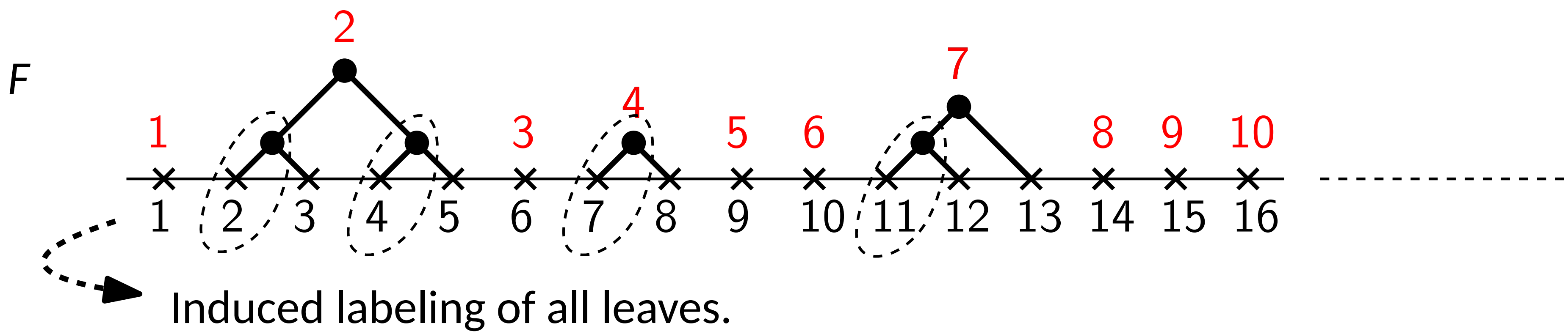

For= set of indexed forests.

- $\mathrm{LTer}(F) =$ the i such that i is the left leaf of a terminal node of F.
  Example $\mathrm{LTer}(F) = \{2, 4, 7, 11\}$ above
- $F \cdot i$ is given by adding a terminal node with left leaf i.
- F/i is the reverse of the above, only defined if $i \in \mathrm{LTer}(F)$.

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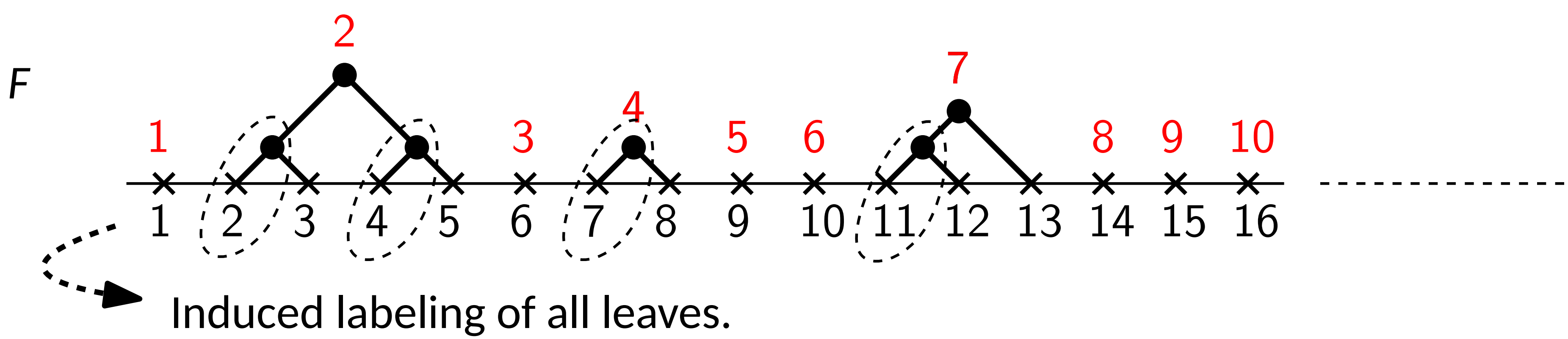

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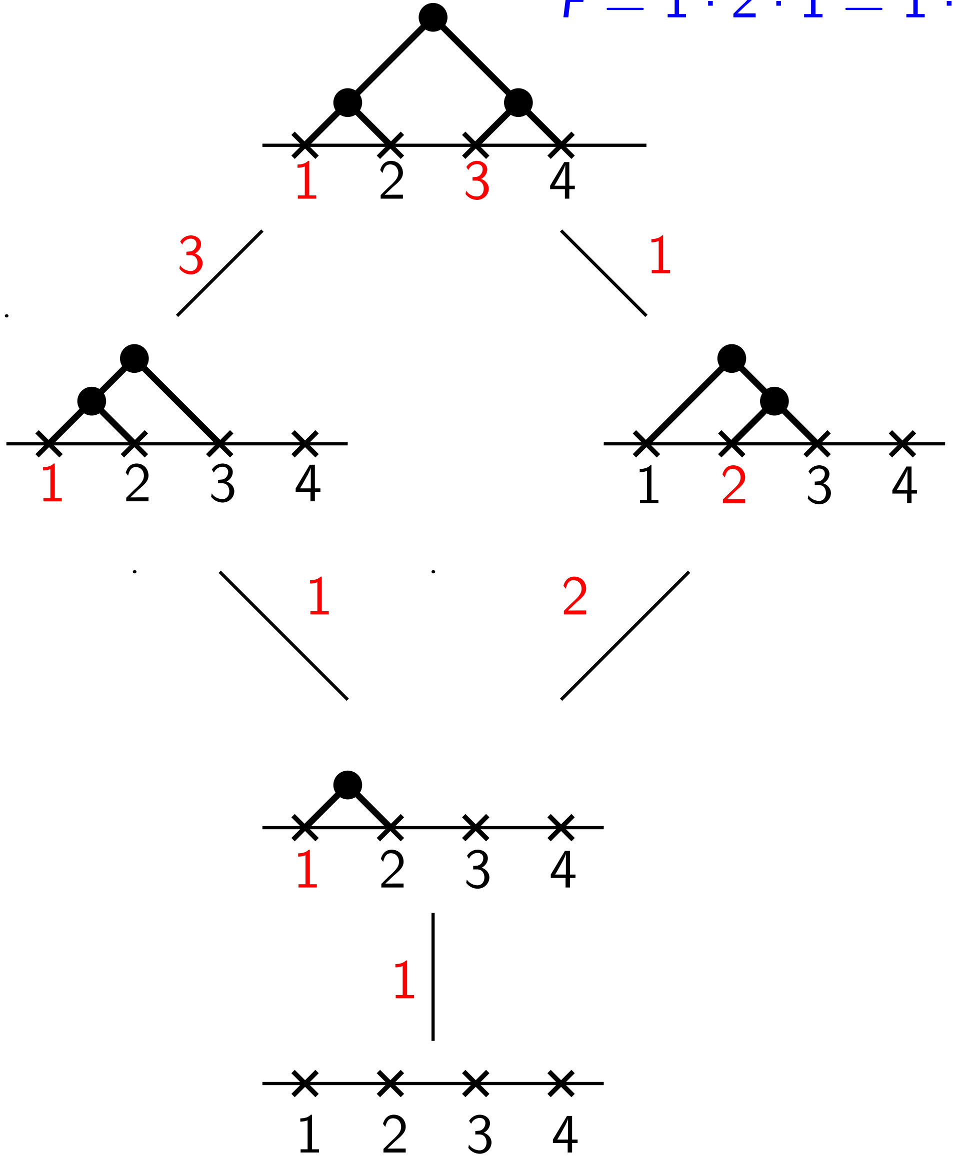
- $\mathrm{LTer}(F)$ = the i such that i is the left leaf of a terminal node of F.
  Example $\mathrm{LTer}(F) = \{2, 4, 7, 11\}$ above
- $F \cdot i$ is given by adding a terminal node with left leaf i.
- F/i is the reverse of the above, only defined if $i \in \mathrm{LTer}(F)$.

Proposition. Define $F \cdot G$ = the forest H obtained by identifying the leaves of F with the roots of G. Then For $\simeq$ Thompson monoid.

$\Rightarrow$ We can define $\mathrm{T}_F = T_{i_1} \cdots T_{i_k}$ by taking any decomposition $F = i_1 \cdots i_k$.

Example

$F = 1 \cdot 2 \cdot 1 = 1 \cdot 1 \cdot 3.$



Forest polynomials

Definition-Theorem The forest polynomials $\mathfrak{P}_F$, $F \in \mathsf{For}$, are the unique family of homogeneous polynomials such that $\mathfrak{P}_\emptyset = 1$ and

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Proof. The uniqueness follows as for Schubert polynomials.
For existence however, we cannot reason similarly, as there is no natural family of "large forests". Instead, we give a direct combinatorial definition in terms of certain colorings of F (omitted here), and check that it works... $\square$

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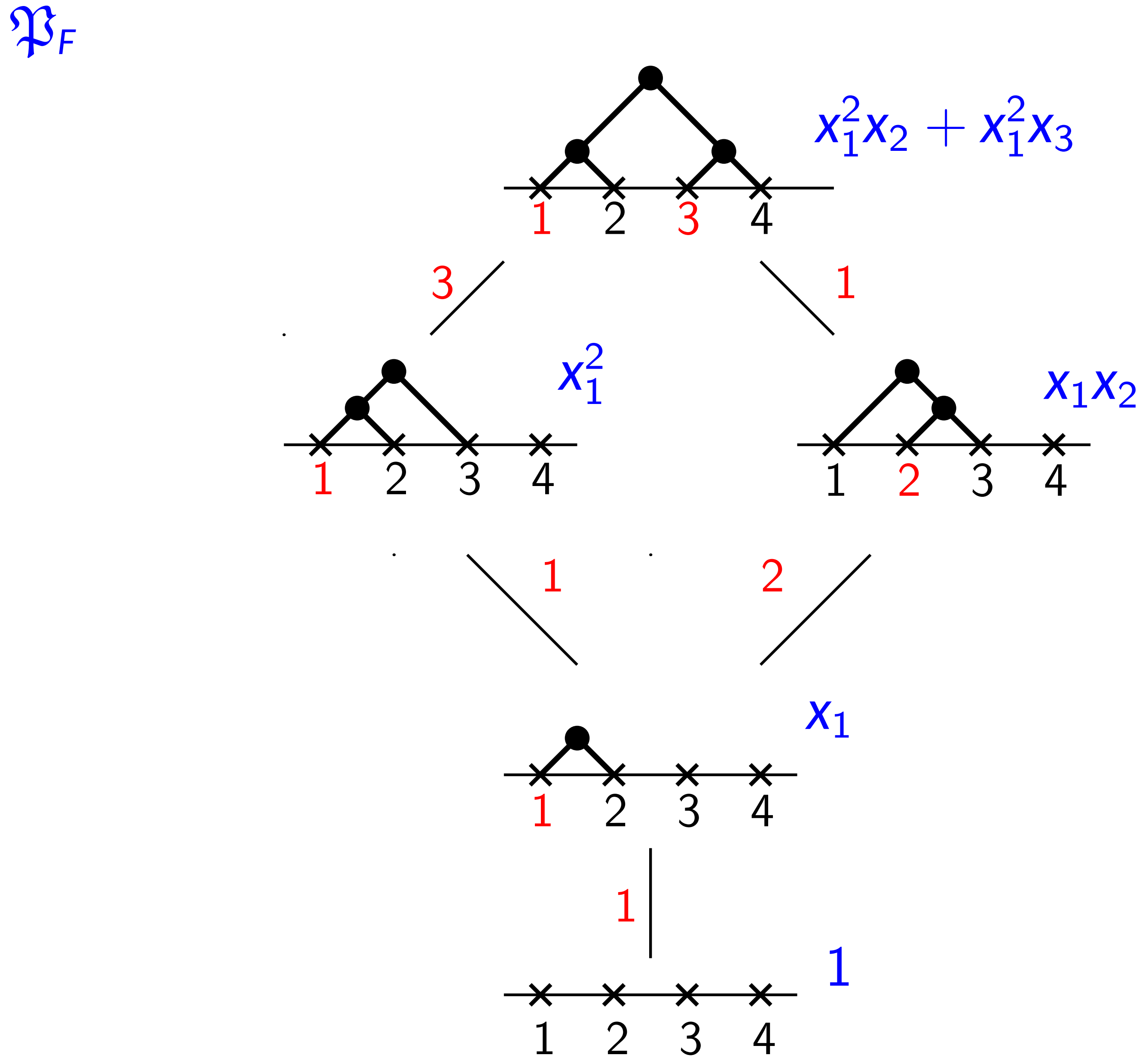
By iteration one gets:

Corollary. (Duality) For $F, G \in \mathrm{For}$, we have

$$\text{Constant term of } \mathrm{T}_F(\mathfrak{P}_G) = \begin{cases} 1 & \text{if } G = F \\ 0 & \text{otherwise.} \end{cases}$$

Back to Example



What do we get ?

Nice bases of various spaces:

- $\mathfrak{P}_F$ is quasisymmetric in $x_1, \ldots, x_n$ if and only F has a unique terminal node at $i = n$.

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Positivity questions

- By their combinatorial definition, the $\mathfrak{P}_F$ have positive coefficients.

- The structure constants $\mathfrak{P}_F\mathfrak{P}_G = \sum_H d_{FG}^H \mathfrak{P}_H$ are positive.
  This can be proved combinatorially.

(**Key**: Leibniz rule $\mathrm{T}_i(fg) = \mathrm{T}_i(f)\mathrm{R}_{i+1}(g) + \mathrm{R}_i(f)\mathrm{T}_i(g)$.)

Bonus: Positivity of Schubert polynomials

A direct check shows:

$$\mathrm{T}_i = \mathrm{R}_i \partial_i$$

Now for $f \in \mathrm{Pol}$ with $f(0) = 0$,

$$\begin{aligned} f &= \sum_{i=1}^{\infty} (\mathrm{R}_{i+1}(f) - \mathrm{R}_i(f)) + \mathrm{R}_1(f) \\ &= \sum_{i=1}^{\infty} x_i \mathrm{T}_i(f) + \mathrm{R}_1(f) = \sum_{i=1}^{\infty} x_i \mathrm{R}_i \partial_i(f) + \mathrm{R}_1(f) \end{aligned}$$

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Choose $f = \mathfrak{S}_w$ with $w \neq \mathrm{id}$

$$\mathfrak{S}_w = \sum_{i \in \mathrm{Des}(w)} x_i \mathrm{R}_i(\mathfrak{S}_{ws_i}) + \mathrm{R}_1(\mathfrak{S}_w).$$

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- This is a **new recurrence**.
- Proves that $\mathfrak{S}_w$ **has positive coefficients.**
- Can be interpreted combinatorially on pipe dreams.