Théorème de Jordan C^1 (Gonnord-Tosel, Calcul diff, p. 95)

$\gamma(0)=0$, $|\gamma'|\equiv 1$, $\gamma'(0)=1$. γ est exactement L périodique $L>0$. $\gamma : \mathbb{R}\to\mathbb{C}$.

Pour $\delta>0$, $\gamma_\delta^\pm(t) := \gamma(t) \pm i\delta\gamma'(t)$ $\quad z_\delta^\pm(0) := \gamma_\delta^\pm(0) = \pm i\delta$.

① $\exists \alpha>0, \forall \delta\in]0,\alpha[, \ \gamma_\delta^\pm(\mathbb{R})\cap\Gamma = \emptyset$ $\quad$ où $\Gamma = \gamma(\mathbb{R})$

② $\mathbb{C}\setminus\Gamma$ admet au plus deux composantes connexes ℓ

③ $I_\gamma(z_\delta^+) - I_\gamma(z_\delta^-) \xrightarrow[\delta\to 0]{} 1$

① Soit $\delta>0$ tq $\gamma(t_1) = \gamma(t_2) + i\delta\gamma'(t_2)$.

alors $|\gamma(t_1)-\gamma(t_2)| = \delta$.

$$\int_s^t \gamma'(u)-\gamma'(s)\,du = \gamma(t)-\gamma(s)-(t-s)\gamma'(s). \quad (*)$$

Soit $\eta>0$ tq $|t-s|\le\eta \Rightarrow |\gamma'(t)-\gamma'(s)|\le 1$.

Par $(*)$, $|\gamma(t)-\gamma(s)-(t-s)\gamma'(s)| \le |t-s|$.

ainsi, $|i\delta-(t_1-t_2)| \le |t_1-t_2|$ abs! donc $|t_1-t_2|>\eta$

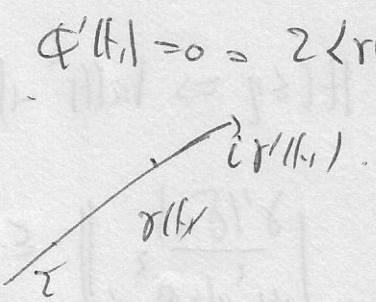
$\alpha := \min\{|\gamma(t)-\gamma(s)| \ / \ t,s\in\underset{\mathbb{R}}{[0,L]},\ d(t-s, L\mathbb{Z}) \ge \eta\} > 0$.

Si $\delta<\alpha$, $|\gamma(t_1)-\gamma(t_2)| = \delta < \alpha$ donc $d(t_1-t_2, L\mathbb{Z}) < \eta$

donc $\exists n\in\mathbb{Z}$, $|t_1-t_2-nL| < \eta$. $\quad t_1 = t_1 - nL$ $\quad$ absurde

② Soit $z\in\mathbb{C}\setminus\Gamma$. $t \overset{\varphi}{\mapsto} |\gamma(t)-z|^2$. $\quad \varphi'(t_1) = 0 = 2\langle\gamma(t_1)-z, \gamma'(t_1)\rangle$

donc $(\gamma(t_1)-z) \,//\, i\gamma'(t_1)$



donc z, $\gamma_\delta^+(t_1)$, $\gamma(t_1)$, $\gamma_\delta^-(t_1)$ sont alignés ainsi, $\gamma(t_1) \notin [z, \underbrace{\gamma_\delta^+(t_1)) \text{ ou } \notin [z, \gamma_\delta^-(t_1))}$.

$$[z, \gamma_\delta^+(t_1)) \subseteq \underbrace{[\gamma_\delta^+(t_1), \gamma(t_1)[} \cup \underbrace{[z, \gamma(t_1)[}$$

$$= \{\gamma_{\delta'}^+(t_1)\}_{0<\delta'\leq\delta} \qquad \cap \Gamma = \emptyset \text{ par def. de } t_1$$

donc $[z, \gamma_\delta^+(t_1)) \cup [\gamma_\delta^+(t_1), \gamma_\delta^+(0))$ convient

$$\overset{\shortparallel}{z_\delta^+}$$

③. $I_\gamma(z_\delta^+) - I_\gamma(z_\delta^-) = \dfrac{1}{2i\pi}\displaystyle\int_\gamma \left(\frac{1}{\xi - i\delta} - \frac{1}{\xi + i\delta}\right) d\xi = \frac{1}{2i\pi}\int_{-L/2}^{L/2} \frac{2i\delta}{\gamma(t)^2+\delta^2}\,\gamma'(t)\,dt$

Soit $\eta > 0$ ($< L/2$).

$$\int_{\eta \leq |t| \leq L/2} \frac{\gamma'(t)}{\gamma(t)+\delta^2}\,dt \underset{\delta\to 0}{=} O(1)$$

$$(*)\ \frac{\delta}{\pi}\int_{|t|\leq\eta} \frac{\gamma'(t)}{\gamma(t)^2+\delta^2}\,dt = \frac{\cancel{\delta}}{\delta^{\cancel{2}}}\int_{|t|\leq\eta} \frac{\gamma'(t)}{\frac{\gamma(t)^2}{\delta^2}+1}\,dt \overset{u = t/\delta}{=} \int_{|u|\leq \eta/\delta} \frac{\gamma'(\delta u)}{\frac{\gamma(u\delta)^2}{\delta^2}+1}\,du$$

$\gamma(0) = 0 \quad \to \gamma(t) = t\,a(t) \qquad a(0) = \gamma'(0)$

$$(*) = \int_{|u|\leq \eta/\delta} \frac{\gamma'(\delta u)}{u^2 a(u\delta)^2+1}\,du = \int_{\mathbb{R}} \underline{\qquad\qquad} \mathbb{1}_{|u|\leq\eta/\delta}\,du$$

$$\xrightarrow[\delta\to 0]{\text{p.s.}} \frac{1}{u^2+1}$$

on prend η tq $|t| \leq \eta \Rightarrow |a(t)^2 - 1| \leq 1/2$. ($\Rightarrow \operatorname{Re} a(t)^2 \geq \frac{1}{2}$).

Si $|u| \leq \eta/\delta$, $\left|\dfrac{\gamma'(\delta u)}{u^2 a(u\delta)^2+1}\right| \leq \dfrac{\|\gamma'\|_\infty}{1+u^2\frac{1}{2}}$.

$I_\gamma(z_\delta^+) - I_\gamma(z_\delta^-) \underset{\delta\to 0}{\longrightarrow} \dfrac{1}{\pi}\displaystyle\int_{\mathbb{R}} \frac{du}{u^2+1} = 1.$

conclut car indice C^0.

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