POSITIVE FORMULA FOR PRODUCT OF CONJUGACY CLASSES IN U_N.

Cinquième Rencontre ARN CORTIPOM

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Multiplicative Horn Problem

$$n \geq 2,\ \alpha = 1 > \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq 0.$$

$$\mathcal{O}(\alpha) = \{UDiag(\mathrm{e}^{2i\pi\alpha_1}, \ldots, \mathrm{e}^{2i\pi\alpha_n})U^*, U \in U(n)\} \subset U(n).$$

Question : Take $(A, B) \in \mathcal{O}(\alpha) \times \mathcal{O}(\beta)$. What are the possible γ s.t $C = AB \in \mathcal{O}(\gamma)$?

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1. $\exists (A, B, C) \in SU(n) : AB = C$ iff
2. $\forall 0 < r < n, d \geq 0, I, J, K \in \binom{n}{[r]} : c_{I,J}^{K,d} > 0$,

$$(IJK)\ \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j \leq \sum_{k \in K} \gamma_k + d.$$

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Today : $(A, B) \sim \mathcal{O}(\alpha) \times \mathcal{O}(\beta)$. Describe $\mathrm{d}\mathbb{P}[\gamma | \alpha, \beta]$.

Discrete hexagon and lozenges

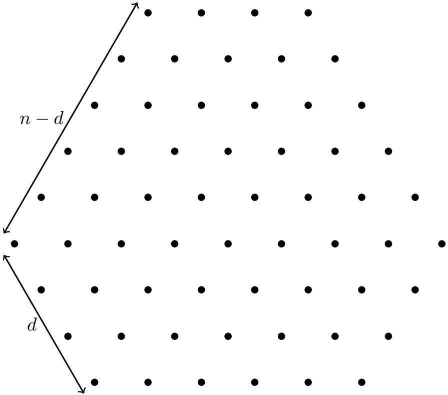

Figure 1: $R_{d,n}$ for $n = 8$ and $d = 3$

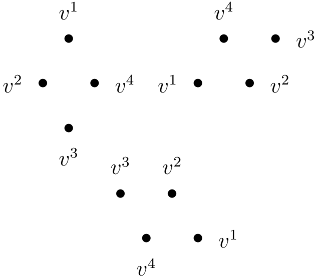

Figure 2: Lozenges in $R_{d,n}$

Regular labeling

Regular labeling

$g : R_{d,n} \to \mathbb{Z}_3$

1. cyclic boundary conditions,
2. $\forall \ell = (v^1, v^2, v^3, v^4)$,

$$(g(v^2) = g(v^4)) \Rightarrow \{g(v^1), g(v^3)\} = \{g(v^2)+1, g(v^2)+2\}.$$

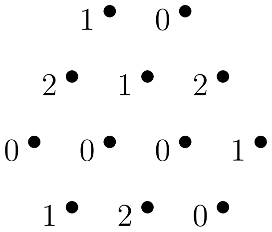

Figure 3: A regular labeling of $R_{1,3}$.

Toric hives

Toric rhombus concave function

$f : R_{d,n} \to \mathbb{R}$, $f(v^2) + f(v^4) \geq f(v^1) + f(v^3)$ with $=$ if

$$(g(v^1), g(v^2), g(v^3), g(v^4)) = (a, a+1, a+2, a+1).$$

$\mathcal{C}_g = \left\{ f_{|Supp(g)}, f : R_{d,n} \to \mathbb{R} \text{ toric rhombus concave with respect to } g \right\}$.

Toric hive polytope

$$P^g_{\alpha,\beta,\gamma} = \{f \in \mathcal{C}_g : f_{|\partial R_{d,n}} = (\alpha, \beta, \gamma)\}.$$

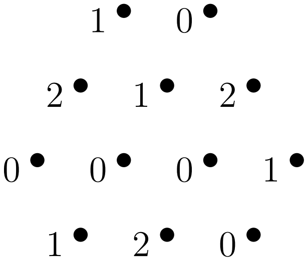

Figure 4: A regular labeling of $R_{1,3}$.

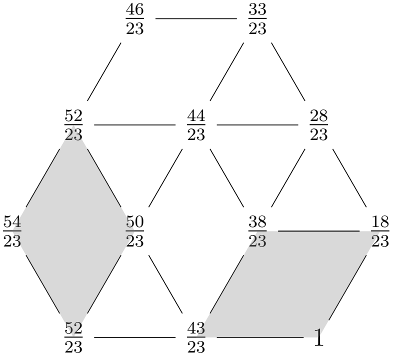

Figure 5: An toric rhombus concave function with boundary conditions
$\alpha = \left(\frac{13}{23} \geq \frac{6}{23} \geq \frac{2}{23}\right)$,
$\beta = \left(\frac{18}{23} \geq \frac{10}{23} \geq \frac{5}{23}\right)$
$\gamma = \left(\frac{20}{23} \geq \frac{9}{23} \geq \frac{2}{23}\right)$.

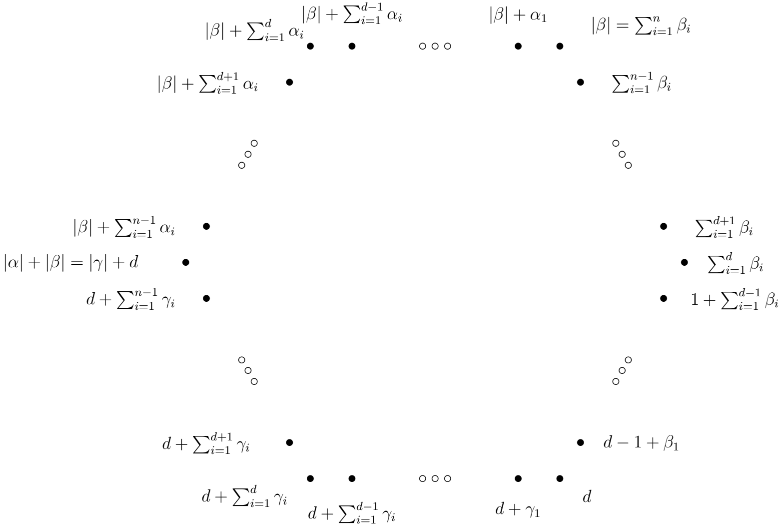

Figure 6: Boundary conditions in $P^{g}_{\alpha,\beta,\gamma}$.

Main Result

Theorem [F. - Tarrago]

$$\mathrm{d}\mathbb{P}[\gamma|\alpha,\beta] = \frac{c_n \Delta(\mathrm{e}^{2i\pi\gamma})}{\Delta(\mathrm{e}^{2i\pi\alpha})\Delta(\mathrm{e}^{2i\pi\beta})} \sum_{g:R_{d,n}\to\mathbb{Z}_3 \text{ regular}} Vol_g(P^g_{\alpha,\beta,\gamma}).$$

1. First Density formula via Fourier Analysis
2. Link with quantum Cohomology
3. Dual hives
4. Convergence to a Volume

First Density formula via Fourier Analysis

Measure m_θ

$$\varphi_\theta : U(n) \to \mathcal{O}(\theta) \subset SU(n)$$
$$U \mapsto U\mathrm{e}^{2i\pi\theta}U^*$$

induces the probability distribution

$$m_\theta = \varphi_\theta \# \mathrm{d}g$$

on $\mathcal{O}(\theta)$ where $\mathrm{d}g$ is the Haar measure on $U(n)$.

Convolution

$(A, B) \in \mathcal{O}(\alpha) \times \mathcal{O}(\beta) \sim m_\alpha \otimes m_\beta$. Then, $C = AB \sim m_{\alpha,\beta}$ where

$$m_{\alpha,\beta} = m_\alpha * m_\beta = Mult\#(m_\alpha \otimes m_\beta).$$

$m_{\alpha,\beta}$: multiplicative convolution of m_α and m_β.

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Fourier Transform on $SU(n)$

$m \in \mathcal{M}(SU(n))$.

$$\widehat{m} : \lambda \in \mathbb{Z}^n_{\geq 0} \mapsto \widehat{m}(\lambda) = \int_{SU(n)} \rho_\lambda(g) \mathrm{d}m(g) \in \mathrm{End}_{V_\lambda}.$$

$$\widehat{m}_{\alpha,\beta}(\lambda) = \widehat{m_\alpha * m}_\beta(\lambda) = \widehat{m}_\alpha(\lambda)\widehat{m}_\beta(\lambda).$$

Fourier Transform of Orbit measures

$$\widehat{m}_\alpha(\lambda) = \frac{\chi_\lambda(\mathrm{e}^{2i\pi\alpha})}{\dim V_\lambda}\mathrm{id}_{V_\lambda}.$$

So

$$\widehat{m}_{\alpha,\beta}(\lambda) = \frac{\chi_\lambda(\mathrm{e}^{2i\pi\alpha})\chi_\lambda(\mathrm{e}^{2i\pi\beta})}{\dim V_\lambda^2}\mathrm{id}_{V_\lambda}.$$

Inverse Fourier Transform

For $f : \lambda \in \mathbb{Z}^n_{\geq 0} \mapsto f(\lambda) \in V_\lambda$,

$$\begin{aligned} f^\vee : SU(n) &\to \mathbb{C} \\ g &\mapsto \sum_{\lambda \in \mathbb{Z}^n_{\geq \mathbf{0}}} \dim V_\lambda \cdot Tr[\rho_\lambda(g^{-1}) f(\lambda)]. \end{aligned}$$

Density of AB

$$(\widehat{m}_{\alpha,\beta})^{\vee}(g) = \sum_{\lambda \in \mathbb{Z}^n_{\geq 0}} \frac{1}{\dim V_\lambda} \chi_\lambda(\mathrm{e}^{2i\pi\alpha})\chi_\lambda(\mathrm{e}^{2i\pi\beta})\chi_\lambda(g^{-1})$$

Density of eigenvalues

$$\mathrm{d}\mathbb{P}[\gamma|\alpha,\beta] = \frac{|\Delta(\mathrm{e}^{2i\pi\gamma})|^2}{(2\pi)^{n-1}n!} \sum_{\lambda \in \mathbb{Z}^n_{\geq 0}} \frac{1}{\dim V_\lambda} \chi_\lambda(\mathrm{e}^{2i\pi\alpha})\chi_\lambda(\mathrm{e}^{2i\pi\beta})\chi_\lambda(\mathrm{e}^{-2i\pi\gamma}).$$

First density expression

Using $\chi_\lambda(\mathrm{e}^{2i\pi\theta}) = \frac{\det[\mathrm{e}^{2i\pi\theta_r\lambda'_s}]_{1\le r,s\le n}}{\Delta(\mathrm{e}^{2i\pi\theta})}$,

$$\mathrm{d}\mathbb{P}[\gamma|\alpha,\beta] = c_n \frac{\Delta(\mathrm{e}^{2i\pi\gamma})}{\Delta(\mathrm{e}^{2i\pi\alpha})\Delta(\mathrm{e}^{2i\pi\beta})} J[\gamma|\alpha,\beta]$$

where

$$J[\gamma|\alpha,\beta] = \sum_{\lambda\in\mathbb{Z}^n_{\ge 0}} \frac{1}{\dim V_\lambda} \det[\mathrm{e}^{2i\pi\alpha_r\lambda'_s}] \det[\mathrm{e}^{2i\pi\beta_r\lambda'_s}] \det[\mathrm{e}^{-2i\pi\gamma_r\lambda'_s}].$$

Link with quantum Cohomology

Quantum Cohomology ring

For $N \geq n$, the ring $QH^\bullet(\mathbb{G}r(n,N))$ has basis
$(q^d \otimes \sigma_\lambda, d \geq 0, (N-n \geq \lambda_1 \geq \cdots \geq \lambda_n \geq 0))$:

$$\sigma_\lambda \cdot \sigma_\mu = \sum_{\substack{\nu, d \geq 0 \\ |\lambda|+|\mu|=|\nu|+Nd}} c_{\lambda,\mu}^{\nu,d} q^d \otimes \sigma_\nu.$$

$c_{\lambda,\mu}^{\nu,d}$ are the Quantum Littlewood-Richardson coefficients.

Link with characters

Using work from [Rietsch 01'],

$$I_{n,N} = \left\{ -\frac{n-1}{2} \leq I_n < \cdots < I_1 \leq N - \frac{n+1}{2} \right\}.$$

$$c_{\lambda,\mu}^{\nu,d} = \frac{1}{N^n} \sum_{I \in I_{n,N}} \frac{\det[\mathrm{e}^{\frac{2i\pi I_r \lambda'_s}{N}}] \det[\mathrm{e}^{\frac{2i\pi I_r \mu'_s}{N}}] \det[\mathrm{e}^{-\frac{2i\pi I_r (\nu^\vee)'_s}{N}}]}{\Delta(\xi^I)}.$$

where $\xi^I = (\xi^{I_1}, \ldots, \xi^{I_n})$ and $\xi = \exp(2i\pi/N)$.

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where $\xi^I = (\xi^{I_1}, \ldots, \xi^{I_n})$ and $\xi = \exp(2i\pi/N)$.
Recall

$$J[\gamma|\alpha,\beta] = \sum_{\lambda \in \mathbb{Z}^n_{\geq 0}} \frac{1}{\dim V_\lambda} \det[\mathrm{e}^{2i\pi\alpha_r\lambda'_s}] \det[\mathrm{e}^{2i\pi\beta_r\lambda'_s}] \det[\mathrm{e}^{-2i\pi\gamma_r\lambda'_s}].$$

Density as limit of Quantum Littlewood Richardson

Theorem [Density = Lim Quantum LR]

$|\lambda_N| + |\mu_N| = |\nu_N| + dN$, $\frac{1}{N}\lambda_N \to \alpha$, $\frac{1}{N}\mu_N \to \beta$, $\frac{1}{N}\nu_N \to \gamma$.

$$\lim_{N\to\infty} N^{-\frac{(n-1)(n-2)}{2}} c^{\nu_N,d}_{\lambda_N,\mu_N} = J[\gamma|\alpha,\beta].$$

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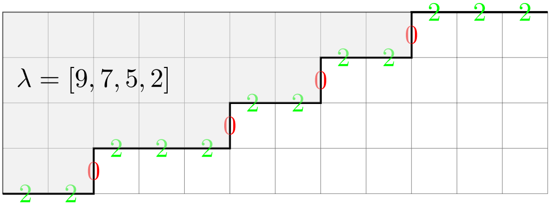
$$\lim_{N\to\infty} N^{-\frac{(n-1)(n-2)}{2}} c_{\lambda_N,\mu_N}^{\nu_N,d} = J[\gamma|\alpha,\beta].$$

Goal : Volume expression for this limit.

Dual hives

Combinatorial expression for $c^{\nu,d}_{\lambda,\mu}$

$$n = 4, N - n = 12, d = 2$$



$$c(\lambda) = 2202220220220222$$

Combinatorial expression for $c_{\lambda,\mu}^{\nu,d}$

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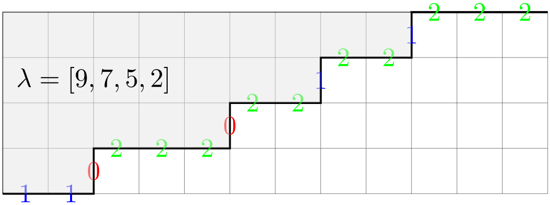


$$c(\lambda) = 1102220221221222$$

Combinatorial expression for $c_{\lambda,\mu}^{\nu,d}$

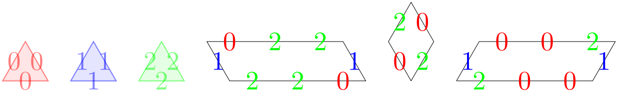


Figure 7: List of puzzle pieces

Theorem [Buch, Kresch, Purbhoo, Tamvakis 2016]

$\lambda, \mu, \nu \subset n \times (N-n)$ s.t $|\lambda| + |\mu| = |\nu| + Nd$. Then,

$$c_{\lambda,\mu}^{\nu,d} = \text{ Nb puzzles with clockwise boundaries } c(\lambda), c(\mu), c(\nu^\vee).$$

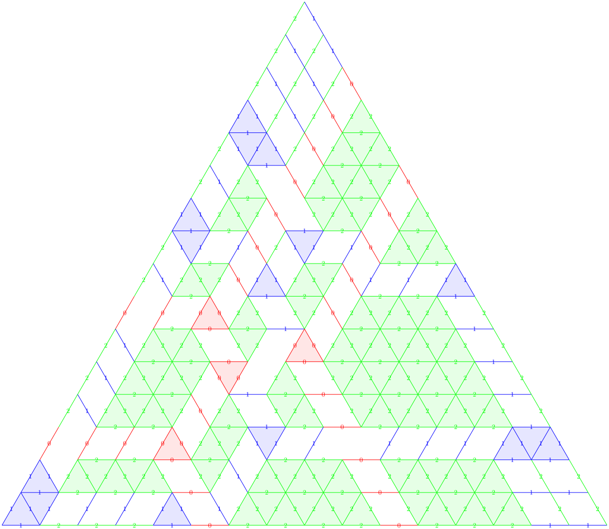
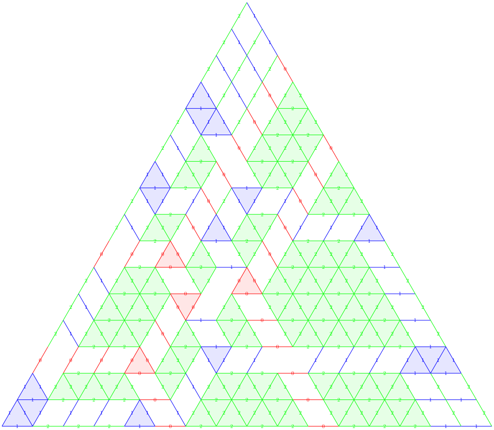
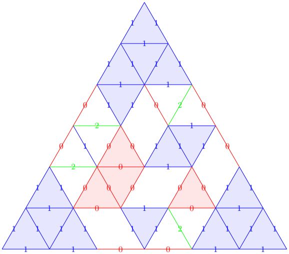


Figure 8: A Puzzle associated to $c_{\lambda,\mu}^{\nu,d}$ with $\lambda = [9, 7, 5, 2], \mu = [10, 6, 4, 2], \nu = [7, 3, 3, 0]$ for $N = 16$, $n = 4$, $d = 2$.





Puzzle

Color Map $C : E_n \to \{0, 1, 3, m\}$.

Edge coordinates

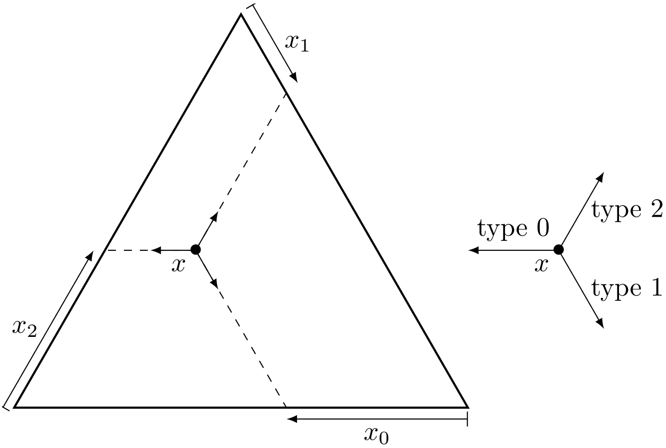
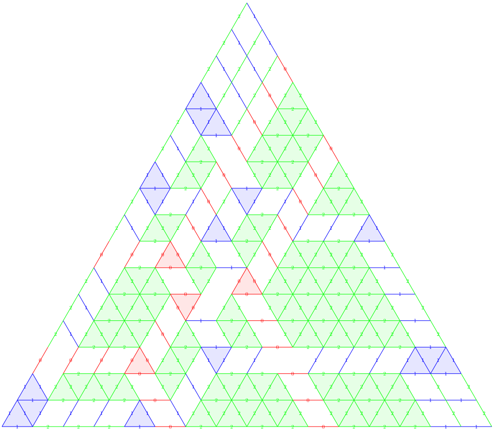
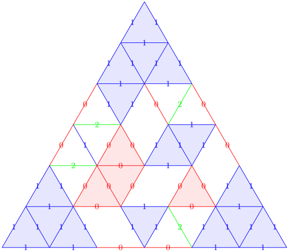


Figure 9: Edge types and coordinates.

Puzzle to Dual hive: Label Map





$$\text{Puzzle} \longleftrightarrow \begin{cases} \text{Color Map } C : E_n \to \{0, 1, 3, m\}. \\ \text{Label Map } L : E_n \to \mathbb{Z}_{\geq 0}. \end{cases}$$

Two-colored Dual Hives

$H = (C, L)$ where

$$C : E_n \to \{0, 1, 3, m\} \text{ Color map}$$

s.t clockwise faces colors are $(0,0,0), (1,1,1), (1,0,3), (0,1,m)$,

$$L : E_n \to \mathbb{Z} \text{ Label map}$$

s.t for (e, e') of same type in l

1. $L(e_0) + L(e_1) + L(e_2) = N - 1$ or $N - 2$,
2. $L(e) = L(e')$ if l is m lozenge,
3. $L(e) \geq L(e')$ or $L(e) > L(e')$.

$$H(\lambda, \mu, \nu, N) = \{ \text{ Dual hives } H = (C, L) : L_{|\partial E_n} = (\lambda, \mu, \nu)\}.$$

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$$P(\lambda, \mu, \nu, N) \leftrightarrow H(\lambda, \mu, \nu, N).$$

Hexagon Rotations

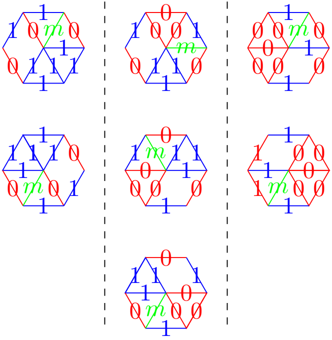


Figure 10: ABC hexagons and their rotations.

Hexagon Rotations

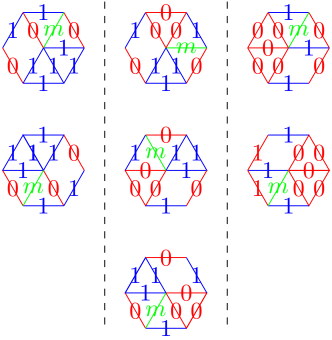


Figure 10: ABC hexagons and their rotations.

Hexagon rotation of color maps

$$Rot : C \mapsto C'$$

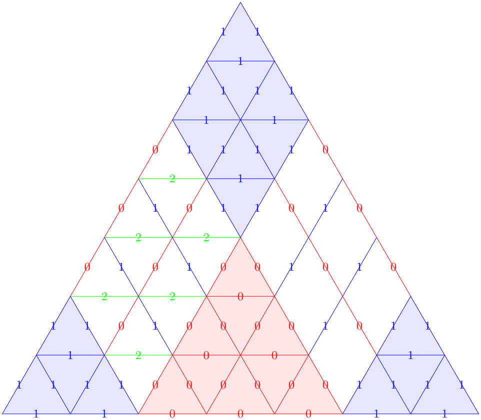
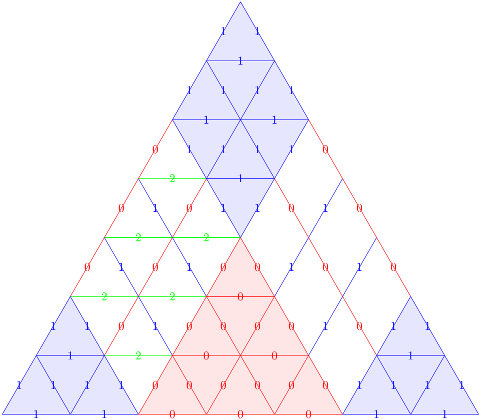


Figure 11: The simple color map $C0$.

Simple Color map



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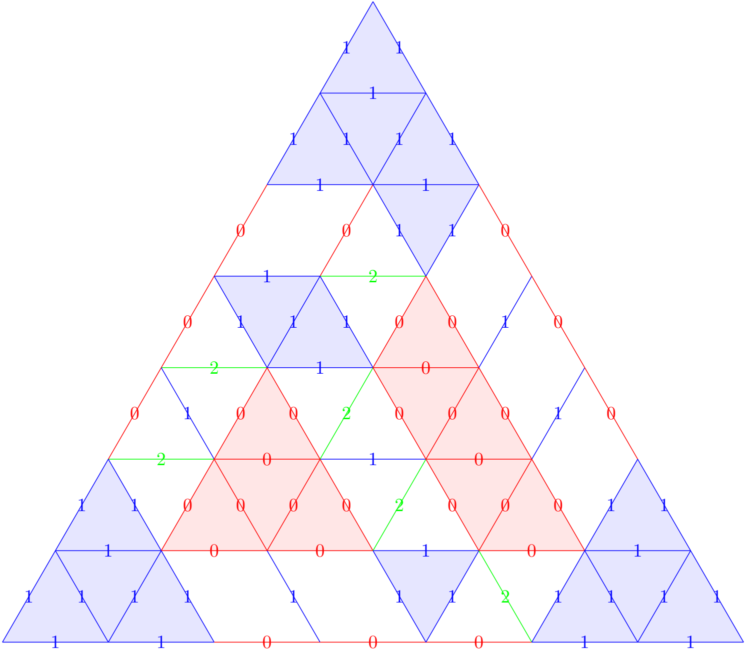
Color maps C can be reduced to $C0$ via hexagon rotations.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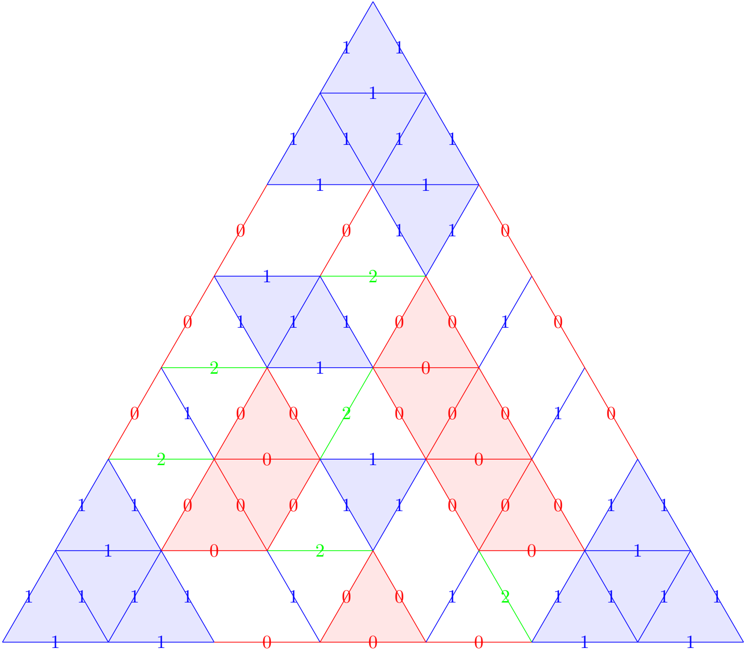
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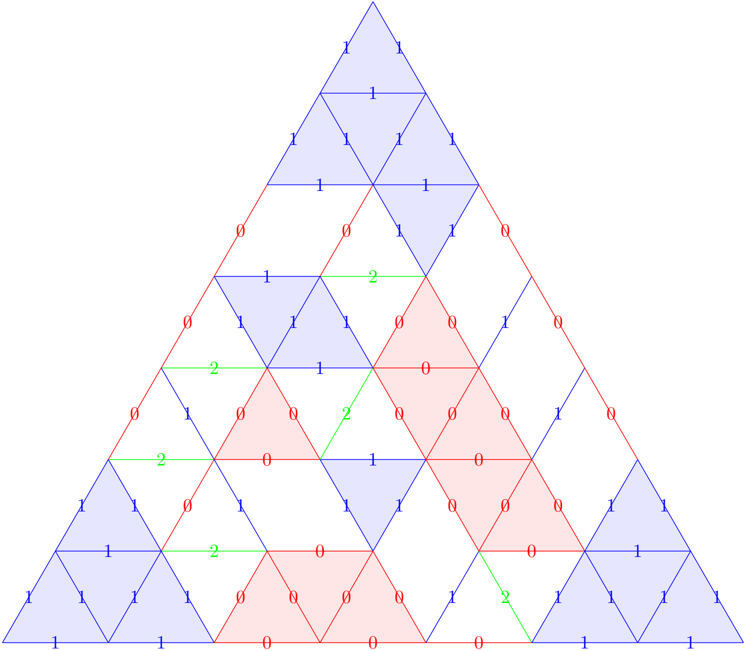
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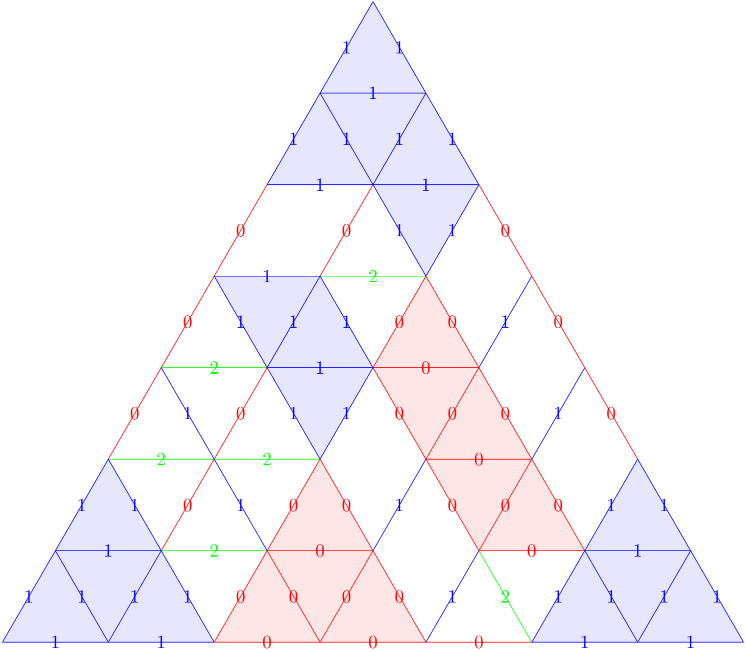
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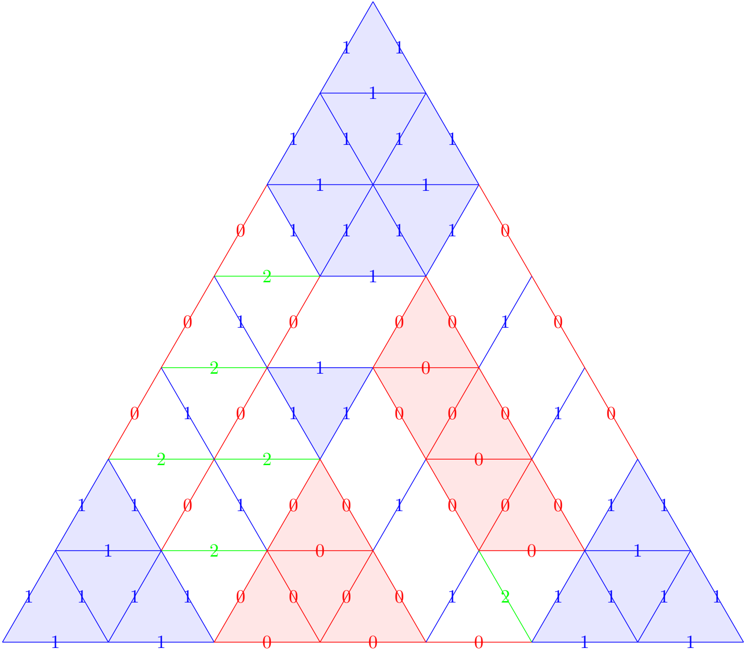

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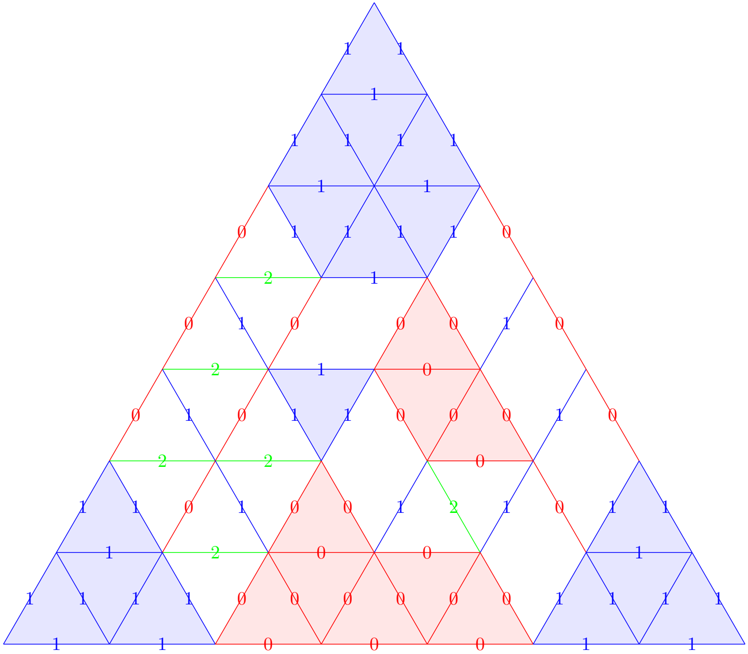

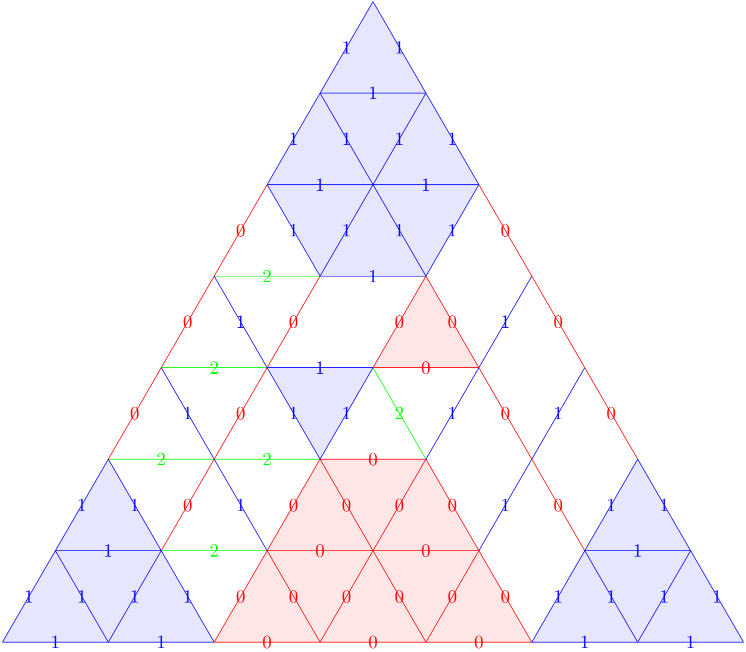
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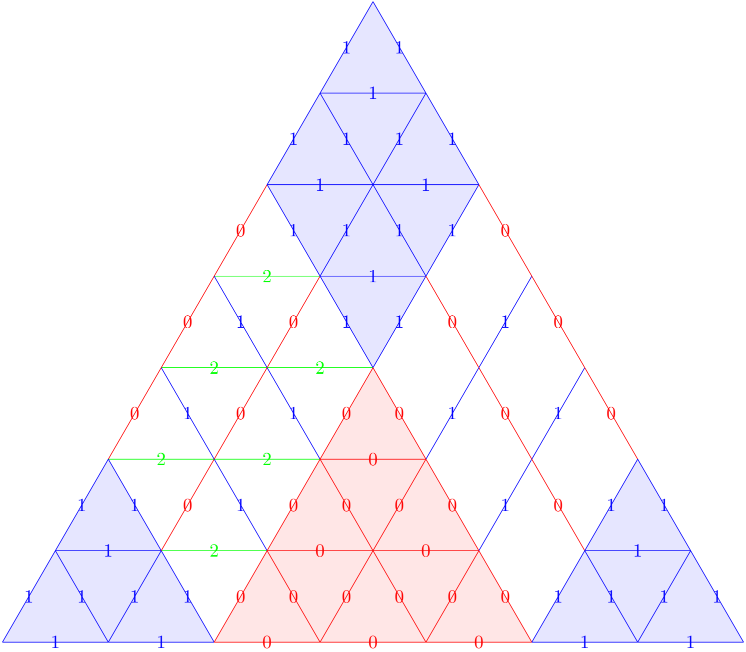
1 2 0 0 1
1 1 1 1 0 0 0 0 0 0 1 1 1 1
1 1 0 0 0 1 1


1 1
1
1 1 1 1
1 1
0 1 1 1 1 0
2 1
0 1 0 1 1 0 1 0
2 2
0 1 0 1 0 0 1 0 1 0
2 2 0
1 1 0 1 0 0 0 0 1 0 1 1
1 2 0 0 1
1 1 1 1 0 0 0 0 0 0 1 1 1 1
1 1 0 0 0 1 1

Quasi dual hives

$H = (C, L)$ where

$$C : E_{n,d} \to \{0, 1, 3, m\} \text{ Color map}$$

s.t clockwise faces colors are $(0,0,0), (1,1,1), (1,0,3), (0,1,m)$,

$$L : E_{n,d} \to \frac{1}{N}\mathbb{Z} \text{ Label map}$$

s.t

1. $L(e_0) + L(e_1) + L(e_2) = 1 - \frac{1}{N}$ or $1 - \frac{2}{N}$
2. $L(e) = L(e')$ if l is m lozenge,
3. ~~$L(e) \geq L(e')$ or $L(e) > L(e')$~~.

$H = (C, L)$ where

$$C : E_{n,d} \to \{0, 1, 3, m\} \text{ Color map}$$

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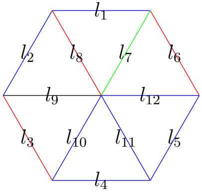
2. $L(e) = L(e')$ if l is m lozenge,
3. ~~$L(e) \geq L(e')$ or $L(e) > L(e')$~~.

$$\widetilde{H}(\lambda, \mu, \nu, N) = \{H = (C, L) : L_{|\partial E_{n,d}} = (\lambda, \mu, \nu)\},$$

$$H(\lambda, \mu, \nu, N) \underset{\times \frac{1}{N}}{\subset} \widetilde{H}(\lambda, \mu, \nu, N).$$

Boundary determine interior edges

$\{L(e), e \in h^o\}$ determined by $\{L(e), e \in \partial h\}$.



$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} l_7 \\ l_8 \\ l_9 \\ l_{10} \\ l_{11} \\ l_{12} \end{pmatrix} = \begin{pmatrix} 1 - \frac{1}{N} - l_1 - l_6 \\ l_6 \\ 1 - \frac{1}{N} - l_2 - l_6 \\ 1 - \frac{2}{N} - l_3 \\ 1 - \frac{1}{N} - l_4 \\ l_1 \end{pmatrix}.$$

Rotation map between quasi hives

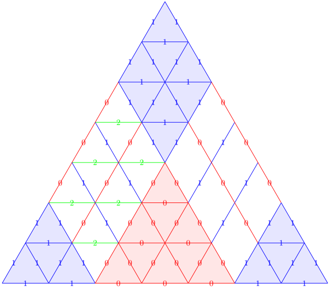
Rotation of quasi label maps

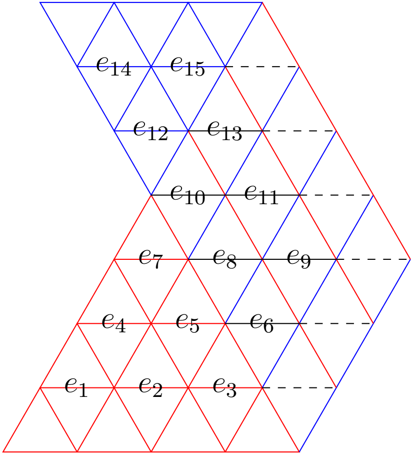
For $C \to_h C'$,

$$Rot[C \to C'] : L \mapsto L'$$

by changing values of $\{L(e), e \in h^o\}$ is an affine bijection with integer coefficients.

Label Maps on the simple Color Map





$$L \text{ determined by } \left\{ z_i = L(e_i), 1 \leq i \leq \frac{(n-1)(n-2)}{2} = D \right\}.$$

Bijection with edge coordinates: $C0$ case

Label map $\Phi^{C_0}(z)$.

$$\Phi^{C0} : \left(\frac{1}{N}\mathbb{Z}\right)^D \to L^{C0}(\lambda, \mu, \nu, N)$$
$$z = (z_1, \dots, z_D) \mapsto \Phi^{C_0}(z) \text{ s.t } \Phi^{C_0}(z)(e_i) = z_i.$$

Bijection with edge coordinates: $C0$ case

Label map $\Phi^{C_0}(z)$.

$$\Phi^{C0} : \left(\frac{1}{N}\mathbb{Z}\right)^D \to L^{C0}(\lambda, \mu, \nu, N)$$
$$z = (z_1, \dots, z_D) \mapsto \Phi^{C_0}(z) \text{ s.t } \Phi^{C_0}(z)(e_i) = z_i.$$

Label map $\Phi^{C}(z)$.

Let C be a color map.

$$\Phi^{C} : \left(\frac{1}{N}\mathbb{Z}\right)^D \to L^{C}(\lambda, \mu, \nu, N)$$
$$z = (z_1, \dots, z_D) \mapsto \Phi^{C}(z) = Rot[C_0 \to C](\Phi^{C_0}(z))$$

affine bijection with integer coefficients.

Convergence to a Volume

Limit dual hive

$$J[\gamma|\alpha,\beta] = \lim_{N\to\infty} N^{-D} c^{\nu_N,d}_{\lambda_N,\mu_N} = \lim_{N\to\infty} N^{-D} |H(\lambda_N,\mu_N,\nu_N,N)|.$$

Limit hive $H(\alpha,\beta,\gamma,\infty)$

$$H = (C, L), \quad C : E_{n,d} \to \{0,1,3,m\}, \quad L : E_{n,d} \to \mathbb{R},$$

1. $L(e_0) + L(e_1) + L(e_2) = 1$
2. $L(e) = L(e')$ in m lozenges,
3. $L(e) \geq L(e')$ or $L(e) > L(e')$.

$H(\alpha,\beta,\gamma,\infty) \subset \widetilde{H}(\alpha,\beta,\gamma,\infty)$ if ~~$L(e) \geq L(e')$ or $L(e) > L(e')$~~ .

Convergence to a volume

$$J[\gamma|\alpha,\beta] = \lim_{N\to\infty} N^{-D} c_{\lambda_N,\mu_N}^{\nu_N,d} = \lim_{N\to\infty} N^{-D} |H(\lambda_N,\mu_N,\nu_N,N)|$$

$$= \lim_{N\to\infty} \sum_C \int_{\mathbb{R}^D} \sum_{z\in(\frac{1}{N}\mathbb{Z})^D:\Phi^C(z)\in H(\lambda_N,\mu_N,\nu_N,N)} 1(u)_{z+[-\frac{1}{2N},\frac{1}{2N}]^D} \mathrm{d}u$$

$$= \lim_{N\to\infty} \sum_C \int_{\mathbb{R}^D} f_N(u)\mathrm{d}u$$

$$= \sum_C \int_{\mathbb{R}^D} f(u)\mathrm{d}u$$

where

$$f(u) = 1(u)_{\{\Phi^C[u]\in H^C(\alpha,\beta,\gamma,\infty)\}}$$

so that

$$J[\gamma|\alpha,\beta] = \sum_C Vol\left(u\in\mathbb{R}^D, \Phi^C[u]\in H^C(\alpha,\beta,\gamma,\infty)\right).$$

From Triangle to Hexagon

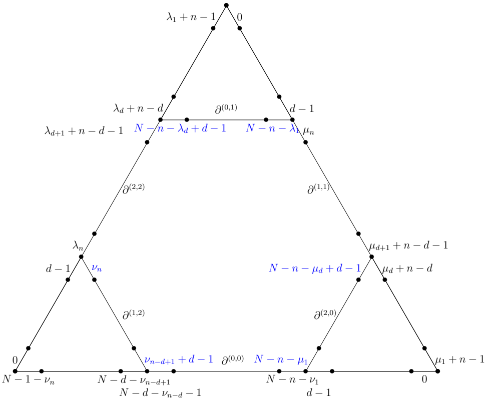


Figure 12: $\partial H(\lambda, \mu, \nu, N)$

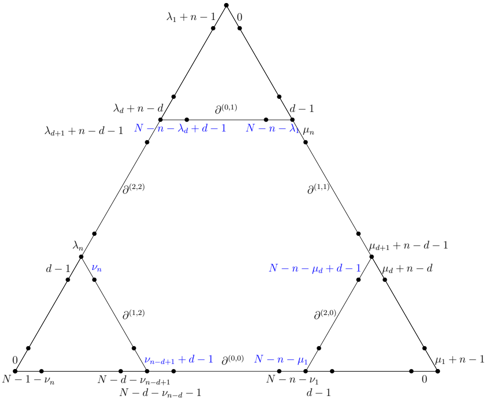


Figure 12: $\partial H(\lambda, \mu, \nu, N)$

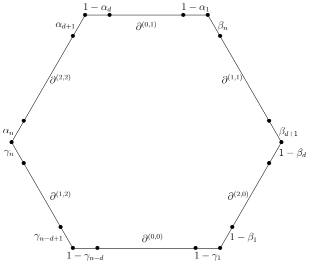


Figure 13: $\partial H(\alpha, \beta, \gamma, \infty)$

$$\left(\frac{1}{N}\lambda^{(N)}, \frac{1}{N}\mu^{(N)}, \frac{1}{N}\nu^{(N)}\right) \to (\alpha, \beta, \gamma).$$

From dual hives to hives

WANTED : $H(\alpha, \beta, \gamma, \infty) \to P^{g}_{\alpha,\beta,\gamma}$.

Color Map C to regular label g

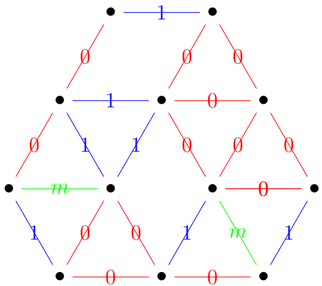


Figure 14: Color map C

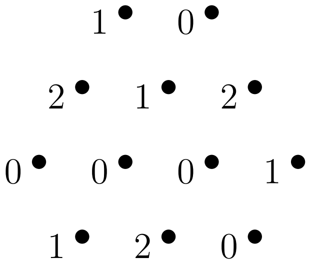


Figure 15: Regular labeling $g[C]$

$$C \mapsto g[C].$$

Edge label map L to toric concave function f

$$\Psi^C : \tilde{H}^C(\alpha, \beta, \gamma, \infty) \to \tilde{P}^C_{\alpha,\beta,\gamma}$$
$$(C, L) \longmapsto \Psi^C[L]$$

by setting $\Psi^C[L](v^*) = d$ and

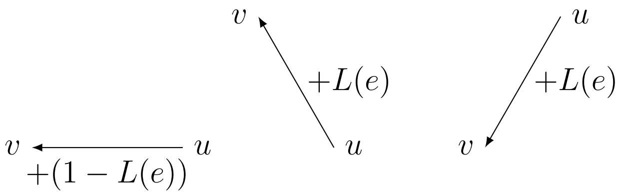
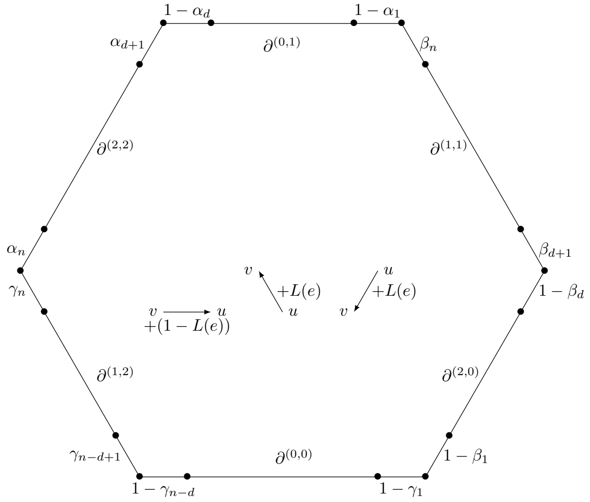
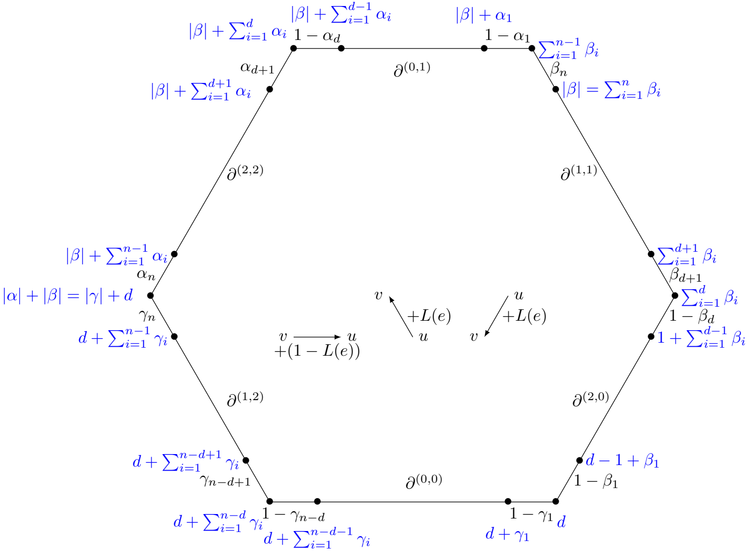


Figure 16: From edge labels to vertex valued function.

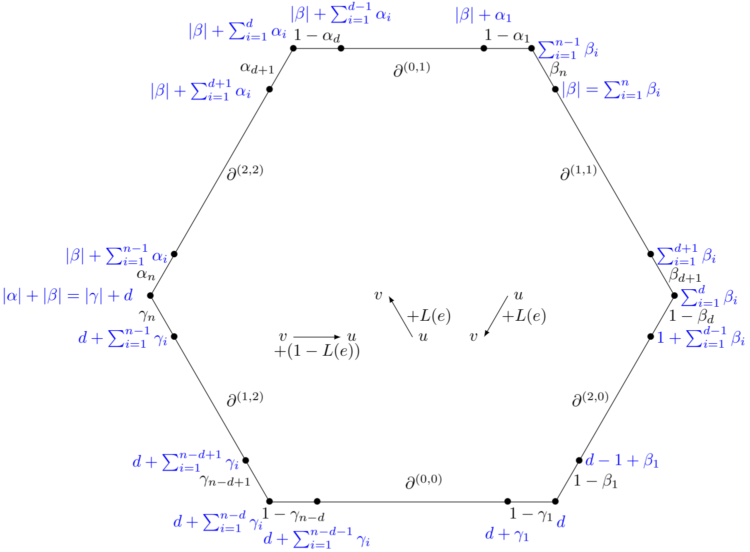
Edge labels to Vertex labels



Edge labels to Vertex labels



Edge labels to Vertex labels



$$\Psi^C(H^C(\alpha, \beta, \gamma, \infty)) = P^{g[C]}_{\alpha,\beta,\gamma}.$$

Volume preservation

$$\Psi^C \circ \Phi^C : \mathbb{R}^D \to \tilde{P}^C_{\alpha,\beta,\gamma} \text{ satisfies } |\det(\Psi^C \circ \Phi^C)| = 1$$

thus

$$Vol(u \in \mathbb{R}^D, \Phi^C(u) \in H^C(\lambda, \mu, \nu, \infty)) = Vol(P^{g[C]}_{\alpha,\beta,\gamma}).$$

Volume expression for the density

$$\mathrm{d}\mathbb{P}[\gamma|\alpha,\beta] = \frac{c_n \Delta(\mathrm{e}^{2i\pi\gamma})}{\Delta(\mathrm{e}^{2i\pi\alpha})\Delta(\mathrm{e}^{2i\pi\beta})} \sum_{g : R_{d,n} \to \mathbb{Z}_3 \text{ regular}} Vol_g(P^g_{\alpha,\beta,\gamma}).$$