Longueur atomique dans les groupes de Weyl
(Rencontre ANR CORTIPOM)

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1. Motivations and historical context
   - Coxeter groups
   - Affine Weyl groups
   - Partitions t-cores

2. The atomic length

3. Λ_0-atomic length and t-cores

Definition of Coxeter groups

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A Coxeter group is a pair (W, S) where W is a group and $S \subset W$ is a set of generators, with the presentation

$$W = \langle S \mid (s_i s_j)^{m_{ij}} = e \rangle$$

where $m_{ii} = 1$ and $m_{ij} = m_{ji} \in \mathbb{N}_{\geq 2} \cup \{\infty\}$. The cardinality of S is called the rank of (W, S).

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Example

If we take the symmetric group $W = \mathfrak{S}_n$ then the adjacent transpositions $S := \{(i, i+1) \mid i = 1 \ldots, n-1\}$ generate W. The pair $(\mathfrak{S}_n, S)$ is a Coxeter group of rank $n-1$.

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- Coxeter groups (non necessarily finite) were then introduced in 1934 by H. S. M. Coxeter as abstractions of real reflections groups.

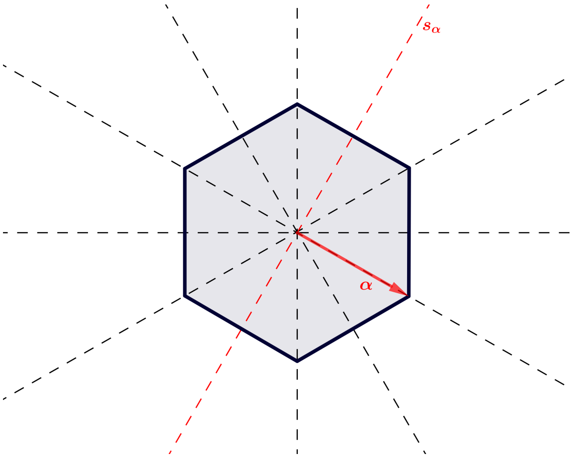

Figure – Here is a 6-gone. If we take the collection of the reflections associated to the hyperplanes on the figure then we get a subgroup of $O(\mathbb{R}^2)$ that is Coxeter group, namely the dihedral group D_6.

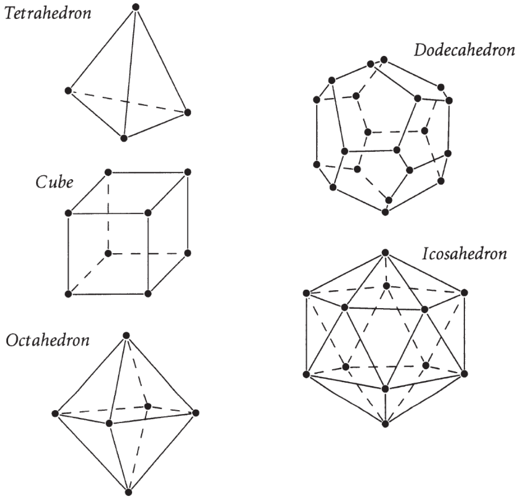

Figure – The symmetry group of each Platonic solid is a Coxeter group.

Words in Coxeter groups

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- If $m_{ij} = 2$ then we have the relation $s_i s_j = s_j s_i$ (commutativity).
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Let (W, S) be a Coxeter group with $S = \{s_1, s_2, s_3, s_4\}$. Assume that $m_{12} = 3$, $m_{13} = 2$, $m_{14} = 2$, $m_{23} = 3$, $m_{24} = 2$ and $m_{34} = 4$. Let $w = s_1 s_3 s_2 s_1 s_4 s_2 s_1$ be an element of W. Then

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Words in Coxeter groups

Let (W, S) be a Coxeter group with $S = \{s_1, s_2, \ldots, s_n\}$. Each element $w \in W$ decomposes as a word on the alphabet S, that is

$$w = s_{i_1} s_{i_2} \cdots s_{i_p}.$$

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Coxeter length via inversion set

Definition

Let (W, S) be a Coxeter group with length function ℓ. Let Φ be a root system of W with $\Phi = \Phi^+ \sqcup \Phi^-$ its usual decomposition into positive roots and negative roots. To any root $\alpha \in \Phi$ one can associate a reflexion $s_\alpha \in W$.
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Proposition

One has $\ell(w) = |N(w)|$.

Graphs of Coxeter groups

Goal : We want to represent Coxeter groups by means of graphs.

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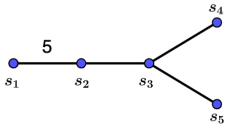

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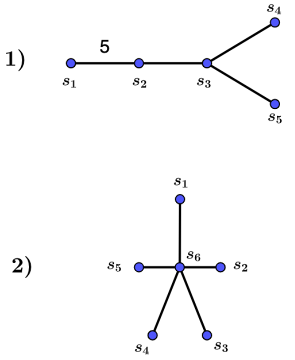

Quick classification of (irreducible) Coxeter groups

Weyl groups	Affine Weyl groups	All other Coxeter groups

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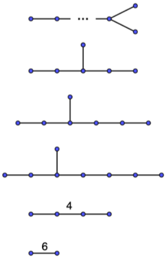
$D_n \, (n \geq 4)$		
E_6		
E_7		
E_8		
F_4		
G_2		



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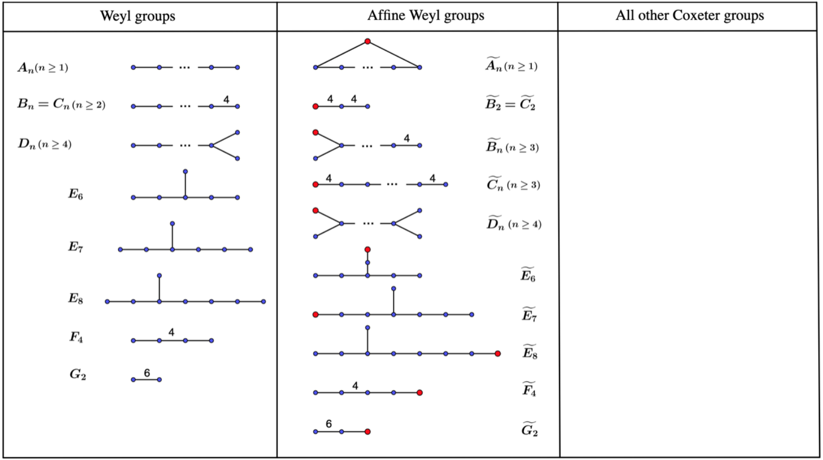

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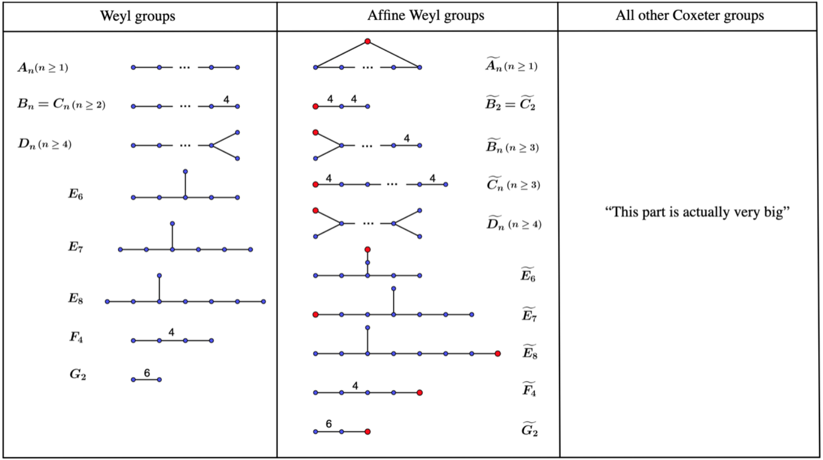

Geometrical construction of (non-twisted) affine Weyl groups

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$$H_{\alpha,k} = \{x \in V \mid (x \mid \alpha) = k\}.$$

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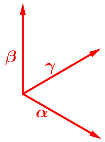
- Let $s_{\alpha,k}$ be the reflection associated to the hyperplane $H_{\alpha,k}$. We define the affine Weyl group W corresponding to Φ by

$$\begin{aligned} W &:= \langle s_{\alpha,k} \mid \alpha \in \Phi, k \in \mathbb{Z}\rangle \\ &\simeq T(M) \rtimes W_0, \end{aligned}$$

where M is the coroot lattice.

Alcoves of $A_2^{(1)}$

Example : Let $\Phi^+ = \{\alpha, \beta, \gamma = \alpha + \beta\}$ be a positive root system of A_2.



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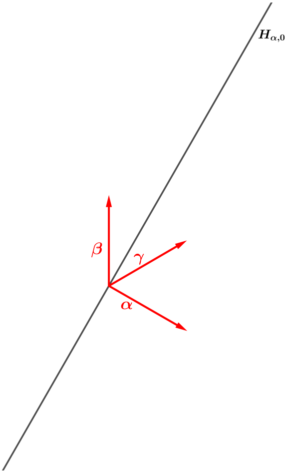

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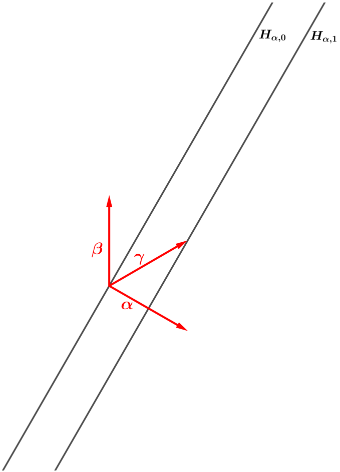

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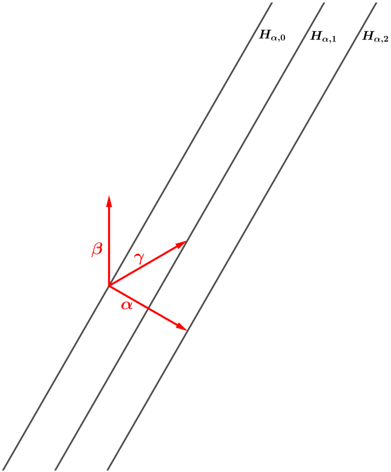

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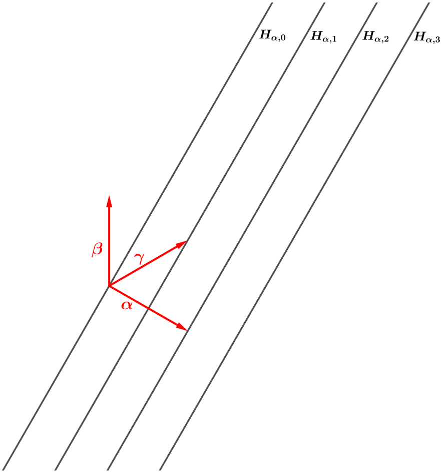

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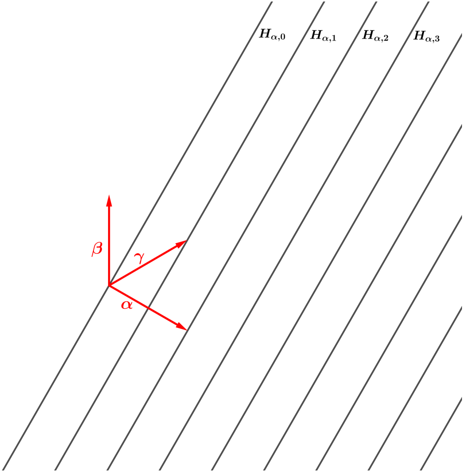

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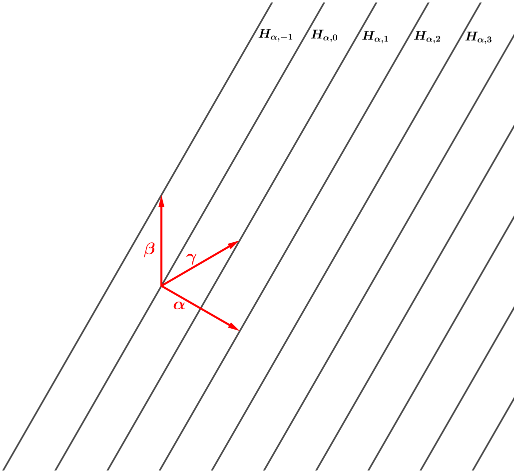

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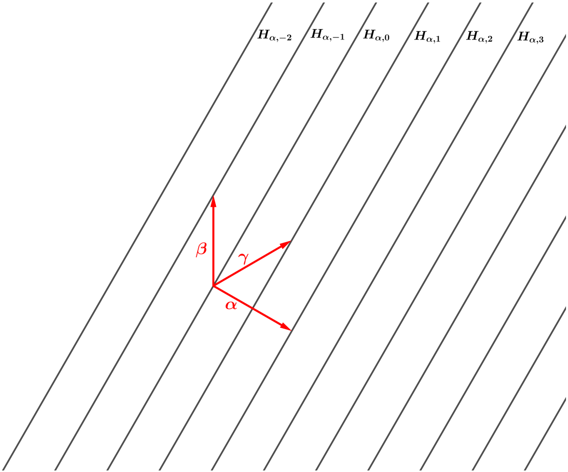

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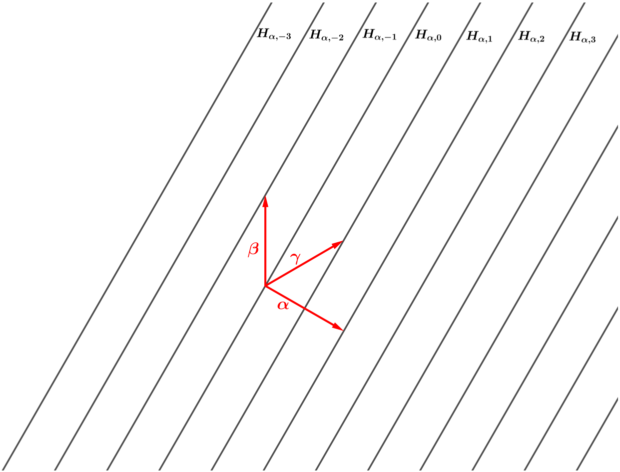

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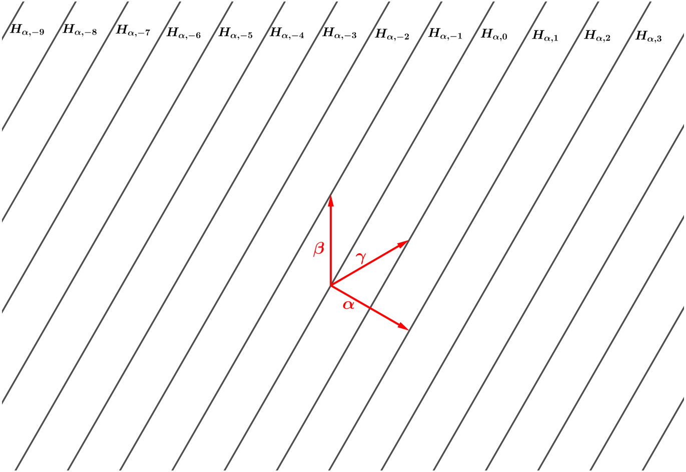

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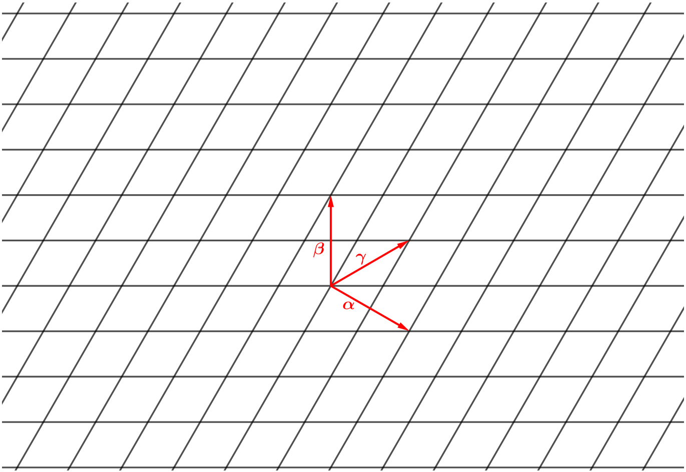

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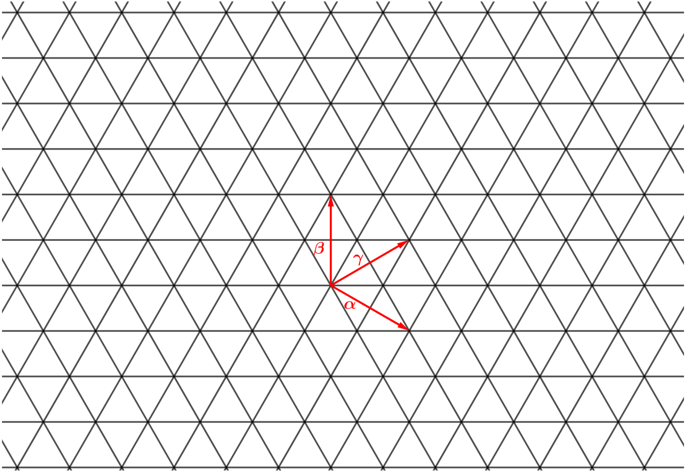

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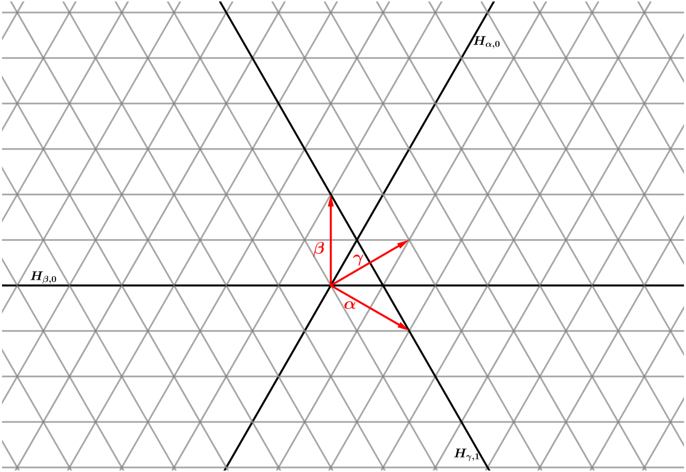

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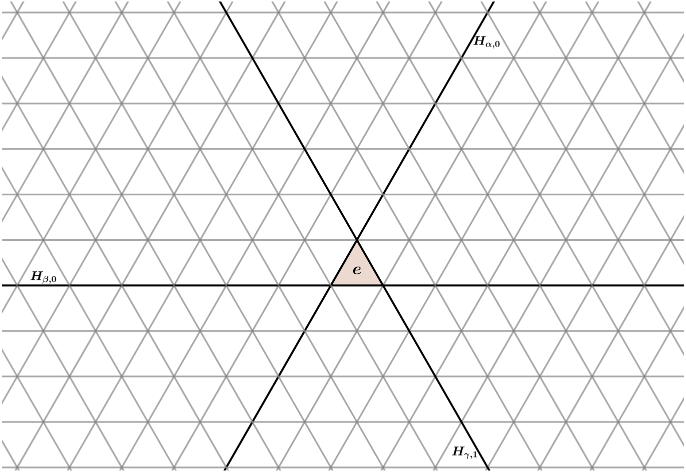

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The affine Weyl group W acts regularly on $\mathcal{A}$. In particular the set of alcoves of W is in bijection with the elements of W. If $w \in W$, we commonly denote by A_w its corresponding alcove.

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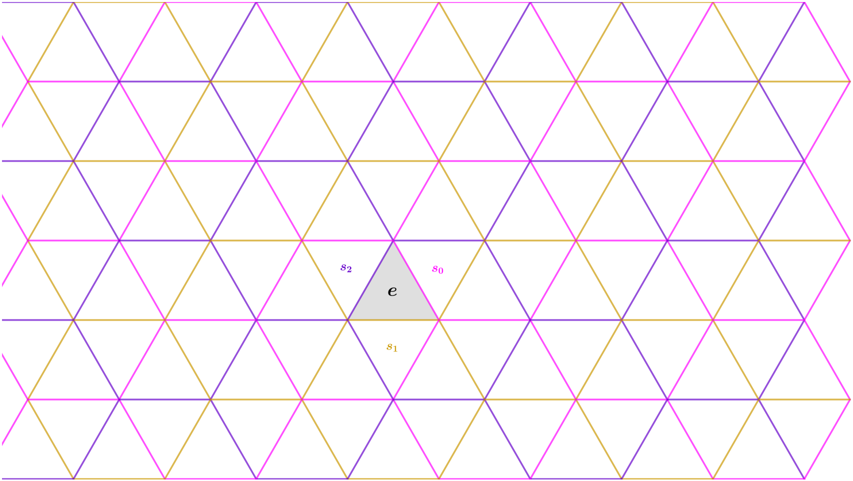
Question : Since W is a Coxeter group, is there a way to see the combinatorial relations in V ?

Relations in $A_2^{(1)}$ using alcoves



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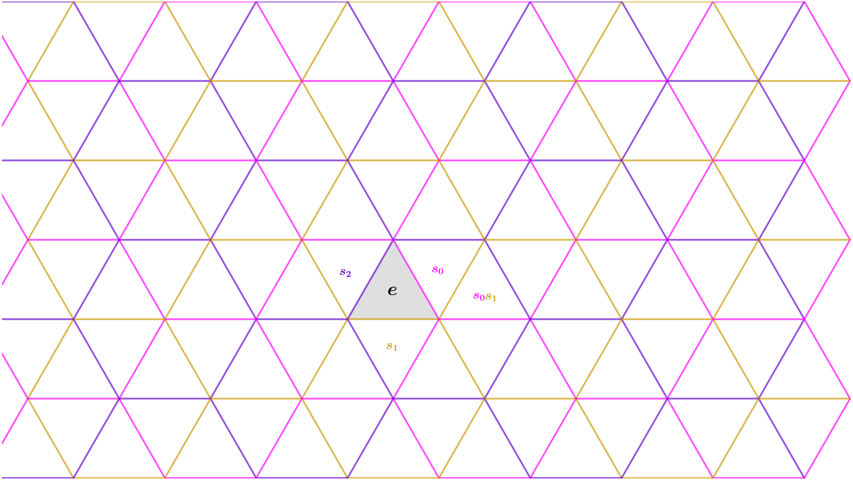

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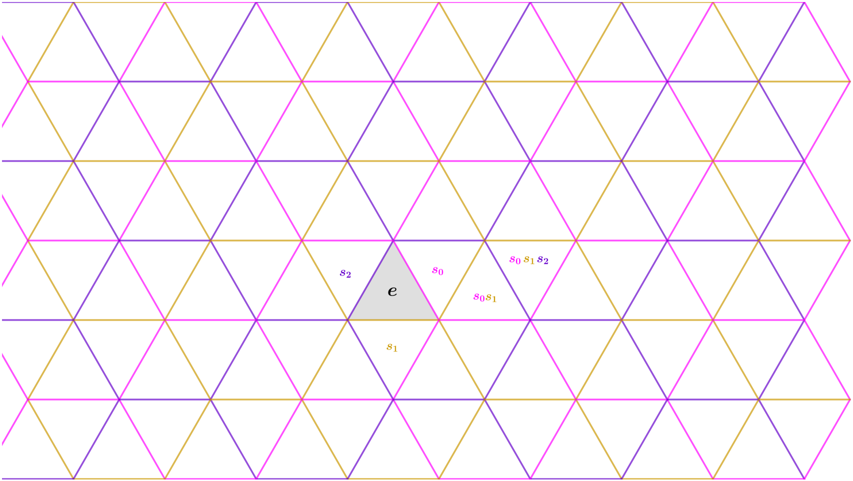

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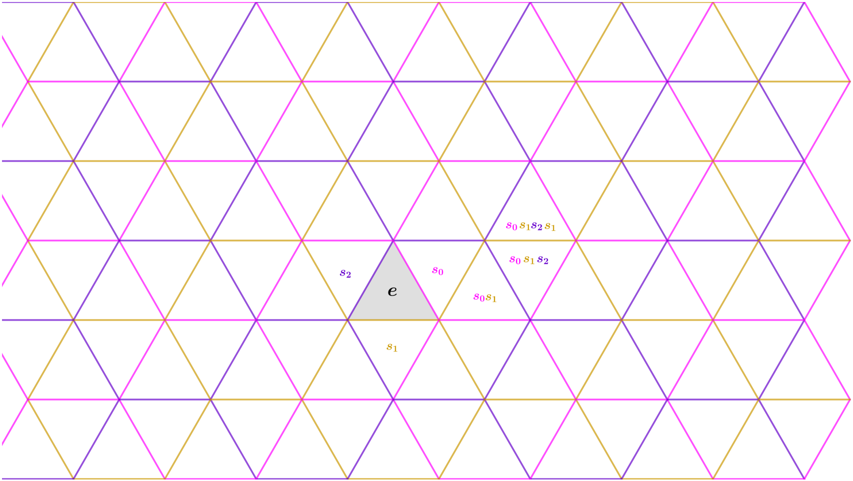

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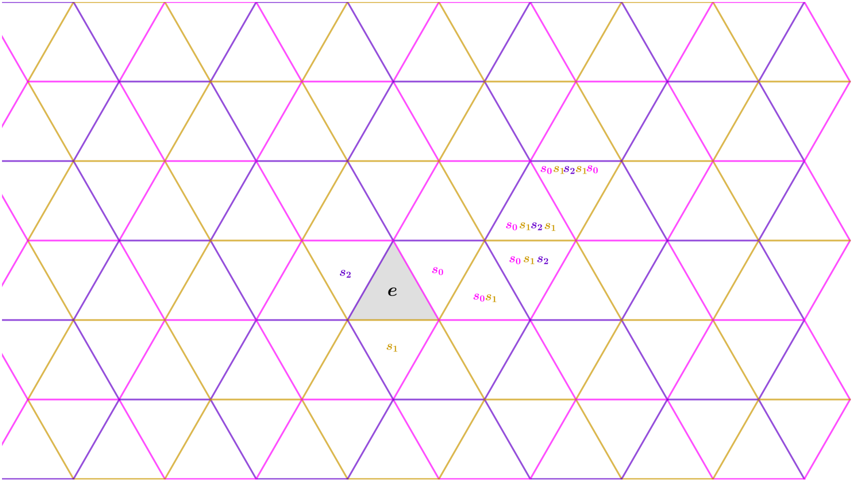

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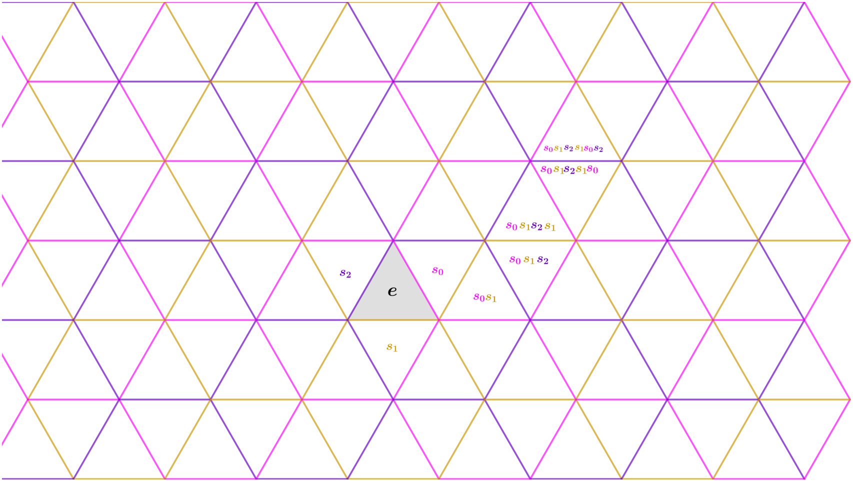

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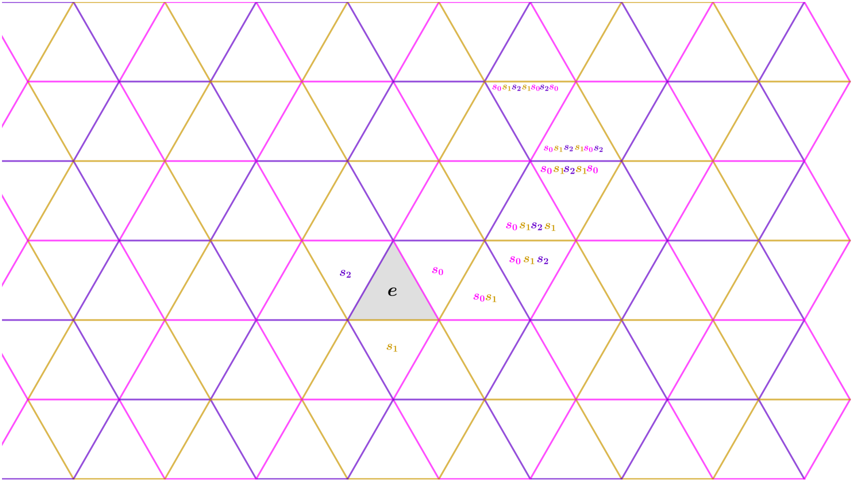

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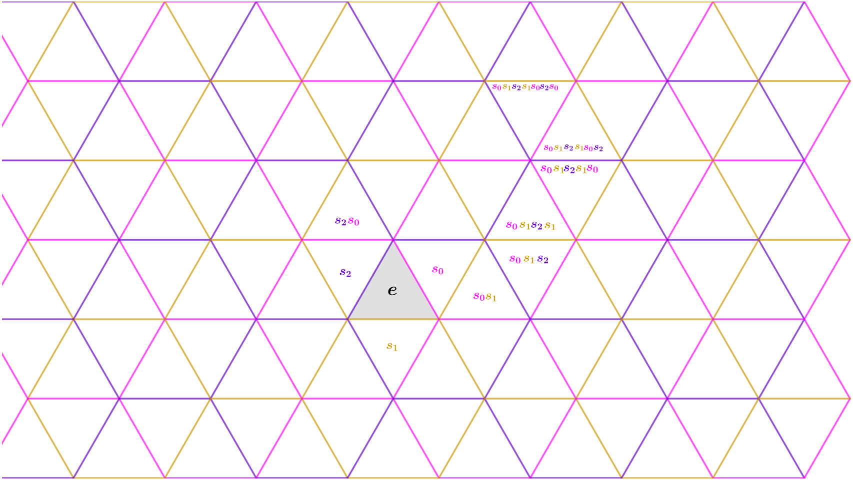

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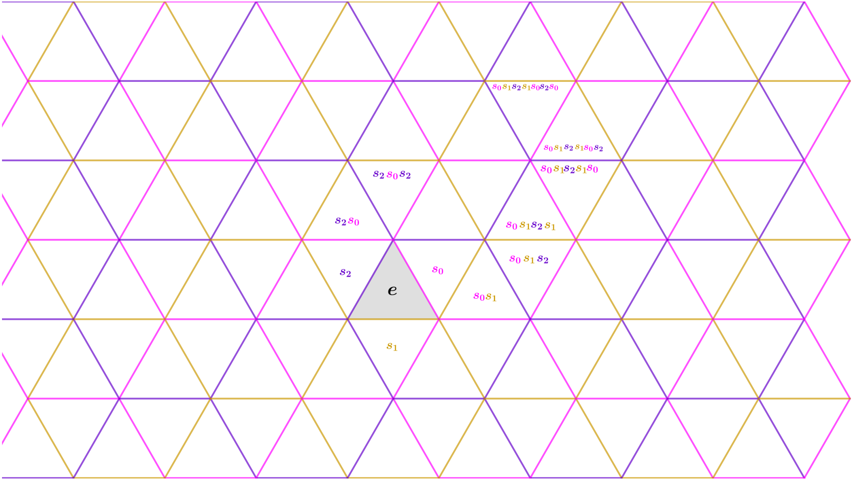

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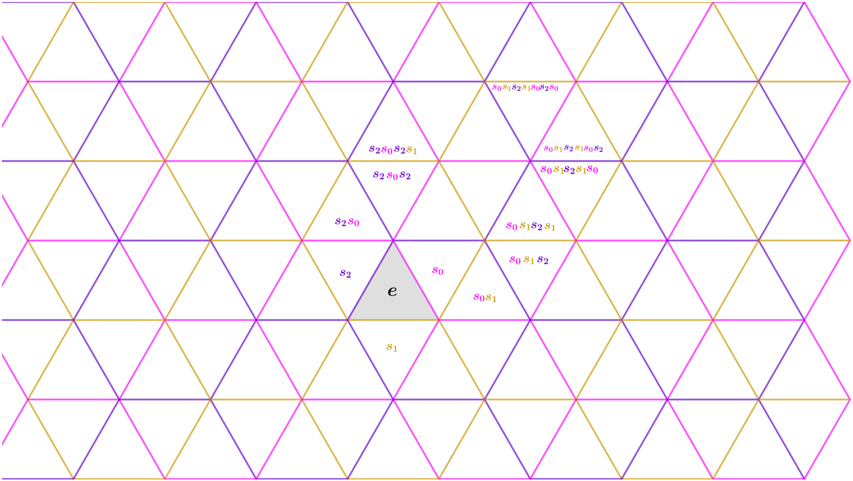

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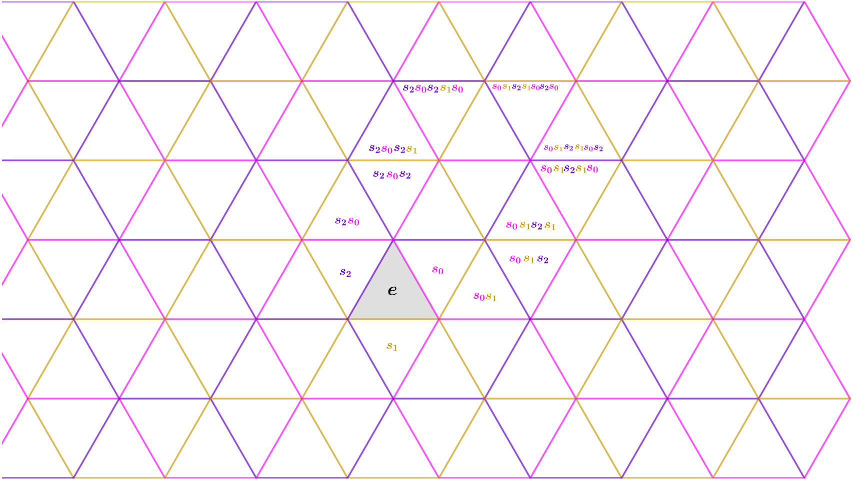

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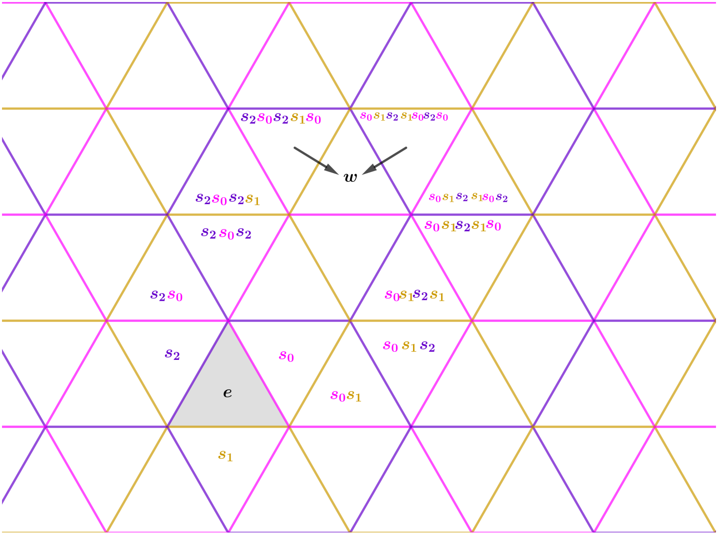

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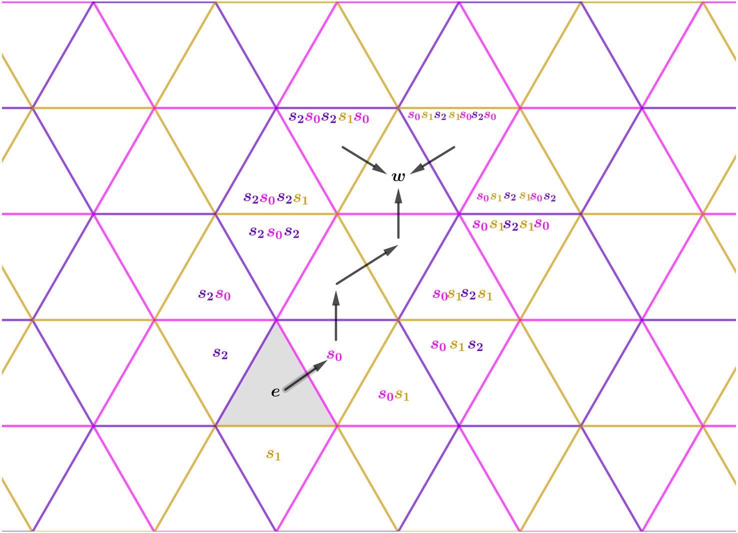


The Granville-Ono theorem

Question

Who are Andrew Granville and Ken Ono ?

Figure – A.Granville, (1962 -), British mathematician. Professor at Université de Montréal since 2002, specialist in number theory.

Known for :

- Infinitude of Carmichael numbers.
- Results on the abc-conjecture, Goldback conjecture, twins conjecture.
- Postdoc advisor of James Maynard (last Fields medallist).
- Postdoc advisor of Lucile Devin (my new officemate in Calais).
- he proved the t-core conjecture.

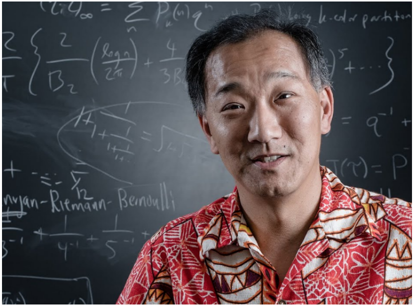


Figure – K.Ono, (1968 -), American mathematician. Professor at the University of Virginia since 2019, specialist in number theory. Former postdoc student of A. Granville.

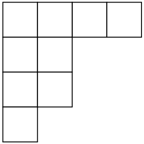
Known for :

- He derived a new theory of Ramanujan congruences.
- Closed formula for the number of partitions on an integer.
- He proved the umbral moonshine conjecture.
- Made an important breakthrough on the Riemann hypothesis.
- he proved the *t*-core conjecture.

Definition of t-cores

Definition

Let λ be a partition n. We say that λ is a t-core of size n if λ does not have any rim-hook of length t.



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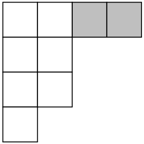

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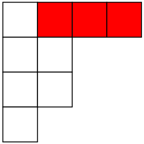

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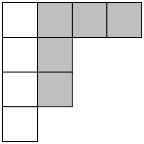

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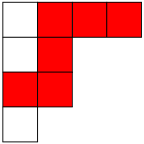

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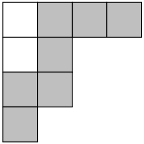

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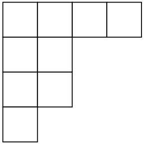

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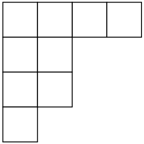
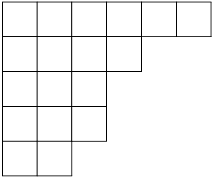


Conclusion : The partition $(4, 2, 2, 1)$ is a 6-core of size 9.

The peeling algorithm (Example for $t = 4$)

From any partition λ, we can peel it off by removing all the rim-hooks of length t. What we are left with is a t-core.



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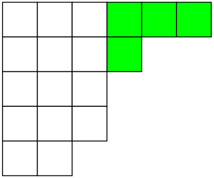

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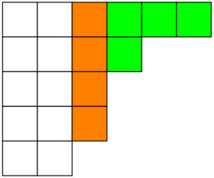

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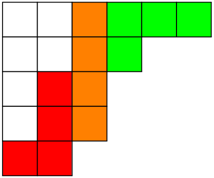

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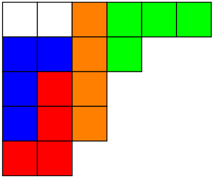

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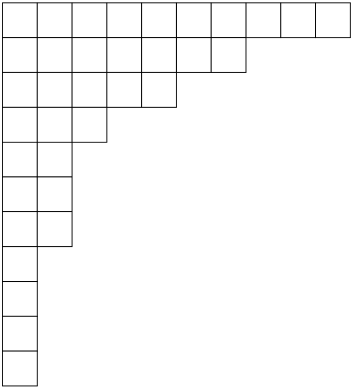

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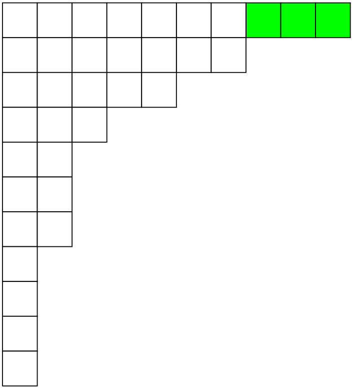

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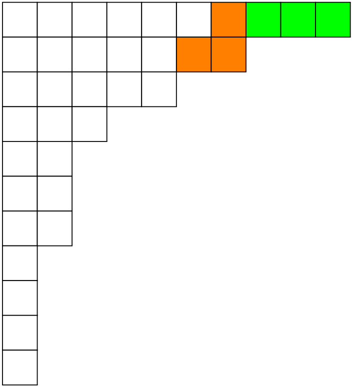

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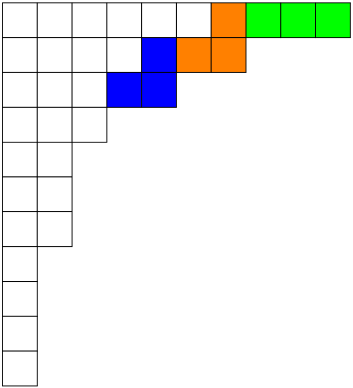

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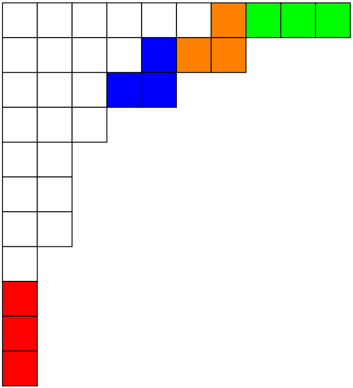

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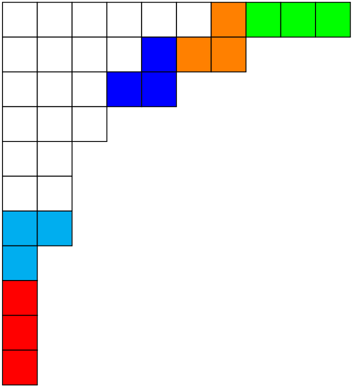

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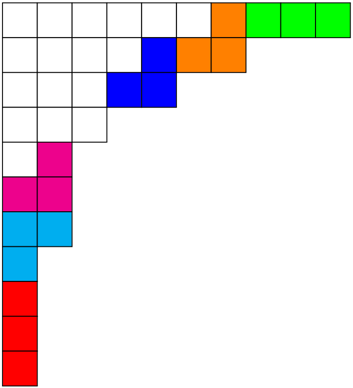

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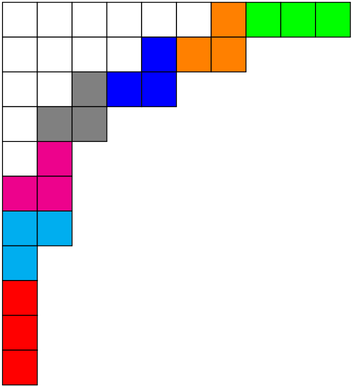

We cannot continue. The white part is then a 3-core of size 14.

The (former) t-core conjecture

Theorem (Granville-Ono,1996)

Denote by $c_t(n)$ the number of t-cores of size n. If $t \geq 4$ then

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Remark

It is actually a difficult question in general, for $t \geq 4$ and $n \in \mathbb{N}$, to find the t-cores of size n. By G-O we know that we always have at least one but we don't have a general way of building them.

Connection with analytic number theory

Definition

Let $n \in \mathbb{N} \sqcup \{\infty\}$. The q-Pochhammer symbol is

$$(a;q)_n = \prod_{k=0}^{n-1} (1 - aq^k).$$

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Theorem (Garvan-Kim-Stanton, 90')

$$\sum_{n=1}^{\infty} p(n) q^n = \frac{1}{(q^t;q^t)_\infty^t} \sum_{n=1}^{\infty} c_t(n) q^n.$$

Question

Why do we care about t-cores ?

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Answer

(1) *It is well known that* $Irr(\mathbb{C}\mathfrak{S}_n) \simeq \{\lambda \mid \lambda \in \mathcal{P}(n)\}$.

(2) *The story is much more complicated for modular representations, that is when the field* $\mathbb{C}$ *is replaced by a field of characteristic* $p > 0$*. In this situation, the notion of t-core plays a crucial role. The t-cores are in bijection with the blocks, and the notion of block is important in the theory of finite groups.*

- *In 1902 Dickson showed that, if p does not divide* $|G|$*, then the representation theory is similar to that of characteristic 0.*
- *The study of modular representations for p dividing* $|G|$ *was started essentially in 1935 with the work of Brauer.*

Main definition

Let $\mathfrak{g}$ be an affine Kac-Moody algebra, $\mathfrak{h}$ a Cartan subalgebra, $\langle -, - \rangle$ the pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$ and W the Weyl group of $\mathfrak{g}$. Let $\{\Lambda_0^\vee, \Lambda_1^\vee, \dots, \Lambda_n^\vee\}$ be the set of affine fundamental coweigts and $\rho^\vee := \sum_{i=0}^{n} \Lambda_i^\vee$. Finally let P be the weight lattice and L the finite-coweight lattice.

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Definition (CL-Gerber, 2022)

Let $\Lambda \in P$. The Λ-atomic length is

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Remark

We have two degrees to play with the definition :

(1) *The weight Λ.*

(2) *The restriction of $\mathscr{L}_\Lambda$ to the subgroups of $GL(\mathfrak{h}^*)$.*

A few results on the atomic length

Theorem (CL-Gerber, 2022)

By specialising $\Lambda = \overline{\rho} := \sum_{i=1}^{n} \omega_i$ *on the finite Weyl group* W_0 *we have*

$$\mathscr{L}_{\overline{\rho}}(w) = \sum_{\alpha \in N(w)} ht(\alpha).$$

This can be seen as a refinement of the usual length since

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Theorem (CL-Gerber, 2022)

Let w_0 *be the longest element of* W_0*. The map* $\mathscr{L}_{\overline{\rho}} : W_0 \to [\![0, \mathscr{L}_{\overline{\rho}}(w_0)]\!]$ *is surjective. (This is the finite version of the theorem of Granville-Ono).*

Atomic length on the extended affine Weyl group

Let $\Sigma = \mathrm{Stab}(A_e)$ be the fundamental group associated fo W. The extended affine Weyl group is defined by $\widehat{W} = \Sigma \ltimes W$. The Coxeter length ℓ extends naturally on $\widehat{W}$ by

$$\ell(\sigma w) = \ell(w) \quad \text{for any} \quad \sigma \in \Sigma, w \in W.$$

Theorem (Brunat-CL-Gerber, 24')

For any $\sigma \in \Sigma$, $w \in W$ one has

$$\mathscr{L}_{\Lambda_0}(\sigma w) = \mathscr{L}_{\Lambda_0}(w).$$

Lascoux's bijection

Proposition (Lascoux, 01')

Let M be the coroot lattice of type A_n. We have the following bijections

$$\{(n+1)\text{-}cores\} \longleftrightarrow M \longleftrightarrow \{alcoves\ in\ the\ fundamental\ chamber\}.$$

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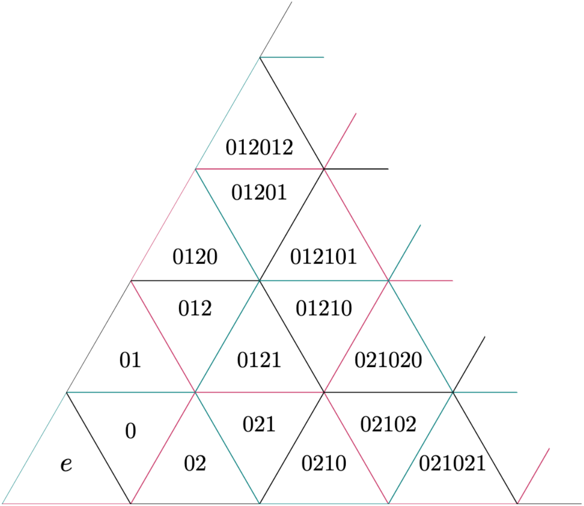
Theorem (CL-Gerber, 22')

Let W be the affine Weyl group of type $A_n^{(1)}$, let $q \in M$ and let $t_q \in W$ be the corresponding translation. One has

$$\mathscr{L}_{\Lambda_0}(t_q) = \text{size of the } (n+1)\text{-core associated to } q$$

.

Example of Lascoux's bijection in type $A_2^{(1)}$



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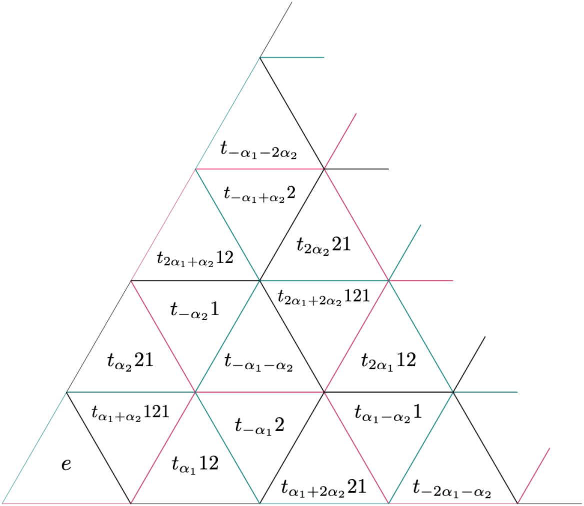

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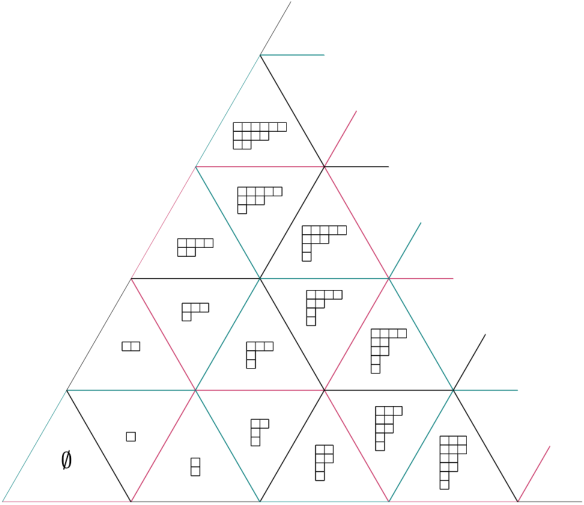

Link with diophantine equations

Theorem (CL-Gerber, 2022)

Let $w = t_v\overline{w} \in W$ with t_v the translation associated to v and $\overline{w}$ the finite part of w. Let $ht(q)$ be the height of q. We have

$$\mathscr{L}_{\Lambda_0}(w) = \frac{h}{2}||q||^2 - ht(q).$$

Example in type $A_3^{(1)}$

Let $w = t_q \overline{w} \in W$ with $q = (q_1, q_2, q_3) \in \mathbb{Z}^3$. By the above theorem we have

$$\mathscr{L}_{\Lambda_0}(w) = 4(q_1^2 + q_2^2 + q_3^2 + q_1 q_2 + q_1 q_3 + q_2 q_3) - (3q_1 + 2q_2 + q_3).$$

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By doing a specific quadratic Gauss reduction on $\mathscr{L}_{\Lambda_0}(w)$ we get

$$\mathscr{L}_{\Lambda_0}(w) = \frac{1}{48}(12q_2 + 4q_3 - 1)^2 + \frac{1}{24}(8q_3 + 1)^2 + \frac{1}{16}(8q_1 + 4q_2 + 4q_3 - 3)^2 - \frac{5}{8}.$$

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We want then to consider the following equation

$$x^2 + 2y^2 + 3z^2 = 48N + 30.$$

The PIG theorem

Let G be the group defined by

$$G = \langle \frac{1}{2} \begin{pmatrix} 1 & 0 & -3 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} ; s_{z=0} \rangle.$$

and let φ be the map defined by

$$\begin{array}{cccc} \varphi & : & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ & & (q_1, q_2, q_3) & \longmapsto & (12q_2 + 4q_3 - 1, 8q_3 + 1, 8q_1 + 4q_2 + 4q_3 - 3). \end{array}$$

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Theorem (Brunat-CL-Gerber, 24')

Let X be an integral solution of $x^2 + 2y^2 + 3z^2 = 48N + 30$. There exists $q \in L$ and $g \in G$ such that

$$g\varphi(q) = X.$$

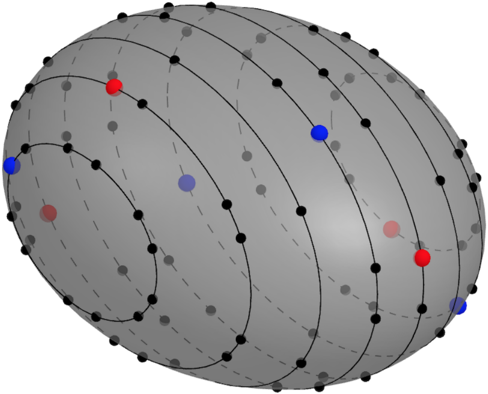

Figure – Integral solutions of $x^2 + 2y^2 + 3z^2 = 48 \cdot 2 + 30$, that is for $N = 2$.

Construction of 4-cores of any size

Corollary (Brunat-CL-Gerber,24')

From any integral solution of the equation $x^2 + 2y^2 + 3z^2 = 48N + 30$*, one can construct a 4-core of size* N*.*

Perspectives

(1) Develop a constructive proof of the Granville-Ono theorem (maybe using the local-global principle on $\mathscr{L}_{\Lambda_0}$).

(2) Study for any weight Λ the map $\mathscr{L}_\Lambda : GL(\mathfrak{h}^*) \to \mathbb{R}$ and in any type.

(3) Study the generating function $T_\Lambda(t,q) = \sum_{w \in W_0} t^{\mathscr{L}_\Lambda(w)} q^{\ell(w)}$.

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