Andrews–Gordon type partition identities and commutative algebra

Frédéric Jouhet

Université Lyon 1

Journées ANR CORTIPOM
Tours, le 2 octobre 2024

(commun avec P. Afsharijoo, J. Dousse, I. Konan et H. Mourtada)

Outline

Integer partitions

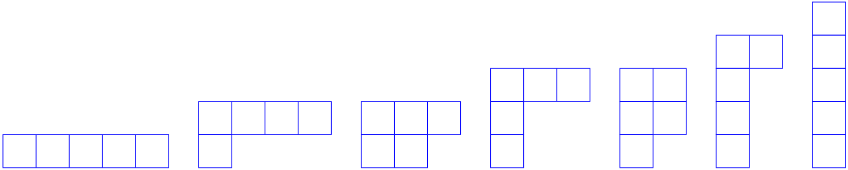
A partition of a non-negative integer n is a non-increasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$.

- The integer $n = |\lambda|$ is the weight of λ.
- The integers λ_k are the parts of λ.
- The integer ℓ is the length of λ.

Example. Partitions of 5 :

$$(5) \quad (4,1) \quad (3,2) \quad (3,1,1) \quad (2,2,1) \quad (2,1,1,1) \quad (1,1,1,1,1)$$

Young diagrams :



Rogers–Ramanujan identities

Theorem (Rogers–Ramanujan, MacMahon, 1916)

Let n be a nonnegative integer and set $i \in \{1;2\}$. Denote by $T_{2,i}(n)$ the number of partitions of n such that the difference between two consecutive parts is at least 2 and the part 1 appears at most $i-1$ times. Let $E_{2,i}(n)$ be the number of partitions of n into parts not congruent to $0, \pm i \mod 5$. Then we have $T_{2,i}(n) = E_{2,i}(n)$.

Example. Among the partitions of 5 :

$$\mathcal{T}_{2,2}(5) = \{(5),(4,1)\} \quad \longleftrightarrow \quad \mathcal{E}_{2,2}(5) = \{(4,1),(1,1,1,1,1)\}$$
$$\mathcal{T}_{2,1}(5) = \{(5)\} \quad \longleftrightarrow \quad \mathcal{E}_{2,1}(5) = \{(3,2)\}$$

Appear in combinatorics (Andrews, Bressoud, Warnaar,...), statistical mechanics (Andrews, Baxter,...), number theory (Ono, Zagier,...), representation theory (Lepowsky, Milne, Wilson,...), algebraic geometry (Mourtada,...),...

Generating functions

Notation : $(a)_n = (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k)$, for $n \in \mathbb{N} \cup \{\infty\}$.

Let $Q(n,k)$ be the number of partitions of n into k distinct parts. Then

$$1 + \sum_{n\geq 1}\sum_{k\geq 1} Q(n,k) z^k q^n = (1+zq)(1+zq^2)(1+zq^3)(1+zq^4)\cdots$$
$$= (-zq)_\infty.$$

Let $p(n,k)$ be the number of partitions of n into k parts. Then

$$1 + \sum_{n\geq 1}\sum_{k\geq 1} p(n,k) z^k q^n = \prod_{n\geq 1}\left(1 + zq^n + z^2q^{2n} + \cdots\right)$$
$$= \frac{1}{(zq)_\infty}.$$

Generating functions for Rogers–Ramanujan identities

If $P_{k,N}(n)$ is the number of partitions of n into parts $\equiv k \mod N$ and if $p_k(n)$ is the number of partitions of n into parts at most k then :

$$\sum_{n\geq 0} P_{k,N}(n)q^n = \frac{1}{(q^k;q^N)_\infty} \quad \text{and} \quad \sum_{n\geq 0} p_k(n)q^n = \frac{1}{(q;q)_k}.$$

Theorem (Rogers–Ramanujan identites, analytic version)

We have

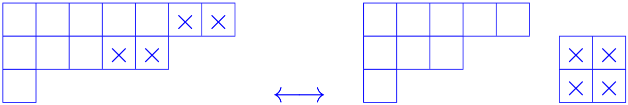
$$\begin{aligned} \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k} &= \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}, \\ \sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(q;q)_k} &= \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \end{aligned}$$

Connecting the analytic and combinatorial versions

The product sides are the generating functions of $E_{2,2}(n)$ and $E_{2,1}(n)$.

As $1 + 3 + \cdots + (2k-1) = k^2$, we have

$$\sum_{n\geq 0} T_{2,2}(n)q^n = \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k}$$



Similarly $2 + 4 + \cdots + 2k = k^2 + k$, so

$$\sum_{n\geq 0} T_{2,1}(n)q^n = \sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(q;q)_k}$$

Generalisation of Rogers–Ramanujan : Gordon's identities

Theorem (Gordon, 1961)

Let r, i be integers with $r \geq 2$, $1 \leq i \leq r$. Denote by $\mathcal{T}_{r,i}$ the set of partitions $\lambda = (\lambda_1, \dots, \lambda_\ell)$ where $\lambda_j - \lambda_{j+r-1} \geq 2$ for all j, and the part 1 appears at most $i-1$ times. Let $\mathcal{E}_{r,i}$ be the set of partitions into parts not congruent to $0, \pm i \mod (2r+1)$.
Let n be a nonnegative integer, and let $T_{r,i}(n)$ (respectively $E_{r,i}(n)$) denote the number of partitions of n which belong to $\mathcal{T}_{r,i}$ (respectively $\mathcal{E}_{r,i}$). Then we have $T_{r,i}(n) = E_{r,i}(n)$.

The Rogers–Ramanujan identities correspond to the cases $r = i = 2$ and $r = i + 1 = 2$, respectively.

Analytic version

Recall $(a)_n := \prod_{j=0}^{n-1}(1-aq^j)$ and write $(a_1,\dots,a_m)_k := (a_1)_k \cdots (a_m)_k$.

Theorem (Andrews–Gordon identities, Andrews, 1974)

Let $r \geq 2$ and $1 \leq i \leq r$ be two integers. We have

$$\sum_{n_1 \geq \cdots \geq n_{r-1} \geq 0} \frac{q^{n_1^2+\cdots+n_{r-1}^2+n_i+\cdots+n_{r-1}}}{(q)_{n_1-n_2}\dots(q)_{n_{r-2}-n_{r-1}}(q)_{n_{r-1}}} = \frac{(q^{2r+1}, q^i, q^{2r-i+1}; q^{2r+1})_\infty}{(q)_\infty}$$

For $r = 2$ we recover the analytic version of the Rogers–Ramanujan identities.

But : how can we see that the left-hand side is the generating series of the set $\mathcal{T}_{r,i}$? Answer by Warnaar (1997) using the multiplicities of partitions $\lambda = 1^{m_1}2^{m_2}\cdots$ and a tricky bijection.

Andrews used a Durfee dissection which does not give the partitions in $\mathcal{T}_{r,i}$.

q-binomial coefficients

For $n \geq 0$, the q-binomial coefficient is defined as :

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q)_n}{(q)_k (q)_{n-k}}.$$

Note that $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$ if $k < 0$ or $k > n$.

It is the generating function for partitions with largest part $\leq k$ and number of parts $\leq n - k$, or equivalently partitions whose Ferrers diagram fits inside a $k \times (n-k)$ rectangle.

The sum-side of the Andrews–Gordon identities can be rewritten as :

$$\sum_{n_1 \geq \cdots \geq n_{r-1} \geq 0} \frac{q^{n_1^2 + \cdots + n_{r-1}^2 + n_i + \cdots + n_{r-1}}}{(q)_{n_1 - n_2} \ldots (q)_{n_{r-2} - n_{r-1}} (q)_{n_{r-1}}}$$

$$= \sum_{n_1 \geq \cdots \geq n_{r-1} \geq 0} \frac{q^{n_1^2 + \cdots + n_{r-1}^2 + n_i + \cdots + n_{r-1}}}{(q)_{n_1}} \begin{bmatrix} n_1 \\ n_1 - n_2 \end{bmatrix}_q \cdots \begin{bmatrix} n_{r-2} \\ n_{r-2} - n_{r-1} \end{bmatrix}_q.$$

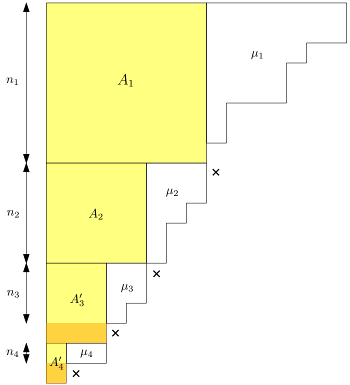
Andrews' Durfee dissection and the set $\mathcal{A}_{r,i}$

Durfee square : largest $A = n \times n$ fitting in the top-left corner of λ.

Vertical Durfee rectangle : largest rectangle $A' = n \times (n+1)$.

Repeat the process until the row below a square/rectangle is empty.

Vertical $(i-1)$-Durfee dissection : the first $i-1$ are squares and all the following ones are rectangles.



$\mathcal{A}_{r,i}$: partitions λ such that in their vertical $(i-1)$-Durfee dissection, all vertical Durfee rectangles below A'_{r-1} are empty, and such that the last row of each non-empty Durfee rectangle is actually a part of λ.

$$\sum_{\lambda \in \mathcal{A}_{r,i}} q^{|\lambda|} = \sum_{n_1 \geq \cdots \geq n_{r-1} \geq 0} \frac{q^{n_1^2 + \cdots + n_{r-1}^2 + n_i + \cdots + n_{r-1}}}{(q)_{n_1}} \begin{bmatrix} n_1 \\ n_1 - n_2 \end{bmatrix}_q \cdots \begin{bmatrix} n_{r-2} \\ n_{r-2} - n_{r-1} \end{bmatrix}_q$$

Outline

A graded algebra of multivariate polynomials

Algebra of polynomials : $\mathcal{R} = K[x_j, j \geq 1]$ over a field K of characteristic 0.

Grading by assigning to x_j the weight j.
Then set $R_0 := K$ and let R_n be the K-vector space with a basis given by the monomials $x_{j_1} \cdots x_{j_\ell}$ such that $j_1 + \cdots + j_\ell = n$ (the variables commute, so we can assume that $j_1 \geq j_2 \geq \cdots \geq j_\ell > 0$). Then

$$\mathcal{R} = \bigoplus_{n \geq 0} R_n.$$

Correspondence between monomials of weight n and partitions of n :

$$x_{j_1} \cdots x_{j_\ell} \longleftrightarrow \lambda = (j_1, \ldots, j_\ell).$$

Therefore the Hilbert–Poincaré series of $\mathcal{R}$ is

$$HP_{\mathcal{R}}(q) := \sum_{n \geq 0} \dim_K(R_n)\, q^n = \sum_{n \geq 0} p(n) q^n = \frac{1}{(q)_\infty}.$$

Interpretation of $T_{2,i}(n)$ in terms of ideals

Consider the ideal $J_{0,i} = (x_1^i, x_k^2, x_k x_{k+1}; k \geq 1)$ of $\mathcal{R} = K[x_j, j \geq 1]$.

Then $\mathcal{R}/J_{0,i}$ corresponds to Rogers–Ramanujan partitions with difference conditions, and

$$HP_{\mathcal{R}/J_{0,i}}(q) = \sum_{n \geq 0} T_{2,i}(n) q^n = \sum_{k \geq 0}^{\infty} \frac{q^{k^2+(2-i)k}}{(q)_k}.$$

Bruschek–Mourtada–Schepers (2011) studied the quotient $\mathcal{R}/[x_1^2]$ (motivated by the theory of arc spaces in algebraic geometry).

Here $[x_1^2]$ is the differential ideal generated by $\{D^k(x_1^2), k \geq 0\}$ where $D(x_j) := x_{j+1}$ and $D(fg) = D(f)g + fD(g)$:

$$[x_1^2] = (x_1^2, 2x_1x_2, 2x_2^2 + 2x_1x_3, 6x_2x_3 + 2x_1x_4, \ldots).$$

Two orders on monomials

Weighted reverse lexicographic order : $x_1^{a_1} x_2^{a_2} \cdots < x_1^{b_1} x_2^{b_2} \cdots$
if $\mathrm{wt}(x_1^{a_1} x_2^{a_2} \cdots) < \mathrm{wt}(x_1^{b_1} x_2^{b_2} \cdots)$
or $\mathrm{wt}(x_1^{a_1} x_2^{a_2} \cdots) = \mathrm{wt}(x_1^{b_1} x_2^{b_2} \cdots)$ and there exists i such that

$$a_1 = b_1, \ldots, a_{i-1} = b_{i-1} \text{ and } a_i < b_i$$
$$\ldots, a_n = b_n, \ldots, a_{i+1} = b_{i+1} \text{ and } a_i > b_i.$$

Leading monomials :

$$[x_1^2] = (x_1^2, 2x_1x_2, 2x_2^2 + 2x_1x_3, 6x_2x_3 + 2x_1x_4, \ldots),$$
$$[x_1^2] = (x_1^2, 2x_1x_2, 2x_2^2 + 2x_1x_3, 6x_2x_3 + 2x_1x_4, \ldots).$$

Interpretation of $T_{2,i}(n)$ in terms of ideals

$$[x_1^2] = (x_1^2, 2x_1x_2, 2x_2^2 + 2x_1x_3, 6x_2x_3 + 2x_1x_4, \ldots).$$

BMS proved that the leading ideal of $J_i := (x_1^i, [x_1^2])$ w.r.t. the weighted reverse lexicographic order (ideal generated by the leading monomials of all the elements in J) is $J_{0,i} = (x_1^i, x_k^2, x_k x_{k+1}; k \geq 1)$. Hence

$$HP_{\mathcal{R}/J_i}(q) = HP_{\mathcal{R}/J_{0,i}}(q) = \sum_{n \geq 0} T_{2,i}(n) q^n = \sum_{k \geq 0} \frac{q^{k^2+(2-i)k}}{(q;q)_k}.$$

Remark : $J_{0,i}$ is in general NOT generated by the leading monomials of a system of generators of J_i. But a system of generators of J_i whose leading monomials generate $J_{0,i}$ is called a Gröbner basis. For different orders, we may have different Gröbner bases.

Question : What happens for the weighted lexicographic order ?

Outline

Back to Gordon's identities

Theorem (Gordon, 1961)

Let r, i be integers with $r \geq 2$, $1 \leq i \leq r$. Denote by $\mathcal{T}_{r,i}$ the set of partitions $\lambda = (\lambda_1, \dots, \lambda_\ell)$ where $\lambda_j - \lambda_{j+r-1} \geq 2$ for all j, and the part 1 appears at most $i-1$ times. Let $\mathcal{E}_{r,i}$ be the set of partitions into parts not congruent to $0, \pm i \mod (2r+1)$.
Let n be a nonnegative integer, and let $T_{r,i}(n)$ (respectively $E_{r,i}(n)$) denote the number of partitions of n which belong to $\mathcal{T}_{r,i}$ (respectively $\mathcal{E}_{r,i}$). Then we have $T_{r,i}(n) = E_{r,i}(n)$.

Bruschek–Mourtada–Schepers (2011) : define $J := (x_1^i, [x_1^r])$, then its leading ideal with respect to the weighted reverse lexicographic order is

$$J_{r,i} = (x_1^i, x_k^{r-s} x_{k+1}^s; k \geq 1; s = 0, \dots, r-1).$$

A conjecture arising from the lexicographic order

Afsharijoo (2019) : guessed the leading ideal $I_{r,i}$ of $J = (x_1^i, [x_1^r])$ with respect to the weighted lexicographic order. Even for $r = 2$ a Gröbner basis is not differentially finite (Afsharijoo–Mourtada, 2020).

For $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, define $N_{r,i}(\lambda) := |\{m \mid p_{i,m}(\lambda) \neq 0\}|$, with

$$p_{i,m}(\lambda) := \begin{cases} \lambda_\ell & \text{if } m = 1, \\ \lambda_{\ell - \sum_{j=1}^{m-1} p_{i,j}(\lambda)} & \text{if } 2 \leq m \leq i, \\ \lambda_{\ell + m - i - \sum_{j=1}^{m-1} p_{i,j}(\lambda)} & \text{if } i < m \leq r-1. \end{cases}$$

Conjecture (Afsharijoo, 2019)

Set $r \geq 2$, $1 \leq i \leq r$ and $\mathcal{C}_{r,i}$ the set of partitions $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ such that at most $i-1$ of the parts are equal to 1 and either $N_{r,i}(\lambda) < r-1$, or $N_{r,i}(\lambda) = r-1$ and $\ell \leq \sum_{j=1}^{r-1} p_{i,j}(\lambda) - (r-i)$. Let n be a nonnegative integer, and denote by $C_{r,i}(n)$ the number of partitions of n which belong to $\mathcal{C}_{r,i}$. Then we have $C_{r,i}(n) = T_{r,i}(n) = E_{r,i}(n)$.

Combinatorial results

Theorem (Afsharijoo–Dousse–J.–Mourtada, 2023)

Afsharijoo's conjecture is true.

We use the set $\mathcal{A}_{r,i}$ from Andrews' Durfee dissection, and define two new sets of partitions $\mathcal{B}_{r,i}$ and $\mathcal{D}_{r,i}$ related to new Durfee-type dissections.

Theorem (ADJM, 2023)

Set $r \geq 2$, $1 \leq i \leq r$. Let n be a nonnegative integer. Then we have

$$A_{r,i}(n) = B_{r,i}(n) = C_{r,i}(n) = D_{r,i}(n) = T_{r,i}(n) = E_{r,i}(n)$$

It is almost immediate that $\mathcal{B}_{r,i} = \mathcal{C}_{r,i}$.
We then prove combinatorially that $\mathcal{B}_{r,i} = \mathcal{D}_{r,i}$.
Finally we show that $\mathcal{D}_{r,r-i}$ and $\mathcal{E}_{r,r-i}$ have the same generating functions : this is the only non-combinatorial part of our proof.

Generating functions

By definition of our sets $\mathcal{D}_{r,i}$:

$$\sum_{\lambda \in \mathcal{D}_{r,r-i}} q^{|\lambda|} = \sum_{n_1 \geq \cdots \geq n_{r-1} \geq 0} \frac{q^{n_1^2+\cdots+n_{r-1}^2 - n_1 - \cdots - n_i}}{(q)_{n_1-n_2} \ldots (q)_{n_{r-2}-n_{r-1}} (q)_{n_{r-1}}} (1-q^{n_i}).$$

We prove that it is equal to the generating function of $\mathcal{E}_{r,r-i}$, that is

$$\frac{(q^{2r+1}, q^{r-i}, q^{r+i+1}; q^{2r+1})_\infty}{(q)_\infty}.$$

Recall the Andrews–Gordon identities :

$$\sum_{n_1 \geq \cdots \geq n_{r-1} \geq 0} \frac{q^{n_1^2+\cdots+n_{r-1}^2 + n_i + \cdots + n_{r-1}}}{(q)_{n_1-n_2} \ldots (q)_{n_{r-2}-n_{r-1}} (q)_{n_{r-1}}} = \frac{(q^{2r+1}, q^i, q^{2r-i+1}; q^{2r+1})_\infty}{(q)_\infty}$$

Extensions : the Bressoud identities

Bressoud : even moduli analogues of the Andrews–Gordon identities.

Theorem (Bressoud 1979)

Let r and i be integers such that $r \geq 2$ and $1 \leq i < r$. Let $\mathcal{U}_{r,i}$ be the set of partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ where $\lambda_j - \lambda_{j+r-1} \geq 2$ for all j, $\lambda_j - \lambda_{j+r-2} \leq 1$ only if $\lambda_j + \lambda_{j+1} + \cdots + \lambda_{j+r-2} \equiv i - 1 \mod 2$, and at most $i - 1$ of the parts λ_j are equal to 1. Let $\mathcal{F}_{r,i}$ be the set of partitions whose parts are not congruent to $0, \pm i \mod (2r)$. Let n be a non-negative integer, and let $U_{r,i}(n)$ (respectively $F_{i,r}(n)$) denote the number of partitions of n which belong to $\mathcal{U}_{r,i}$ (respectively $\mathcal{F}_{r,i}$). Then we have

$$U_{r,i}(n) = F_{r,i}(n).$$

$$\sum_{n_1 \geq \cdots \geq n_{r-1} \geq 0} \frac{q^{n_1^2 + \cdots + n_{r-1}^2 + n_i + \cdots + n_{r-1}}}{(q)_{n_1 - n_2} \ldots (q)_{n_{r-2} - n_{r-1}} (q^2; q^2)_{n_{r-1}}} = \frac{(q^{2r}, q^i, q^{2r-i}; q^{2r})_\infty}{(q)_\infty}.$$

Question : is it possible to express the generating function for $\mathcal{U}_{r,i}$ in terms

Outline

Partitions allowing 0 parts by multiplicities

Now $\lambda = (\lambda_1 \geq \ldots \geq \lambda_\ell \geq 0) = (f_u)_{u \geq 0}$ the multiplicity sequence. Then

$$|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell = \sum_{u \geq 0} u f_u.$$

Example : $(4, 4, 3, 1, 0) = (1, 1, 0, 1, 2, 0, \ldots)$.

Recall that $\mathcal{T}_{r,i}$ consists of $\lambda = (\lambda_1 \geq \ldots \geq \lambda_\ell > 0)$ where $\lambda_j - \lambda_{j+r-1} \geq 2$ for all j, and the part 1 appears at most $i - 1$ times.

Equivalently, it consists of partitions $(f_u)_{u \geq 0}$ such that

$$\begin{cases} f_0 = 0, \\ f_1 \leq i - 1, \\ \text{for all } u \geq 1, \; f_u + f_{u+1} \leq r - 1. \end{cases}$$

Set

$$\mathcal{A}_r := \{(f_u)_{u \geq 0} \,|\, f_0 \leq r - 1 \text{ and } f_u + f_{u+1} \leq r - 1 \text{ for all } u\}.$$

A particular element in $\mathcal{A}_r$

For integers $n_1 \geq \cdots \geq n_{r-1} \geq 0$, $n_0 := \infty,\ n_r := 0$, define the partition $\mu(n_1, \ldots, n_{r-1}) = (f_u)_{u \geq 0}$ by

$$(f_{2u}, f_{2u+1}) = (j, 0) \text{ for all } n_{j+1} \leq u < n_j \text{ and } 0 \leq j \leq r-1.$$

Note that its multiplicity sequence is

$$(\underbrace{r-1, 0, \ldots, r-1, 0}_{n_{r-1} \text{ pairs}}, \ldots, \underbrace{j, 0, \ldots, j, 0}_{n_j - n_{j+1} \text{ pairs}}, \ldots, \underbrace{1, 0, \ldots, 1, 0}_{n_1 - n_2 \text{ pairs}}, 0, \ldots),$$

that $\mu(n_1, \ldots, n_{r-1}) \in \mathcal{A}_r$ and

$$\begin{aligned} \ell(\mu(n_1, \ldots, n_{r-1})) &= n_1 + \cdots + n_{r-1} \\ |\mu(n_1, \ldots, n_{r-1})| &= n_1^2 + \cdots + n_{r-1}^2 - n_1 - \cdots - n_{r-1}. \end{aligned}$$

The bijection

Let $\mathcal{P}(n_1, \dots, n_{r-1})$ be the set of sequences $\lambda = (\lambda_0, \dots, \lambda_{n_1-1})$ of non-negative integers such that for all $j \in \{1, \dots, r-1\}$, the sequence $\lambda^{(j)} := (\lambda_{n_j-1}, \dots, \lambda_{n_{j+1}})$ is a partition.

Set

$$\mathcal{P}_r := \bigsqcup_{n_1 \geq \cdots \geq n_{r-1} \geq 0} \{\mu(n_1, \dots, n_{r-1})\} \times \mathcal{P}(n_1, \dots, n_{r-1}).$$

Weight : $|\mu(n_1, \dots, n_{r-1})| + |\lambda^{(1)}| + \cdots + |\lambda^{(r-1)}|$.

Length : $\ell(\mu(n_1, \dots, n_{r-1})) = n_1 + \cdots + n_{r-1}$.

Theorem (Dousse–J.–Konan, 2024)

For all $r \geq 2$, there is an explicit weight- and length-preserving bijection between the sets $\mathcal{P}_r$ and $\mathcal{A}_r$.

Consequences

Simplification of Warnaar's result :

$$\sum_{\lambda \in \mathcal{T}_{r,i}} q^{|\lambda|} = \sum_{n_1 \geq \cdots \geq n_{r-1} \geq 0} \frac{q^{n_1^2+\cdots+n_{r-1}^2+n_i+\cdots+n_{r-1}}}{(q)_{n_1-n_2} \ldots (q)_{n_{r-2}-n_{r-1}} (q)_{n_{r-1}}}.$$

Extension to Bressoud's set of partitions :

$$\sum_{\lambda \in \mathcal{U}_{r,i}} q^{|\lambda|} = \sum_{n_1 \geq \cdots \geq n_{r-1} \geq 0} \frac{q^{n_1^2+\cdots+n_{r-1}^2+n_i+\cdots+n_{r-1}}}{(q)_{n_1-n_2} \ldots (q)_{n_{r-2}-n_{r-1}} (q^2;q^2)_{n_{r-1}}}.$$

Restricting our bijection to subsets of $\mathcal{T}_{r,i}$ and using Andrews–Gordon and Bressoud's identities as starting points, we prove combinatorially several known and new identities, including the formula for the generating function of $\mathcal{D}_{r,r-i}$ which we used to conclude the proof of Afsharijoo's conjecture :

$$\mathcal{D}_{r,r-i} \leftrightarrow \mathcal{T}_{r,r-i}.$$