Cristaux parfaits et partitions

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Outline

1. Basics on Lie algebras
2. Character formulas
3. Crystals and grounded partitions
4. From perfect crystals to partition identities
5. From partitions to character formulas
   - Generalisation of Primc's identity and characters for $A_n^{(1)}$ at level 1
   - The case of non-constant ground state paths

Lie algebras

Definition

A *Lie algebra* $\mathfrak{g}$ is a vector space together with a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the Lie bracket, satisfying:

- alternativity : for all $x \in \mathfrak{g}$, $[x, x] = 0$,
- the Jacobi identity: for all $x, y, z \in \mathfrak{g}$,
  $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$.

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Example

The *special linear Lie algebra* of order n, denoted A_{n-1} or $\mathfrak{sl}_n(\mathbb{C})$, is the Lie algebra of $n \times n$ matrices with trace zero and with the Lie bracket $[X, Y] = XY - YX$.

Representations

Definition

A *representation (or module)* of $\mathfrak{g}$ is a vector space V together with a linear map $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, such that

$$\rho([X, Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X).$$

By abuse of notation, V is often called a $\mathfrak{g}$-module and $\rho(X)(v)$ is often written $X \cdot v$.

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Examples

- trivial representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ such that $\rho(X) = 0$ for all $X \in \mathfrak{g}$,
- adjoint representation $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ such that $ad(X)(Y) = [X, Y]$ for all $X, Y \in \mathfrak{g}$.

Infinite dimensional Lie algebras

Let $\mathfrak{g}$ be a finite dimensional semi-simple Lie algebra.
It is possible to define an *affine Kac-Moody Lie algebra* $\hat{\mathfrak{g}}$ corresponding to $\mathfrak{g}$ as

$$\hat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c,$$

where $\mathbb{C}[t, t^{-1}]$ is the complex vector space of Laurent polynomials in the indeterminate t, and $\mathbb{C}c$ is $\hat{\mathfrak{g}}$'s center (one-dimensional) which satisfies $[c, g] = 0$ for all $g \in \mathfrak{g}$.

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Kac-Moody Lie algebras can also be described in terms of generators and relations.

Weights

Definition

Let V be a module and μ be a linear functional on $\mathfrak{h}$, the Cartan subalgebra. The *weight space* of V with weight μ is $V_\mu := \{v \in V : \forall H \in \mathfrak{h}, \quad H \cdot v = \mu(H)v\}$. A *weight* is a linear functional μ such that V_μ is non-zero.
If V is a direct sum $V = \bigoplus_\mu V_\mu$ of its weight spaces, then it is called a *weight module*.

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The *roots* are weights for the adjoint representation. They can be written as a linear combination of *simple roots*.

A weight λ is *higher* than another weight μ if $\lambda - \mu$ can be written as a sum of positive roots, and λ is a *highest weight* if it is higher than any other weight in V.

Outline

Characters

Definition

Let $L(\lambda) = \bigoplus_\mu V_\mu$ be an irreducible highest weight module with highest weight λ. The *character* $\mathrm{ch}L(\lambda)$ of $L(\lambda)$ is defined as

$$\mathrm{ch}L(\lambda) = \sum_\mu \dim(V_\mu) e^\mu,$$

where e^μ is a formal exponential satisfying $e^\mu e^{\mu'} = e^{\mu+\mu'}$.

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By definition of a highest weight,

$$e^{-\lambda} \mathrm{ch} L(\lambda) = \sum_\mu \dim(V_\mu) e^{\mu-\lambda}$$

is a **series with positive coefficients** in $\mathbb{Z}[[e^{-\alpha_0}, \dots, e^{-\alpha_n}]]$, where $\alpha_0, \dots, \alpha_n$ are the simple roots.

Character formulas

Theorem (Weyl–Kac character formula)

$$\mathrm{ch}(L(\lambda)) = \frac{\sum_{w\in W} \mathrm{sgn}(w) e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})^{dim\mathfrak{g}_\alpha}},$$

where W is the Weyl group of $\mathfrak{g}$, Δ^+ the set of positive roots of $\mathfrak{g}$, $\mathrm{sgn}(w)$ the signature of w, $\rho \in \mathfrak{h}^*$ the Weyl vector, and $\mathfrak{g}_\alpha$ the α root space of $\mathfrak{g}$.

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Beautiful formula but **does not exhibit the positivity** of the coefficients.

The **principal specialisation** ($e^{-\alpha_i} \mapsto q$ for all i) gives an infinite product.

Example: $A_1^{(1)}$ at level 3 (Lepowsky–Wilson)

$$e^{-(\Lambda_0+2\Lambda_1)}\mathrm{ch}L(\Lambda_0+2\Lambda_1) = \frac{(-q;q)_\infty}{(q,q^4;q^5)_\infty}, e^{-3\Lambda_1}\mathrm{ch}L(3\Lambda_1) = \frac{(-q;q)_\infty}{(q^2,q^3;q^5)_\infty},$$

where $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$ and $(a,b;q)_n = (a;q)_n(b;q)_n$.

The Rogers–Ramanujan identities

Definition

A *partition* λ of a positive integer n is a finite non-increasing sequence of positive integers $(\lambda_1, \dots, \lambda_m)$ such that $\lambda_1 + \dots + \lambda_m = n$. The integers $\lambda_1, \dots, \lambda_m$ are called the *parts* of the partition λ.

Example

There are 5 partitions of 4: $(4), (3,1), (2,2), (2,1,1)$ and $(1,1,1,1)$.

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Example

There are 5 partitions of 4: $(4), (3,1), (2,2), (2,1,1)$ and $(1,1,1,1)$.

- The generating function for partitions into distinct parts congruent to $k \mod N$ is
$$(-zq^k; q^N)_\infty.$$
- The generating function for partitions into parts congruent to $k \mod N$ is
$$\frac{1}{(zq^k; q^N)_\infty}.$$

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Example

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Theorem (Rogers 1894, Rogers–Ramanujan 1919)

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty},$$

For every positive integer n, the number of partitions of n such that the difference between two consecutive parts is at least 2 is equal to the number of partitions of n into parts congruent to 1 or 4 modulo 5.

Representation theoretic interpretation

Lepowsky and Wilson 1984: representation theoretic interpretation

$$(-q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = (-q;q)_\infty \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}$$

Obtained by giving two different formulations for the principal specialisation of $e^{-(\Lambda_0+2\Lambda_1)}\mathrm{ch}L(\Lambda_0+2\Lambda_1)$, where $L(\Lambda_0+2\Lambda_1)$ is the irreducible highest weight $A_1^{(1)}$-module of level 3 with highest weight $\Lambda_0+2\Lambda_1$.

RHS: principal specialisation of the Weyl–Kac character formula

LHS: comes from the construction of a basis of $L(\Lambda_0+2\Lambda_1)$ using vertex operators

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LHS: comes from the construction of a basis of $L(\Lambda_0 + 2\Lambda_1)$ using vertex operators.

Very rough idea:

- Start with a spanning set of $L(\Lambda_0 + 2\Lambda_1)$: here, monomials of the form $Z_1^{f_1} \dots Z_s^{f_s}$ for $s, f_1, \dots, f_s \in \mathbb{N}_{\geq 0}$.
- Using Lie theory, reduce this spanning set: here, one should remove all monomials containing Z_j^2 or $Z_j Z_{j+1}$.
- Show that the obtained set is a basis of the representation (difficult).

Partition identities and characters

With Lepowsky and Wilson's approach (vertex operators + Weyl–Kac character formula): discovery of many new partition identities yet unknown to combinatorialists

- Meurman–Primc 1987: higher levels of $A_1^{(1)}$
- Capparelli 1993: level 3 standard modules of $A_2^{(2)}$
- Siladić 2002: twisted level 1 modules of $A_2^{(2)}$
- Nandi 2014: level 4 standard modules of $A_2^{(2)}$
- Primc and Šikić 2016: level k standard modules of $C_n^{(1)}$

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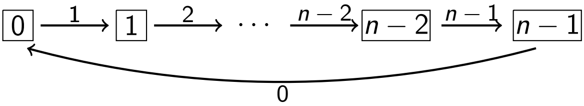
Combinatorics can also be used to find **explicitly positive formulas** for characters, for example with the theory of **perfect crystals**.

Outline

Crystals: "combinatorial representations" of Lie algebras

Crystal for the affine Lie algebra $A_{n-1}^{(1)}$ at level 1:

$\mathcal{B}$:



If $b_1 \xrightarrow{i} b_2$, we write $\tilde{f}_i b_1 = b_2$, or equivalently $b_1 = \tilde{e}_i b_2$.

Let $\varphi_i(b)$ (resp. $\varepsilon_i(b)$) denote the length of the maximal chain of i-arrows coming out of (resp. arriving in) b. Example: $\varphi_1(0) = 1$, $\varphi_2(0) = 0$.

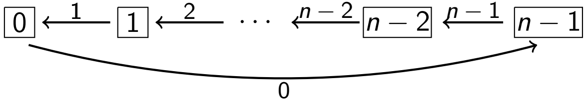
To each vertex $b \in \mathcal{B}$ is associated a weight $\overline{\mathrm{wt}}(b)$.

A *perfect crystal* satisfies a few additional properties and one can associate to it a so-called *energy function*.

Crystals: "combinatorial representations" of Lie algebras

The dual of $\mathcal{B}$ is also a crystal:

$\mathcal{B}^\vee$:



We have $\tilde{f}_i b_1 = b_2$ in $\mathcal{B}$ if and only if $\tilde{e}_i b_1^\vee = b_2^\vee$, and $\overline{\mathrm{wt}}(b^\vee) = -\overline{\mathrm{wt}}(b)$.

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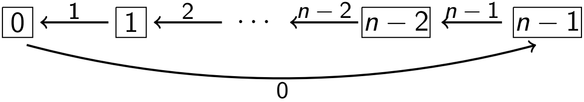

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If $\mathcal{B}_1$ and $\mathcal{B}_2$ are crystals, then we can define a crystal $\mathcal{B}_1 \otimes \mathcal{B}_2$ with the following arrows:

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$$

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Then $\overline{\mathrm{wt}}(b_1 \otimes b_2) = \overline{\mathrm{wt}}(b_1) + \overline{\mathrm{wt}}(b_2)$.

Example: $A_1^{(1)}$ at level 1

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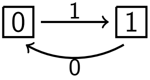
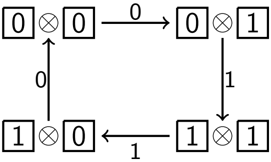


$\mathcal{B} \otimes \mathcal{B}^\vee$:



Energy functions

Let $\mathcal{B}$ be a perfect crystal.

Definition

An *energy function* on $\mathcal{B} \otimes \mathcal{B}$ is a map $H : \mathcal{B} \otimes \mathcal{B} \to \mathbb{Z}$ satisfying for all i,

$$H\left(\tilde{e}_i(b_1 \otimes b_2)\right) = \begin{cases} H(b_1 \otimes b_2) & \text{if } \ i \neq 0, \\ H(b_1 \otimes b_2) + 1 & \text{if } \ i = 0 \text{ and } \varphi_0(b_1) \geq \varepsilon_0(b_2) \\ H(b_1 \otimes b_2) - 1 & \text{if } \ i = 0 \text{ and } \varphi_0(b_1) < \varepsilon_0(b_2). \end{cases}$$

By definition, the value of $H(b_1 \otimes b_2)$ determines the values $H(b_1' \otimes b_2')$ of all the vertices $b_1' \otimes b_2'$ which are in the same connected component as $b_1 \otimes b_2$.

The $(\mathrm{KMN})^2$ crystal base character formula (1992)

To each dominant integral weight λ, one can associate a **ground state path**

$$\mathfrak{p}_\lambda = (g_k)_{k=0}^{\infty} = \cdots \otimes g_{k+1} \otimes g_k \otimes \cdots \otimes g_1 \otimes g_0,$$

where $g_i \in \mathcal{B}$ for all i.
A tensor product $\mathfrak{p} = (p_k)_{k=0}^{\infty} = \cdots \otimes p_{k+1} \otimes p_k \otimes \cdots \otimes p_1 \otimes p_0$ of elements $p_k \in \mathcal{B}$ is said to be a λ-*path* if $p_k = g_k$ for k large enough. Let $\mathcal{P}(\lambda)$ denote the set of λ-paths .

Theorem (Kang–Kashiwara–Misra–Miwa–Nakashima–Nakayashiki)

Let $L(\lambda)$ be an irreducible highest weight module of weight λ. We have

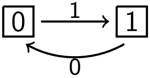
$$\mathrm{ch}(L(\lambda)) = \sum_{\mathfrak{p} \in \mathcal{P}(\lambda)} e^{\mathrm{wt}\,\mathfrak{p}},$$

where $\mathrm{wt}\,\mathfrak{p}$ is defined in terms of the energy function and the simple roots.

Outline

Example: Primc's identity on $A_1^{(1)}$ at level 1

$\mathcal{B}$:



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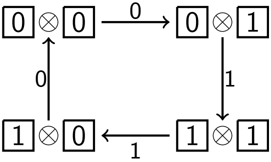

$$
\begin{aligned}
0 \otimes 1 &\longleftrightarrow a,\\
0 \otimes 0 &\longleftrightarrow b,\\
1 \otimes 1 &\longleftrightarrow c,\\
1 \otimes 0 &\longleftrightarrow d.
\end{aligned}
$$

For Λ_0, the ground state path is $\mathfrak{p}_{\Lambda_0} = \cdots \otimes b \otimes b$.

Primc's identity (conjectured using KMN^2 and Weyl–Kac)

From Weyl–Kac, the principal specialisation of $e^{-\Lambda_0}\mathrm{ch}L(\Lambda_0)$ is equal to $\frac{1}{(q;q)_\infty}$.

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Let P be the energy function in $(\mathcal{B} \otimes \mathcal{B}^\vee) \otimes (\mathcal{B} \otimes \mathcal{B}^\vee)$ for $A_1^{(1)}$ at level 1.
Consider partitions in four colours a, b, c, d, with difference conditions

$$P = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 2 & 1 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix} \end{array}.$$

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Primc (1998) conjectured that after performing the dilations

$$k_a \to 2k-1, k_b \to 2k, k_c \to 2k, k_d \to 2k+1,$$

equivalent to the **principal specialisation**, the generating function for these partitions (not keeping track of the colours) also gives $e^{-\Lambda_0}\mathrm{ch}L(\Lambda_0)$.

Refinement of Primc's identity

Theorem (D.–Lovejoy 2017)

Let $P(n;k,\ell,m)$ denote the number of partitions satisfying the difference conditions of matrix P, with k parts coloured a, ℓ parts coloured c and m parts coloured d. Then

$$\sum_{n,k,\ell,m\geq 0} P(n;k,\ell,m)q^n a^k c^\ell d^m = \frac{(-aq;q^2)_\infty(-dq;q^2)_\infty}{(q;q)_\infty(cq;q^2)_\infty}.$$

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We we perform the principal specialisation, the product side indeed becomes $\frac{1}{(q;q)_\infty}$.

Connecting the KMN^2 character formula to partitions

Reminder: (KMN)^2 character formula

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In $A_{n-1}^{(1)}$, the fundamental weights are $\Lambda_0, \ldots, \Lambda_{n-1}$. With respect to the crystal $\mathcal{B} \otimes \mathcal{B}^\vee$, they all have *constant ground state paths*.

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where $\mathcal{P}(\lambda)$ is the set of λ-paths.

In $A_{n-1}^{(1)}$, the fundamental weights are $\Lambda_0, \ldots, \Lambda_{n-1}$. With respect to the crystal $\mathcal{B} \otimes \mathcal{B}^\vee$, they all have *constant ground state paths*.

Goal: relate λ-paths to coloured partitions to give a purely combinatorial character formula in terms of partition generating functions.

Grounded partitions

Definition

Let $\mathcal{C}$ be a set of colours and $c_g \in \mathcal{C}$. Let $\succ$ be a binary relation defined on the coloured integers $\mathbb{Z}_\mathcal{C} = \{k_c : k \in \mathbb{Z}, c \in \mathcal{C}\}$.
A *grounded partition* with ground c_g and relation $\succ$ is a finite sequence $(\pi_0, \ldots, \pi_s)$ of coloured integers, such that

- for all $i \in \{0, \ldots, s-1\}$, $\pi_i \succ \pi_{i+1}$,
- $\pi_s = 0_{c_g}$,
- $\pi_{s-1} \neq 0_{c_g}$.

Let $\mathcal{P}_{c_g}^{\succ}$ denote the set of such partitions.

Example

Let $\mathcal{C} = c_1, c_2, c_3$, and for all $k \in \mathbb{Z}, c, c' \in \mathcal{C}$, $k_c \succ k'_{c'} \Leftrightarrow k = k' + 1$.
The sequence $(4_{c_1}, 3_{c_3}, 2_{c_2}, 1_{c_2}, 0_{c_1})$ is a grounded partition with ground c_1 and relation $\succ$.

Connection with ground state paths

Let $\mathcal{B}$ a perfect crystal and λ be a highest weight with constant ground state path $\mathfrak{p}_\lambda = \cdots \otimes g \otimes g \otimes g$.
Let H be an energy function on $\mathcal{B} \otimes \mathcal{B}$ such that $H(g \otimes g) = 0$.
Let $\mathcal{C}_{\mathcal{B}} = \{c_b : b \in \mathcal{B}\}$ be the set of colours indexed by the vertices of $\mathcal{B}$.
We define the binary relations $\gtrdot$ and $\gg$ on $\mathbb{Z}_{\mathcal{C}_{\mathcal{B}}}$ by

$$k_{c_b} \gtrdot k'_{c_{b'}} \text{ if and only if } k - k' = H(b' \otimes b),$$

$$k_{c_b} \gg k'_{c_{b'}} \text{ if and only if } k - k' \geq H(b' \otimes b).$$

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Theorem (D.–Konan 2019)

The set of λ-paths is in bijection with the set of grounded partitions $\mathcal{P}^{\gtrdot}_{c_g}$.

Theorem (D.–Konan 2019)

There is a bijection between $\mathcal{P}^{\gg}_{c_g}$ and $\mathcal{P}^{\gtrdot}_{c_g} \times \mathcal{P}_{c_g}$, where $\mathcal{P}_{c_g}$ is the set of coloured partitions where all parts have colour c_g.

New combinatorial character formula

Theorem (D.–Konan 2019)

Let $L(\lambda)$ *be an irreducible highest weight module of weight* λ *with constant ground state path. Denoting by* $C(\pi)$ *the colour sequence of* π *and setting* $q = e^{-\delta/d_0}$ *and* $c_b = e^{\mathrm{wt}\, b}$ *for all* $b \in \mathcal{B}$*, we have*

$$\sum_{\pi \in \mathcal{P}_{c_g}^{\gtrdot}} C(\pi) q^{|\pi|} = e^{-\lambda} \mathrm{ch}(L(\lambda)),$$

$$\sum_{\pi \in \mathcal{P}_{c_g}^{\gg}} C(\pi) q^{|\pi|} = \frac{e^{-\lambda} \mathrm{ch}(L(\lambda))}{(q;q)_\infty}.$$

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Primc's original conjecture can be deduced by using on one hand the principal specialisation in this theorem, and on the other hand the principal specialisation in the Weyl–Kac character formula.

Another example of application (from $A_1^{(1)}$ at level n)

Theorem (D.–Hardiman–Konan 2022)

Let n be a non-negative integer. Let $\mathcal{C}_n$ denote the set of $(n+1)$-coloured partitions $(\lambda_1, \dots, \lambda_s)$, where each part is a non-negative integer indexed by a colour taken from $\{c_0, c_1, \dots, c_n\}$, such that for all $1 \leq i \leq s-1$,

$$\lambda_i - \lambda_{i+1} = |u_i - u_{i+1}|,$$

where for all $i \in \{1, \dots, s\}$, λ_i has colour c_{u_i}. Let $C_{i,n}(m)$ be the number of $(n+1)$-coloured partitions of m in $\mathcal{C}_n$ such that the last part is 0_{c_i} and the penultimate part has colour different from c_i. We have

$$\sum_{m \geq 0} C_{i,n}(m) q^m = \frac{(q^{i+1}, q^{n-i+1}, q^{n+2}; q^{n+2})_\infty}{(q;q^2)_\infty (q;q)_\infty}.$$

Outline

The idea

- We have seen how to combine the principal specialisation of the Weyl–Kac character formula and of our combinatorial character formula to obtain partition identities (losing information on the colours because of the specialisation).

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- We have seen how to combine the principal specialisation of the Weyl–Kac character formula and of our combinatorial character formula to obtain partition identities (losing information on the colours because of the specialisation).
- If we do not perform the principal specialisation, we can study the grounded partitions combinatorially (keeping as many colours as possible) and deduce **non-specialised** character formulas. They also have the advantage of having **manifestly positive coefficients** in the $e^{-\alpha_i}$'s.

Outline

Another identity of Primc

Studying the crystal of $A_2^{(1)}$ at level 1, Primc proved that, after performing the principal specialisation, the generating function for coloured partitions satisfying the difference conditions given by the energy

$$\begin{array}{c|ccccccccc} & a_2b_0 & a_2b_1 & a_1b_0 & a_0b_0 & a_2b_2 & a_1b_1 & a_0b_1 & a_1b_2 & a_0b_2 \\ \hline a_2b_0 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 \\ a_2b_1 & 1 & 2 & 1 & 1 & 2 & 1 & 2 & 2 & 2 \\ a_1b_0 & 1 & 1 & 2 & 1 & 1 & 2 & 2 & 2 & 2 \\ a_0b_0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ a_2b_2 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 2 & 2 \\ a_1b_1 & 0 & 1 & 0 & 1 & 1 & 0 & 2 & 1 & 2 \\ a_0b_1 & 0 & 1 & 0 & 1 & 1 & 0 & 2 & 1 & 2 \\ a_1b_2 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 2 & 2 \\ a_0b_2 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 2 \end{array}$$

becomes

$$\frac{1}{(q;q)_\infty}.$$

The energy function in a level 1 perfect crystal for $A_n^{(1)}$

Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences of symbols. Define the set of colours: $\{a_i b_k : i, k \in \mathbb{N}\}$.

Definition

For all $i, k, i', k' \in \mathbb{N}$, let

$$\Delta(a_i b_k, a_{i'} b_{k'}) = \chi(i \geq i') - \chi(i = k = i') + \chi(k \leq k') - \chi(k = i' = k'),$$

where $\chi(prop)$ equals 1 if *prop* is true and 0 otherwise.

Theorem (D.–Konan (2022))

The energy of the crystal $\mathcal{B} \otimes \mathcal{B}^\vee$ *of* $A_n^{(1)}$ *at level* 1 *such that* $H((0 \otimes 0^\vee) \otimes (0 \otimes 0^\vee)) = 0$ *satisfies for all* $k, \ell, k', \ell' \in \{0, \ldots, n\}$,

$$H((\ell' \otimes k'^\vee) \otimes (\ell \otimes k^\vee)) = \Delta(a_k b_\ell; a_{k'} b_{\ell'}).$$

Generalisation of Primc's identity

For every positive integer n, let $\mathcal{P}_n$ denote the set of partitions with colours $\{a_i b_k : 0 \leq i, k \leq n-1\}$, satisfying the difference conditions Δ.

Let $P_n(m; u_0, \ldots, u_{n-1}; v_0, \ldots, v_{n-1})$ denote the number of n^2-coloured partitions of m which belong to $\mathcal{P}_n$, where for $i \in \{0, \ldots, n-1\}$, the symbol a_i (resp. b_i) appears u_i (resp. v_i) times in its colour sequence.

Theorem (D.–Konan (2022))

For every positive integer n, we have, after setting $a_i = b_i^{-1}$ for all i,

$$\sum_{m, u_0, \ldots, u_{n-1}, v_0, \ldots, v_{n-1} \geq 0} P_n(m; u_0, \ldots, u_{n-1}; v_0, \ldots, v_{n-1}) q^m b_0^{v_0 - u_0} \cdots b_{n-1}^{v_{n-1} - u_{n-1}}$$

$$= [x^0] \prod_{i=0}^{n-1} (-b_i^{-1} x q; q)_\infty (-b_i x^{-1}; q)_\infty .$$

Purely combinatorial proof by studying the coloured partitions.

Principal specialisation

In his paper, Primc used the principal specialisation, which corresponds to:

$$\begin{cases} q & \mapsto & q^n \\ b_i & \mapsto & q^i \quad \text{for all } i \in \{0, \dots, n-1\}. \end{cases}$$

Corollary (D.–Konan (2022))

Let n be a positive integer. By performing the dilations above, the generating function for the coloured partitions in $\mathcal{P}_n$ becomes:

$$\begin{aligned} [x^0] \prod_{i=0}^{n-1} (-q^{n-i}x; q^n)_\infty (-q^i x^{-1}; q^n)_\infty &= [x^0](-qx; q)_\infty (-x^{-1}; q)_\infty \\ &= \frac{1}{(q;q)_\infty}. \end{aligned}$$

The cases $n = 2$ and $n = 3$ recover Primc's original results.

Non-specialised character formula for $A_n^{(1)}$

Combining our new character formula with our generalisation of Primc's identity, we obtain:

Theorem (D.–Konan)

Let n be a positive integer, and let $\Lambda_0, \ldots, \Lambda_n$ be the fundamental weights of $A_n^{(1)}$. By setting $e^{\mathrm{wt}\, v_i} = b_i$ and $e^{-\delta} = q$, we have:

$$\frac{e^{-\Lambda_\ell}\mathrm{ch}(L(\Lambda_\ell))}{(q;q)_\infty} = [x^0]\Bigg(\prod_{i=0}^{\ell-1}(-b_i^{-1}x;q)_\infty(-b_i x^{-1}q;q)_\infty \times \prod_{i=\ell}^{n-1}(-b_i^{-1}xq;q)_\infty(-b_i x^{-1};q)_\infty\Bigg).$$

This recovers a character formula of Kac–Peterson (1984) and gives a new expression as a sum of infinite products with positive coefficients.

Outline

Multi-grounded partitions

Goal: extend the idea of grounded partitions to treat the cases of crystals where the ground state paths are not constant.

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Definition

Let $\mathcal{C}$ be a set of colors and $\succ$ a binary relation defined on $\mathbb{Z}_{\mathcal{C}}$. Suppose that there exist some colors $c_{g_0}, \dots, c_{g_{t-1}}$ in $\mathcal{C}$ and *unique* coloured integers $u^{(0)}_{c_{g_0}}, \dots, u^{(t-1)}_{c_{g_{t-1}}}$ such that

$$u^{(0)} + \cdots + u^{(t-1)} = 0,$$
$$u^{(0)}_{c_{g_0}} \succ u^{(1)}_{c_{g_1}} \succ \cdots \succ u^{(t-1)}_{c_{g_{t-1}}} \succ u^{(0)}_{c_{g_0}}.$$

Then a *multi-grounded partition* with ground $c_{g_0}, \dots, c_{g_{t-1}}$ and relation $\succ$ is a finite sequence $\pi = (\pi_0, \cdots, \pi_{s-1}, u^{(0)}_{c_{g_0}}, \dots, u^{(t-1)}_{c_{g_{t-1}}})$ of coloured integers such that $\pi_i \succ \pi_{i+1}$ for all i, and
$(\pi_{s-t}, \cdots, \pi_{s-1}) \neq (u^{(0)}_{c_{g_0}}, \dots, u^{(t-1)}_{c_{g_{t-1}}})$ in terms of coloured integers.
The set of these multi-grounded partitions is denoted by $\mathcal{P}^{\succ}_{c_{g_0} \cdots c_{g_{t-1}}}$.

Example

Take $\mathcal{C} = \{c_1, c_2, c_3\}$,

$$M = \begin{pmatrix} 2 & 2 & 2 \\ 0 & 0 & 2 \\ -2 & 0 & 2 \end{pmatrix},$$

and define the relation $\succ$ on $\mathbb{Z}_\mathcal{C}$ by $k_{c_b} \succ k'_{c_{b'}}$ if and only if $k - k' \geq M_{b,b'}$.
If we choose $(g_0, g_1) = (1, 3)$, the pair $(u^{(0)}, u^{(1)}) = (1, -1)$ is the unique pair satisfying the conditions

$$u^{(0)} + u^{(1)} = 0,$$
$$u^{(0)}_{c_1} \succ u^{(1)}_{c_3} \succ u^{(0)}_{c_1}.$$

The sequences $(3_{c_3}, 3_{c_2}, 3_{c_1}, -1_{c_3}, 1_{c_1}, -1_{c_3})$ and $(1_{c_3}, 3_{c_1}, 1_{c_3}, 3_{c_1}, -1_{c_3}, 1_{c_1}, -1_{c_3})$ are multi-grounded partitions with ground c_1, c_3 and relation $\succ$,
$(1_{c_1}, -1_{c_3}, 1_{c_1}, -1_{c_3})$ and $(2_{c_1}, 1_{c_1}, -1_{c_3})$ are not.

Non-constant ground state paths

Let $\mathcal{B}$ be a crystal of level ℓ, let λ be a dominant weight, and let

$$\mathfrak{p}_\lambda = (g_k)_{k=0}^{\infty} = \quad \cdots \otimes g_{k+1} \otimes g_k \otimes \cdots \otimes g_1 \otimes g_0$$

be the corresponding ground state path. It is always periodic. Let t denote the period of $\mathfrak{p}_\lambda$, i.e. the smallest positive integer k such that $g_{i+k} = g_i$ for all $i \geq 0$.

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Let H be an energy function on $\mathcal{B} \otimes \mathcal{B}$, and define

$$H_\lambda(b \otimes b') := H(b \otimes b') - \frac{1}{t} \sum_{k=0}^{t-1} H(g_{k+1} \otimes g_k).$$

Thus we have

$$\sum_{k=0}^{t-1} H_\lambda(g_{k+1} \otimes g_k) = 0.$$

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Thus we have

$$\sum_{k=0}^{t-1} H_\lambda(g_{k+1} \otimes g_k) = 0.$$

Let D be a positive integer such that $DH_\lambda(\mathcal{B} \otimes \mathcal{B}) \subset \mathbb{Z}$ and $\frac{1}{t} \sum_{k=0}^{t-1} (k+1) DH_\lambda(g_{k+1} \otimes g_k) \in \mathbb{Z}$.

Non-constant ground state paths

Let us define the relations on $\mathbb{Z}_{\mathcal{C}_\mathcal{B}}$:

$$k_{c_b} \gtrdot k'_{c_{b'}} \iff k - k' = DH_\lambda(b' \otimes b),$$
$$k_{c_b} \gg k'_{c_{b'}} \iff k - k' \geq DH_\lambda(b' \otimes b).$$

Theorem (D.–Konan 2022)

There is a bijection between the set of λ-paths $\mathcal{P}(\lambda)$ and the set ${}_t\mathcal{P}^{\gtrdot}_{c_{g_0}\cdots c_{g_{t-1}}}$ of multi-grounded partitions of $\mathcal{P}^{\gtrdot}_{c_{g_0}\cdots c_{g_{t-1}}}$ whose number of parts is divisible by t.

Theorem (D.–Konan 2022)

Let ${}^d\mathcal{P}$ be the set of partitions where all parts are divisible by d. There is a bijection between ${}_t\mathcal{P}^{\gtrdot}_{c_{g_0}\cdots c_{g_{t-1}}} \times {}^d\mathcal{P}$ and ${}^d_t\mathcal{P}^{\gg}_{c_{g_0}\cdots c_{g_{t-1}}}$, where ${}^d_t\mathcal{P}^{\gg}_{c_{g_0}\cdots c_{g_{t-1}}}$ is the set of $\pi \in {}_t\mathcal{P}^{\gg}_{c_{g_0}\cdots c_{g_{t-1}}}$ such that for all k,

$\pi_k - \pi_{k+1} - DH_\lambda(p_{k+1} \otimes p_k) \in d\mathbb{Z}_{\geq 0}$, where $c(\pi_k) = c_{p_k}$ and $\pi_s = u^{(0)}_{c_{g_0}}$.

A general character formula

Theorem (D.–Konan 2022)

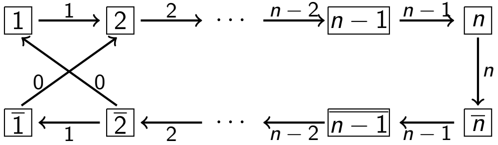
Let $L(\lambda)$ be an irreducible highest weight module of weight λ ~~with constant ground state path~~. Setting $q = e^{-\delta/(d_0 D)}$ and $c_b = e^{\overline{\mathrm{wt}} b}$ for all $b \in \mathcal{B}$, we have $c_{g_0} \cdots c_{g_{t-1}} = 1$, and the character of the irreducible highest weight module $L(\lambda)$ is given by the following expressions:

$$\sum_{\mu \in {}_t\mathcal{P}^{>}_{c_{g_0} \cdots c_{g_{t-1}}}} C(\pi) q^{|\pi|} = e^{-\lambda} \mathrm{ch}(L(\lambda)),$$

$$\sum_{\pi \in {}^d_t\mathcal{P}^{\gg}_{c_{g_0} \cdots c_{g_{t-1}}}} C(\pi) q^{|\pi|} = \frac{e^{-\lambda} \mathrm{ch}(L(\lambda))}{(q^d; q^d)_\infty}.$$

Example: character of Λ_0 in $A_{2n-1}^{(2)} (n \geq 3)$

$\mathcal{B}$:



Ground state path: $\mathfrak{p}_{\Lambda_0} = \cdots \otimes \overline{1} \otimes 1 \otimes \overline{1} \otimes 1 \otimes \overline{1}$,

$$\text{energy } H = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ n \\ \overline{n} \\ \vdots \\ \overline{2} \\ \overline{1} \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & \cdots & n & \overline{n} & \cdots & \overline{2} & \overline{1} \end{array} \\ \begin{pmatrix} 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ 0 & \ddots & & & & & & \vdots \\ \vdots & \ddots & \ddots & & & 1^* & & \vdots \\ \vdots & & \ddots & \ddots & & & & \vdots \\ \vdots & & & \ddots & \ddots & & & \vdots \\ \vdots & & 0^* & & \ddots & \ddots & & \vdots \\ 0 & 0 & & & & \ddots & \ddots & \vdots \\ -1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix} \end{array}.$$

Example: character of Λ_0 in $A_{2n-1}^{(2)}(n \geq 3)$

We have $H(1 \otimes \overline{1}) + H(\overline{1} \otimes 1) = 0$, so $H_{\Lambda_0} = H$.

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We have $H(1 \otimes \overline{1}) + H(\overline{1} \otimes 1) = 0$, so $H_{\Lambda_0} = H$.

We apply our character formula with $d = 2$ and $D = 2$ and obtain

$$\sum_{\pi \in {}_2^2\mathcal{P}^{\gg}_{c_{\overline{1}} c_1}} C(\pi) q^{|\pi|} = \frac{e^{-\Lambda_0} \mathrm{ch}(L(\Lambda_0))}{(q^2; q^2)_\infty},$$

where $q = e^{-\delta/2}$ and $c_b = e^{\overline{\mathrm{wt}} b}$ for all $b \in \mathcal{B}$.

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$$\sum_{\pi \in {}^2_2\mathcal{P}^{\gg}_{c_{\overline{1}} c_1}} C(\pi) q^{|\pi|} = \frac{e^{-\Lambda_0} \mathrm{ch}(L(\Lambda_0))}{(q^2; q^2)_\infty},$$

where $q = e^{-\delta/2}$ and $c_b = e^{\overline{\mathrm{wt}} b}$ for all $b \in \mathcal{B}$.

Thus we must compute the generating function for ${}^2_2\mathcal{P}^{\gg}_{c_{\overline{1}} c_1}$, the set of multi-grounded partitions $\pi = (\pi_0, \ldots, \pi_{2s-1}, -1_{c_{\overline{1}}}, 1_{c_1})$ with relation $\gg$ and ground $c_{\overline{1}}, c_1$, **having an even number of parts**, such that for all $k \in \{0, \ldots, 2s-1\}$,

$$\pi_k - \pi_{k+1} - 2H(p_{k+1} \otimes p_k) \in 2\mathbb{Z}_{\geq 0},$$

where $c(\pi_k) = c_{p_k}$ and $\pi_{2s} = -1_{c_{\overline{1}}}$.

Example: character of Λ_0 in $A_{2n-1}^{(2)} (n \geq 3)$

$$H = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ n \\ \overline{n} \\ \vdots \\ \overline{2} \\ \overline{1} \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & \cdots & n & \overline{n} & \cdots & \overline{2} & \overline{1} \end{array} \\ \begin{pmatrix} 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ 0 & \ddots & & & & & & \vdots \\ \vdots & \ddots & \ddots & & & 1^* & & \vdots \\ \vdots & & \ddots & \ddots & & & & \vdots \\ \vdots & & & \ddots & \ddots & & & \vdots \\ \vdots & & 0^* & & \ddots & \ddots & & \vdots \\ 0 & 0 & & & & \ddots & \ddots & \vdots \\ -1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix} \end{array}.$$

By the values of H, the condition $\pi_k - \pi_{k+1} - 2H(p_{k+1} \otimes p_k) \in 2\mathbb{Z}_{\geq 0}$, and the fact that $u^{(0)} = -1$, the multi-grounded partitions of ${}^2_2\mathcal{P}^{\gg}_{c_{\overline{1}}c_1}$ have parts with odd sizes.

Example: character of Λ_0 in $A_{2n-1}^{(2)} (n \geq 3)$

$$H = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ n \\ \overline{n} \\ \vdots \\ \overline{2} \\ \overline{1} \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & \cdots & n & \overline{n} & \cdots & \overline{2} & \overline{1} \end{array} \\ \begin{pmatrix} 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ 0 & \ddots & & & & & & \vdots \\ \vdots & \ddots & \ddots & & & 1^* & & \vdots \\ \vdots & & \ddots & \ddots & & & & \vdots \\ \vdots & & & \ddots & \ddots & & & \vdots \\ \vdots & & 0^* & & \ddots & \ddots & & \vdots \\ 0 & 0 & & & & \ddots & \ddots & \vdots \\ -1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix} \end{array}.$$

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The relation $\gg$ corresponds to the following partial order on the set of coloured odd integers:

$$\begin{array}{c} (-1)_{c_{\overline{1}}} \\ 1_{c_1} \end{array} \ll 1_{c_2} \ll \cdots \ll 1_{c_n} \ll 1_{c_{\overline{n}}} \ll \cdots \ll 1_{c_{\overline{2}}} \ll \begin{array}{c} 1_{c_{\overline{1}}} \\ 3_{c_1} \end{array} \ll 3_{c_2} \ll \cdots .$$

Example: character of Λ_0 in $A_{2n-1}^{(2)} (n \geq 3)$

$$H = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ n \\ \overline{n} \\ \vdots \\ \overline{2} \\ \overline{1} \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & \cdots & n & \overline{n} & \cdots & \overline{2} & \overline{1} \end{array} \\ \left(\begin{array}{cccccccc} 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ 0 & \ddots & & & & & & \vdots \\ \vdots & \ddots & \ddots & & & 1^* & & \vdots \\ \vdots & & \ddots & \ddots & & & & \vdots \\ \vdots & & & \ddots & \ddots & & & \vdots \\ \vdots & & 0^* & & \ddots & \ddots & & \vdots \\ 0 & 0 & & & & \ddots & \ddots & \vdots \\ -1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{array} \right) \end{array} .$$

By the values of H, the condition $\pi_k - \pi_{k+1} - 2H(p_{k+1} \otimes p_k) \in 2\mathbb{Z}_{\geq 0}$, and the fact that $u^{(0)} = -1$, the multi-grounded partitions of ${}^2_2\mathcal{P}^{\gg}_{c_{\overline{1}} c_1}$ have parts with odd sizes.

The relation $\gg$ corresponds to the following partial order on the set of coloured odd integers:

$$\begin{array}{c} (-1)_{c_{\overline{1}}} \\ 1_{c_1} \end{array} \ll 1_{c_2} \ll \cdots \ll 1_{c_n} \ll 1_{c_{\overline{n}}} \ll \cdots \ll 1_{c_{\overline{2}}} \ll \begin{array}{c} 1_{c_{\overline{1}}} \\ 3_{c_1} \end{array} \ll 3_{c_2} \ll \cdots .$$

Only parts coloured c_1 and $c_{\overline{1}}$ can appear several times, in sequences of the form

$$\cdots \ll (2k-1)_{c_{\overline{1}}} \ll (2k+1)_{c_1} \ll (2k-1)_{c_{\overline{1}}} \ll \cdots \ll (2k-1)_{c_{\overline{1}}} \ll \cdots .$$

Example: character of Λ_0 in $A^{(2)}_{2n-1} (n \geq 3)$

$$\begin{matrix}(-1)_{c_{\overline{1}}} \\ 1_{c_1}\end{matrix} \ll 1_{c_2} \ll \cdots \ll 1_{c_n} \ll 1_{c_{\overline{n}}} \ll \cdots \ll 1_{c_{\overline{2}}} \ll \begin{matrix}1_{c_{\overline{1}}} \\ 3_{c_1}\end{matrix} \ll 3_{c_2} \ll \cdots ,$$

where parts coloured c_1 and $c_{\overline{1}}$ can repeat in sequences

$$\cdots \ll (2k-1)_{c_{\overline{1}}} \ll (2k+1)_{c_1} \ll (2k-1)_{c_{\overline{1}}} \ll \cdots \ll (2k-1)_{c_{\overline{1}}} \ll \cdots .$$

For fixed $k \geq 1$, sequences of parts coloured c_1 and $c_{\overline{1}}$ are generated by

$$\frac{(1+c_{\overline{1}}q^{2k-1})(1+c_1q^{2k+1})}{(1-c_{\overline{1}}c_1q^{4k})}.$$

For $k=0$, the sequence $(1_{c_1}, (-1)_{c_{\overline{1}}}, 1_{c_1})$ can occur at the end of the partitions grounded in $c_{\overline{1}}, c_1$, but $((-1)_{c_{\overline{1}}}, 1_{c_1}, (-1)_{c_{\overline{1}}}, 1_{c_1})$ cannot.
So, if we temporarily forgot the condition on the even number of parts in ${}^2_2\mathcal{P}^{\gg}_{c_{\overline{1}}c_1}$, the generation function would be

$$(1+c_1q) \cdot \frac{(-c_1q^3, -c_{\overline{1}}q, -c_2q, -c_{\overline{2}}q, \ldots, -c_nq, -c_{\overline{n}}q; q^2)_\infty}{(c_{\overline{1}}c_1q^4; q^4)_\infty}.$$

Example: character of Λ_0 in $A_{2n-1}^{(2)} (n \geq 3)$

Remark

$$\sum_{n,k\geq 0} a_{n,k} x^k q^n + \sum_{n,k\geq 0} a_{n,k} (-x)^k q^n = 2 \sum_{n,k\geq 0} a_{n,2k} x^{2k} q^n$$

Thus, the generating function for multi-grounded partitions in ${}^{2}_{2}\mathcal{P}^{\gg}_{c_{\overline{1}}c_1}$ is

$$\sum_{\pi \in {}^{2}_{2}\mathcal{P}^{\gg}_{c_{\overline{1}}c_1}} C(\pi) q^{|\pi|} = \frac{1}{2(c_{\overline{1}} c_1 q^4; q^4)_\infty} \Big((-c_1 q, -c_{\overline{1}} q, \ldots, -c_n q, -c_{\overline{n}} q; q^2)_\infty + (c_1 q, c_{\overline{1}} q, \ldots, c_n q, c_{\overline{n}} q; q^2)_\infty \Big)$$

$$= \frac{e^{-\Lambda_0} \mathrm{ch}(L(\Lambda_0))}{(q^2; q^2)_\infty},$$

where $\delta = \alpha_0 + \alpha_1 + 2\alpha_2 \cdots + 2\alpha_{n-1} + \alpha_n$,
$q = e^{-\delta/2}$ and $c_i = e^{\alpha_i + \cdots + \alpha_{n-1} + \alpha_n/2}$ for all $i \in \{1, \ldots, n\}$.

Conclusion

What we know:

- Non-specialised character formulas for level 1 standard modules of types $A_n^{(1)}$, $C_n^{(1)}$, $A_{2n}^{(2)}$, $D_{n+1}^{(2)}$, $A_{2n-1}^{(2)}$, $B_n^{(1)}$, and $D_n^{(1)}$.
- Partition identities and specialised characters for higher levels of $A_n^{(1)}$.
- Partition identities from level 1 standard modules of $C_n^{(1)}$, partially proving the Capparelli–Meurman–Primc–Primc conjecture (D.–Konan 2023).

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What we want to know:

- Non-specialised character formulas for all types and all levels.
- New partition identities arising from these crystals (Dombos 2024+ : partition identity coming from $G_2^{(2)}$ at level 1).

Thank you very much

(KMN)^2

$$\mathrm{wt}\,\mathfrak{p} = \lambda + \sum_{k=0}^{\infty} \left(\left(\overline{\mathrm{wt}}\, p_k - \overline{\mathrm{wt}}\, g_k\right) - \frac{\delta}{d_0} \sum_{j=k}^{\infty} \left(H(p_{j+1} \otimes p_j) - H(g_{j+1} \otimes g_j)\right) \right),$$

$$\mathrm{ch}(L(\lambda)) = \sum_{\mathfrak{p} \in \mathcal{P}(\lambda)} e^{\mathrm{wt}\,\mathfrak{p}}.$$