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Algèbres de Hecke carquois et algèbres de Iwahori–Hecke généralisées

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Contents

Introduction

Iwahori–Hecke algebras appeared first in the context of finite Chevalley groups, as centraliser algebras of the induced representation from the trivial representation on a Borel subgroup. In type A, this corresponds to inducing the trivial representation on the subgroup of upper triangular matrices to the whole group of invertible matrices. Since then, both the structure and the representation theory of Iwahori–Hecke algebras have been intensively studied. In particular, they have been defined independently as deformations of the group algebra of finite Coxeter groups. Further, connections with many other objects and theories have been established (this includes, for instance, the theory of quantum groups and knot theory). Many variations and generalisations of the "classical" Iwahori–Hecke algebras have already been defined. Among these, we will be interested in the following ones: Ariki–Koike algebras, Yokonuma–Hecke algebras and finally quiver Hecke algebras.

Generalising real reflection groups, which are the same as finite Coxeter groups, complex reflection groups are finite groups generated by complex reflections, that is, by endomorphisms of $\mathbb{C}^n$ that fix a hyperplane. As in the case of finite Coxeter groups, there is a classification of irreducible complex reflection groups ([ShTo]). This classification is given by an infinite series $\{G(r,p,n)\}$ where r,p,n are positive integers with $r = dp$ for $d \in \mathbb{N}^*$, together with 34 exceptional groups. More precisely, the group $G(r,p,n)$ can be seen as the group consisting of all $n \times n$ monomial matrices such that each non-zero entry is a complex rth root of unity, and the product of all non-zero entries is a dth root of unity. If $\xi \in \mathbb{C}^\times$ is a primitive rth root of unity, the latter group is generated by the elements

$$\begin{aligned}
s &\coloneqq \xi^p E_{1,1} + \sum_{k=2}^{n} E_{k,k},\\
\widetilde{t}_1 &\coloneqq \xi E_{1,2} + \xi^{-1} E_{2,1} + \sum_{k=3}^{n} E_{k,k},\\
t_a &\coloneqq E_{a,a+1} + E_{a+1,a} + \sum_{\substack{1\leq k\leq n\\ k\neq a,a+1}} E_{k,k},
\end{aligned}$$

for all $a \in \{1,\dots,n-1\}$, where $E_{k,\ell}$ is the elementary $n \times n$ matrix with 1 as the (k,ℓ)-entry and 0 everywhere else.

Aiming at generalising the construction of Iwahori–Hecke algebras, Broué–Malle [BrMa] and Broué, Malle and Rouquier [BMR] defined such a deformation for every complex reflection group, also known as Hecke algebra. Ariki and Koike [ArKo] defined such a Hecke algebra $\mathrm{H}_n(q,\boldsymbol{u})$ in the particular case of $G(r,1,n)$, the so-called "Ariki–Koike algebra", where q and $\boldsymbol{u} = (u_1,\dots,u_r)$ are some parameters. Later on, Ariki [Ar95] also defined a Hecke algebra for $G(r,p,n)$. For a suitable choice of parameter $\boldsymbol{u}$ and weight $\boldsymbol{\Lambda}$ of level r (that is, a tuple of non-negative integers of sum r), the Hecke algebra $\mathrm{H}^{\boldsymbol{\Lambda}}_{p,n}(q)$ of $G(r,p,n)$ can be seen as a subalgebra of $\mathrm{H}^{\boldsymbol{\Lambda}}_n(q) \coloneqq \mathrm{H}_n(q,\boldsymbol{u})$.

On the other hand, shortly after the introduction of Iwahori–Hecke algebras, Yokonuma [Yo] defined the Yokonuma–Hecke algebras in the study of finite Chevalley groups as well. These algebras arise as centraliser algebras of the induced representation from the trivial representation, but on a maximal *unipotent* subgroup, contrary to Iwahori–Hecke algebras. In type A, this corresponds to inducing the trivial representation on the subgroup of upper *uni*triangular matrices to the whole group of invertible matrices. Similarly to Ariki–Koike algebras, the Yokonuma–Hecke algebra $\mathrm{Y}_{d,n}(q)$ of type A, where $d \in \mathbb{N}^*$, can be viewed as a deformation of the group algebra of $G(d, 1, n)$. However, the wreath product structure $G(d, 1, n) \simeq (\mathbb{Z}/d\mathbb{Z}) \wr \mathfrak{S}_n$ now appears in the definition by generators and relations.

We now focus to the representation theory of the algebras introduced above. Let us first recall some results about the symmetric group $\mathfrak{S}_n$ on n letters. We know since Frobenius that the irreducible representations $\{\mathcal{D}^\lambda\}_\lambda$ of $\mathfrak{S}_n$ over a field of characteristic 0 are parametrised by the *partitions* of n, that is, sequences $\lambda = (\lambda_0 \geq \cdots \geq \lambda_{h-1} > 0)$ of positive integers with $|\lambda| := \lambda_0 + \cdots + \lambda_{h-1} = n$. When the ground field is of prime characteristic p, the irreducible representations $\{\mathcal{D}^\lambda\}_\lambda$ are now indexed by the *p-regular* partitions of n, that is, the partitions of n with no part repeated p times or more. However, in this case some representations may not be written as a direct sum of irreducible ones. We say that the representation theory is *non-semisimple*. Hence, we are also interested in the *blocks* of the group algebra, that is, indecomposable two-sided ideals. Blocks also partition both sets of irreducible and indecomposable representations. Brauer and Robinson proved that these blocks are parametrised by the *p-cores* of the partitions of n, proving the so-called "Nakayama's Conjecture". We refer to [JamKe] for more details about the representation theory of the symmetric group.

If $\mathbf{\Lambda}$ is a weight of level $r = 1$, the representation theory of $\mathrm{H}_n^{\mathbf{\Lambda}}(q)$ is similar to the one of the symmetric group: if $\mathrm{H}_n^{\mathbf{\Lambda}}(q)$ is semisimple then its irreducible modules $\{\mathcal{D}^\lambda\}_\lambda$ are parametrised by the partitions of n. Otherwise, they are parametrised by the e-regular partitions of n while the blocks of $\mathrm{H}_n^{\mathbf{\Lambda}}(q)$ are parametrised by the e-cores of the partitions of n, where $e \in \mathbb{N}$ is the smallest non-negative integer such that $1 + q + \cdots + q^{e-1} = 0$. We now consider a weight $\mathbf{\Lambda}$ of arbitrary level r. In the semisimple case, Ariki and Koike have determined all irreducible modules for $\mathrm{H}_n^{\mathbf{\Lambda}}(q)$. They are parametrised by the *r-partitions* of n, that is, by the r-tuples $\boldsymbol{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(r-1)})$ of partitions with $|\boldsymbol{\lambda}| := |\lambda^{(0)}| + \cdots + |\lambda^{(r-1)}| = n$. The modular case was treated by Ariki and Mathas [ArMa, Ar01], and also by Graham and Lehrer [GrLe] and Dipper, James and Mathas [DJM], using the theory of cellular algebras. This theory provides a collection of *cell modules*, also called in this case *Specht modules*. These modules allow to construct a complete family of irreducible $\mathrm{H}_n^{\mathbf{\Lambda}}(q)$-modules $\{\mathcal{D}^{\boldsymbol{\lambda}}\}_{\boldsymbol{\lambda}}$. This family can be indexed by a non-trivial generalisation of e-regular partitions, known as *Kleshchev* r-partitions (see [ArMa, Ar01]). Similarly, the naive generalisation of e-cores to r-partition, the *e-multicores*, do not provide in general a parametrisation of the blocks of $\mathrm{H}_n^{\mathbf{\Lambda}}(q)$. In fact, Lyle and Mathas [LyMa] proved that the blocks of $\mathrm{H}_n^{\mathbf{\Lambda}}(q)$ are parametrised by the multisets of κ-residues modulo e of the r-partitions of n, where $\kappa \in (\mathbb{Z}/e\mathbb{Z})^r$ is a multicharge corresponding to $\mathbf{\Lambda}$. Finally, an important result in the representation theory of $\mathrm{H}_n^{\mathbf{\Lambda}}(q)$ is a theorem of Ariki [Ar96], proving a conjecture of Lascoux, Leclerc and Thibon [LLT]. The theorem has the following consequence in characteristic 0: determining the decomposition matrix of $\mathrm{H}_n^{\mathbf{\Lambda}}(q)$ or the canonical basis of a certain integrable highest weight $\widehat{\mathfrak{sl}}_e$-module $L(\mathbf{\Lambda})$, where $\widehat{\mathfrak{sl}}_e$ denotes the Kac–Moody algebra of type $A_{e-1}^{(1)}$, are equivalent problems. Together with the works of Lascoux, Leclerc and Thibon [LLT] and Jacon [Jac05], which compute this canonical basis, we are thus able to explicitly describe the decomposition matrix of $\mathrm{H}_n^{\mathbf{\Lambda}}(q)$ (see also Uglov [Ug]).

In the semisimple case, Ariki [Ar95] used Clifford theory to determine all irreducible modules for $\mathrm{H}_{p,n}^{\mathbf{\Lambda}}(q)$. In the modular case, Genet and Jacon [GeJac] and Chlouveraki and Jacon [ChJac] gave a parametrisation of the simple modules of $\mathrm{H}_{p,n}^{\mathbf{\Lambda}}(q)$ over $\mathbb{C}$, and Hu [Hu04, Hu07] classified them over a field containing a primitive pth root of unity. Furthermore, Hu and Mathas [HuMa09, HuMa12] gave a procedure to compute the decomposition matrix of $\mathrm{H}_{p,n}^{\boldsymbol{\Lambda}}(q)$ in characteristic 0 under a separation condition (where the Hecke algebra is not semisimple in general). We also mention the work of Geck [Ge00], who deals with the case of type D (corresponding to $r = p = 2$). All these works studying the representation theory of $\mathrm{H}_{p,n}^{\boldsymbol{\Lambda}}(q)$ use the *shift* map on r-partitions, defined by

$$^{\sigma}\boldsymbol{\lambda} := (\lambda^{(r-d)}, \dots, \lambda^{(r-1)}, \lambda^{(0)}, \dots, \lambda^{(r-d-1)}),$$

for any r-partition $\boldsymbol{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(r-1)})$, where $r = dp$. If $\boldsymbol{\lambda}$ is a Kleshchev r-partition of n, the restriction of the irreducible $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q)$-module $\mathcal{D}^{\boldsymbol{\lambda}}$ to a $\mathrm{H}_{p,n}^{\boldsymbol{\Lambda}}(q)$-module is isomorphic to a sum of irreducible modules, whose number depends on the cardinality of the orbit $[\boldsymbol{\lambda}]$ of $\boldsymbol{\lambda}$ under the action of σ.

Concerning Yokonuma–Hecke algebras, their natural presentation in type A has been transformed since the original work of Yokonuma (see [Ju98, Ju04, JuKa, ChPA14, ChPou]). The representation theory of Yokonuma–Hecke algebras has been first studied by Thiem [Th04, Th05, Th07], while a combinatorial approach to this representation theory in type A has been given in [ChPA14, ChPA15]. In this latter paper [ChPA15], Chlouveraki and Poulain d'Andecy introduced and studied generalisations of these algebras: the affine Yokonuma–Hecke algebras and their cyclotomic quotients, which generalise affine Hecke algebras of type A and Ariki–Koike algebras respectively. The interest in Yokonuma–Hecke algebras has grown recently: in [CJKL] (see also [PAWag]), the authors defined a link invariant from Yokonuma–Hecke algebras which is stronger than the famous ones obtained from classical Iwahori–Hecke algebras of type A, such as the HOMFLYPT polynomial, and Ariki–Koike algebras. Another topologically interesting object is the subalgebra of the Yokonuma–Hecke algebra known as the *algebra of braids and ties*. This algebra has been introduced by Juyumaya [Ju99] and the original definition has been generalised to all finite complex reflection groups by Marin [Mar].

A new aspect of the representation theory of Ariki–Koike algebras was developped around the 2010s. Partially motivated by Ariki's theorem, Khovanov and Lauda [KhLau09, KhLau11] and Rouquier [Rou] independently introduced the algebra $\mathrm{R}_n(\Gamma)$, known as a *quiver Hecke algebra* or *KLR algebra*. This led to the categorification result

$$\mathcal{U}_v^-(\mathfrak{g}_\Gamma) \simeq \bigoplus_{n \geq 0} [\mathrm{Proj}(\mathrm{R}_n(\Gamma))]\,,$$

where $\mathcal{U}_v^-(\mathfrak{g}_\Gamma)$ is the negative part of the quantum group of $\mathfrak{g}_\Gamma$, the Kac–Moody algebra associated with the quiver Γ, and $[\mathrm{Proj}(\mathrm{R}_n(\Gamma))]$ denotes the Grothendieck group of the additive category of finitely generated graded projective $\mathrm{R}_n(\Gamma)$-modules. Moreover, considering some cyclotomic quotients $\mathrm{R}_n^{\boldsymbol{\Lambda}}(\Gamma)$ of the quiver Hecke algebra, Kang and Kashiwara [KanKa] also proved a categorification result for the highest weight $\mathcal{U}_v(\mathfrak{g}_\Gamma)$-modules, as conjectured by [KhLau09]. More specifically, for each dominant weight $\boldsymbol{\Lambda}$ the algebra $\mathrm{R}_n(\Gamma)$ has a cyclotomic quotient $\mathrm{R}_n^{\boldsymbol{\Lambda}}(\Gamma)$ that categorifies the corresponding highest weight module $L(\boldsymbol{\Lambda})$.

When $\Gamma = \Gamma_e$ is the quiver of type $A_{e-1}^{(1)}$, we thus obtain a connection between the Ariki–Koike algebra $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q)$ and $\mathrm{R}_n^{\boldsymbol{\Lambda}}(\Gamma)$. A big step towards understanding this connection, and thus cyclotomic quiver Hecke algebras, was made by Brundan and Kleshchev [BrKl-a] and independently by Rouquier [Rou]. The first two authors proved that, over a field, Ariki–Koike algebras are particular cases of cyclotomic quiver Hecke algebras, providing a family of explicit isomorphisms. Rouquier also gave an affine version of this isomorphism. Brundan and Kleshchev noticed that the Ariki–Koike algebra inherits the natural $\mathbb{Z}$-grading of the cyclotomic quiver Hecke algebra, a fact that allows us to study the *graded* representation theory of Ariki–Koike algebras (see for example [BrKl-b]). Moreover, they established a graded version of Ariki's categorification theorem, where the whole quantum group $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$ appears (and not only its specialisation $\widehat{\mathfrak{sl}}_e$ at $v = 1$). Note that, inspired by the work of Brundan and Kleshchev [BrKl-a], Hu and Mathas [HuMa10] constructed a *graded* cellular basis for $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q)$. This was the first example of a homogeneous basis for $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q)$.

Our aim in this thesis is to generalise some of these previous results. The results we give are a compilation of the papers [Ro16, Ro17-a, Ro17-b]. First, in Chapter 1 we prove some results on cyclotomic quiver Hecke algebras. More specifically, we study cyclotomic "disjoint quiver" Hecke algebras and fixed point subalgebras for automorphisms built on quiver automorphisms of finite order. Then, in Chapter 2 we give a cyclotomic quiver Hecke-like presentation for $\mathrm{H}_{p,n}^{\boldsymbol{\Lambda}}(q)$, in particular this algebra is a graded subalgebra of $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q)$. In Chapter 3 we show that cyclotomic Yokonuma–Hecke algebras are a particular case of cyclotomic quiver Hecke algebras, where the quiver is in fact the same as for Ariki–Koike algebras. Chapter 4 is largely independent from the others. It is mainly concerned with a purely combinatorial problem, namely the link between orbit cardinalities for the shift action on multipartitions and on their multisets of residues. We then apply this result to the representation theory of $\mathrm{H}_{p,n}^{\boldsymbol{\Lambda}}(q)$. Finally, in the very short Chapter 5, we present some works in progress (both in collaboration).

We now give the content of each chapter in more details. In Chapter 1, given a quiver Γ and $n \in \mathbb{N}^*$ we define the quiver Hecke algebra $\mathrm{R}_n(\Gamma)$ and its cyclotomic quotient $\mathrm{R}_n^{\boldsymbol{\Lambda}}(\Gamma)$, where $\boldsymbol{\Lambda}$ is a weight. Note that we also define these algebras in the more general setting of [Rou], where the quiver is replaced by a matrix of bivariate polynomials. In Section 1.3 we study the decomposition of $\mathrm{R}_n(\Gamma)$ and its cyclotomic quotients when Γ is not connected. For instance, if $\Gamma = \Gamma^1 \amalg \Gamma^2$ where Γ^1, Γ^2 are full subquivers of Γ then

$$\mathrm{R}_n(\Gamma) \simeq \bigoplus_{k=0}^{n} \mathrm{Mat}_{\binom{n}{k}}\Big(\mathrm{R}_k(\Gamma^1) \otimes \mathrm{R}_{n-k}(\Gamma^2)\Big).$$

The general statements are given at Theorems 1.3.47 and 1.3.57. Although similar situations have already been studied in the literature (see, for instance, [SVV, Theorem 3.15] or [RSVV, Lemma 5.33]), the result we obtain in our context seems to be new and of independent interest. In Section 1.4 we start with an automorphism σ of finite order p of the quiver Γ. The automorphism σ naturally induces a homogeneous automorphism of $\mathrm{R}_n(\Gamma)$. The subalgebra $\mathrm{R}_n(\Gamma)^\sigma$ of fixed points is automatically $\mathbb{Z}$-graded. The interesting point is that we are able to give a homogeneous presentation for $\mathrm{R}_n(\Gamma)^\sigma$ that looks like the presentation of $\mathrm{R}_n(\Gamma)$, see Corollary 1.4.18. We then extend these results to certain cyclotomic quotients, the homogeneous presentation being given at Theorem 1.4.36. Note that these results in [Ro16] were proved when the order of σ is invertible in the base ring of $\mathrm{R}_n(\Gamma)$, in contrast, the proofs here are characteristic-free.

Now let q be a non-zero element of the base field and let $e \in \mathbb{N}^* \cup \{\infty\}$ be minimal such that $1 + q + \cdots + q^{e-1} = 0$. In Chapter 2 we recall the definition of the Hecke algebra $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q)$ (respectively $\mathrm{H}_{p,n}^{\boldsymbol{\Lambda}}(q)$) of type $G(r,1,n)$ (resp. $G(r,p,n)$), where $\boldsymbol{\Lambda}$ is a weight of level r and where $n, p \in \mathbb{N}^*$ with p dividing r. We prove in §2.2.3 that the two different presentations that we find in the literature for $\mathrm{H}_{p,n}^{\boldsymbol{\Lambda}}(q)$, namely [BMR, Ar95], are isomorphic. Recall that Γ_e denotes the cyclic quiver with e vertices (a two-sided infinite line when $e = \infty$). In Section 2.3, we prove that the isomorphism of Brundan and Kleshchev [BrKl-a] between $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q)$ and $\mathrm{R}_n^{\boldsymbol{\Lambda}}(\Gamma_e)$ can easily be generalised to the general algebra $\mathrm{H}_n(q, \boldsymbol{u})$, see Theorem 2.3.6. We find a family of isomorphisms between $\mathrm{H}_n(q, \boldsymbol{u})$ and a cyclotomic quotient of $\mathrm{R}_n(\Gamma_{e,p'})$, where p' is the number of q-orbits in $\boldsymbol{u}$ and $\Gamma_{e,p'}$ is the quiver given by p' disjoint copies of Γ_e. Using the results of Section 1.4 and a particular element of the above isomorphisms family, we find a cyclotomic quiver Hecke-like presentation for $\mathrm{H}_{p,n}^{\boldsymbol{\Lambda}}(q)$. Using Section 1.3, we can also deduce another result: the above isomorphism gives a new proof of a well-known Morita equivalence of Ariki–Koike algebras (proved by Dipper–Mathas [DiMa]), see Theorem 2.3.19. This Morita equivalence theorem is an important result in the representation theory of Ariki–Koike algebras: it shows that it suffices to study the case where $\boldsymbol{u}$ consists of a single q-orbit. The proof of [DiMa] uses a general argument, namely the existence of a projective generator for $\mathrm{H}_n(q, \boldsymbol{u})$. It is thus interesting to find out that this Morita equivalence comes in fact from an explicit isomorphism.

In Chapter 3, given $n, d \in \mathbb{N}^*$ and a weight $\boldsymbol{\Lambda}$, we define the cyclotomic Yokonuma–Hecke algebra $\mathrm{Y}_{d,n}^{\boldsymbol{\Lambda}}(q)$. It is a generalisation of the Ariki-Koike algebra $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q)$. Then, we follow the proof of [BrKl-a] to obtain a family of isomorphisms between $\mathrm{Y}_{d,n}^{\boldsymbol{\Lambda}}(q)$ and $\mathrm{R}_n^{\boldsymbol{\Lambda}}(\Gamma_{e,d})$, see Theorem 3.5.1 and (3.5.2). We also treat the degenerate case, see Theorem 3.6.20. In both cases, the calculations are either the same as in [BrKl-a] or without difficulty. Moreover, as in the Ariki–Koike case, the isomorphisms are over a field. The chapter ends with Section 3.7, where we show that an isomorphism of Lusztig [Lu] (when $\boldsymbol{\Lambda}$ has level 1) and Poulain d'Andecy [PA], expressing $\mathrm{Y}_{d,n}^{\boldsymbol{\Lambda}}(q)$ in terms of Ariki–Koike algebras, is a particular case of the isomorphism of Section 1.3. We also prove that we recover the explicit version of this isomorphism, as used in [JacPA, PA]. Note that all the isomorphisms we obtain are over a field, while in [Lu, JacPA, PA] the isomorphisms are over a ring.

In Chapter 4, we study the shift map $\boldsymbol{\lambda} \mapsto {}^{\sigma}\boldsymbol{\lambda}$ on r-partitions of n, as defined above. More precisely, let $e, p \in \mathbb{N}$ with $p \neq 0$ dividing e. Recall that the blocks of $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q)$ are indexed by the multisets of κ-residues modulo e of the r-partitions of n, where $\boldsymbol{\Lambda}$ is a weight of level r associated to the multicharge $\kappa \in (\mathbb{Z}/e\mathbb{Z})^r$. This multiset can be viewed as a subset of $Q^+ := \mathbb{N}^{\mathbb{Z}/e\mathbb{Z}}$, and for any r-partition $\boldsymbol{\lambda}$ of n we denote by $\alpha_\kappa(\boldsymbol{\lambda})$ the corresponding element of Q^+. We can define a shift map $\alpha \mapsto \sigma \cdot \alpha$ on Q^+, which we again denote by σ. This shift is compatible with the shift on r-partitions of n in the following way:

$$\alpha_\kappa({}^{\sigma}\boldsymbol{\lambda}) = \sigma \cdot \alpha_\kappa(\boldsymbol{\lambda}),$$

for any r-partition $\boldsymbol{\lambda}$ of n. If $[\alpha]$ denotes the orbit of $\alpha \in Q^+$ under the action of σ, the above equality shows that $\#[\alpha_\kappa(\boldsymbol{\lambda})] \leq \#[\boldsymbol{\lambda}]$. The aim of Chapter 4 is to prove a converse statement. More precisely, there always exists an r-multipartition $\boldsymbol{\mu}$ of n such that $\#[\boldsymbol{\mu}] = \#[\alpha_\kappa(\boldsymbol{\lambda})]$, see Theorem 4.2.31 and Corollary 4.2.34. To that extent, we first recall in Section 4.2 the abacus representation of a partition. We use it to parametrise the subset of Q^+ given by κ-residues of r-partitions. As a consequence, the main theorem reduces to a convex optimisation problem with integral variables and linear constraints. Section 4.3 is devoted to technical tools that we need to prove the main theorem. The main result of this section is an existence theorem for a binary matrix with prescribed row, column and block sums. Without the block sums condition, we recover a particular case of a theorem of Gale [Ga] and Ryser [Ry]. Finally, in Section 4.4 we prove the main theorem of the chapter, first using a naive approach and then making it work using Section 4.3. In Section 4.5, we first quickly recall the theory of cellular algebras. Then we study the cellularity of $\mathrm{H}_{p,n}^{\boldsymbol{\Lambda}}(q)$ and we give a direct application of the main theorem to the maximal number of "Specht modules" of $\mathrm{H}_{p,n}^{\boldsymbol{\Lambda}}(q)$ when restricting Specht modules of the blocks of $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q)$.

Chapter 5 is devoted to some work in progress. We first quickly describe a effort to endow $\mathrm{H}_{p,n}^{\boldsymbol{\Lambda}}(q)$ with a cellular structure, where we have to slightly change one cellularity axiom. The point is to consider a particular graded cellular basis, diagrammatically constructed by Webster [We] and Bowman [Bow]. This work is in collaboration with Jun Hu and Andrew Mathas. In a second part, we give the idea of a work with Loïc Poulain d'Andecy and Ruari Walker, who proved a type B version of the graded isomorphism theorem of Section 2.3. Our aim is now to give a type B version of the Morita equivalence of Ariki–Koike algebras, as we did in §2.3.4.

Notation

Here, we introduce some notation that we will extensively use throughout this thesis. Let $\mathcal{K}$ be a set and $n \in \mathbb{N}^*$. A *$\mathcal{K}$-composition of n* is a (finitely-supported) tuple of non-negative integers indexed by $\mathcal{K}$ with sum n. We write $\alpha \models_{\mathcal{K}} n$ if $\alpha = (\alpha_k) \in \mathbb{N}^{(\mathcal{K})}$ is a $\mathcal{K}$-composition of n. For any $d \in \mathbb{N}^*$, we will write $\models_d$ instead of $\models_{\{1,\dots,d\}}$. A *weight* is a finitely-supported tuple $\mathbf{\Lambda} = (\Lambda_k) \in \mathbb{N}^{(\mathcal{K})}$ of non-negative integers indexed by $\mathcal{K}$. The *level* of a weight $\mathbf{\Lambda} = (\Lambda_k)_{k\in\mathcal{K}} \in \mathbb{N}^{(\mathcal{K})}$ is $\ell(\mathbf{\Lambda}) \coloneqq \sum_{k\in\mathcal{K}} \Lambda_k$.

Let $\alpha \models_{\mathcal{K}} n$. We denote by $\mathcal{K}^\alpha$ the subset of $\mathcal{K}^n$ formed by the elements $\boldsymbol{k} = (k_1, \dots, k_n) \in \mathcal{K}^n$ such that

$$\#\{a \in \{1, \dots, n\} : k_a = k\} = \alpha_k,$$

for all $k \in \mathcal{K}$, that is, $(k_1, \dots, k_n) \in \mathcal{K}^\alpha$ if and only if for all $k \in \mathcal{K}$, there are exactly α_k integers $a \in \{1, \dots, n\}$ such that $k_a = k$. The subsets $\mathcal{K}^\alpha$ are the orbits of $\mathcal{K}^n$ under the natural action of the symmetric group $\mathfrak{S}_n$ on n letters; in particular, each $\mathcal{K}^\alpha$ is finite.

We denote by F a field and we consider $q \in F^\times$. Except in Section 3.6, we always have $q \neq 1$. We consider the minimal $e \in \mathbb{N}^* \cup \{\infty\}$ such that $1 + q + \cdots + q^{e-1} = 0$. If $q \neq 1$ and $e \neq \infty$ then q is a primitive eth root of unity. We define

$$I \coloneqq \begin{cases} \mathbb{Z}/e\mathbb{Z}, & \text{if } e \neq \infty, \\ \mathbb{Z}, & \text{otherwise.} \end{cases}$$

Chapter 1

Quiver Hecke algebras

This chapter is an adapted and revised version of [Ro16, Ro17-a].

1.1 Overview

This chapter consists of three sections, the last two of them being independent from each other. We first define in Section 1.2 quiver Hecke algebras and their cyclotomic quotients. Then, we study in Section 1.3 the decomposition of a cyclotomic quiver Hecke algebra according to the connected components of the underlying quiver. Finally, we give in Section 1.4 a presentation of the fixed point subalgebra for an algebra automorphism that comes from a quiver automorphism of finite order. Note that, contrary to [Ro16], the proofs are characteristic-free. The presentation that we obtain is similar to the one of a cyclotomic quiver Hecke algebra.

We now give an brief overview of the chapter. Let K be a set and $Q = (Q_{k,k'})_{k,k' \in K}$ be a family of bivariate polynomials satisfying the condition (1.2.1). An important example of such a family is given at (1.2.14), where Q is defined by a loop-free quiver of vertex set K. Let $\boldsymbol{\Lambda} \in \mathbb{N}^{(K)}$ be a weight and $\mathfrak{a}$ be a family of monic polynomials. We define in Section 1.2 the quiver Hecke algebra $\mathrm{R}_n(Q)$ and its cyclotomic quotient $\mathrm{R}_n^{\boldsymbol{\Lambda},\mathfrak{a}}(Q)$. We then give a basis for $\mathrm{R}_n(Q)$ as a module and a generating family for $\mathrm{R}_n^{\boldsymbol{\Lambda},\mathfrak{a}}(Q)$. We begin Section 1.3 by some quick calculations about the minimal length representatives of the cosets of a Young subgroup in the symmetric group $\mathfrak{S}_n$ on n letters. The main results of the section are given in Theorems 1.3.47 and 1.3.57, where we prove an isomorphism about (cyclotomic) "disjoint quiver" Hecke algebras. In Section 1.4, we consider a permutation of K of finite order p for which Q is invariant (see (1.4.1)). When Q is associated to a loop-free quiver, this permutation is a quiver automorphism. To this permutation, we associate in Theorem 1.4.5 an automorphism of $\mathrm{R}_n(Q)$ of finite order p and we easily give in §1.4.1 a presentation of the fixed point subalgebra (Corollary 1.4.18). In contrast, we need a little more work in §1.4.2 to do the same thing for the cyclotomic quotient $\mathrm{R}_n^{\boldsymbol{\Lambda},\mathfrak{a}}(Q)$. We give a presentation for the fixed point subalgebra in Theorem 1.4.36. Note that in [Ro16] we assumed that p was invertible in the base ring, assumption that we here drop.

1.2 Definition

We define quiver Hecke algebras and their cyclotomic quotients. We end the section by recalling some properties of the underlying modules.

1.2.1 General definition

Let K be a set and A a commutative ring. Let u and v be two indeterminates over A and $Q = (Q_{k,k'})_{k,k' \in K}$ be a matrix of bivariate polynomials. We assume that the polynomials $Q_{k,k'} \in A[u,v]$ satisfy

$$\begin{cases} Q_{k,k'}(u,v) = Q_{k',k}(v,u), \\ Q_{k,k} = 0, \end{cases} \tag{1.2.1}$$

for all $k, k' \in K$.

Let $\alpha \models_K n$. The *quiver Hecke algebra* $\mathrm{R}_\alpha(Q)$ associated with $(Q_{k,k'})_{k,k' \in K}$ at α is the unitary associative A-algebra with generating set

$$\{e(\boldsymbol{k})\}_{\boldsymbol{k} \in K^\alpha} \cup \{y_1, \dots, y_n\} \cup \{\psi_1, \dots, \psi_{n-1}\} \tag{1.2.2}$$

and the following relations:

$$\sum_{\boldsymbol{k} \in K^\alpha} e(\boldsymbol{k}) = 1, \tag{1.2.3a}$$

$$e(\boldsymbol{k}) e(\boldsymbol{k}') = \delta_{\boldsymbol{k},\boldsymbol{k}'} e(\boldsymbol{k}), \tag{1.2.3b}$$

$$y_a e(\boldsymbol{k}) = e(\boldsymbol{k}) y_a, \tag{1.2.3c}$$

$$\psi_b e(\boldsymbol{k}) = e(s_b \cdot \boldsymbol{k}) \psi_b, \tag{1.2.3d}$$

$$y_a y_{a'} = y_{a'} y_a, \tag{1.2.3e}$$

$$\psi_b y_a = y_a \psi_b, \qquad \text{if } a \neq b, b+1, \tag{1.2.3f}$$

$$\psi_b \psi_{b'} = \psi_{b'} \psi_b, \qquad \text{if } |b - b'| > 1, \tag{1.2.3g}$$

$$\psi_b y_{b+1} e(\boldsymbol{k}) = \begin{cases} (y_b \psi_b + 1) e(\boldsymbol{k}), & \text{if } k_b = k_{b+1}, \\ y_b \psi_b e(\boldsymbol{k}), & \text{if } k_b \neq k_{b+1}, \end{cases} \tag{1.2.3h}$$

$$y_{b+1} \psi_b e(\boldsymbol{k}) = \begin{cases} (\psi_b y_b + 1) e(\boldsymbol{k}), & \text{if } k_b = k_{b+1}, \\ \psi_b y_b e(\boldsymbol{k}), & \text{if } k_b \neq k_{b+1}, \end{cases} \tag{1.2.3i}$$

together with

$$\psi_b^2 e(\boldsymbol{k}) = Q_{k_b, k_{b+1}}(y_b, y_{b+1}) e(\boldsymbol{k}), \tag{1.2.4a}$$

$$\psi_{c+1} \psi_c \psi_{c+1} e(\boldsymbol{k}) = \begin{cases} \psi_c \psi_{c+1} \psi_c e(\boldsymbol{k}) + \frac{Q_{k_c, k_{c+1}}(y_c, y_{c+1}) - Q_{k_{c+2}, k_{c+1}}(y_{c+2}, y_{c+1})}{y_c - y_{c+2}} e(\boldsymbol{k}), & \text{if } k_c = k_{c+2}, \\ \psi_c \psi_{c+1} \psi_c e(\boldsymbol{k}), & \text{otherwise,} \end{cases} \tag{1.2.4b}$$

for all $\boldsymbol{k} \in K^\alpha$, $a, a' \in \{1, \dots, n\}$, $b, b' \in \{1, \dots, n-1\}$ and $c \in \{1, \dots, n-2\}$, where s_b is the transposition $(b, b+1) \in \mathfrak{S}_n$.

Remark 1.2.5. Let $\boldsymbol{k} \in K^\alpha, a \in \{1, \dots, n-2\}$ and let $P := Q_{k_a, k_{a+1}}$. The relation (1.2.4b) for $k_a = k_{a+2}$ is

$$\psi_{a+1} \psi_a \psi_{a+1} e(\boldsymbol{k}) = \psi_a \psi_{a+1} \psi_a e(\boldsymbol{k}) + \frac{P(y_a, y_{a+1}) - P(y_{a+2}, y_{a+1})}{y_a - y_{a+2}} e(\boldsymbol{k}). \tag{1.2.6}$$

Writing $P(u,v) = \sum_{m \geq 0} u^m P_m(v)$, we get that the right side of (1.2.6) is well-defined and is an element of $A[y_a, y_{a+1}, y_{a+2}] e(\boldsymbol{k})$.

Remark 1.2.7. The generators in [Rou] are given by $1_{\boldsymbol{k}} := e(\boldsymbol{k})$, $x_{a,\boldsymbol{k}} := y_a e(\boldsymbol{k})$ and $\tau_{a,\boldsymbol{k}} := \psi_a e(\boldsymbol{k})$.

When the set K is finite, in a similar way we can define the quiver Hecke algebra $\mathrm{R}_n(Q)$ as the unitary associative A-algebra with generating set

$$\{e(\boldsymbol{k})\}_{\boldsymbol{k}\in K^n} \cup \{y_1, \dots, y_n\} \cup \{\psi_1, \dots, \psi_{n-1}\} \tag{1.2.8}$$

together with the same relations (1.2.3)–(1.2.4), where (1.2.3a) is replaced by

$$\sum_{\boldsymbol{k}\in K^n} e(\boldsymbol{k}) = 1. \tag{1.2.9}$$

Defining for any $\alpha \models_K n$ the central idempotent

$$e(\alpha) \coloneqq \sum_{\boldsymbol{k}\in K^\alpha} e(\boldsymbol{k}) \in \mathrm{R}_n(Q), \tag{1.2.10}$$

we have:

$$e(\alpha)\mathrm{R}_n(Q) \simeq \mathrm{R}_\alpha(Q),$$

thus:

$$\mathrm{R}_n(Q) \simeq \bigoplus_{\alpha\models_K n} \mathrm{R}_\alpha(Q). \tag{1.2.11}$$

Note that this equality can be seen as a definition of $\mathrm{R}_n(Q)$ is K is infinite.

We conclude this paragraph by introducing cyclotomic quotients of these quiver Hecke algebras. Let $\boldsymbol{\Lambda} = (\Lambda_k)_{k\in K} \in \mathbb{N}^{(K)}$ be a weight and $\alpha \models_K n$. Let $\mathrm{Pol}_K^{\boldsymbol{\Lambda}}$ be the set of families $(a_k)_{k\in K}$ of monic polynomials with coeffficients in A such that a_k has degree Λ_k and let $\mathfrak{a} = (a_k)_{k\in K} \in \mathrm{Pol}_K^{\boldsymbol{\Lambda}}$. Following [KanKa, §4.1], we define the *cyclotomic* quiver Hecke algebra $\mathrm{R}_\alpha^{\boldsymbol{\Lambda},\mathfrak{a}}(Q)$ at α as the quotient of the quiver Hecke algebra $\mathrm{R}_\alpha(Q)$ by the ideal $\mathcal{I}_\alpha^{\boldsymbol{\Lambda},\mathfrak{a}}$ generated by the relations

$$a_{k_1}(y_1)e(\boldsymbol{k}) = 0, \tag{1.2.12}$$

for all $\boldsymbol{k} = (k_1, \dots, k_n) \in K^\alpha$. Similarly, if K is finite we define the cyclotomic quiver Hecke algebra $\mathrm{R}_n^{\boldsymbol{\Lambda},\mathfrak{a}}(Q)$.

1.2.2 Case of quivers

Let Γ be a loop-free quiver with vertex set K. For any $k \neq k' \in K$, we write $d_{k,k'}$ for the number of arrows from k to k'. The *Cartan matrix* of Γ is the matrix $C = (c_{k,k'})_{k,k'\in K}$ defined by

$$c_{k,k'} \coloneqq \begin{cases} 2, & \text{if } k = k', \\ -d_{k,k'} - d_{k',k}, & \text{otherwise}, \end{cases} \tag{1.2.13}$$

for all $k, k' \in K$. Following [Rou, §3.2.4] we associate with Γ the matrix $(Q_{k,k'})_{k,k'\in K}$ of bivariate polynomials given by $Q_{k,k} \coloneqq 0$ and

$$Q_{k,k'}(u, v) \coloneqq (-1)^{d_{k,k'}}(u - v)^{-c_{k,k'}}, \tag{1.2.14}$$

for any $k, k' \in K$ with $k \neq k'$. Moreover, we define:

$$\mathrm{R}_\alpha(\Gamma) \coloneqq \mathrm{R}_\alpha(Q),$$

and if K is finite we also set $\mathrm{R}_n(\Gamma) \coloneqq \mathrm{R}_n(Q)$.

We will be particularly interested in the case where Γ has no multiple edges. In this case, the definition (1.2.14) reads

$$Q_{k,k'}(u,v) = \begin{cases} 0, & \text{if } k = k', \\ 1, & \text{if } k \not\to k', \\ v - u, & \text{if } k \to k', \\ u - v, & \text{if } k \leftarrow k', \\ -(u-v)^2, & \text{if } k \leftrightarrows k', \end{cases} \tag{1.2.15}$$

where:

- we write $k \not\to k'$ when $k \neq k'$ and neither (k, k') or (k', k) is an edge of Γ;
- we write $k \to k'$ when (k, k') is an edge of Γ and (k', k) is not;
- we write $k \leftarrow k'$ when (k', k) is an edge of Γ and (k, k') is not;
- we write $k \leftrightarrows k'$ when both (k, k') and (k', k) are edges of Γ.

In this case, the defining relations (1.2.4) become in $\mathrm{R}_\alpha(\Gamma)$

$$\psi_b^2 e(\boldsymbol{k}) = \begin{cases} 0, & \text{if } k_b = k_{b+1}, \\ e(\boldsymbol{k}), & \text{if } k_b \not\to k_{b+1}, \\ (y_{b+1} - y_b)e(\boldsymbol{k}), & \text{if } k_b \to k_{b+1}, \\ (y_b - y_{b+1})e(\boldsymbol{k}), & \text{if } k_b \leftarrow k_{b+1}, \\ (y_{b+1} - y_b)(y_b - y_{b+1})e(\boldsymbol{k}), & \text{if } k_b \leftrightarrows k_{b+1}, \end{cases} \tag{1.2.16a}$$

$$\psi_{c+1}\psi_c\psi_{c+1} e(\boldsymbol{k}) = \begin{cases} (\psi_c\psi_{c+1}\psi_c - 1)e(\boldsymbol{k}), & \text{if } k_{c+2} = k_c \to k_{c+1}, \\ (\psi_c\psi_{c+1}\psi_c + 1)e(\boldsymbol{k}), & \text{if } k_{c+2} = k_c \leftarrow k_{c+1}, \\ (\psi_c\psi_{c+1}\psi_c + 2y_{c+1} - y_c - y_{c+2})e(\boldsymbol{k}), & \text{if } k_{c+2} = k_c \leftrightarrows k_{c+1}, \\ \psi_c\psi_{c+1}\psi_c e(\boldsymbol{k}), & \text{otherwise}, \end{cases} \tag{1.2.16b}$$

for all $\boldsymbol{k} \in K^\alpha$, $b \in \{1, \dots, n-1\}$ and $c \in \{1, \dots, n-2\}$.

We now give a remarkable fact about quiver Hecke algebras. Its proof only requires a simple check of the different defining relations.

Proposition 1.2.17. *Let Γ be a loop-free quiver with vertex set K. The quiver Hecke algebra $\mathrm{R}_\alpha(\Gamma)$ is $\mathbb{Z}$-graded via*

$$\begin{aligned} \deg e(\boldsymbol{k}) &= 0, \\ \deg y_a e(\boldsymbol{k}) &= 2, && \textit{for all } a \in \{1, \dots, n\}, \\ \deg \psi_a e(\boldsymbol{k}) &= -c_{k_a, k_{a+1}}, && \textit{for all } a \in \{1, \dots, n-1\}, \end{aligned}$$

for all $\boldsymbol{k} \in K^\alpha$.

Now let $\boldsymbol{\Lambda} = (\Lambda_k)_{k \in K} \in \mathbb{N}^{(K)}$ be a weight. We define a particular case of cyclotomic quotient of $\mathrm{R}_\alpha(\Gamma)$, which will be important in Chapters 2 and 3.

Definition 1.2.18. The cyclotomic quiver Hecke algebra $\mathrm{R}_\alpha^{\mathbf{\Lambda}}(\Gamma)$ is the quotient of the quiver Hecke algebra $\mathrm{R}_\alpha(\Gamma)$ by the two-sided ideal $\mathcal{I}_\alpha^{\mathbf{\Lambda}}$ generated by the relations

$$y_1^{\Lambda_{k_1}} e(\boldsymbol{k}) = 0, \tag{1.2.19}$$

for all $\boldsymbol{k} \in K^\alpha$. In other words, if $\mathfrak{a}^0 \in \mathrm{Pol}_K^{\mathbf{\Lambda}}$ is given by $a_k^0(y) := y^{\Lambda_k}$ for any $k \in K$, then $\mathrm{R}_\alpha^{\mathbf{\Lambda}}(\Gamma) = \mathrm{R}_\alpha^{\mathbf{\Lambda},\mathfrak{a}^0}(\Gamma)$.

Note that $\mathrm{R}_\alpha^{\mathbf{\Lambda}}(\Gamma)$ inherits the grading of $\mathrm{R}_\alpha(\Gamma)$. If K is finite, the cyclotomic quotient $\mathrm{R}_n^{\mathbf{\Lambda}}(\Gamma)$ is obtained by quotienting $\mathrm{R}_n(\Gamma)$ by the relations

$$y_1^{\Lambda_{k_1}} e(\boldsymbol{k}) = 0,$$

for all $\boldsymbol{k} \in K^n$. Moreover, we have $e(\alpha)\mathrm{R}_n^{\mathbf{\Lambda}}(\Gamma) \simeq \mathrm{R}_\alpha^{\mathbf{\Lambda}}(\Gamma)$ and

$$\mathrm{R}_n^{\mathbf{\Lambda}}(\Gamma) \simeq \bigoplus_{\alpha \models_K n} \mathrm{R}_\alpha^{\mathbf{\Lambda}}(\Gamma),$$

and this can be considered as a definition of $\mathrm{R}_n^{\mathbf{\Lambda}}(\Gamma)$ if K is infinite.

1.2.3 Properties of the underlying modules

For each $w \in \mathfrak{S}_n$, we now choose a reduced expression $w = s_{a_1} \cdots s_{a_r}$ and we set:

$$\psi_w := \psi_{a_1} \cdots \psi_{a_r} \in \mathrm{R}_n(Q). \tag{1.2.20}$$

Because of (1.2.4b), we cannot apply Matsumoto's theorem (see, for instance, [GePf, Theorem 1.2.2]), as in the case of Iwahori–Hecke algebras: the element ψ_w may depend on the chosen reduced expression for $w \in \mathfrak{S}_n$. However, we can still use these elements to give a basis of $\mathrm{R}_n(Q)$. In fact, we have the following theorem ([Rou, Theorem 3.7], [KhLau09, Theorem 2.5]).

Theorem 1.2.21. *The family*

$$\left\{\psi_w y_1^{m_1} \cdots y_n^{m_n} e(\boldsymbol{k}) : w \in \mathfrak{S}_n, m_a \in \mathbb{N}, \boldsymbol{k} \in K^\alpha\right\},$$

is a basis of the free A-module $\mathrm{R}_\alpha(Q)$.

We do not have an explicit basis for the cyclotomic quotient, however we have the following finiteness result of [KanKa, Corollary 4.4].

Theorem 1.2.22. *For any weight $\mathbf{\Lambda} \in \mathbb{N}^{(K)}$ and family $\mathfrak{a} \in \mathrm{Pol}_K^{\mathbf{\Lambda}}$, the A-module $\mathrm{R}_\alpha^{\mathbf{\Lambda},\mathfrak{a}}(Q)$ is finitely generated.*

Now assume that Γ is a quiver with no loop. Let $\alpha \models_K n$ and let $\mathbf{\Lambda} \in \mathbb{N}^{(K)}$ be a weight. We will give a more precise version of Theorem 1.2.22. The following lemma appears at [BrKl-a, Lemma 2.1] in the case where Γ has (no loop and) only single edges, but the proof straight away generalises to our case.

Lemma 1.2.23. *The elements $y_a \in \mathrm{R}_\alpha^{\mathbf{\Lambda}}(\Gamma)$ are nilpotent for any $a \in \{1, \ldots, n\}$.*

We thus recover a particular case of Theorem 1.2.22.

Theorem 1.2.24. *The family*

$$\mathcal{B}_\alpha^{\mathbf{\Lambda}} := \left\{\psi_w y_1^{m_1} \cdots y_n^{m_n} e(\boldsymbol{k}) : w \in \mathfrak{S}_n, m_a \in \mathbb{N}, \boldsymbol{k} \in K^\alpha\right\},$$

is finite and spans $\mathrm{R}_\alpha^{\mathbf{\Lambda}}(\Gamma)$ over A. In particular, if $A = F$ is a field then $\mathrm{R}_\alpha^{\mathbf{\Lambda}}(\Gamma)$ is a finite-dimensional F-vector space.

Note that it is not clear how to extract an F-basis from the generating family of Theorem 1.2.24, or even to explicit a basis. Let us mention the works of Hu and Mathas [HuMa10] or Hu and Liang [HuLi], for instance, who gave examples of explicit bases in very particular cases.

1.3 Disjoint quiver isomorphism

In this section, we prove a general result on quiver Hecke algebras in the case where the quiver is given by a disjoint union of full subquivers, see Theorems 1.3.47 and 1.3.57. The isomorphism is built from the following map (see (1.3.50)):

$$\Psi_{\mathfrak{t}',\mathfrak{t}} : w \mapsto (\psi_{\pi_{\mathfrak{t}'}} w \psi_{\pi_{\mathfrak{t}}^{-1}}) E_{\mathfrak{t}',\mathfrak{t}},$$

for any $w \in e(\mathfrak{t}')\mathrm{R}_n(Q)e(\mathfrak{t})$, where $E_{\mathfrak{t}',\mathfrak{t}}$ are elementary matrices. In §1.3.1.2 we introduce the elements $\pi_{\mathfrak{t}}$, the minimal-length representatives of the right cosets of $\mathfrak{S}_n$ under the action of the Young subgroup $\mathfrak{S}_\lambda$ where $\lambda \models_d n$. This will lead to some calculations which will only be needed to explicit our homomorphism $\Psi_{\mathfrak{t}',\mathfrak{t}}$. Then, in §1.3.2 we will study the elements $\psi_{\pi_{\mathfrak{t}}}$, and we will go on with the previous calculations. In §1.3.3 we prove the theorem for the quiver Hecke algebra and in §1.3.4 we prove it for the cyclotomic quotients.

1.3.1 Setting

For simplicity we assume here that the set K is finite. We define $J = \mathbb{Z}/d\mathbb{Z} \simeq \{1, \dots, d\}$. We consider a partition of K into d parts $K = \sqcup_{j \in J} K_j$. We recall that the left action of $w \in \mathfrak{S}_n$ on tuples is given by $w \cdot (x_1, \dots, x_n) := (x_{w^{-1}(1)}, \dots, x_{w^{-1}(n)})$. We may use some elementary theory about Coxeter groups: we refer for instance to [GePf] or [Hum]. In particular, in that context we will denote by ℓ the usual length function $\mathfrak{S}_n \to \mathbb{N}$.

1.3.1.1 Labellings and shapes

Let $\lambda \models_d n$. We define the integers $\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_d$, given by $\boldsymbol{\lambda}_j := \lambda_1 + \dots + \lambda_j$ for any $j \in J$. In particular, we have $\boldsymbol{\lambda}_1 = \lambda_1$ and $\boldsymbol{\lambda}_d = n$. We also set $\boldsymbol{\lambda}_0 := 0$. In this section, from now on the letter λ always stands for a d-composition of n.

Definition 1.3.1. Let $\boldsymbol{k} \in K^n$ and $\mathfrak{t} \in J^n$.

- We say that $\boldsymbol{k}$ is a *labelling* of $\mathfrak{t}$ when

  $$k_a \in K_{\mathfrak{t}_a},$$

  for all $a \in \{1, \dots, n\}$, that is,

  $$k_a \in K_j \iff \mathfrak{t}_a = j,$$

  for all $a \in \{1, \dots, n\}$ and $j \in J$. We write $K^{\mathfrak{t}}$ for the elements K^n which are labellings of $\mathfrak{t}$.

- We say that $\mathfrak{t}$ has *shape* $\lambda \models_d n$ and we write $[\mathfrak{t}] = \lambda$ if for all $j \in J$ there are exactly λ_j components of $\mathfrak{t}$ equal to j, that is,

  $$\#\{a \in \{1, \dots, n\} : \mathfrak{t}_a = j\} = \lambda_j,$$

  for all $j \in J$. We write J^λ for the elements J^n with shape λ.

The sets J^λ are exactly the orbits of J^n under the action of $\mathfrak{S}_n$, in particular $[w \cdot \mathfrak{t}] = [\mathfrak{t}]$ for every $w \in \mathfrak{S}_n$ and $\mathfrak{t} \in J^n$. Moreover, the cardinality of J^λ is

$$m_\lambda := \frac{n!}{\lambda_1! \dots \lambda_d!}. \tag{1.3.2}$$

We write $\mathfrak{t}^\lambda \in J^\lambda$ for the trivial element of shape λ, given by

$$\mathfrak{t}^\lambda_a = j \iff \boldsymbol{\lambda}_{j-1} < a \leq \boldsymbol{\lambda}_j, \tag{1.3.3}$$

for all $a \in \{1, \dots, n\}$ and $j \in J$, that is,

$$\mathfrak{t}^\lambda := (1, \dots, 1, \dots, d, \dots, d),$$

where each $j \in J$ appears exactly λ_j times. Note that $K^{\mathfrak{t}^\lambda} \simeq K_1^{\lambda_1} \times \cdots \times K_d^{\lambda_d}$.

1.3.1.2 Young subgroups

Most results of this section are well-known. However, since in the literature they are stated either for a left or a right action (see Remark 1.3.16), for the convenience of the reader we state all of them with a left action. We remind the reader that some calculations made here will only be used in §3.7, namely with Lemmas 1.3.19 and 1.3.20.

Let $\lambda \vDash_d n$. The following group:

$$\mathfrak{S}_\lambda := \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_d},$$

can be seen as a subgroup of $\mathfrak{S}_n$ (the "Young subgroup"), where we consider that $\mathfrak{S}_{\lambda_j} \simeq \mathfrak{S}(\{\boldsymbol{\lambda}_{j-1} + 1, \dots, \boldsymbol{\lambda}_j\})$. Recall that:

- the group $\mathfrak{S}_n$ (resp. $\mathfrak{S}_{\lambda_j}$) is generated by $s_1, \dots, s_{n-1}$ (resp. $s_{\boldsymbol{\lambda}_{j-1}+1}, \dots, s_{\boldsymbol{\lambda}_j - 1}$);
- the subgroup $\mathfrak{S}_\lambda$ is generated by the elements s_a for all $a \in \{1, \dots, n\} \setminus \{\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_d\}$.

In particular:

$$w_j w_{j'} = w_{j'} w_j \text{ in } \mathfrak{S}_n, \tag{1.3.4}$$

for all $j \neq j'$ and $(w_j, w_{j'}) \in \mathfrak{S}_{\lambda_j} \times \mathfrak{S}_{\lambda_{j'}}$.

Remark 1.3.5. If $w = s_{a_1} \cdots s_{a_r} \in \mathfrak{S}_\lambda$ is a reduced expression, by (1.3.4) we can assume that there is a sequence $0 =: r_0 \leq r_1 \leq \cdots \leq r_{d-1} \leq r_d := r$ such that for each $j \in J$, the word $s_{a_{r_{j-1}+1}} \cdots s_{a_{r_j}}$ is reduced and lies in $\mathfrak{S}_{\lambda_j}$. The converse is also true: if for each $j \in J$ we have a reduced word $s_{a_{r_{j-1}+1}} \cdots s_{a_{r_j}} \in \mathfrak{S}_{\lambda_j}$ then their concatenation $s_{a_1} \cdots s_{a_r} \in \mathfrak{S}_\lambda$ is reduced.

The following proposition is straightforward.

Proposition 1.3.6. *The stabiliser of $\mathfrak{t}^\lambda$ under the action of $\mathfrak{S}_n$ is exactly $\mathfrak{S}_\lambda$.*

We now study the right cosets in $\mathfrak{S}_n$ for the (left) action of $\mathfrak{S}_\lambda$.

Lemma 1.3.7. *Two words $w, w' \in \mathfrak{S}_n$ are in the same right coset if and only if $w^{-1} \cdot \mathfrak{t}^\lambda = w'^{-1} \cdot \mathfrak{t}^\lambda$.*

The proof is straightforward from Proposition 1.3.6. An element $C \in \mathfrak{S}_\lambda \backslash \mathfrak{S}_n$ is thus determined by the constant value $\mathfrak{t} := w^{-1} \cdot \mathfrak{t}^\lambda \in J^\lambda$ for any $w \in C$. We write $C_\mathfrak{t}$ for the coset C (as each $\mathfrak{t} \in J^n$ has a unique shape, we do not need to precise the underlying composition in the indexation). Noticing that $m_\lambda = |\mathfrak{S}_n|/|\mathfrak{S}_\lambda|$, we conclude that the cosets are parametrised by the whole set J^λ, that is, $\mathfrak{S}_\lambda \backslash \mathfrak{S}_n = \{C_\mathfrak{t}\}_{\mathfrak{t} \in J^\lambda}$.

We know by [GePf, Proposition 2.1.1] that each coset $C_\mathfrak{t}$ has a unique minimal length element: we write $\pi_\mathfrak{t} \in C_\mathfrak{t}$ for this unique element. In particular, since Lemma 1.3.7 gives

$$w \in C_\mathfrak{t} \iff w \cdot \mathfrak{t} = \mathfrak{t}^\lambda, \tag{1.3.8}$$

for all $w \in \mathfrak{S}_n$, we obtain the following proposition.

Proposition 1.3.9. *The element $\pi_\mathfrak{t}$ is the unique minimal length element of $\mathfrak{S}_n$ such that*

$$\pi_\mathfrak{t} \cdot \mathfrak{t} = \mathfrak{t}^\lambda. \tag{1.3.10}$$

Remark 1.3.11. The decomposition into right cosets is obtained in the following way. Given $w \in \mathfrak{S}_n$, we know that w belongs to the coset $C_\mathfrak{t}$ with $\mathfrak{t} := w^{-1} \cdot \mathfrak{t}^\lambda$. The element $\widetilde{w}^{-1} := \pi_\mathfrak{t} w^{-1}$ stabilises $\mathfrak{t}^\lambda$, thus lies in $\mathfrak{S}_\lambda$ and we have $w = \widetilde{w}\pi_\mathfrak{t}$.

Proposition 1.3.12. *The elements $\pi_\mathfrak{t}$ are given by*

$$\pi_\mathfrak{t}(a) = \boldsymbol{\lambda}_{\mathfrak{t}_a - 1} + \#\{b \leq a : \mathfrak{t}_b = \mathfrak{t}_a\},$$

for any $a \in \{1, \dots, n\}$.

An example is given in Figure 1.1. To prove Proposition 1.3.12, we will use the vocabulary

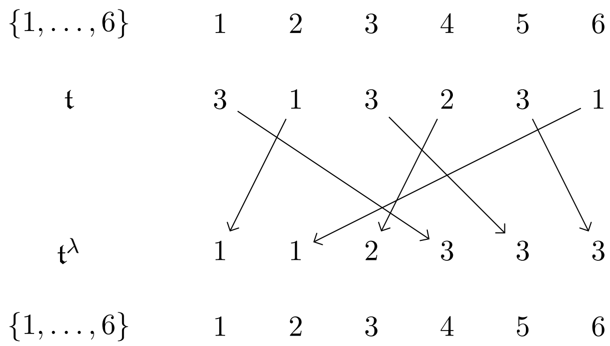


Figure 1.1: The permutation $\pi_\mathfrak{t}$ for $\lambda := (2,1,3) \models_3 6$ and $\mathfrak{t} := (3,1,3,2,3,1)$.

of "tableaux" (see for example [Ma99, §3.1]). As a quick reminder, a λ-*tableau* $\mathcal{T}$ is a bijection $\{(j,m) \in \mathbb{N}^2 : 1 \leq j \leq d \text{ and } 1 \leq m \leq \lambda_j\} \to \{1, \dots, n\}$; the tableau $\mathcal{T}$ is *row-standard* if in each rows, its entries increase from left to right. Here are two examples of λ-tableaux, with $\lambda := (2,1,3) \models_3 6$:

2	6	
4		
1	3	5

,

3	2	
5		
6	1	4

,

the first only being row-standard.

To any $\mathfrak{t} \in J^\lambda$, we associate the λ-tableau $\mathcal{T}_\mathfrak{t}$ given by the following rule: for any $j \in J$ and $m \in \{1, \dots, \lambda_j\}$, we label the node (j,m) by the index a of the mth occurrence of j in $\mathfrak{t}$, that is, by the integer $a \in \{1, \dots, n\}$ determined by:

$$\mathfrak{t}_a = j \text{ and } \#\{b \leq a : \mathfrak{t}_b = \mathfrak{t}_a\} = m. \tag{1.3.13}$$

In particular, the tableau $\mathcal{T}_\mathfrak{t}$ is row-standard; conversely, each row-standard λ-tableau is a $\mathcal{T}_\mathfrak{t}$ for a unique $\mathfrak{t} \in J^\lambda$. With the notation of Figure 1.1, here are two examples of row-standard λ-tableaux:

$\mathcal{T}_\mathfrak{t} =$

2	6	
4		
1	3	5

,

$\mathcal{T}_{\mathfrak{t}^\lambda} =$

1	2	
3		
4	5	6

.

We consider the natural left action of the symmetric group $\mathfrak{S}_n$ on the set of λ-tableaux: if $w \in \mathfrak{S}_n$ and $\mathcal{T}$ is a λ-tableau, the tableau $w \cdot \mathcal{T}$ is obtained by applying w in each box of $\mathcal{T}$. If now $\mathcal{T}$ and $\mathcal{T}'$ are two λ-tableaux, we write $\mathcal{T} \sim \mathcal{T}'$ if for all $j \in J$, the labels of the jth row of $\mathcal{T}$ are a permutation of the labels of the jth row of $\mathcal{T}'$.

Lemma 1.3.14. *For any $w \in \mathfrak{S}_n$ and $\mathfrak{t} \in J^\lambda$ we have*

$$w \cdot \mathcal{T}_{\mathfrak{t}} \sim \mathcal{T}_{\mathfrak{t}^\lambda} \iff w \cdot \mathfrak{t} = \mathfrak{t}^\lambda.$$

Proof. If $j \in J$ and $m \in \{1, \dots, \lambda_j\}$, we denote by $a[j,m]$ the label of the box (j,m) of $\mathcal{T}_{\mathfrak{t}}$. By (1.3.13) we have $\mathfrak{t}_{a[j,m]} = j$. We get:

$$\begin{aligned}
w \cdot \mathcal{T}_{\mathfrak{t}} \sim \mathcal{T}_{\mathfrak{t}^\lambda} &\iff \forall j \in J, \forall m \in \{1, \dots, \lambda_j\}, w(a[j,m]) \in \{\boldsymbol{\lambda}_{j-1} + 1, \dots, \boldsymbol{\lambda}_j\} \\
&\iff \forall j \in J, \forall m \in \{1, \dots, \lambda_j\}, \mathfrak{t}^\lambda_{w(a[j,m])} = j \\
&\iff \forall j \in J, \forall m \in \{1, \dots, \lambda_j\}, \mathfrak{t}^\lambda_{w(a[j,m])} = \mathfrak{t}_{a[j,m]} \\
&\iff \forall a \in \{1, \dots, n\}, \mathfrak{t}^\lambda_{w(a)} = \mathfrak{t}_a \\
&\iff \forall a \in \{1, \dots, n\}, \mathfrak{t}^\lambda_a = \mathfrak{t}_{w^{-1}(a)} \\
w \cdot \mathcal{T}_{\mathfrak{t}} \sim \mathcal{T}_{\mathfrak{t}^\lambda} &\iff \mathfrak{t}^\lambda = w \cdot \mathfrak{t},
\end{aligned}$$

as desired. □

Proof of Proposition 1.3.12. Let $\mathfrak{t} \in J^\lambda$. There is a unique element $d(\mathfrak{t}) \in \mathfrak{S}_n$ such that $\mathcal{T}_{\mathfrak{t}} = d(\mathfrak{t}) \cdot \mathcal{T}_{\mathfrak{t}^\lambda}$, that is, $d(\mathfrak{t})^{-1} \cdot \mathcal{T}_{\mathfrak{t}} = \mathcal{T}_{\mathfrak{t}^\lambda}$. By the equation of the coset $C_{\mathfrak{t}}$ given at (1.3.8) and Lemma 1.3.14, we get that $d(\mathfrak{t})^{-1} \in C_{\mathfrak{t}}$. Applying [Ma99, Proposition 3.3], we know that $d(\mathfrak{t})^{-1}$ is the unique minimal length element of $C_{\mathfrak{t}}$. As a consequence, we have $d(\mathfrak{t})^{-1} = \pi_{\mathfrak{t}}$ and thus:

$$\pi_{\mathfrak{t}} \cdot \mathcal{T}_{\mathfrak{t}} = \mathcal{T}_{\mathfrak{t}^\lambda}. \tag{1.3.15}$$

Let $j \in J$ and $m \in \{1, \dots, \lambda_j\}$, and let a (respectively α) be the label of the box (j,m) in $\mathcal{T}_{\mathfrak{t}}$ (resp. $\mathcal{T}_{\mathfrak{t}^\lambda}$). In particular, by (1.3.13) we have $\alpha = \boldsymbol{\lambda}_{j-1} + m$. Moreover, by (1.3.15) we have $\pi_{\mathfrak{t}}(a) = \alpha$. We conclude that the announced formula is satisfied, since, by a last use of (1.3.13), we have $j = \mathfrak{t}_a$ and $m = \#\{b \leq a : \mathfrak{t}_b = \mathfrak{t}_a\}$. □

Remark 1.3.16. In [Ma99], the author considers the elements of $\mathfrak{S}_n$ as acting on $\{1, \dots, n\}$ from the *right*, by $iw := w(i)$ where $i \in \{1, \dots, n\}$ and $w \in \mathfrak{S}_n$ is a permutation. This is the right action of $\mathfrak{S}_n^{\mathrm{op}}$. In such a setting, we read products of permutations from left to right.

Lemma 1.3.17. *Let $\mathfrak{t} \in J^\lambda$ and let $\pi_{\mathfrak{t}} = s_{a_1} \cdots s_{a_r}$ be a reduced expression. Then*

$$s_{a_m} \cdot (w_m \cdot \mathfrak{t}) \neq w_m \cdot \mathfrak{t},$$

for all $m \in \{1, \dots, r\}$, where $w_m := s_{a_{m+1}} \cdots s_{a_r}$ (with $w_m = 1$ if $m = r$).

Proof. Let us assume that $s_{a_m} \cdot (w_m \cdot \mathfrak{t}) = w_m \cdot \mathfrak{t}$ and define $\widetilde{\pi}_{\mathfrak{t}} := s_{a_1} \cdots s_{a_{m-1}} s_{a_{m+1}} \cdots s_{a_r}$. Using the assumption and the equality $\pi_{\mathfrak{t}} \cdot \mathfrak{t} = \mathfrak{t}^\lambda$, we see that the element $\widetilde{\pi}_{\mathfrak{t}}$ satisfies $\widetilde{\pi}_{\mathfrak{t}} \cdot \mathfrak{t} = \mathfrak{t}^\lambda$ too. As the element $\widetilde{\pi}_{\mathfrak{t}}$ is strictly shorter that $\pi_{\mathfrak{t}}$ (since $s_{a_1} \cdots s_{a_r}$ is reduced), this is in contradiction with Proposition 1.3.9. □

Remark 1.3.18. Using $\mathfrak{t} = \pi_{\mathfrak{t}}^{-1} \cdot \mathfrak{t}^\lambda$ in Lemma 1.3.17, we get the following similar result for $\pi_{\mathfrak{t}}^{-1}$. If $\pi_{\mathfrak{t}}^{-1} = s_{a_r} \cdots s_{a_1}$ is a reduced expression, then

$$w'_m \cdot \mathfrak{t}^\lambda \neq s_{a_m} \cdot (w'_m \cdot \mathfrak{t}^\lambda),$$

for all $m \in \{1, \dots, r\}$, where $w'_m := s_{a_{m-1}} \cdots s_{a_1}$.

The next two lemmas are not essential to the proof of the main theorem of this section, Theorem 1.3.47; however, they will allow us to relate our construction to the one of [JacPA, PA]. Let $\mathfrak{t} \in J^n$ and $a \in \{1, \dots, n-1\}$. We give in the next lemma the decomposition of Remark 1.3.11 for the element $\pi_\mathfrak{t} s_a$. This is in fact a particular case of Deodhar's lemma (see, for instance, [GePf, Lemma 2.1.2]).

Lemma 1.3.19. *Let $\mathfrak{t} \in J^n$ and $a \in \{1, \dots, n-1\}$. The element $\pi_\mathfrak{t} s_a$ belongs to the coset $C_{s_a \cdot \mathfrak{t}}$, more precisely we have:*

$$\pi_\mathfrak{t} s_a = \begin{cases} s_{\pi_\mathfrak{t}(a)} \pi_\mathfrak{t} & \text{if } \mathfrak{t}_a = \mathfrak{t}_{a+1}, \\ \pi_{s_a \cdot \mathfrak{t}} & \text{if } \mathfrak{t}_a \neq \mathfrak{t}_{a+1}. \end{cases}$$

Proof. First, from (1.3.8) we have $\pi_\mathfrak{t} s_a \cdot (s_a \cdot \mathfrak{t}) = \mathfrak{t}^\lambda$ (where $\lambda \models_d n$ is the shape of $\mathfrak{t}$) thus $\pi_\mathfrak{t} s_a$ lies in the coset $C_{s_a \cdot \mathfrak{t}}$.

We assume that $\mathfrak{t}_a = \mathfrak{t}_{a+1}$. We have $\pi_\mathfrak{t} s_a \pi_\mathfrak{t}^{-1} = (\pi_\mathfrak{t}(a), \pi_\mathfrak{t}(a+1))$, and we conclude since $\pi_\mathfrak{t}(a+1) = \pi_\mathfrak{t}(a) + 1$ by Proposition 1.3.12.

We now assume that $\mathfrak{t}_a \neq \mathfrak{t}_{a+1}$. Using the same Proposition 1.3.12, we know that the permutation $w := \pi_\mathfrak{t}^{-1} \pi_{s_a \cdot \mathfrak{t}} \in \mathfrak{S}_n$ is supported by $\{a, a+1\}$. Thus, either $w = s_a$ or $w = \mathrm{id}$. Since $\mathfrak{t} \neq s_a \cdot \mathfrak{t}$ we have $\pi_\mathfrak{t} \neq \pi_{s_a \cdot \mathfrak{t}}$, hence $w \neq \mathrm{id}$. Hence, we get $w = s_a$, that is, $\pi_\mathfrak{t} s_a = \pi_{s_a \cdot \mathfrak{t}}$. ☐

We now generalise the result of Lemma 1.3.19 in the case $\mathfrak{t}_a = \mathfrak{t}_{a+1}$.

Lemma 1.3.20. *Let $\mathfrak{t} \in J^n$ and $a \in \{1, \dots, n-1\}$ with $\mathfrak{t}_a = \mathfrak{t}_{a+1}$. Let $s_{b_1} \cdots s_{b_r}$ be a reduced expression of $\pi_\mathfrak{t}$ and set $w_m := s_{b_{m+1}} \cdots s_{b_r}$. If $b \in \{1, \dots, n-1\}$ satisfies $s_b = w_m s_a w_m^{-1}$ for some $m \in \{0, \dots, r\}$, then:*

$$\pi_\mathfrak{t} s_a = s_{b_1} \cdots s_{b_m} s_b s_{b_{m+1}} \cdots s_{b_r}$$

is a reduced expression. Moreover, every reduced expression of $\pi_\mathfrak{t} s_a$ is as above.

Proof. We first make an observation. As $\mathfrak{t}_a = \mathfrak{t}_{a+1}$, we deduce from (1.3.8) that the element $\pi_\mathfrak{t} s_a$ remains in $C_\mathfrak{t}$. Hence, by minimality of $\pi_\mathfrak{t}$ we have:

$$\ell(\pi_\mathfrak{t} s_a) > \ell(\pi_\mathfrak{t}). \tag{1.3.21}$$

Let now $s_{b_1} \cdots s_{b_r}$ be a reduced expression of $\pi_\mathfrak{t}$ and let $b \in \{1, \dots, n-1\}$ and $m \in \{0, \dots r\}$ such that $s_b = w_m s_a w_m^{-1}$. We have:

$$\pi_\mathfrak{t} s_a = s_{b_1} \cdots s_{b_m} w_m s_a = s_{b_1} \cdots s_{b_m} s_b w_m = s_{b_1} \cdots s_{b_m} s_b s_{b_{m+1}} \cdots s_{b_r},$$

and this expression is reduced since $\ell(\pi_\mathfrak{t} s_a) = \ell(\pi_\mathfrak{t}) + 1$. Conversely, let $s_{b'_0} \cdots s_{b'_r}$ be a reduced expression of $\pi_\mathfrak{t} s_a$. Since $\ell((\pi_\mathfrak{t} s_a) s_a) < \ell(\pi_\mathfrak{t} s_a)$, we can apply [Hum, §5.8 Theorem]: we know that there is some $m \in \{0, \dots r\}$ such that $s_{b'_0} \cdots \hat{s}_{b'_m} \cdots s_{b'_r}$ is a reduced expression of $\pi_\mathfrak{t}$, where the hat denotes the omission. We have:

$$s_{b'_0} \cdots s_{b'_r} = s_{b'_0} \cdots \hat{s}_{b'_m} \cdots s_{b'_r} s_a,$$

thus:

$$s_{b'_m} = w_m s_a w_m^{-1},$$

where $w_m := s_{b'_{m+1}} \cdots s_{b'_r}$. We now set $b := b'_m$ and:

$$b_p := \begin{cases} b'_{p-1} & \text{if } p \in \{1, \dots, m\}, \\ b'_p & \text{if } p \in \{m+1, \dots r\}. \end{cases}$$

Moreover:

- the expression $s_{b_1} \cdots s_{b_r} = s_{b'_0} \cdots \hat{s}_{b'_m} \cdots s_{b'_r} = \pi_{\mathfrak{t}}$ is reduced;
- we have $w_m = s_{b'_{m+1}} \cdots s_{b'_r} = s_{b_{m+1}} \cdots s_{b_r}$;
- we have $s_b = s_{b'_m} = w_m s_a w_m^{-1}$;

thus the reduced expression $\pi_{\mathfrak{t}} s_a = s_{b'_0} \cdots s_{b'_r} = s_{b_1} \cdots s_{b_m} s_b s_{b_{m+1}} \cdots s_{b_r}$ is of the desired form. □

Remark 1.3.22. Let $s_{b_1} \cdots s_{b_r} = \pi_{\mathfrak{t}}$ be a reduced expression and set $w_m := s_{b_{m+1}} \cdots s_{b_r}$. For any $m \in \{0, \ldots, r\}$, there exists $b' \in \{1, \ldots, n-1\}$ such that $s_{b'} = w_m s_a w_m^{-1}$ if and only if $w_m(a+1) = w_m(a) \pm 1$. Moreover, as Lemma 1.3.17 ensures that $w_m(a+1) > w_m(a)$, we have $w_m(a+1) = w_m(a) \pm 1 \iff w_m(a+1) = w_m(a) + 1$.

We end this subsection by introducing a notation. If $\mathfrak{t} \in J^\lambda$ and $\boldsymbol{k} \in K^{\mathfrak{t}^\lambda}$, we define:

$$\boldsymbol{k}^{\mathfrak{t}} := \pi_{\mathfrak{t}}^{-1} \cdot \boldsymbol{k} \in K^{\mathfrak{t}}; \tag{1.3.23}$$

in particular, we may denote by $\boldsymbol{k}^{\mathfrak{t}}$ the elements of $K^{\mathfrak{t}}$.

1.3.1.3 A "disjoint quiver" Hecke algebra

We consider the setting of §1.2.1. Note that since K is finite, we can consider the quiver Hecke algebra $\mathrm{R}_n(Q)$. Recall that the generators of $\mathrm{R}_n(Q)$ are given at (1.2.8) and are subject to the relations (1.2.3)–(1.2.4), where (1.2.3a) is replaced by (1.2.9). As we noticed in §1.2.3, the elements ψ_a do not satisfy the same braid relations as the elements $s_a \in \mathfrak{S}_n$. In particular, if $s_{b_1} \cdots s_{b_r}$ is a reduced expression different from the chosen one for $w \in \mathfrak{S}_n$, we may have $\psi_{b_1} \cdots \psi_{b_r} \neq \psi_w$. However, according to Remark 1.3.5 we can assume that we chose the reduced expressions such that

$$\psi_w = \psi_{w_1} \cdots \psi_{w_d}, \tag{1.3.24}$$

for all $w = (w_1, \ldots, w_d) \in \mathfrak{S}_\lambda$. To that extent, we can first choose some reduced expressions for the elements of the subgroups $\mathfrak{S}_{\lambda_j}$ for all $j \in J$ and then by product we obtain the reduced expressions of the element of $\mathfrak{S}_\lambda$. We now assume that we did such choices. Note that, concerning the elements of $\mathfrak{S}_n \setminus \mathfrak{S}_\lambda$, we can arbitrarily choose their reduced expressions.

We now assume that the matrix Q satisfies

$$Q_{k,k'} = 1, \tag{1.3.25}$$

for all $j \neq j'$ and $(k, k') \in K_j \times K_{j'}$. When the matrix Q is associated with a quiver Γ (recall §1.2.2), the condition (1.3.25) is satisfied when Γ is the disjoint union of d proper full subquivers $\Gamma^1, \ldots, \Gamma^d$. It means that:

- if v is a vertex in Γ then there is a unique $1 \leq j \leq d$ such that v is a vertex of Γ^j;
- if (v, w) is an edge in Γ then there is a (unique) $1 \leq j \leq d$ such that:
  - the vertices v and w are vertices of Γ^j,
  - the edge (v, w) is an edge of Γ^j.

Such a disjoint union in d proper subquivers will be encountered in Chapters 2 and 3. Moreover, regarding the Cartan matrix of Γ (defined at (1.2.13)) we have

$$c_{k,k'} = 0, \tag{1.3.26}$$

for all $j \neq j'$ and $(k, k') \in K_j \times K_{j'}$, that is, up to a permutation of the indexing set, the Cartan matrix of Γ is block diagonal. Finally, for any $j \in J$ we define

$$\forall k, k' \in K_j, Q^j_{k,k'} := Q_{k,k'}. \tag{1.3.27}$$

For each $j \in J$ and $n' \in \mathbb{N}$, we have an associated quiver Hecke algebra $\mathrm{R}_{n'}(Q^j)$.

1.3.1.4 Useful idempotents

We define in this section some idempotents of $\mathrm{R}_n(Q)$ which are essential for our proof. Thanks to the defining relations (1.2.3b)–(1.2.3d) and (1.2.9), for each $\lambda \models_d n$ the following element:

$$e(\lambda) := \sum_{\mathfrak{t} \in J^\lambda} \sum_{\boldsymbol{k} \in K^{\mathfrak{t}}} e(\boldsymbol{k}), \tag{1.3.28}$$

is a central idempotent in $\mathrm{R}_n(Q)$, that is, $e(\lambda) = e(\lambda)^2$ commutes with every element of $\mathrm{R}_n(Q)$. Moreover:

- if $\lambda' \models_d n$ is different from λ then $e(\lambda)e(\lambda') = 0$;
- we have $\sum_{\lambda \models_d n} e(\lambda) = 1$;

hence we have the following decomposition into subalgebras:

$$\mathrm{R}_n(Q) = \bigoplus_{\lambda \models_d n} e(\lambda)\mathrm{R}_n(Q). \tag{1.3.29}$$

For any $\mathfrak{t} \in J^\lambda$, we also define the following idempotent:

$$e(\mathfrak{t}) := \sum_{\boldsymbol{k} \in K^{\mathfrak{t}}} e(\boldsymbol{k}).$$

We can note that $e(\lambda) = \sum_{\mathfrak{t} \in J^\lambda} e(\mathfrak{t})$. Moreover, we have $e(\mathfrak{t})e(\mathfrak{t}') = 0$ if $\mathfrak{t}' \in J^n \setminus \{\mathfrak{t}\}$. We now give some lemmas which involve these elements $e(\mathfrak{t})$.

Lemma 1.3.30. *Let $\mathfrak{t} \in J^n$. We have the following relations:*

$$\begin{aligned}
\psi_a y_{a+1} e(\mathfrak{t}) &= y_a \psi_a e(\mathfrak{t}), && \text{if } \mathfrak{t}_a \neq \mathfrak{t}_{a+1},\\
\psi_a y_a e(\mathfrak{t}) &= y_{a+1} \psi_a e(\mathfrak{t}), && \text{if } \mathfrak{t}_a \neq \mathfrak{t}_{a+1},\\
\psi_a^2 e(\mathfrak{t}) &= e(\mathfrak{t}), && \text{if } \mathfrak{t}_a \neq \mathfrak{t}_{a+1},\\
\psi_{a+1}\psi_a\psi_{a+1} e(\mathfrak{t}) &= \psi_a\psi_{a+1}\psi_a e(\mathfrak{t}), && \text{if } \mathfrak{t}_a \neq \mathfrak{t}_{a+2}.
\end{aligned}$$

Proof. Let us first prove the first one. Let $\boldsymbol{k} \in K^{\mathfrak{t}}$. We have $k_a \in K_{\mathfrak{t}_a}$ and $k_{a+1} \in K_{\mathfrak{t}_{a+1}}$ with $\mathfrak{t}_a \neq \mathfrak{t}_{a+1}$ thus $k_a \neq k_{a+1}$. Hence, we get the result using the defining relation (1.2.3h) by summing over all $\boldsymbol{k} \in K^{\mathfrak{t}}$. The proofs of the second and the last equalities are similar.

Let us now prove $\psi_a^2 e(\mathfrak{t}) = e(\mathfrak{t})$ if $\mathfrak{t}_a \neq \mathfrak{t}_{a+1}$. Let $\boldsymbol{k} \in K^{\mathfrak{t}}$. We have $k_a \in K_{\mathfrak{t}_a}$ and $k_{a+1} \in K_{\mathfrak{t}_{a+1}}$ with $\mathfrak{t}_a \neq \mathfrak{t}_{a+1}$ thus $Q_{k_a,k_{a+1}} = 1$ (see (1.3.25)). Hence, the defining relation (1.2.4a) gives $\psi_a^2 e(\boldsymbol{k}) = e(\boldsymbol{k})$, and we again conclude by summing over all $\boldsymbol{k} \in K^{\mathfrak{t}}$. □

1.3.2 About the elements $\psi_{\pi_\mathfrak{t}}$

Here we prove some identities which are satisfied by the elements we have just introduced. Some of them will be essential to the proof of Theorem 1.3.47, while others will only be used in §3.7, namely with Lemmas 1.3.39 to 1.3.41. We first study some properties about the elements $\psi_{\pi_\mathfrak{t}}$ for any $\mathfrak{t} \in J^n$. We begin by the most important one, which is mentioned in the proof of [SVV, Lemma 3.17].

Lemma 1.3.31. *Let $\mathfrak{t} \in J^n$. If $s_{a_1} \cdots s_{a_r}$ and $s_{b_1} \cdots s_{b_r}$ are two reduced expressions of $\pi_\mathfrak{t}$, then:*

$$\psi_{a_1} \cdots \psi_{a_r} e(\mathfrak{t}) = \psi_{b_1} \cdots \psi_{b_r} e(\mathfrak{t}).$$

In other words the element $\psi_{\pi_\mathfrak{t}} e(\mathfrak{t}) \in \mathrm{R}_n(Q)$ does not depend on the choice of a reduced expression for $\pi_\mathfrak{t}$.

Proof. By Matsumoto's theorem, it suffices to check that every braid relation in $s_{a_1} \cdots s_{a_r}$ also occurs in $\psi_{a_1} \cdots \psi_{a_r} e(\mathfrak{t})$. By (1.2.3g), it is true for length 2-braids so it remains to check the case of the braids of length 3.

Suppose that we have a braid of length 3 in $s_{a_1} \cdots s_{a_r}$, at rank m. Thus, we have $a_m = a_{m+2} = a_{m+1} \pm 1$. We set $a := \min(a_m, a_{m+1})$. With $w_\mathrm{l} := s_{a_1} \cdots s_{a_{m-1}}$ and $w_\mathrm{r} := s_{a_{m+3}} \cdots s_{a_r}$, we have:

$$w_\mathrm{l}(s_a s_{a+1} s_a) w_\mathrm{r} = w_\mathrm{l}(s_{a+1} s_a s_{a+1}) w_\mathrm{r},$$

and we have to prove, with $\psi_\mathrm{l} := \psi_{a_1} \cdots \psi_{a_{m-1}}$ and $\psi_\mathrm{r} := \psi_{a_{m+3}} \cdots \psi_{a_r}$,

$$\psi_\mathrm{l}(\psi_a \psi_{a+1} \psi_a) \psi_\mathrm{r} e(\mathfrak{t}) = \psi_\mathrm{l}(\psi_{a+1} \psi_a \psi_{a+1}) \psi_\mathrm{r} e(\mathfrak{t}).$$

Using (1.2.3d), this becomes, where $\mathfrak{s} := w_\mathrm{r} \cdot \mathfrak{t}$,

$$\psi_\mathrm{l}(\psi_a \psi_{a+1} \psi_a) e(\mathfrak{s}) \psi_\mathrm{r} = \psi_\mathrm{l}(\psi_{a+1} \psi_a \psi_{a+1}) e(\mathfrak{s}) \psi_\mathrm{r}. \tag{1.3.32}$$

By Lemma 1.3.17, we have $s_{a_{m+1}} \cdot (s_{a_{m+2}} \cdot \mathfrak{s}) \neq s_{a_{m+2}} \cdot \mathfrak{s}$. Thus, we have either $s_a \cdot (s_{a+1} \cdot \mathfrak{s}) \neq s_{a+1} \cdot \mathfrak{s}$ or $s_{a+1} \cdot (s_a \cdot \mathfrak{s}) \neq s_a \cdot \mathfrak{s}$. Both cases give $\mathfrak{s}_a \neq \mathfrak{s}_{a+2}$, hence, applying Lemma 1.3.30 we know that (1.3.32) holds. ☐

Remark 1.3.33. In particular, if $\boldsymbol{k} \in K^\mathfrak{t}$ then $\psi_{\pi_\mathfrak{t}} e(\boldsymbol{k}) \in \mathrm{R}_n(Q)$ does not depend on the choice of a reduced expression for $\pi_\mathfrak{t}$ (note that $\psi_{\pi_\mathfrak{t}} e(\boldsymbol{k}) = \psi_{\pi_\mathfrak{t}} e(\mathfrak{t}) e(\boldsymbol{k})$).

Similarly to Lemma 1.3.31, using Remark 1.3.18 we prove that for any $\mathfrak{t} \in J^\lambda$ the element

$$e(\mathfrak{t}) \psi_{\pi_\mathfrak{t}^{-1}} = \psi_{\pi_\mathfrak{t}^{-1}} e(\mathfrak{t}^\lambda) \in \mathrm{R}_n(Q), \tag{1.3.34}$$

does not depend on the chosen reduced expression for $\pi_\mathfrak{t}^{-1}$. We now give some analogues of the results of Lemma 1.3.30.

Proposition 1.3.35. *Let $\mathfrak{t} \in J^\lambda$. We have:*

$$\psi_{\pi_\mathfrak{t}^{-1}} \psi_{\pi_\mathfrak{t}} e(\mathfrak{t}) = e(\mathfrak{t}),$$
$$\psi_{\pi_\mathfrak{t}} \psi_{\pi_\mathfrak{t}^{-1}} e(\mathfrak{t}^\lambda) = e(\mathfrak{t}^\lambda).$$

Remark 1.3.36. Both factors $\psi_{\pi_\mathfrak{t}^{-1}}$ and $\psi_{\pi_\mathfrak{t}}$ do not depend on the choices of reduced expressions: for instance, using (1.2.3d) we have $\psi_{\pi_\mathfrak{t}^{-1}} \psi_{\pi_\mathfrak{t}} e(\mathfrak{t}) = \psi_{\pi_\mathfrak{t}^{-1}} e(\mathfrak{t}^\lambda) \psi_{\pi_\mathfrak{t}}$ thus we can apply Lemma 1.3.31 and (1.3.34).

Proof. We only prove the first equality, the proof of the second one being entirely similar. Let $s_{a_1} \cdots s_{a_r}$ be a reduced expression for $\pi_{\mathfrak{t}}$. We prove by induction that for every $m \in \{1, \dots, r+1\}$ we have

$$\psi_{\pi_{\mathfrak{t}}^{-1}} \psi_{\pi_{\mathfrak{t}}} e(\mathfrak{t}) = \psi_{a_r} \cdots \psi_{a_m} \psi_{a_m} \cdots \psi_{a_r} e(\mathfrak{t}). \tag{1.3.37}$$

First, the case $a = 1$ comes with the definition of $\psi_{\pi_{\mathfrak{t}}^{-1}} e(\mathfrak{t}^\lambda)$ and $\psi_{\pi_{\mathfrak{t}}} e(\mathfrak{t})$. Now, if (1.3.37) is true for some $m \in \{1, \dots, r\}$ we have, using (1.2.3d),

$$\psi_{\pi_{\mathfrak{t}}^{-1}} \psi_{\pi_{\mathfrak{t}}} e(\mathfrak{t}) = \psi_{a_r} \cdots \psi_{a_{m+1}} \psi_{a_m}^2 e(w_m \cdot \mathfrak{t}) \psi_{a_{m+1}} \cdots \psi_{a_r}, \tag{1.3.38}$$

where $w_m \coloneqq s_{a_{m+1}} \cdots s_{a_r}$. By Lemma 1.3.17, we know that $(w_m \cdot \mathfrak{t})_{a_m} \neq (w_m \cdot \mathfrak{t})_{a_m+1}$. Hence, by Lemma 1.3.30 we have $\psi_{a_m}^2 e(w_m \cdot \mathfrak{t}) = e(w_m \cdot \mathfrak{t})$ thus (1.3.38) becomes

$$\psi_{\pi_{\mathfrak{t}}^{-1}} \psi_{\pi_{\mathfrak{t}}} e(\mathfrak{t}) = \psi_{a_r} \cdots \psi_{a_{m+1}} e(w_m \cdot \mathfrak{t}) \psi_{a_{m+1}} \cdots \psi_{a_r},$$

which becomes, with a last use of (1.2.3d),

$$\psi_{\pi_{\mathfrak{t}}^{-1}} \psi_{\pi_{\mathfrak{t}}} e(\mathfrak{t}) = \psi_{a_r} \cdots \psi_{a_{m+1}} \psi_{a_{m+1}} \cdots \psi_{a_r} e(\mathfrak{t}).$$

Thus (1.3.37) holds for every $m \in \{1, \dots, r+1\}$, in particular for $m = r+1$ we get the statement of the Proposition. □

Once again, what follows is not essential to the proof of the main result Theorem 1.3.47. However, it will allow us to relate our construction to the one of [JacPA, PA]. With a similar proof as Proposition 1.3.35, we obtain the following lemma.

Lemma 1.3.39. *Let $a \in \{1, \dots, n\}$ and $\mathfrak{t} \in J^\lambda$. We have*

$$y_a \psi_{\pi_{\mathfrak{t}}} e(\mathfrak{t}) = \psi_{\pi_{\mathfrak{t}}} y_{\pi_{\mathfrak{t}}^{-1}(a)} e(\mathfrak{t}),$$
$$y_a \psi_{\pi_{\mathfrak{t}}^{-1}} e(\mathfrak{t}^\lambda) = \psi_{\pi_{\mathfrak{t}}^{-1}} y_{\pi_{\mathfrak{t}}(a)} e(\mathfrak{t}^\lambda).$$

We now want to see what is happening with Lemma 1.3.19 for the associated elements ψ_w.

Lemma 1.3.40. *Let $\mathfrak{t} \in J^n$ and $a \in \{1, \dots, n-1\}$ such that $\mathfrak{t}_a \neq \mathfrak{t}_{a+1}$. We have*

$$e(\mathfrak{t}) \psi_{\pi_{\mathfrak{t}}^{-1}} \psi_{\pi_{s_a \cdot \mathfrak{t}}} = e(\mathfrak{t}) \psi_a.$$

Proof. By Lemma 1.3.19 we have $\ell(\pi_{s_a \cdot \mathfrak{t}}) = \ell(\pi_{\mathfrak{t}}) \pm 1$. We now simply distinguish cases.

- We first assume that $\ell(\pi_{s_a \cdot \mathfrak{t}}) = \ell(\pi_{\mathfrak{t}}) + 1$. Hence, applying Lemma 1.3.31 for the elements $\psi_{\pi_{s_a \cdot \mathfrak{t}}}$ and $\psi_{\pi_{\mathfrak{t}}}$ we have $e(\mathfrak{t}^\lambda) \psi_{\pi_{s_a \cdot \mathfrak{t}}} = e(\mathfrak{t}^\lambda) \psi_{\pi_{\mathfrak{t}}} \psi_a$. Finally, using Proposition 1.3.35 we have:

$$e(\mathfrak{t}) \psi_{\pi_{\mathfrak{t}}^{-1}} \psi_{\pi_{s_a \cdot \mathfrak{t}}} = \psi_{\pi_{\mathfrak{t}}^{-1}} e(\mathfrak{t}^\lambda) \psi_{\pi_{\mathfrak{t}}} \psi_a = \psi_{\pi_{\mathfrak{t}}^{-1}} \psi_{\pi_{\mathfrak{t}}} e(\mathfrak{t}) \psi_a = e(\mathfrak{t}) \psi_a.$$

- We now assume that $\ell(\pi_{s_a \cdot \mathfrak{t}}) = \ell(\pi_{\mathfrak{t}}) - 1$. Hence, we have $\ell(\pi_{\mathfrak{t}}^{-1}) = \ell(\pi_{s_a \cdot \mathfrak{t}}^{-1}) + 1$. Recall that $\pi_{\mathfrak{t}}^{-1} = s_a \pi_{s_a \cdot \mathfrak{t}}^{-1}$. Using the extension (1.3.34) of Lemma 1.3.31, we get $e(\mathfrak{t}) \psi_{\pi_{\mathfrak{t}}^{-1}} = e(\mathfrak{t}) \psi_a \psi_{\pi_{s_a \cdot \mathfrak{t}}^{-1}}$ and finally:

$$e(\mathfrak{t}) \psi_{\pi_{\mathfrak{t}}^{-1}} \psi_{\pi_{s_a \cdot \mathfrak{t}}} = \psi_a e(s_a \cdot \mathfrak{t}) \psi_{\pi_{s_a \cdot \mathfrak{t}}^{-1}} \psi_{\pi_{s_a \cdot \mathfrak{t}}} = \psi_a e(s_a \cdot \mathfrak{t}) = e(\mathfrak{t}) \psi_a.$$

□

Lemma 1.3.41. *Let* $\mathfrak{t} \in J^n$ *and* $a \in \{1, \dots, n-1\}$ *such that* $\mathfrak{t}_a = \mathfrak{t}_{a+1}$. *The element* $\psi_{\pi_\mathfrak{t} s_a} e(\mathfrak{t}) \in \mathrm{R}_n(Q)$ *does not depend on the choice of a reduced expression for* $\pi_\mathfrak{t} s_a$. *In particular,*

$$\psi_{\pi_\mathfrak{t}} \psi_a e(\mathfrak{t}) = \psi_{\pi_\mathfrak{t}(a)} \psi_{\pi_\mathfrak{t}} e(\mathfrak{t}).$$

Proof. As in the proof of Lemma 1.3.31, it suffices to prove that every 3-braid relation which occurs in a reduced expression of $\pi_\mathfrak{t} s_a$ is also satisfied in the corresponding element of $\mathrm{R}_n(Q)e(\mathfrak{t})$. By Lemma 1.3.20, we know that any reduced expression of $\pi_\mathfrak{t} s_a$ can be written $s_{b_0} \cdots s_{b_r}$, where there is $m \in \{0, \dots r\}$ such that:

- the word $s_{b_0} \cdots \hat{s}_{b_m} \cdots s_{b_r}$ is a reduced expression of $\pi_\mathfrak{t}$;
- we have $s_{b_m} = w_m s_a w_m^{-1}$ with $w_m := s_{b_{m+1}} \cdots s_{b_r}$.

We suppose that a 3-braid appears in $s_{b_0} \cdots s_{b_r}$ at index l, that is, we have $b_l = b_{l+2} = b_{l+1} \pm 1$. We set $b := \min(b_l, b_{l+1})$. We want to prove that, with $\psi_\mathrm{l} := \psi_{b_0} \cdots \psi_{b_{l-1}}$ and $\psi_\mathrm{r} := \psi_{b_{l+3}} \cdots \psi_{b_r}$,

$$\psi_\mathrm{l}(\psi_b \psi_{b+1} \psi_b) \psi_\mathrm{r} e(\mathfrak{t}) = \psi_\mathrm{l}(\psi_{b+1} \psi_b \psi_{b+1}) \psi_\mathrm{r} e(\mathfrak{t}).$$

To that extent, as in the proof of Lemma 1.3.31, thanks to Lemma 1.3.30 it suffices to prove that $\mathfrak{s}_b \neq \mathfrak{s}_{b+2}$ where $\mathfrak{s} := s_{b_{l+3}} \cdots s_{b_r} \cdot \mathfrak{t}$.

- If $m < l+1$ then applying Lemma 1.3.17 we have $s_{b_{l+1}} \cdot (s_{b_{l+2}} \cdot \mathfrak{s}) \neq s_{b_{l+2}} \cdot \mathfrak{s}$ thus $\mathfrak{s}_b \neq \mathfrak{s}_{b+2}$.
- The case $m = l+1$ is impossible: as $b_l = b_{l+2}$, if $m = l+1$ we would have $b_{m-1} = b_{m+1}$ and this is nonsense since the expression $s_{b_{m-1}} s_{b_{m+1}}$ is reduced (as a subexpression of the reduced expression $s_{b_0} \cdots \hat{s}_{b_m} \cdots s_{b_r} = \pi_\mathfrak{t}$).
- If $m = l+2$ then by Lemma 1.3.17 we get $s_{b_l} \cdot (s_{b_{l+1}} \cdot \mathfrak{s}) \neq s_{b_l} \cdot \mathfrak{s}$ thus $\mathfrak{s}_b \neq \mathfrak{s}_{b+2}$.
- Finally, if $m > l+2$ then we can notice that

  $$\mathfrak{s} = s_{b_{l+3}} \cdots \hat{s}_{b_m} \cdots s_{b_r} \cdot \mathfrak{t}$$

  (since $s_{b_m} \cdot (w_m \cdot \mathfrak{t}) = w_m \cdot (s_a \cdot \mathfrak{t}) = w_m \cdot \mathfrak{t}$; recall that $s_{b_m} = w_m s_a w_m^{-1}$ and $\mathfrak{t}_a = \mathfrak{t}_{a+1}$). Hence, we deduce once again the result from Lemma 1.3.17.

The last statement of the lemma is now immediate. As $\mathfrak{t}_a = \mathfrak{t}_{a+1}$, we can use (1.3.21) hence $\psi_{\pi_\mathfrak{t} s_a} e(\mathfrak{t}) = \psi_{\pi_\mathfrak{t}} \psi_a e(\mathfrak{t})$. Moreover, applying Lemma 1.3.19 another consequence of (1.3.21) is $\ell(s_{\pi_\mathfrak{t}(a)} \pi_\mathfrak{t}) = \ell(\pi_\mathfrak{t}) + 1$, thus we get $\psi_{\pi_\mathfrak{t} s_a} e(\mathfrak{t}) = \psi_{\pi_\mathfrak{t}(a)} \psi_{\pi_\mathfrak{t}} e(\mathfrak{t})$. Finally, we have $\psi_{\pi_\mathfrak{t} s_a} e(\mathfrak{t}) = \psi_{\pi_\mathfrak{t}} \psi_a e(\mathfrak{t}) = \psi_{\pi_\mathfrak{t}(a)} \psi_{\pi_\mathfrak{t}} e(\mathfrak{t})$. □

1.3.3 Decomposition along the subquiver Hecke algebras

We are now ready to prove the main result of this section: we will give in Theorem 1.3.47 a decomposition of $\mathrm{R}_n(Q)$ involving the algebras $\mathrm{R}_{n_j}(Q^j)$ for $j \in J$.

1.3.3.1 A distinguished subalgebra

In this paragraph, we prove the key of Theorem 1.3.47. Recall that we have set in (1.3.27)

$$Q^j_{k,k'} := Q_{k,k'},$$

for any $j \in J$ and $k, k' \in K_j$. Let $\lambda \models_d n$ be a d-composition of n. We define the following algebra:

$$\mathrm{R}_\lambda(Q) := \mathrm{R}_{\lambda_1}(Q^1) \otimes \cdots \otimes \mathrm{R}_{\lambda_d}(Q^d).$$

With $e_\lambda := e(\mathbf{t}^\lambda)$, we prove here that we can identify $\mathrm{R}_\lambda(Q)$ with the subalgebra $e_\lambda \mathrm{R}_n(Q) e_\lambda$ (with unit e_λ). We reindex the generators $\psi_1, \dots, \psi_{\lambda_j - 1}$ and $y_1, \dots, y_{\lambda_j}$ of $\mathrm{R}_{\lambda_j}(Q^j)$, respectively by $\psi_{\boldsymbol{\lambda}_{j-1}+1}, \dots, \psi_{\boldsymbol{\lambda}_j - 1}$ and $y_{\boldsymbol{\lambda}_{j-1}+1}, \dots, y_{\boldsymbol{\lambda}_j}$. In particular, we set

$$\psi_w^\otimes := \psi_{w_1} \otimes \cdots \otimes \psi_{w_d} \in \mathrm{R}_\lambda(Q),$$

for any $w = (w_1, \dots, w_d) \in \mathfrak{S}_\lambda$ and

$$e^\otimes(\boldsymbol{k}^1, \dots, \boldsymbol{k}^d) := e(\boldsymbol{k}^1) \otimes \cdots \otimes e(\boldsymbol{k}^d) \in \mathrm{R}_\lambda(Q), \tag{1.3.42}$$

for any $\boldsymbol{k} = (\boldsymbol{k}^1, \dots, \boldsymbol{k}^d) \in K_1^{\lambda_1} \times \cdots \times K_d^{\lambda_d}$.

Lemma 1.3.43. *The following family:*

$$\left\{ \psi_w y_1^{m_1} \cdots y_n^{m_n} e(\boldsymbol{k}), w \in \mathfrak{S}_\lambda, m_a \in \mathbb{N}, \boldsymbol{k} \in K^{\mathbf{t}^\lambda} \right\}, \tag{1.3.44}$$

is an A-basis of $e_\lambda \mathrm{R}_n(Q) e_\lambda$. Moreover, the algebra $e_\lambda \mathrm{R}_n(Q) e_\lambda$ is exactly the (non-unitary) subalgebra of $\mathrm{R}_n(Q)$ generated by:

- *the elements $\psi_a e_\lambda$ for all $a \in \{1, \dots, n\} \setminus \{\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_d\}$;*
- *the elements $y_a e_\lambda$ for all $1 \leq a \leq n$;*
- *the elements $e(\boldsymbol{k})$ for all $\boldsymbol{k} \in K^{\mathbf{t}^\lambda}$.*

Proof. The first part is a immediate application of Theorem 1.2.21, (1.2.3c), (1.2.3d) and Proposition 1.3.6. It remains to check that $e_\lambda \mathrm{R}_n(Q) e_\lambda$ is the described subalgebra, subalgebra that we temporarily denote by R. First, all the listed elements belong to $e_\lambda \mathrm{R}_n(Q) e_\lambda$. Note that to prove $\psi_a e_\lambda \in e_\lambda \mathrm{R}_n(Q) e_\lambda$ for any $a \neq \boldsymbol{\lambda}_j$, we can either use the above basis, or we can simply use (1.2.3d) to get $\psi_a e_\lambda = \psi_a e_\lambda^2 = e_\lambda \psi_a e_\lambda$. Hence, we have $\mathrm{R} \subseteq e_\lambda \mathrm{R}_n(Q) e_\lambda$. Finally, we conclude since every element of the basis (1.3.44) lies in R. ☐

Lemma 1.3.45. *There is a unitary algebra homomorphism from $\mathrm{R}_\lambda(Q)$ to $e_\lambda \mathrm{R}_n(Q) e_\lambda$.*

Proof. We define the algebra homomorphism from $\mathrm{R}_\lambda(Q)$ to $e_\lambda \mathrm{R}_n(Q) e_\lambda$ by sending:

- the generators $\psi_a^\otimes \in \mathrm{R}_\lambda(Q)$ for any $a \in \{1, \dots, n\} \setminus \{\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_d\}$ to $\psi_a e_\lambda \in e_\lambda \mathrm{R}_n(Q) e_\lambda$;
- the generators $y_b \in \mathrm{R}_\lambda(Q)$ for any $b \in \{1, \dots, n\}$ to $y_b e_\lambda \in e_\lambda \mathrm{R}_n(Q) e_\lambda$;
- the generators $e^\otimes(\boldsymbol{k}) \in \mathrm{R}_\lambda(Q)$ for any $\boldsymbol{k} \in K^{\mathbf{t}^\lambda}$ to $e(\boldsymbol{k}) \in e_\lambda \mathrm{R}_n(Q) e_\lambda$.

It suffices now to check the defining relations of $\mathrm{R}_\lambda(Q)$. We will only check (1.2.3h)–(1.2.4b), the remaining ones being straightforward.

(1.2.3h). Let $a \notin \{\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_d\}$ and $\boldsymbol{k} \in K^{\mathbf{t}^\lambda}$. If $k_a \in K_j$, as $a \neq \boldsymbol{\lambda}_{j'}$ for any j' we have $k_{a+1} \in K_j$ (cf. (1.3.3)). Hence, in $\mathrm{R}_\lambda(Q)$ the relation (1.2.3h),

$$\psi_a^\otimes y_{a+1} e^\otimes(\boldsymbol{k}) = \begin{cases} (y_a \psi_a^\otimes + 1) e^\otimes(\boldsymbol{k}), & \text{if } k_a = k_{a+1}, \\ y_a \psi_a^\otimes e^\otimes(\boldsymbol{k}), & \text{if } k_a \neq k_{a+1}, \end{cases}$$

comes from the corresponding relation in $\mathrm{R}_{\lambda_j}(Q^j)$. The same relation,

$$\psi_a y_{a+1} e(\boldsymbol{k}) = \begin{cases} (y_a \psi_a + 1) e(\boldsymbol{k}), & \text{if } k_a = k_{a+1}, \\ y_a \psi_a e(\boldsymbol{k}), & \text{if } k_a \neq k_{a+1}, \end{cases}$$

is satisfied in $e_\lambda \mathrm{R}_n(Q) e_\lambda$, as a relation in $\mathrm{R}_n(Q)$.

(1.2.3i). Similar.

(1.2.4a). Similarly, the indices k_a, k_{a+1} are in a same K_j. Hence, the relation (1.2.4a) in $\mathrm{R}_\lambda(Q)$ comes from a relation in $\mathrm{R}_{\lambda_j}(Q^j)$, and the same relation is satisfied in $e_\lambda \mathrm{R}_n(Q) e_\lambda$.

(1.2.4b). For any $j \in J$ and $a \in \{\boldsymbol{\lambda}_{j-1}+1, \dots, \boldsymbol{\lambda}_j - 2\}$, the relation (1.2.4b) is a relation from $\mathrm{R}_{\lambda_j}(Q^j)$, and this same relation is satisfied in $e_\lambda \mathrm{R}_n(Q) e_\lambda$.

□

Proposition 1.3.46. *The previous algebra homomorphism $\mathrm{R}_\lambda(Q) \to e_\lambda \mathrm{R}_n(Q) e_\lambda$ is an isomorphism. In particular, we can identify $\mathrm{R}_\lambda(Q)$ to a (non-unitary) subalgebra of $\mathrm{R}_n(Q)$.*

Proof. We know by Theorem 1.2.21 that $\mathrm{R}_{\lambda_j}(Q^j)$ has for basis

$$\left\{\psi_{w_j} y_{\boldsymbol{\lambda}_{j-1}+1}^{m_{\boldsymbol{\lambda}_{j-1}+1}} \cdots y_{\boldsymbol{\lambda}_j}^{m_{\boldsymbol{\lambda}_j}} e(\boldsymbol{k}^j) : w_j \in \mathfrak{S}_{\lambda_j}, m_a \in \mathbb{N}, \boldsymbol{k}^j \in K_j^{\lambda_j}\right\}.$$

Hence, the family

$$\left\{\psi_w^{\otimes} y_1^{m_1} \cdots y_n^{m_n} e^{\otimes}(\boldsymbol{k}) : w \in \mathfrak{S}_\lambda, m_a \in \mathbb{N}, \boldsymbol{k} \in K_1^{\lambda_1} \times \cdots \times K_d^{\lambda_d}\right\},$$

is a basis of $\mathrm{R}_\lambda(Q)$. We conclude since by (1.3.24) the homomorphism of Lemma 1.3.45 sends this basis onto the basis of $e_\lambda \mathrm{R}_n(Q) e_\lambda$ given in Lemma 1.3.43. (In particular, note that $\psi_w^{\otimes} \in \mathrm{R}_\lambda(Q)$ is sent to $\psi_w e_\lambda \in e_\lambda \mathrm{R}_n(Q) e_\lambda$ for any $w \in \mathfrak{S}_\lambda$.) □

1.3.3.2 Decomposition theorem

We recall the notation m_λ introduced at (1.3.2) for any $\lambda \vDash_d n$.

Theorem 1.3.47. *We have an A-algebra isomorphism*

$$\mathrm{R}_n(Q) \simeq \bigoplus_{\lambda \vDash_d n} \mathrm{Mat}_{m_\lambda} \mathrm{R}_\lambda(Q).$$

Note that, with a similar proof, we can also give such an isomorphism if K is infinite, decomposing the algebra $\mathrm{R}_\alpha(Q)$ for any $\alpha \vDash_K n$. The remaining part of this paragraph is devoted to the proof of Theorem 1.3.47. Due to (1.3.29), it suffices to prove that we have an A-algebra isomorphism

$$e(\lambda)\mathrm{R}_n(Q) \simeq \mathrm{Mat}_{m_\lambda} \mathrm{R}_\lambda(Q). \tag{1.3.48}$$

Let us label the rows and the columns of the elements of $\mathrm{Mat}_{m_\lambda} \mathrm{R}_\lambda(Q)$ by $(\mathfrak{t}', \mathfrak{t}) \in (J^\lambda)^2$, and let us write $E_{\mathfrak{t}',\mathfrak{t}}$ for the elementary matrix with one 1 at position $(\mathfrak{t}', \mathfrak{t})$ and 0 everywhere else. Recall the following property satisfied by the $E_{\mathfrak{t}',\mathfrak{t}}$:

$$\forall \mathfrak{t}, \mathfrak{t}', \mathfrak{s}, \mathfrak{s}' \in J^\lambda, E_{\mathfrak{t}',\mathfrak{t}} E_{\mathfrak{s}',\mathfrak{s}} = \delta_{\mathfrak{t},\mathfrak{s}'} E_{\mathfrak{t}',\mathfrak{s}}. \tag{1.3.49}$$

We have the following A-module isomorphism, where $\mathfrak{t}, \mathfrak{t}' \in J^\lambda$,

$$e(\mathfrak{t}')\mathrm{R}_n(Q)e(\mathfrak{t}) \simeq \mathrm{R}_\lambda(Q) E_{\mathfrak{t}',\mathfrak{t}}.$$

Indeed, let us define

$$\begin{aligned} \Phi_{\mathfrak{t}',\mathfrak{t}} &: \mathrm{R}_\lambda(Q) E_{\mathfrak{t}',\mathfrak{t}} \to e(\mathfrak{t}')\mathrm{R}_n(Q)e(\mathfrak{t}), \\ \Psi_{\mathfrak{t}',\mathfrak{t}} &: e(\mathfrak{t}')\mathrm{R}_n(Q)e(\mathfrak{t}) \to \mathrm{R}_\lambda(Q) E_{\mathfrak{t}',\mathfrak{t}}, \end{aligned} \tag{1.3.50}$$

by:

$$\Phi_{\mathfrak{t}',\mathfrak{t}}(vE_{\mathfrak{t}',\mathfrak{t}}) := \psi_{\pi_{\mathfrak{t}'}^{-1}} v \psi_{\pi_{\mathfrak{t}}}, \qquad \text{for all } v \in \mathrm{R}_\lambda(Q),$$
$$\Psi_{\mathfrak{t}',\mathfrak{t}}(w) := (\psi_{\pi_{\mathfrak{t}'}} w \psi_{\pi_{\mathfrak{t}}^{-1}}) E_{\mathfrak{t}',\mathfrak{t}}, \qquad \text{for all } w \in e(\mathfrak{t}')\mathrm{R}_n(Q)e(\mathfrak{t}).$$

The goal sets of (1.3.50) are respected, according to (1.2.3d), (1.3.10) and Proposition 1.3.46. Indeed, for instance we have, for any $v \in \mathrm{R}_\lambda(Q) \simeq e_\lambda \mathrm{R}_n(Q) e_\lambda$,

$$\begin{aligned}
\Phi_{\mathfrak{t}',\mathfrak{t}}(vE_{\mathfrak{t}',\mathfrak{t}}) &= \psi_{\pi_{\mathfrak{t}'}^{-1}} v \psi_{\pi_{\mathfrak{t}}} \\
&= \psi_{\pi_{\mathfrak{t}'}^{-1}} e(\mathfrak{t}^\lambda) v e(\mathfrak{t}^\lambda) \psi_{\pi_{\mathfrak{t}}} \\
&= e(\pi_{\mathfrak{t}'}^{-1} \cdot \mathfrak{t}^\lambda) \psi_{\pi_{\mathfrak{t}'}^{-1}} v \psi_{\pi_{\mathfrak{t}}} e(\pi_{\mathfrak{t}}^{-1} \cdot \mathfrak{t}^\lambda) \\
&= e(\mathfrak{t}') \psi_{\pi_{\mathfrak{t}'}^{-1}} v \psi_{\pi_{\mathfrak{t}}} e(\mathfrak{t}) \in e(\mathfrak{t}')\mathrm{R}_n(Q)e(\mathfrak{t}).
\end{aligned}$$

Remark 1.3.51. Our map $\Phi_{\mathfrak{t}',\mathfrak{t}}$ is similar to [SVV, (17)].

Furthermore, these two maps $\Phi_{\mathfrak{t}',\mathfrak{t}}$ and $\Psi_{\mathfrak{t}',\mathfrak{t}}$ are clearly A-linear and by Proposition 1.3.35 these are inverse isomorphisms. We now set:

$$\begin{aligned}
\Phi_\lambda &:= \bigoplus_{\mathfrak{t},\mathfrak{t}' \in J^\lambda} \Phi_{\mathfrak{t}',\mathfrak{t}} : \mathrm{Mat}_{m_\lambda} \mathrm{R}_\lambda(Q) \to e(\lambda)\mathrm{R}_n(Q), \\
\Psi_\lambda &:= \bigoplus_{\mathfrak{t},\mathfrak{t}' \in J^\lambda} \Psi_{\mathfrak{t}',\mathfrak{t}} : e(\lambda)\mathrm{R}_n(Q) \to \mathrm{Mat}_{m_\lambda} \mathrm{R}_\lambda(Q).
\end{aligned} \tag{1.3.52}$$

From the properties of $\Phi_{\mathfrak{t}',\mathfrak{t}}$ and $\Psi_{\mathfrak{t}',\mathfrak{t}}$, the above maps are inverse A-module isomorphisms; it now suffices to check that Ψ_λ is an A-algebra homomorphism. This property comes from the following one:

$$\Psi_{\mathfrak{t}',\mathfrak{t}}(w_{\mathfrak{t}',\mathfrak{t}}) \Psi_{\mathfrak{s}',\mathfrak{s}}(w_{\mathfrak{s}',\mathfrak{s}}) = \Psi_{\mathfrak{t}',\mathfrak{s}}(w_{\mathfrak{t}',\mathfrak{t}} w_{\mathfrak{s}',\mathfrak{s}}), \tag{1.3.53}$$

where $\mathfrak{t}, \mathfrak{t}', \mathfrak{s}, \mathfrak{s}' \in J^\lambda$, $w_{\mathfrak{t}',\mathfrak{t}} \in e(\mathfrak{t}')\mathrm{R}_n(Q)e(\mathfrak{t})$ and $w_{\mathfrak{s}',\mathfrak{s}} \in e(\mathfrak{s}')\mathrm{R}_n(Q)e(\mathfrak{s})$. The equality (1.3.53) is obviously satisfied when $\mathfrak{t} \neq \mathfrak{s}'$ since both sides are zero, thus we assume $\mathfrak{t} = \mathfrak{s}'$. We have, using Proposition 1.3.35 and noticing that $w_{\mathfrak{t},\mathfrak{s}} = e(\mathfrak{t}) w_{\mathfrak{t},\mathfrak{s}}$,

$$\begin{aligned}
\Psi_{\mathfrak{t}',\mathfrak{t}}(w_{\mathfrak{t}',\mathfrak{t}}) \Psi_{\mathfrak{s}',\mathfrak{s}}(w_{\mathfrak{s}',\mathfrak{s}}) &= \Psi_{\mathfrak{t}',\mathfrak{t}}(w_{\mathfrak{t}',\mathfrak{t}}) \Psi_{\mathfrak{t},\mathfrak{s}}(w_{\mathfrak{t},\mathfrak{s}}) \\
&= (\psi_{\pi_{\mathfrak{t}'}} w_{\mathfrak{t}',\mathfrak{t}} \psi_{\pi_{\mathfrak{t}}^{-1}})(\psi_{\pi_{\mathfrak{t}}} w_{\mathfrak{t},\mathfrak{s}} \psi_{\pi_{\mathfrak{s}}^{-1}}) E_{\mathfrak{t}',\mathfrak{t}} E_{\mathfrak{t},\mathfrak{s}} \\
&= \psi_{\pi_{\mathfrak{t}'}} w_{\mathfrak{t}',\mathfrak{t}} [\psi_{\pi_{\mathfrak{t}}^{-1}} \psi_{\pi_{\mathfrak{t}}} e(\mathfrak{t})] w_{\mathfrak{t},\mathfrak{s}} \psi_{\pi_{\mathfrak{s}}^{-1}} E_{\mathfrak{t}',\mathfrak{s}} \\
&= \psi_{\pi_{\mathfrak{t}'}} w_{\mathfrak{t}',\mathfrak{t}} w_{\mathfrak{t},\mathfrak{s}} \psi_{\pi_{\mathfrak{s}}^{-1}} E_{\mathfrak{t}',\mathfrak{s}} \\
&= \Psi_{\mathfrak{t}',\mathfrak{s}}(w_{\mathfrak{t}',\mathfrak{t}} w_{\mathfrak{s}',\mathfrak{s}}).
\end{aligned}$$

Finally, the maps Φ_λ and Ψ_λ are inverse A-algebra isomorphisms. We deduce the isomorphism (1.3.48) and thus Theorem 1.3.47.

Remark 1.3.54. For any $\boldsymbol{k} \in K^{\mathfrak{t}}$, we write here $\boldsymbol{k}^* := \pi_{\mathfrak{t}} \cdot \boldsymbol{k} \in K^{\mathfrak{t}^\lambda}$. Using Proposition 1.3.35 and Lemmas 1.3.39, 1.3.41, we can give the images of the generators of $e(\lambda)\mathrm{R}_n(Q)$ for each $\mathfrak{t} \in J^\lambda$ and $\boldsymbol{k} \in K^{\mathfrak{t}}$:

$$\begin{aligned}
\Psi_\lambda(e(\boldsymbol{k})) &= e(\boldsymbol{k}^*) E_{\mathfrak{t},\mathfrak{t}}, \\
\Psi_\lambda(y_a e(\boldsymbol{k})) &= y_{\pi_{\mathfrak{t}}(a)} e(\boldsymbol{k}^*) E_{\mathfrak{t},\mathfrak{t}}, && \text{for all } a \in \{1, \dots, n\}, \\
\Psi_\lambda(\psi_a e(\boldsymbol{k})) &= \psi_{\pi_{s_a \cdot \mathfrak{t}} s_a \pi_{\mathfrak{t}}^{-1}} e(\boldsymbol{k}^*) E_{s_a \cdot \mathfrak{t},\mathfrak{t}}, \\
&= \begin{cases} e(\boldsymbol{k}^*) E_{s_a \cdot \mathfrak{t},\mathfrak{t}}, & \text{if } \mathfrak{t}_a \neq \mathfrak{t}_{a+1}, \\ \psi_{\pi_{\mathfrak{t}}(a)} e(\boldsymbol{k}^*) E_{\mathfrak{t},\mathfrak{t}}, & \text{if } \mathfrak{t}_a = \mathfrak{t}_{a+1}, \end{cases} && \text{for all } a \in \{1, \dots, n-1\},
\end{aligned}$$

(note that $\pi_{s_a\cdot\mathfrak{t}} s_a \pi_{\mathfrak{t}}^{-1} = \begin{cases} \mathrm{id} & \text{if } \mathfrak{t}_a \neq \mathfrak{t}_{a+1} \\ s_{\pi_{\mathfrak{t}}(a)} & \text{if } \mathfrak{t}_a = \mathfrak{t}_{a+1} \end{cases}$, cf. Lemma 1.3.19). We observe that these images look like those described in [JacPA, (22)] and [PA, (3.2)–(3.4)].

Remark 1.3.55. We consider the setting of Proposition 1.2.17. We can prove that the algebra isomorphism $\mathrm{R}_n(Q) \simeq \oplus_{\lambda\models_d n} \mathrm{Mat}_{m_\lambda} \mathrm{R}_\lambda(Q)$ is a graded isomorphism (with the canonical gradings on the direct sum, matrix algebras and tensor products). In particular, we have (recall the notation $\boldsymbol{k}^{\mathfrak{t}}$ of (1.3.23))

$$\deg \psi_{\pi_{\mathfrak{t}'}^{-1}} e(\boldsymbol{k}) = \deg \psi_{\pi_{\mathfrak{t}}} e(\boldsymbol{k}^{\mathfrak{t}}) = 0,$$

as a consequence of Lemma 1.3.17 and Remark 1.3.18. Indeed, if for $\mathfrak{s} \in J^n$ and $a \in \{1, \dots, n-1\}$ we have $\mathfrak{s}_a \neq \mathfrak{s}_{a+1}$ then for any $\boldsymbol{k} \in K^{\mathfrak{s}}$ we have $c_{k_a, k_{a+1}} = 0$ (cf. (1.3.26)).

1.3.4 Cyclotomic version

Let us consider a weight $\boldsymbol{\Lambda} = (\Lambda_k)_{k\in K} \in \mathbb{N}^K$ and a family $\mathfrak{a} = (a_k)_{k\in K} \in \mathrm{Pol}_K^{\boldsymbol{\Lambda}}$. For any $j \in J$, we write $\boldsymbol{\Lambda}^j \in \mathbb{N}^{K_j}$ (respectively $\mathfrak{a}^j \in \mathrm{Pol}_{K_j}^{\boldsymbol{\Lambda}^j}$) the restriction of $\boldsymbol{\Lambda}$ (resp. a) to K_j. We show here how the isomorphism of Theorem 1.3.47 is compatible with cyclotomic quotients, as defined in (1.2.12).

1.3.4.1 Factorisation theorem

For any $\lambda \models_d n$, we define the cyclotomic quotient $\mathrm{R}_\lambda^{\boldsymbol{\Lambda},\mathfrak{a}}(Q)$ of $\mathrm{R}_\lambda(Q)$ by

$$\mathrm{R}_\lambda^{\boldsymbol{\Lambda},\mathfrak{a}}(Q) \coloneqq \mathrm{R}_{\lambda_1}^{\boldsymbol{\Lambda}^1,\mathfrak{a}^1}(Q^1) \otimes \cdots \otimes \mathrm{R}_{\lambda_d}^{\boldsymbol{\Lambda}^d,\mathfrak{a}^d}(Q^d),$$

that is, $\mathrm{R}_\lambda^{\boldsymbol{\Lambda},\mathfrak{a}}(Q)$ is the quotient of $\mathrm{R}_\lambda(Q)$ by the two-sided ideal generated by the relations

$$a_{k_a}(y_a) e(\boldsymbol{k}) = 0, \tag{1.3.56}$$

for all $\boldsymbol{k} = (k_1, \dots, k_n) \in K^{\mathfrak{t}^\lambda}$ and $a \in \{\boldsymbol{\lambda}_0 + 1, \dots, \boldsymbol{\lambda}_{d-1} + 1\}$.

Theorem 1.3.57. *The isomorphism of Theorem 1.3.47 factors through the cyclotomic quotients, in other words we have*

$$\mathrm{R}_n^{\boldsymbol{\Lambda},\mathfrak{a}}(Q) \simeq \bigoplus_{\lambda\models_d n} \mathrm{Mat}_{m_\lambda} \mathrm{R}_\lambda^{\boldsymbol{\Lambda},\mathfrak{a}}(Q).$$

Proof. Let $\lambda \models_d n$ and let $\mathfrak{I}$ (resp. $\mathfrak{I}_\lambda$) be the two-sided ideal of $e(\lambda)\mathrm{R}_n(Q)$ (resp. $\mathrm{R}_\lambda(Q)$) generated by the elements in (1.2.12) (resp. (1.3.56)). It suffices to prove that $\Psi_\lambda(\mathfrak{I}) = \mathrm{Mat}_{m_\lambda}\mathfrak{I}_\lambda$.

We first prove $\Psi_\lambda(\mathfrak{I}) \subseteq \mathrm{Mat}_{m_\lambda}\mathfrak{I}_\lambda$. Each element of $\mathfrak{I}$ can be written as

$$\sum_{v,w\in\mathrm{R}_n(Q)} \sum_{\mathfrak{t}\in J^\lambda} \sum_{\boldsymbol{k}^{\mathfrak{t}}\in K^{\mathfrak{t}}} v \left[\sum_{m=0}^{\Lambda_{k_1^{\mathfrak{t}}}} c_{k_1^{\mathfrak{t}} m} y_1^m e(\boldsymbol{k}^{\mathfrak{t}})\right] w.$$

By (1.2.3d), Proposition 1.3.35 and Lemma 1.3.39, the previous element becomes

$$\sum_{v,w\in\mathrm{R}_n(Q)} \sum_{\mathfrak{t}\in J^\lambda} \sum_{\boldsymbol{k}^{\mathfrak{t}}\in K^{\mathfrak{t}}} v\psi_{\pi_{\mathfrak{t}}^{-1}} \left[\sum_{m=0}^{\Lambda_{k_1^{\mathfrak{t}}}} c_{k_1^{\mathfrak{t}} m} y_{\pi_{\mathfrak{t}}(1)}^m e(\pi_{\mathfrak{t}} \cdot \boldsymbol{k}^{\mathfrak{t}})\right] \psi_{\pi_{\mathfrak{t}}} w.$$

As Ψ_λ is an algebra homomorphism and $\mathrm{Mat}_{m_\lambda}\mathfrak{I}_\lambda$ is a two-sided ideal, it suffices to prove that for any $\boldsymbol{k} \in K^{\mathfrak{t}^\lambda}$ and $\mathfrak{t} \in J^\lambda$ we have, with $a := \pi_{\mathfrak{t}}(1)$,

$$\sum_{m=0}^{\Lambda_{k_a}} c_{k_a m} \Psi_\lambda(y_a^m e(\boldsymbol{k})) \in \mathrm{Mat}_{m_\lambda}(\mathfrak{I}_\lambda).$$

The above element is exactly

$$\sum_{m=0}^{\Lambda_{k_a}} c_{k_a m} y_a^m e(\boldsymbol{k}) E_{\mathfrak{t}^\lambda, \mathfrak{t}^\lambda}$$

(recall that $\pi_{\mathfrak{t}^\lambda} = \mathrm{id}$), hence we are done since the element $\sum_{m=0}^{\Lambda_{k_a}} c_{k_a m} y_a^m e(\boldsymbol{k})$ lies in $\mathfrak{I}_\lambda$, in particular, there is a $j \in J$ such that $a = \boldsymbol{\lambda}_{j-1} + 1$, cf. Proposition 1.3.12.

We now prove $\mathfrak{I} \supseteq \Phi_\lambda(\mathrm{Mat}_{m_\lambda}\mathfrak{I}_\lambda)$. Each element of $\mathrm{Mat}_{m_\lambda}\mathfrak{I}_\lambda$ can be written

$$\sum_{\mathfrak{t},\mathfrak{t}' \in J^\lambda} \sum_{v,w \in \mathrm{R}_\lambda(Q)} \sum_{\boldsymbol{k} \in K^{\mathfrak{t}^\lambda}} \sum_{a \in \{\boldsymbol{\lambda}_0+1,\dots,\boldsymbol{\lambda}_{d-1}+1\}} v \left[\sum_{m=0}^{\Lambda_{k_a}} c_{k_a m} y_a^m e(\boldsymbol{k}) E_{\mathfrak{t}',\mathfrak{t}} \right] w.$$

As Φ_λ is an algebra homomorphism and $\mathfrak{I}$ is a two-sided ideal, it suffices to prove that for any $\mathfrak{t}, \mathfrak{t}' \in J^\lambda, \boldsymbol{k} \in K^{\mathfrak{t}^\lambda}$ and $a = \boldsymbol{\lambda}_{j-1} + 1$ for $j \in J$ we have

$$\sum_{m=0}^{\Lambda_{k_a}} c_{k_a m} \Phi_\lambda(y_a^m e(\boldsymbol{k}) E_{\mathfrak{t}',\mathfrak{t}}) \in \mathfrak{I}.$$

We consider an element $\mathfrak{s} \in J^\lambda$ which satisfies $\mathfrak{s}_1 = j$ (we can take for instance $\mathfrak{s} := (1, a) \cdot \mathfrak{t}^\lambda$); note that $\pi_{\mathfrak{s}}(1) = \boldsymbol{\lambda}_{j-1} + 1 = a$. Using (1.3.49), we can write the above element

$$\sum_{m=0}^{\Lambda_{k_a}} c_{k_a m} \Phi_\lambda(y_a^m e(\boldsymbol{k}) E_{\mathfrak{t}',\mathfrak{s}}) \Phi_\lambda(E_{\mathfrak{s},\mathfrak{t}}),$$

hence it suffices to prove that

$$\alpha := \sum_{m=0}^{\Lambda_{k_a}} c_{k_a m} \Phi_\lambda(y_a^m e(\boldsymbol{k}) E_{\mathfrak{t}',\mathfrak{s}}),$$

belongs to the ideal $\mathfrak{I}$. We obtain

$$\begin{aligned} \alpha &= \sum_{m=0}^{\Lambda_{k_a}} c_{k_a m} \psi_{\pi_{\mathfrak{t}'}^{-1}} y_a^m e(\boldsymbol{k}) \psi_{\pi_{\mathfrak{s}}} \\ &= \sum_{m=0}^{\Lambda_{k_a}} c_{k_a m} \psi_{\pi_{\mathfrak{t}'}^{-1}} \psi_{\pi_{\mathfrak{s}}} y_{\pi_{\mathfrak{s}}^{-1}(a)}^m e(\boldsymbol{k}^{\mathfrak{s}}) \\ &= \psi_{\pi_{\mathfrak{t}'}^{-1}} \psi_{\pi_{\mathfrak{s}}} \sum_{m=0}^{\Lambda_{k_a}} c_{k_a m} y_1^m e(\boldsymbol{k}^{\mathfrak{s}}), \end{aligned}$$

where we recall the notation $\boldsymbol{k}^{\mathfrak{s}} \in K^{\mathfrak{s}}$ from (1.3.23). We have $k_1^{\mathfrak{s}} = k_a$, thus we obtain

$$\alpha = \psi_{\pi_{\mathfrak{t}'}^{-1}} \psi_{\pi_{\mathfrak{s}}} \underbrace{\sum_{m=0}^{\Lambda_{k_1^{\mathfrak{s}}}} c_{k_1^{\mathfrak{s}} m} y_1^m e(\boldsymbol{k}^{\mathfrak{s}})}_{\in \mathfrak{I}},$$

and we are done. □

1.3.4.2 An alternative proof

We explain here how we can get Theorem 1.3.57 from [SVV, §3.2.3]. We assume that Q is associated to a (finite) quiver Γ with no loops. As we saw in §1.3.1.3, for any $j \in J$ the matrix Q^j is associated to a full subquiver Γ^j of Γ. Any Γ^j is a union of connected components of Γ and $\Gamma = \amalg_{j\in J}\Gamma^j$. We assume that $\mathbf{\Lambda} = \sum_{k\in K} \Lambda_k \omega_k$ where the ω_k are the fundamental weights related to the ambient Cartan datum (we refer to [SVV, §3.2.2] for more details). We define the set

$$K_{\mathbf{\Lambda}} \coloneqq \{(k, m) \in K \times \mathbb{N} : k \in K, m \in \{0, \dots, \Lambda_k\}\}.$$

For any $t = (k, m) \in K_{\mathbf{\Lambda}}$, we write $k_t \coloneqq k$ and $\omega_t \coloneqq \omega_k$. Finally, for convenience we introduce some notation:

- we write $\models_{\mathbf{\Lambda}}$ instead of $\models_{K_{\mathbf{\Lambda}}}$;
- if μ (respectively ν) is an r-composition (resp. s-composition), we set $m_\mu^\nu \coloneqq \frac{\nu_1!\dots\nu_s!}{\mu_1!\dots\mu_r!}$; in particular with the 1-partition $\nu \coloneqq (\mu_1 + \dots + \mu_r)$ we recover $m_\mu^\nu = m_\mu$.

Theorem 1.3.58 ([SVV, Theorem 3.15])**.** *There is an algebra isomorphism*

$$\mathrm{R}_n^{\mathbf{\Lambda}}(\Gamma) \simeq \bigoplus_{(n_t)_t \models_{\mathbf{\Lambda}} n} \mathrm{Mat}_{m_{(n_t)}} \left(\bigotimes_{t\in K_{\mathbf{\Lambda}}} \mathrm{R}_{n_t}^{\omega_t}(\Gamma) \right).$$

If we look closer to the proof of [SVV], comparing to the proof of Theorem 1.3.47 the idea is still to consider some elements $e(\mathbf{t})$ but with idempotents which “refine” the idempotents $e(\boldsymbol{k})$ (these idempotents are indexed by $K_{\mathbf{\Lambda}}^n$, which is much bigger than K^n). In particular, the following isomorphism for any $\lambda \models_d n$, writing $(n_t^j)_t$ for the restriction of $(n_t)_{t\in K_{\mathbf{\Lambda}}}$ to $K_{\mathbf{\Lambda}^j}$,

$$\mathrm{Mat}_{m_\lambda} \mathrm{R}_\lambda^{\mathbf{\Lambda}}(\Gamma) \simeq \bigoplus_{\substack{(n_t)_t \models_{\mathbf{\Lambda}} n \\ \text{with } (n_t^j)_t \models_{\mathbf{\Lambda}^j} \lambda_j \\ \text{for all } j}} \mathrm{Mat}_{m_{(n_t)}} \left(\bigotimes_{t\in K_{\mathbf{\Lambda}}} \mathrm{R}_{n_t}^{\omega_t}(\Gamma) \right) \tag{1.3.59}$$

implies our Theorem 1.3.57 by summing over all $\lambda \models_d n$. In order to prove (1.3.59), we can simply apply Theorem 1.3.58 to the factors $\mathrm{R}_{\lambda_j}^{\mathbf{\Lambda}^j}(\Gamma^j)$ of $\mathrm{R}_\lambda^{\mathbf{\Lambda}}(\Gamma)$. We have, for any $j \in J$,

$$\mathrm{R}_{\lambda_j}^{\mathbf{\Lambda}^j}(\Gamma^j) \simeq \bigoplus_{(n_t^j)_t \models_{\mathbf{\Lambda}^j} \lambda_j} \mathrm{Mat}_{m_{(n_t^j)}} \left(\bigotimes_{t\in K_{\mathbf{\Lambda}^j}} \mathrm{R}_{n_t^j}^{\omega_t}(\Gamma^j) \right). \tag{1.3.60}$$

Before going further, we give the following lemma. Let us mention that we can find a non-cyclotomic statement in [Rou, Corollary 3.8]; see also [Ma15, Proposition 2.4.6] and [BoyMa, Lemma 1.16].

Lemma 1.3.61. *Let $j \in J$. If $k \in K_j$ then $\mathrm{R}_n^{\omega_k}(\Gamma) \simeq \mathrm{R}_n^{\omega_k}(\Gamma^j)$.*

Proof. It suffices to prove that every $e(\boldsymbol{k}) \in \mathrm{R}_n(\Gamma)$ with $\boldsymbol{k} \in K^n \setminus K_j^n$ vanishes in $\mathrm{R}_n^{\omega_k}(\Gamma)$. To that extent, we prove by induction on $a \in \{1, \dots, n\}$ that for all $\boldsymbol{k} \in K^n$,

$$\text{there exists } b \in \{1, \dots, a\}, k_b \notin K_j \implies e(\boldsymbol{k}) = 0 \text{ in } \mathrm{R}_n^{\omega_k}(\Gamma). \tag{1.3.62}$$

First, we shall verify this proposition for $a = 1$. Let $\boldsymbol{k} \in K^n$ such that $\exists b \in \{1, \dots, 1\}, k_b \notin K_j$. We obviously have $k_1 \notin K_j$, in particular $k_1 \neq k$ thus it follows directly from the cyclotomic condition (1.2.12) that $e(\boldsymbol{k}) = 0$ in $\mathrm{R}_n^{\omega_k}(\Gamma)$. We now assume that (1.3.62) is satisfied for some $a \in \{1, \ldots, n-1\}$ and we let $\boldsymbol{k} \in K^n$ such that there exists $b \in \{1, \ldots, a+1\}$ with $k_b \notin K_j$. We know from the induction hypothesis that $e(\boldsymbol{k}) = 0$ in $\mathrm{R}_n^{\omega_k}(\Gamma)$ if $k_a \notin K_j$ or $k_{a+1} \in K_j$, hence it remains to deal with the case $k_a \in K_j$ and $k_{a+1} \notin K_j$. Recalling (1.3.25), this implies that $Q_{k_a,k_{a+1}} = 1$. In particular, the defining relation (1.2.4a) becomes

$$\psi_a^2 e(\boldsymbol{k}) = e(\boldsymbol{k}). \tag{1.3.63}$$

Besides, using (1.2.3d) and the induction hypothesis, we have

$$\psi_a e(\boldsymbol{k}) = \underbrace{e(s_a \cdot \boldsymbol{k})}_{=0} \psi_a = 0 \text{ in } \mathrm{R}_n^{\omega_k}(\Gamma).$$

Thus, left-multiplying by ψ_a and using (1.3.63) we get $e(\boldsymbol{k}) = 0$ in $\mathrm{R}_n^{\omega_k}(\Gamma)$ which ends the induction. Finally, for $a = n$, we get that if $\boldsymbol{k} \in K^n \setminus K_j^n$ then $e(\boldsymbol{k}) = 0$ in $\mathrm{R}_n^{\omega_k}(\Gamma)$. □

We now go back to the proof of (1.3.59). Using Lemma 1.3.61, the isomorphism (1.3.60) gives

$$\mathrm{R}_\lambda^{\boldsymbol{\Lambda}}(\Gamma) \simeq \bigotimes_{j\in J} \mathrm{R}_{\lambda_j}^{\boldsymbol{\Lambda}_j}(\Gamma^j) \simeq \bigotimes_{j \in J} \bigoplus_{(n_t^j)_t \vDash_{\boldsymbol{\Lambda}^j} \lambda_j} \mathrm{Mat}_{m_{(n_t^j)}}\left(\bigotimes_{t \in K_{\boldsymbol{\Lambda}^j}} \mathrm{R}_{n_t^j}^{\omega_t}(\Gamma)\right).$$

We obtain

$$\begin{aligned}
\mathrm{R}_\lambda^{\boldsymbol{\Lambda}}(\Gamma) &\simeq \bigoplus_{\substack{(n_t)_t \vDash_{\boldsymbol{\Lambda}} n \\ \text{with } (n_t^j)_t \vDash_{\boldsymbol{\Lambda}^j} \lambda_j \\ \text{for all } j}} \bigotimes_{j\in J} \mathrm{Mat}_{m_{(n_t^j)}}\left(\bigotimes_{t\in K_{\boldsymbol{\Lambda}^j}} \mathrm{R}_{n_t^j}^{\omega_t}(\Gamma)\right) \\
&\simeq \bigoplus_{\substack{(n_t)_t \vDash_{\boldsymbol{\Lambda}} n \\ \text{with } (n_t^j)_t \vDash_{\boldsymbol{\Lambda}^j} \lambda_j \\ \text{for all } j}} \mathrm{Mat}_{m_{(n_t^1)} \cdots m_{(n_t^d)}}\left(\bigotimes_{j \in J} \bigotimes_{t\in K_{\boldsymbol{\Lambda}^j}} \mathrm{R}_{n_t^j}^{\omega_t}(\Gamma)\right) \\
\mathrm{R}_\lambda^{\boldsymbol{\Lambda}}(\Gamma) &\simeq \bigoplus_{\substack{(n_t)_t \vDash_{\boldsymbol{\Lambda}} n \\ \text{with } (n_t^j)_t \vDash_{\boldsymbol{\Lambda}^j} \lambda_j \\ \text{for all } j}} \mathrm{Mat}_{m^\lambda_{(n_t)}}\left(\bigotimes_{t\in K_{\boldsymbol{\Lambda}}} \mathrm{R}_{n_t}^{\omega_t}(\Gamma)\right).
\end{aligned}$$

Finally, we deduce (1.3.59) and thus Theorem 1.3.57 from the equality $m_\lambda m^\lambda_{(n_t)} = m_{(n_t)}$.

1.4 Fixed point subalgebra

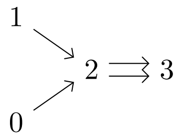
Let $Q = (Q_{k,k'})_{k,k' \in K}$ be a family of bivariates polynomials with coefficients in A as in §1.2.1. Let σ be a permutation of K of finite order $p \in \mathbb{N}^*$ such that

$$Q_{\sigma(k),\sigma(k')} = Q_{k,k'}, \tag{1.4.1}$$

for all $k, k' \in K$.

Remark 1.4.2. In the particular case where Q is associated to a loop-free quiver Γ (cf. §1.2.2), this condition means that for any $k, k' \in K$ with $k \neq k'$, there are so many arrows from k to k' as from $\sigma(k)$ to $\sigma(k')$, that is, we have $d_{\sigma(k),\sigma(k')} = d_{k,k'}$ and thus $c_{\sigma(k),\sigma(k')} = c_{kk'}$. In other words, the map σ is a quiver automorphism of Γ.

Example 1.4.3. Let Γ be the following quiver:



The permutation σ of the vertex set $\{0, 1, 2, 3\}$ given by

$$\sigma(0) \coloneqq 1, \qquad \sigma(1) \coloneqq 0, \qquad \sigma(2) \coloneqq 2, \qquad \sigma(3) \coloneqq 3,$$

satisfies the σ-invariance condition (1.4.1) (see Remark 1.4.2).

The permutation σ naturally induces a map $\sigma : K^n \to K^n$, defined by $\sigma(\boldsymbol{k}) \coloneqq (\sigma(k_1), \ldots, \sigma(k_n))$ for any $k = (k_1, \ldots, k_n) \in K^n$. Note that this map commutes with the action of $\mathfrak{S}_n$ on K^n. For any $\alpha \models_K n$, the following lemma explains how $\sigma : K^n \to K^n$ restricts to K^α (compare to [Boy, after Lemma 5.3.2]).

Lemma 1.4.4. *For any $\alpha \models_K n$, the map $\sigma : K^n \to K^n$ maps K^α onto $K^{\sigma\cdot\alpha}$, where $\sigma \cdot \alpha$ is the K-composition of n given by:*

$$(\sigma \cdot \alpha)_k \coloneqq \alpha_{\sigma^{-1}(k)},$$

for all $k \in K$.

Proof. Let $\boldsymbol{k} \in K^n$. We have:

$$\begin{aligned}
&\boldsymbol{k} \text{ has } \alpha_k \text{ components equal to } k \text{ for all } k \in K \\
\iff\ &\sigma(\boldsymbol{k}) \text{ has } \alpha_k \text{ components equal to } \sigma(k) \text{ for all } k \in K, \\
\iff\ &\sigma(\boldsymbol{k}) \text{ has } \alpha_{\sigma^{-1}(k)} \text{ components equal to } k \text{ for all } k \in K \\
\iff\ &\sigma(\boldsymbol{k}) \text{ has } (\sigma \cdot \alpha)_k \text{ components equal to } k \text{ for all } k \in K.
\end{aligned}$$

We conclude that $\boldsymbol{k} \in K^\alpha \iff \sigma(\boldsymbol{k}) \in K^{\sigma\cdot\alpha}$. □

We can now explain how σ induces an isomorphism between (cyclotomic) quiver Hecke algebras. We will also give a presentation for the fixed point subalgebras.

1.4.1 Affine case

In the affine case, we will be able to give a basis for the subalgebra of the fixed points of σ. As an easy consequence, we will give a presentation of this subalgebra.

Theorem 1.4.5. *Let $\alpha \models_K n$. There is a well-defined algebra homomorphism $\sigma : \mathrm{R}_\alpha(Q) \to \mathrm{R}_{\sigma\cdot\alpha}(Q)$ given by:*

$$\begin{aligned}
\sigma(e(\boldsymbol{k})) &\coloneqq e(\sigma(\boldsymbol{k})), && \textit{for all } \boldsymbol{k} \in K^\alpha, \\
\sigma(y_a) &\coloneqq y_a, && \textit{for all } a \in \{1, \ldots, n\}, \\
\sigma(\psi_a) &\coloneqq \psi_a, && \textit{for all } a \in \{1, \ldots, n-1\}.
\end{aligned}$$

Proof. We check the different relations (1.2.3) and (1.2.4), thanks to the σ invariance (1.4.1) of Q and the following fact:

$$\sigma(\boldsymbol{k})_a = \sigma(k_a),$$

for all $a \in \{1, \ldots, n\}$ and $\boldsymbol{k} = (k_1, \ldots, k_n) \in K^n$. Note that to prove (1.2.3a) we use the additional fact that $\sigma : K^\alpha \to K^{\sigma\cdot\alpha}$ is a bijection. □

Remark 1.4.6. If Γ is a loop-free quiver, the homomorphism $\sigma : \mathrm{R}_\alpha(\Gamma) \to \mathrm{R}_{\sigma\cdot\alpha}(\Gamma)$ is homogeneous with respect to the grading given in Proposition 1.2.17.

We want to study the fixed points of σ. To that extent, we first need to find an algebra which is stable under σ. Let $[\alpha]$ be the orbit of α under the action of $\langle\sigma\rangle$. Note that since $\sigma^p = \mathrm{id}_K$, the cardinality of $[\alpha]$ divides p. For any $\alpha \models_K n$ we define the following finite subset of K^n:

$$K^{[\alpha]} \coloneqq \bigsqcup_{\beta\in[\alpha]} K^\beta, \tag{1.4.7}$$

and similarly we define the following unitary algebra:

$$\mathrm{R}_{[\alpha]}(Q) \coloneqq \bigoplus_{\beta\in[\alpha]} \mathrm{R}_\beta(Q). \tag{1.4.8}$$

We obtain an *automorphism* $\sigma : \mathrm{R}_{[\alpha]}(Q) \to \mathrm{R}_{[\alpha]}(Q)$ of order p.

Remark 1.4.9. We have $\mathrm{R}_n(Q) \simeq \oplus_{[\alpha]}\mathrm{R}_{[\alpha]}(Q)$, in particular, for any $\boldsymbol{k} \in K^n$ the idempotent $e(\boldsymbol{k})$ of $\mathrm{R}_n(Q)$ belongs to $\mathrm{R}_{[\alpha]}(Q)$ if and only if $\boldsymbol{k} \in K^{[\alpha]}$.

We consider the equivalence relation $\sim$ on K generated by

$$k \sim \sigma(k), \tag{1.4.10}$$

for all $k \in K$. We extend it to $K^{[\alpha]}$ by:

$$\boldsymbol{k} \sim \sigma(\boldsymbol{k}), \tag{1.4.11}$$

for all $\boldsymbol{k} \in K^{[\alpha]}$.

Definition 1.4.12. We write $K_\sigma^{[\alpha]}$ for the quotient set $K^{[\alpha]}/\!\sim$.

For any element $\gamma \in K_\sigma^{[\alpha]}$, its cardinality o_γ divides p and we have

$$\gamma = \left\{\boldsymbol{k}, \sigma(\boldsymbol{k}), \dots, \sigma^{o_\gamma-1}(\boldsymbol{k})\right\}, \tag{1.4.13}$$

for any $\boldsymbol{k} \in \gamma$.

Definition 1.4.14. For any $\gamma \in K_\sigma^{[\alpha]}$, we define

$$e(\gamma) \coloneqq \sum_{\boldsymbol{k}\in\gamma} e(\boldsymbol{k}).$$

These elements $e(\gamma)$ have the property of being fixed by σ. Note that for any $\boldsymbol{k} \in \gamma$, by (1.4.13) we have

$$e(\gamma) = \sum_{j=0}^{o_\gamma-1} e(\sigma^j(\boldsymbol{k})). \tag{1.4.15}$$

We now give the analogue of Proposition 2.2.11, by describing all the fixed points of σ.

Theorem 1.4.16. *The following family:*

$$\mathcal{B}_{[\alpha]}^\sigma \coloneqq \left\{\psi_w y_1^{m_1}\cdots y_n^{m_n} e(\gamma) : w \in \mathfrak{S}_n, m_1, \dots, m_n \in \mathbb{N}, \gamma \in K_\sigma^{[\alpha]}\right\},$$

is an A-basis of $\mathrm{R}_{[\alpha]}(Q)^\sigma$, the A-module of the σ-fixed points of $\mathrm{R}_{[\alpha]}(Q)$.

Proof. First, by Theorem 1.2.21 we know that $\mathcal{B}_{[\alpha]} := \sqcup_{\beta\in[\alpha]}\mathcal{B}_\beta$ is an A-basis of $\mathrm{R}_{[\alpha]}(Q)$. Hence, the family $\mathcal{B}^\sigma_{[\alpha]}$ is A-free. Moreover, each element of $\mathcal{B}^\sigma_{[\alpha]}$ is fixed by σ. Define

$$\mathcal{F} := \{\psi_w y_1^{m_1}\dots y_n^{m_n} : w \in \mathfrak{S}_n, m_1,\dots,m_n \in \mathbb{N}\} \subseteq \mathrm{R}_{[\alpha]}(Q)^\sigma,$$

so that

$$\begin{aligned}\mathcal{B}_{[\alpha]} &= \left\{fe(\boldsymbol{k}) : f \in \mathcal{F}, \boldsymbol{k} \in K^{[\alpha]}\right\},\\ \mathcal{B}^\sigma_{[\alpha]} &= \left\{fe(\gamma) : f \in \mathcal{F}, \gamma \in K^{[\alpha]}_\sigma\right\}.\end{aligned}$$

Now let $h \in \mathrm{R}_{[\alpha]}(Q)$ be fixed by σ. We want to prove that h lies in $\mathrm{span}_A(\mathcal{B}^\sigma_{[\alpha]})$. Using Theorem 1.2.21, we can write

$$h = \sum_{f\in\mathcal{F}}\sum_{\boldsymbol{k}\in K^{[\alpha]}} h_{f,\boldsymbol{k}} fe(\boldsymbol{k}),$$

where $h_{f,\boldsymbol{k}} \in A$. We have

$$\begin{aligned}h &= \sigma^{-1}(h)\\ &= \sum_{f\in\mathcal{F}}\sum_{\boldsymbol{k}\in K^{[\alpha]}} h_{f,\boldsymbol{k}} fe(\sigma^{-1}(\boldsymbol{k}))\\ &= \sum_{f\in\mathcal{F}}\sum_{\boldsymbol{k}\in K^{[\alpha]}} h_{f,\sigma(\boldsymbol{k})} fe(\boldsymbol{k}).\end{aligned}$$

Thus, since $\mathcal{B}_{[\alpha]}$ is A-free, we have

$$h_{f,\boldsymbol{k}} = h_{f,\sigma(\boldsymbol{k})},$$

for all $f \in \mathcal{F}$ and $\boldsymbol{k} \in K^{[\alpha]}$. In particular, for each $f \in \mathcal{F}$ and $\gamma \in K^{[\alpha]}_\sigma$ there is a well-defined scalar $h_{f,\gamma}$. We obtain

$$\begin{aligned}h &= \sum_{f\in\mathcal{F}}\sum_{\gamma\in K^{[\alpha]}_\sigma}\sum_{\boldsymbol{k}\in\gamma} h_{f,\gamma} fe(\boldsymbol{k})\\ &= \sum_{f\in\mathcal{F}}\sum_{\gamma\in K^{[\alpha]}_\sigma} h_{f,\gamma} fe(\gamma),\end{aligned}$$

thus h lies in $\mathrm{span}_A(\mathcal{B}^\sigma_{[\alpha]})$. □

Theorem 1.4.16 allows us to give a presentation of the algebra $\mathrm{R}_{[\alpha]}(Q)^\sigma$. First, let us note that for any $a, b \in \{1,\dots,n\}$ and $\gamma \in K^{[\alpha]}_\sigma$, the expressions

$$\gamma_a = \gamma_b, \qquad\qquad \gamma_a \neq \gamma_b,$$

together with the quantity $s_c \cdot \gamma$ for any $c \in \{1,\dots,n-1\}$, are well-defined. Moreover, for any $a, b \in \{1,\dots,n\}$ and $\gamma \in K^{[\alpha]}_\sigma$, the bivariate polynomial Q_{γ_a,γ_b} is also well-defined, by the σ-invariance condition (1.4.1).

Remark 1.4.17. In contrast, if γ' is another element of $K^{[\alpha]}_\sigma$ then the expression $\gamma_a = \gamma'_a$ (for instance) is not well-defined.

Corollary 1.4.18. *The algebra* $\mathrm{R}_{[\alpha]}(Q)^\sigma$ *has the following presentation. The generating set is*

$$\{e(\gamma)\}_{\gamma\in K^{[\alpha]}_\sigma} \cup \{y_1,\dots,y_n\} \cup \{\psi_1,\dots,\psi_{n-1}\}, \tag{1.4.19}$$

and the relations are

$$\sum_{\gamma \in K_\sigma^{[\alpha]}} e(\gamma) = 1, \tag{1.4.20a}$$

$$e(\gamma)e(\gamma') = \delta_{\gamma,\gamma'} e(\gamma), \tag{1.4.20b}$$

$$y_a e(\gamma) = e(\gamma) y_a, \tag{1.4.20c}$$

$$\psi_b e(\gamma) = e(s_b \cdot \gamma)\psi_b, \tag{1.4.20d}$$

$$y_a y_{a'} = y_{a'} y_a, \tag{1.4.20e}$$

$$\psi_b y_a = y_a \psi_b, \qquad \text{if } a \neq b, b+1, \tag{1.4.20f}$$

$$\psi_b \psi_{b'} = \psi_{b'} \psi_b, \qquad \text{if } |b - b'| > 1, \tag{1.4.20g}$$

$$\psi_b y_{b+1} e(\gamma) = \begin{cases} (y_b \psi_b + 1) e(\gamma), & \text{if } \gamma_b = \gamma_{b+1}, \\ y_b \psi_b e(\gamma), & \text{if } \gamma_b \neq \gamma_{b+1}, \end{cases} \tag{1.4.20h}$$

$$y_{b+1} \psi_b e(\gamma) = \begin{cases} (\psi_b y_b + 1) e(\gamma), & \text{if } \gamma_b = \gamma_{b+1}, \\ \psi_b y_b e(\gamma), & \text{if } \gamma_b \neq \gamma_{b+1}, \end{cases} \tag{1.4.20i}$$

and

$$\psi_b^2 e(\gamma) = Q_{\gamma_b, \gamma_{b+1}}(y_b, y_{b+1}) e(\gamma), \tag{1.4.21a}$$

$$\psi_{c+1}\psi_c\psi_{c+1} e(\gamma) = \begin{cases} \psi_c \psi_{c+1} \psi_c e(\gamma) + \frac{Q_{\gamma_c,\gamma_{c+1}}(y_c, y_{c+1}) - Q_{\gamma_{c+2},\gamma_{c+1}}(y_{c+2}, y_{c+1})}{y_c - y_{c+2}} e(\gamma), & \text{if } \gamma_c = \gamma_{c+2}, \\ \psi_c \psi_{c+1} \psi_c e(\gamma), & \text{otherwise,} \end{cases} \tag{1.4.21b}$$

for all $\gamma \in K_\sigma^{[\alpha]}$*,* $a, a' \in \{1, \dots, n\}$*,* $b, b' \in \{1, \dots, n-1\}$ *and* $c \in \{1, \dots, n-2\}$*.*

Proof. Let us temporary write $e(\gamma)^\sigma, y_a^\sigma$ and ψ_a^σ for the generators of Corollary 1.4.18, and write R^σ for the algebra which admits this presentation. Given the defining relations (1.2.3) and (1.2.16) of $\mathrm{R}_{[\alpha]}(Q)$ and the σ-invariance condition (1.4.1), there is a well-defined algebra homomorphism $f : \mathrm{R}^\sigma \to \mathrm{R}_{[\alpha]}(Q)$ given by

$$\begin{aligned} f(e(\gamma)^\sigma) &\coloneqq e(\gamma), && \text{for all } \gamma \in K_\sigma^{[\alpha]}, \\ f(y_a^\sigma) &\coloneqq y_a, && \text{for all } a \in \{1, \dots, n\}, \\ f(\psi_a^\sigma) &\coloneqq \psi_a, && \text{for all } a \in \{1, \dots, n-1\}. \end{aligned}$$

We can notice that the family

$$\mathcal{B}^\sigma \coloneqq \left\{ \psi_w^\sigma (y_1^\sigma)^{m_1} \cdots (y_n^\sigma)^{m_n} e(\gamma)^\sigma : w \in \mathfrak{S}_n, m_a \in \mathbb{N}, \gamma \in K_\sigma^{[\alpha]} \right\},$$

spans R^σ over A, where the elements ψ_w^σ are defined as in (1.2.20), with the same reduced expressions. We recall from Theorem 1.4.16 that the family

$$\mathcal{B}_{[\alpha]}^\sigma = \left\{ \psi_w y_1^{m_1} \cdots y_n^{m_n} e(\gamma) : w \in \mathfrak{S}_{\mathfrak{n}}, m_a \in \mathbb{N}, \gamma \in K_\sigma^{[\alpha]} \right\}$$

is an A-basis of $\mathrm{R}_{[\alpha]}^\sigma(Q)$. Noticing that the algebra homomorphism f maps $\mathcal{B}^\sigma$ onto $\mathcal{B}_{[\alpha]}^\sigma$, we deduce that

- the family $\mathcal{B}^\sigma$ is linearly independent;
- the map f surjects onto $\mathrm{R}_{[\alpha]}(Q)^\sigma$.

Finally, the family $\mathcal{B}^\sigma$ is a basis of R^σ. In particular, the homomorphism f sends a basis to a basis hence f is an isomorphism. □

The reader may have noticed the similarity between the relations (1.4.20) and (1.4.21) defining $\mathrm{R}_{[\alpha]}(Q)^\sigma$ and the relations (1.2.3) together with (1.2.4) defining $\mathrm{R}_\alpha(Q)$. However, now the indexing set for the idempotents is generally not an $\mathfrak{S}_n$-stable subset of $\mathcal{I}^n$ for $\mathcal{I}$ an indexing set.

In the case where Γ is a loop-free quiver with simple edges, the relations (1.4.21) become, with the notation of §1.2.2,

$$\psi_b^2 e(\gamma) = \begin{cases} 0, & \text{if } \gamma_b = \gamma_{b+1}, \\ e(\gamma), & \text{if } \gamma_b \not- \gamma_{b+1}, \\ (y_{b+1} - y_b) e(\gamma), & \text{if } \gamma_b \to \gamma_{b+1}, \\ (y_b - y_{b+1}) e(\gamma), & \text{if } \gamma_b \leftarrow \gamma_{b+1}, \\ (y_{b+1} - y_b)(y_b - y_{b+1}) e(\gamma), & \text{if } \gamma_b \leftrightarrows \gamma_{b+1}, \end{cases} \tag{1.4.22a}$$

$$\psi_{c+1}\psi_c\psi_{c+1} e(\gamma) = \begin{cases} (\psi_c\psi_{c+1}\psi_c - 1) e(\gamma), & \text{if } \gamma_{c+2} = \gamma_c \to \gamma_{c+1}, \\ (\psi_c\psi_{c+1}\psi_c + 1) e(\gamma), & \text{if } \gamma_{c+2} = \gamma_c \leftarrow \gamma_{c+1}, \\ (\psi_c\psi_{c+1}\psi_c + 2y_{c+1} - y_c - y_{c+2}) e(\gamma), & \text{if } \gamma_{c+2} = \gamma_c \leftrightarrows \gamma_{c+1}, \\ \psi_c\psi_{c+1}\psi_c e(\gamma), & \text{otherwise}, \end{cases} \tag{1.4.22b}$$

for all $\gamma \in K_\sigma^{[\alpha]}$, $b \in \{1, \ldots, n-1\}$ and $c \in \{1, \ldots, n-2\}$.

Remark 1.4.23. If Γ is a loop-free quiver then $\sigma : \mathrm{R}_{[\alpha]}(\Gamma) \to \mathrm{R}_{[\alpha]}(\Gamma)$ is homogeneous (cf. Remark 1.4.6) and the subalgebra $\mathrm{R}_{[\alpha]}(\Gamma)^\sigma$ is a graded subalgebra of $\mathrm{R}_{[\alpha]}(\Gamma)$. More precisely, we can give an analogue of Proposition 1.2.17: there is a unique $\mathbb{Z}$-grading on $\mathrm{R}_{[\alpha]}(\Gamma)^\sigma$ such that $e(\gamma)$ is of degree 0, the element y_a is of degree 2 and $\psi_a e(\gamma)$ is of degree $-c_{\gamma_a,\gamma_{a+1}}$ (this quantity is well-defined, recall Remark 1.4.2).

1.4.2 Cyclotomic case

Recall the Definition 1.2.18 of a cyclotomic quiver Hecke algebra. For any $\alpha \models_K n$, we want the algebra homomorphism $\sigma : \mathrm{R}_\alpha(Q) \to \mathrm{R}_{\sigma\cdot\alpha}(Q)$ to factor through cyclotomic quotients. Contrary to the affine case, it will be more difficult to get a presentation for the fixed point subalgebra of the cyclotomic quiver Hecke algebra (recall that we do not have an analogue of Theorem 1.2.21). Contrary to [Ro16], we will give here a characteristic-free proof.

Let $\mathbf{\Lambda} \in \mathbb{N}^{(K)}$ be a weight and $\mathfrak{a} = (a_k) \in \mathrm{Pol}_K^{\mathbf{\Lambda}}$. Recall from §1.2.1 that a_k is a monic polynomial of degree Λ_k with coefficients in A. As for K-compositions, we define the weight $\sigma \cdot \mathbf{\Lambda} \in \mathbb{N}^{(K)}$ by

$$(\sigma \cdot \mathbf{\Lambda})_k \coloneqq \Lambda_{\sigma^{-1}(k)},$$

for all $k \in K$, and the family $\sigma \cdot \mathfrak{a} \in \mathrm{Pol}_K^{\sigma\cdot\mathbf{\Lambda}}$ by

$$(\sigma \cdot \mathfrak{a})_k \coloneqq a_{\sigma^{-1}(k)},$$

for all $k \in K$.

Lemma 1.4.24. *We have $\sigma(\mathcal{I}_\alpha^{\mathbf{\Lambda},\mathfrak{a}}) = \mathcal{I}_{\sigma\cdot\alpha}^{\sigma\cdot\mathbf{\Lambda},\sigma\cdot\mathfrak{a}}$. In particular, the algebra homomorphism $\sigma : \mathrm{R}_\alpha(Q) \to \mathrm{R}_{\sigma\cdot\alpha}(Q)$ induces an algebra homomorphism $\sigma^{\mathbf{\Lambda},\mathfrak{a}} : \mathrm{R}_\alpha^{\mathbf{\Lambda},\mathfrak{a}}(Q) \to \mathrm{R}_{\sigma\cdot\alpha}^{\sigma\cdot\mathbf{\Lambda},\sigma\cdot\mathfrak{a}}(Q)$.*

Proof. We notice that for any $\boldsymbol{k} \in K^\alpha$ the quantity

$$\sigma\Big(a_{k_1}(y_1)e(\boldsymbol{k})\Big) = a_{k_1}(y_1)e(\sigma(\boldsymbol{k})),$$

lies in $\mathcal{I}_{\sigma\cdot\alpha}^{\sigma\cdot\boldsymbol{\Lambda},\sigma\cdot\boldsymbol{a}}$ since $\Lambda_{k_1} = (\sigma \cdot \boldsymbol{\Lambda})_{\sigma(\boldsymbol{k})_1}$ and $a_{k_1} = (\sigma \cdot \mathfrak{a})_{\sigma(\boldsymbol{k})_1}$. Hence $\sigma(\mathcal{I}_\alpha^{\boldsymbol{\Lambda},\mathfrak{a}}) \subseteq \mathcal{I}_{\sigma\cdot\alpha}^{\sigma\cdot\boldsymbol{\Lambda},\sigma\cdot\mathfrak{a}}$ and we have equality by repeating the argument with σ^{-1}. ☐

Until the end of this section, we make the following σ-stability assumption on our weight $\boldsymbol{\Lambda} \in \mathbb{N}^{(K)}$,

$$\Lambda_k = \Lambda_{\sigma(k)}, \tag{1.4.25}$$

for all $k \in K$, and also the corresponding σ-stability assumption on $\mathfrak{a} \in \mathrm{Pol}_K^{\boldsymbol{\Lambda}}$,

$$a_k = a_{\sigma(k)}, \tag{1.4.26}$$

for all $k \in K$. In other words, we assume that $\boldsymbol{\Lambda} = \sigma \cdot \boldsymbol{\Lambda}$ and $\mathfrak{a} = \sigma \cdot \mathfrak{a}$. Equivalently, the weight $\boldsymbol{\Lambda} \in \mathbb{N}^{(K)}$ (respectively the family $\mathfrak{a} \in \mathrm{Pol}_K^{\boldsymbol{\Lambda}}$) factors to an element $\widetilde{\boldsymbol{\Lambda}}$ of $\mathbb{N}^{(K/\sim)}$ (resp. $\mathrm{Pol}_{K/\sim}^{\widetilde{\boldsymbol{\Lambda}}}$), with the notation of (1.4.10).

Remark 1.4.27. If Γ is a loop-free quiver, if the σ-stability condition (1.4.25) is satisfied then the σ-stability condition (1.4.26) is automatically satisfied, in the setting of a cyclotomic quotient $\mathrm{R}_\alpha^{\boldsymbol{\Lambda}}(\Gamma)$ (recall Definition 1.2.18).

Similarly to (1.4.8), we define:

$$\mathrm{R}_{[\alpha]}^{\boldsymbol{\Lambda},\mathfrak{a}}(Q) \coloneqq \bigoplus_{\beta\in[\alpha]} \mathrm{R}_\beta^{\boldsymbol{\Lambda},\mathfrak{a}}(Q).$$

This algebra is the quotient of $\mathrm{R}_{[\alpha]}(\Gamma)$ by the two sided ideal

$$\mathcal{I}_{[\alpha]}^{\boldsymbol{\Lambda},\mathfrak{a}} \coloneqq \bigoplus_{\beta\in[\alpha]} \mathcal{I}_\beta^{\boldsymbol{\Lambda},\mathfrak{a}}$$

generated by the elements $a_{k_1}(y_1)e(\boldsymbol{k})$ for all $\boldsymbol{k} \in K^{[\alpha]}$. We deduce from Lemma 1.4.24 the following statement.

Lemma 1.4.28. *We have* $\sigma(\mathcal{I}_{[\alpha]}^{\boldsymbol{\Lambda},\mathfrak{a}}) = \mathcal{I}_{[\alpha]}^{\boldsymbol{\Lambda},\mathfrak{a}}$. *Moreover,* $\sigma : \mathrm{R}_{[\alpha]}(Q) \to \mathrm{R}_{[\alpha]}(Q)$ *induces an algebra homomorphism* $\sigma^{\boldsymbol{\Lambda},\mathfrak{a}} : \mathrm{R}_{[\alpha]}^{\boldsymbol{\Lambda},\mathfrak{a}}(Q) \to \mathrm{R}_{[\alpha]}^{\boldsymbol{\Lambda},\mathfrak{a}}(Q)$.

If $\pi_{[\alpha]} : \mathrm{R}_{[\alpha]}(Q) \twoheadrightarrow \mathrm{R}_{[\alpha]}^{\boldsymbol{\Lambda},\mathfrak{a}}(Q)$ is the canonical projection, by definition the induced automorphism $\sigma^{\boldsymbol{\Lambda},\mathfrak{a}}$ satisfies

$$\sigma^{\boldsymbol{\Lambda},\mathfrak{a}} \circ \pi_{[\alpha]} = \pi_{[\alpha]} \circ \sigma. \tag{1.4.29}$$

We will often write σ as well for the automorphism $\sigma^{\boldsymbol{\Lambda},\mathfrak{a}}$.

Definition 1.4.30. We define $\mathrm{R}_{[\alpha]}^{\boldsymbol{\Lambda},\mathfrak{a}}(Q)^\sigma$ as the A-algebra of the fixed points of $\mathrm{R}_{[\alpha]}^{\boldsymbol{\Lambda},\mathfrak{a}}(Q)$ under the automorphism $\sigma^{\boldsymbol{\Lambda},\mathfrak{a}}$.

The following lemma is an easy consequence of [Ro16, Lemma 2.22] when p is invertible in A. We prove here a characteristic-free version.

Lemma 1.4.31. *The family*

$$\left\{\psi_w y_1^{m_1} \dots y_n^{m_n} e(\gamma) : w \in \mathfrak{S}_n, m_1, \dots, m_n \in \mathbb{N}, \gamma \in K_\sigma^{[\alpha]}\right\},$$

spans $\mathrm{R}_{[\alpha]}^{\boldsymbol{\Lambda},\mathfrak{a}}(Q)^\sigma$ *over* A.

Proof. By Theorem 1.2.21, we know that the family

$$\mathcal{F}^{\mathbf{\Lambda},\mathfrak{a}} := \{\psi_w y_1^{m_1} \dots y_n^{m_n} : w \in \mathfrak{S}_n, m_1, \dots, m_n \in \mathbb{N}\} \subseteq \mathrm{R}_{[\alpha]}^{\mathbf{\Lambda},\mathfrak{a}}(Q)^\sigma,$$

is such that

$$\left\{ f e(\boldsymbol{k}) : f \in \mathcal{F}^{\mathbf{\Lambda},\mathfrak{a}}, \boldsymbol{k} \in K^{[\alpha]} \right\},$$

spans $\mathrm{R}_{[\alpha]}^{\mathbf{\Lambda},\mathfrak{a}}(Q)$ over A. Let $h \in \mathrm{R}_{[\alpha]}^{\mathbf{\Lambda},\mathfrak{a}}(Q)^\sigma$ and write

$$h = \sum_{f \in \mathcal{F}^{\mathbf{\Lambda},\mathfrak{a}}} \sum_{\boldsymbol{k} \in K^{[\alpha]}} h_{f,\boldsymbol{k}} f e(\boldsymbol{k}),$$

where $h_{f,\boldsymbol{k}} \in A$. For any $j \in \mathbb{Z}$ we have

$$\sigma^{-j}(h) = \sum_{f \in \mathcal{F}^{\mathbf{\Lambda},\mathfrak{a}}} \sum_{\boldsymbol{k} \in K^{[\alpha]}} h_{f,\sigma^j(\boldsymbol{k})} f e(\boldsymbol{k}).$$

Since h is σ-invariant, by (1.2.3b) we obtain

$$\sum_{f \in \mathcal{F}^{\mathbf{\Lambda},\mathfrak{a}}} h_{f,\boldsymbol{k}} f e(\boldsymbol{k}) = \sum_{f \in \mathcal{F}^{\mathbf{\Lambda},\mathfrak{a}}} h_{f,\sigma^j(\boldsymbol{k})} f e(\boldsymbol{k}), \tag{1.4.32}$$

for any $\boldsymbol{k} \in K^{[\alpha]}$. For each $\gamma \in K_\sigma^{[\alpha]}$, fix a representative $\boldsymbol{k}_\gamma \in \gamma$, so that

$$\gamma = \left\{ \boldsymbol{k}_\gamma, \sigma(\boldsymbol{k}_\gamma), \dots, \sigma^{o_\gamma - 1}(\boldsymbol{k}_\gamma) \right\},$$

as in (1.4.13). We deduce from (1.4.32) that

$$\begin{aligned} h &= \sum_{f \in \mathcal{F}^{\mathbf{\Lambda},\mathfrak{a}}} \sum_{\gamma \in K_\sigma^{[\alpha]}} \sum_{j=0}^{o_\gamma - 1} h_{f,\sigma^j(\boldsymbol{k}_\gamma)} f e(\sigma^j(\boldsymbol{k}_\gamma)) \\ &= \sum_{f \in \mathcal{F}^{\mathbf{\Lambda},\mathfrak{a}}} \sum_{\gamma \in K_\sigma^{[\alpha]}} \sum_{j=0}^{o_\gamma - 1} h_{f,\boldsymbol{k}_\gamma} f e(\sigma^j(\boldsymbol{k}_\gamma)) \\ &= \sum_{f \in \mathcal{F}^{\mathbf{\Lambda},\mathfrak{a}}} \sum_{\gamma \in K_\sigma^{[\alpha]}} h_{f,\boldsymbol{k}_\gamma} f e(\gamma). \end{aligned}$$

Finally, we prove that the family $\{f e(\gamma) : f \in \mathcal{F}^{\mathbf{\Lambda},\mathfrak{a}}, \gamma \in K_\sigma^{[\alpha]}\}$ spans $\mathrm{R}_{[\alpha]}^{\mathbf{\Lambda},\mathfrak{a}}(Q)^\sigma$ over A thus we are done. □

Since $\mathbf{\Lambda}$ (respectively $\mathfrak{a}$) satisfies the σ-stability assumption (1.4.25) (resp. (1.4.26)) and considering the canonical map $K_\sigma^n = K^n/\sim \to (K/\sim)^n$, we may also consider the algebra $(\mathrm{R}_{[\alpha]}(Q)^\sigma)^{\mathbf{\Lambda},\mathfrak{a}}$, the quotient of $\mathrm{R}_{[\alpha]}(Q)^\sigma$ by the two-sided ideal $\mathcal{I}_{[\alpha],\sigma}^{\mathbf{\Lambda},\mathfrak{a}}$ generated by the following relations:

$$a_{\gamma_1}(y_1) e(\gamma) = 0, \tag{1.4.33}$$

for all $\gamma \in K_\sigma^{[\alpha]}$. In order to give a presentation of $\mathrm{R}_{[\alpha]}^{\mathbf{\Lambda},\mathfrak{a}}(Q)^\sigma$, we want to prove that this algebra is isomorphic to

$$(\mathrm{R}_{[\alpha]}(Q)^\sigma)^{\mathbf{\Lambda},\mathfrak{a}} = \mathrm{R}_{[\alpha]}(Q)^\sigma \big/ \mathcal{I}_{[\alpha],\sigma}^{\mathbf{\Lambda},\mathfrak{a}},$$

for which we know a presentation. The following lemma was proved in [Ro16, Lemma 2.24] under the assumption that p is invertible in A. Once again we now drop this assumption.

Lemma 1.4.34. *We have*
$$\mathcal{I}_{[\alpha]}^{\mathbf{\Lambda},\mathfrak{a}} \cap \mathrm{R}_{[\alpha]}(Q)^{\sigma} = \mathcal{I}_{[\alpha],\sigma}^{\mathbf{\Lambda},\mathfrak{a}}.$$

Proof. Since for any $\gamma \in K_{\sigma}^{[\alpha]}$ we have $a_{\gamma_1}(y_1)e(\gamma) \in \mathcal{I}_{[\alpha]}^{\mathbf{\Lambda},\mathfrak{a}} \cap \mathrm{R}_{[\alpha]}(Q)^{\sigma}$, we obtain $\mathcal{I}_{[\alpha]}^{\mathbf{\Lambda},\mathfrak{a}} \cap \mathrm{R}_{[\alpha]}(Q)^{\sigma} \supseteq \mathcal{I}_{[\alpha],\sigma}^{\mathbf{\Lambda},\mathfrak{a}}$. We now want to prove the converse inclusion. The idea of the proof is the same as for Lemma 1.4.31. Let

$$\mathcal{F} \coloneqq \{\psi_w y_1^{m_1} \dots y_n^{m_n} : w \in \mathfrak{S}_n, m_1, \dots, m_n \in \mathbb{N}\} \subseteq \mathrm{R}_{[\alpha]}(Q)^{\sigma}.$$

For any $f \in \mathcal{F}$, choose $w \in \mathfrak{S}_n$ such that $f = \psi_w y_1^{m_1} \dots y_n^{m_n}$ for some $m_1, \dots, m_n \in \mathbb{N}$ and define $w_f \coloneqq w$. By Theorem 1.2.21, we know that the family

$$\left\{ fe(\boldsymbol{k}) : f \in \mathcal{F}, \boldsymbol{k} \in K^{[\alpha]} \right\},$$

is an A-basis of $\mathrm{R}_{[\alpha]}(Q)$. Moreover, since

$$fe(\boldsymbol{k}) = e(w_f \cdot \boldsymbol{k})f,$$

for any $f \in \mathcal{F}$ and $\boldsymbol{k} \in K^{[\alpha]}$, the family

$$\left\{ e(\boldsymbol{k})g : \boldsymbol{k} \in K^{[\alpha]}, g \in \mathcal{F} \right\},$$

is also an A-basis of $\mathrm{R}_{[\alpha]}(Q)$. We now consider an arbitrary element h of $\mathcal{I}_{[\alpha]}^{\mathbf{\Lambda},\mathfrak{a}} \cap \mathrm{R}_{[\alpha]}(Q)^{\sigma}$. We can write

$$\begin{aligned} h &= \sum_{f,g \in \mathcal{F}} \sum_{\boldsymbol{k} \in K^{[\alpha]}} h_{f,g,\boldsymbol{k}} f a_{k_1}(y_1) e(\boldsymbol{k}) g \\ &= \sum_{f,g \in \mathcal{F}} \sum_{\boldsymbol{k} \in K^{[\alpha]}} h_{f,g,\boldsymbol{k}} f a_{k_1}(y_1) g e(w_g^{-1} \cdot \boldsymbol{k}) \\ &= \sum_{f,g \in \mathcal{F}} \sum_{\boldsymbol{k} \in K^{[\alpha]}} h_{f,g,w_g \cdot \boldsymbol{k}} f a_{(w_g \cdot \boldsymbol{k})_1}(y_1) g e(\boldsymbol{k}), \end{aligned}$$

with $h_{f,g,\boldsymbol{k}} \in A$. For any $\boldsymbol{k} \in K^{[\alpha]}$, we define

$$\phi(\boldsymbol{k}) \coloneqq \sum_{f,g \in \mathcal{F}} h_{f,g,w_g \cdot \boldsymbol{k}} f a_{(w_g \cdot \boldsymbol{k})_1}(y_1) g \in \mathrm{R}_{[\alpha]}(Q)^{\sigma},$$

so that

$$h = \sum_{\boldsymbol{k} \in K^{[\alpha]}} \phi(\boldsymbol{k}) e(\boldsymbol{k}).$$

For any $j \in \mathbb{Z}$ we have

$$\sigma^j(h) = \sum_{\boldsymbol{k} \in K^{[\alpha]}} \phi(\sigma^{-j}(\boldsymbol{k})) e(\boldsymbol{k}).$$

Since h is σ-invariant, by (1.2.3b) we obtain

$$\phi(\boldsymbol{k}) e(\boldsymbol{k}) = \phi(\sigma^{-j}(\boldsymbol{k})) e(\boldsymbol{k}), \tag{1.4.35}$$

for any $\boldsymbol{k} \in K^{[\alpha]}$. For each $\gamma \in K_\sigma^{[\alpha]}$, fix a representative $\boldsymbol{k}_\gamma \in \gamma$. We obtain from (1.4.15), (1.4.35) and the σ-invariance of $\mathfrak{a}$ that

$$\begin{aligned}
h &= \sum_{\gamma \in K_\sigma^{[\alpha]}} \sum_{j=0}^{o_\gamma - 1} \phi(\sigma^j(\boldsymbol{k}_\gamma)) e(\sigma^j(\boldsymbol{k}_\gamma)) \\
&= \sum_{\gamma \in K_\sigma^{[\alpha]}} \sum_{j=0}^{o_\gamma - 1} \phi(\boldsymbol{k}_\gamma) e(\sigma^j(\boldsymbol{k}_\gamma)) \\
&= \sum_{\gamma \in K_\sigma^{[\alpha]}} \sum_{j=0}^{o_\gamma - 1} \sum_{f,g \in \mathcal{F}} h_{f,g,w_g \cdot \boldsymbol{k}_\gamma} f a_{(w_g \cdot \boldsymbol{k}_\gamma)_1}(y_1) g e(\sigma^j(\boldsymbol{k}_\gamma)) \\
&= \sum_{\gamma \in K_\sigma^{[\alpha]}} \sum_{j=0}^{o_\gamma - 1} \sum_{f,g \in \mathcal{F}} h_{f,g,w_g \cdot \boldsymbol{k}_\gamma} f a_{(w_g \cdot \gamma)_1}(y_1) e(\sigma^j(w_g \cdot \boldsymbol{k}_\gamma)) g \\
&= \sum_{\gamma \in K_\sigma^{[\alpha]}} \sum_{f,g \in \mathcal{F}} h_{f,g,w_g \cdot \boldsymbol{k}_\gamma} f a_{(w_g \cdot \gamma)_1}(y_1) e(w_g \cdot \gamma) g,
\end{aligned}$$

thus $h \in \mathcal{I}_{[\alpha],\sigma}^{\boldsymbol{\Lambda},\mathfrak{a}}$ since $f, g \in \mathrm{R}_{[\alpha]}(Q)^\sigma$. □

We are now ready to state the main theorem of this section. Recall that this is a characteristic-free version of [Ro16, Theorem 2.26].

Theorem 1.4.36. *The algebras $\mathrm{R}_{[\alpha]}^{\boldsymbol{\Lambda},\mathfrak{a}}(Q)^\sigma$ and $(\mathrm{R}_{[\alpha]}(Q)^\sigma)^{\boldsymbol{\Lambda},\mathfrak{a}}$ are isomorphic. In particular, the generators* (1.4.19) *together with the relations* (1.4.20), (1.4.21) *and* (1.4.33) *give a presentation of $\mathrm{R}_{[\alpha]}^{\boldsymbol{\Lambda},\mathfrak{a}}(Q)^\sigma$.*

Proof. Recalling Corollary 1.4.18, we begin by noticing that the given presentation is a presentation of $(\mathrm{R}_{[\alpha]}(Q)^\sigma)^{\boldsymbol{\Lambda},\mathfrak{a}}$. In particular, we can define a homomorphism of algebras $f : (\mathrm{R}_{[\alpha]}(Q)^\sigma)^{\boldsymbol{\Lambda},\mathfrak{a}} \to \mathrm{R}_{[\alpha]}^{\boldsymbol{\Lambda},\mathfrak{a}}(Q)^\sigma$ by

$$\begin{aligned}
f(e(\gamma)) &:= e(\gamma), && \text{for all } \gamma \in K_\sigma^{[\alpha]}, \\
f(y_a) &:= y_a, && \text{for all } a \in \{1, \dots, n\}, \\
f(\psi_a) &:= \psi_a, && \text{for all } a \in \{1, \dots, n-1\}.
\end{aligned}$$

The algebra homomorphism f is surjective by Theorem 1.4.16 and Lemma 1.4.31, and injective by Lemma 1.4.34. Thus f is an algebra isomorphism and we are done. □

Remark 1.4.37. Let Γ be a loop-free quiver. The grading of Remark 1.4.23 on $\mathrm{R}_{[\alpha]}(\Gamma)^\sigma$ thus gives a grading on $\mathrm{R}_{[\alpha]}^{\boldsymbol{\Lambda},\mathfrak{a}}(\Gamma)^\sigma$, for which $\sigma^{\boldsymbol{\Lambda},\mathfrak{a}}$ is homogeneous (recall Remark 1.4.6). Moreover, the algebra $\mathrm{R}_{[\alpha]}^{\boldsymbol{\Lambda},\mathfrak{a}}(\Gamma)^\sigma$ is a graded subalgebra of $\mathrm{R}_{[\alpha]}^{\boldsymbol{\Lambda},\mathfrak{a}}(\Gamma)$.

Chapter 2

Hecke algebras of complex reflection groups

This chapter is adapted from [Ro16].

2.1 Overview

In this chapter, we generalise an isomorphism of Brundan and Kleshchev between the Hecke algebra of type $G(r,1,n)$ and the cyclotomic quiver Hecke algebra of type A. Then, we use the results of Chapter 1 to give a cyclotomic quiver Hecke-like presentation for the Hecke algebra of type $G(r,p,n)$, that is, for the complex reflection groups of the infinite series. In addition, we give an explicit isomorphism which realises a well-known Morita equivalence between Ariki–Koike algebras.

We now give an overview of the chapter. Let $r, p, d \in \mathbb{N}^*$ be some integers with $r = dp$. Let $\zeta \in F^\times$ be a primitive pth root of unity and $J := \mathbb{Z}/p\mathbb{Z} \simeq \langle\zeta\rangle$. Recall that $e \in \mathbb{N}^* \cup \{\infty\}$ is the order in $F^\times$ of $q \in F \setminus \{0,1\}$ and that $I = \mathbb{Z}/e\mathbb{Z}$ (with $I = \mathbb{Z}$ if $e = \infty$). Finally, we consider a tuple $\boldsymbol{\Lambda} = (\Lambda_{i,j})$ where $(i,j) \in I \times J$ and $\Lambda_{i,j} \in \mathbb{N}$. We begin Section 2.2 by defining in §2.2.1 the Hecke algebra $\mathrm{H}_n(q, \boldsymbol{u})$ of type $G(r,1,n)$, also known as Ariki–Koike algebra, which we write $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q,\zeta)$ when each u_k is of the form $\zeta^j q^i$. The algebra $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q,\zeta)$ is generated by some elements $S, T_1, \dots, T_{n-1}$, subject to relations (2.2.2b)–(2.2.2f) and the "cyclotomic" one:

$$\prod_{i\in I}\prod_{j\in J}\left(S - \zeta^j q^i\right)^{\Lambda_{i,j}} = 0$$

(see (2.2.6)). We define in Proposition 2.2.9 an important object of this chapter: the *shift automorphism* σ of $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q,\zeta)$. It maps S to ζS and is the identity on the remaining generators $T_1, \dots, T_{n-1}$. We then start §2.2.2 by defining the Hecke algebra $\mathrm{H}_{p,n}^{\boldsymbol{\Lambda}}(q)$ of type $G(r,p,n)$. Our definition, from [BrMa], differs from Ariki's [Ar95]. However, in §2.2.3 we prove that if $p \geq 2$ then the two definitions are equivalent (this fact is mentionned in [BMR] but we did not find any proof in the literature). We then prove in Corollary 2.2.19 that $\mathrm{H}_{p,n}^{\boldsymbol{\Lambda}}(q)$ is the fixed point subalgebra of $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q,\zeta)$ under the shift automorphism. We introduce in §2.2.4 a divisor p' of p, together with the set $J' := \{0, \dots, p'-1\}$, such that the map $I \times J' \ni (i,j) \mapsto \zeta^j q^i$ is one-to-one and has the same image as $I \times J \ni (i,j) \mapsto \zeta^j q^i$. We then define a finitely-supported tuple $\mathbf{\Lambda}$, indexed by $I \times J'$, associated with the tuple $\boldsymbol{\Lambda} \in \mathbb{N}^{(I\times J)}$ (see Proposition 2.2.39). We also introduce the notation $\mathrm{H}_n^{\mathbf{\Lambda}}(q,\zeta)$ and $\mathrm{H}_{p,n}^{\mathbf{\Lambda}}(q)$. The reader should not get afraid of confusing the two notations $\boldsymbol{\Lambda}$ and $\mathbf{\Lambda}$: we will not use $\boldsymbol{\Lambda}$ after Section 2.2.

We generalise in Section 2.3 the main result of [BrKl-a]. The calculations are entirely similar, hence we do not write them down. More precisely, we prove the following F-isomorphism (Theorem 2.3.6):

$$\mathrm{H}_n(q, \boldsymbol{u}) \simeq \mathrm{R}_n^{\boldsymbol{\Lambda}}(\Gamma_{e,p'}).$$

The quiver $\Gamma_{e,p'}$ is given by p' copies of the cyclic quiver Γ_e with e vertices (which is a two-sided infinite line if $e = \infty$), where $\boldsymbol{u}$ is such that the set $\{u_1, \dots, u_r\}$ is a union of p' orbits for the action of $\langle q \rangle$ on $F^\times$. In particular, in the setting of [BrKl-a] we have $p' = 1$. Moreover, we deduce that this isomorphism realises the well-known Morita equivalence of [DiMa] involving Ariki–Koike algebras, see §2.3.4.

In Section 2.4, we show with Theorem 2.4.15 that the isomorphism of Theorem 2.3.6 can be chosen such that the shift automorphism of $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q, \zeta)$ corresponds to an automorphism of $\mathrm{R}_n^{\boldsymbol{\Lambda}}(\Gamma_{e,p'})$ coming from an automorphism of the quiver $\Gamma_{e,p'}$, as described in Section 1.4. The automorphism of $\Gamma_{e,p'}$ maps a vertex $v = \zeta^j q^i$ for any $(i, j) \in K \coloneqq I \times J'$ to $\sigma(v) \coloneqq \zeta v$. We then see how it "shifts" the vertices. Using the results of Section 1.4, we finally deduce a cyclotomic quiver Hecke-like presentation for $\mathrm{H}_{p,n}^{\boldsymbol{\Lambda}}(q)$ in Corollary 2.4.17. In particular, this implies that $\mathrm{H}_{p,n}^{\boldsymbol{\Lambda}}(q)$ is a graded subalgebra of $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q, \zeta)$ (Corollary 2.4.18) and that $\mathrm{H}_{p,n}^{\boldsymbol{\Lambda}}(q)$ does not depend on q but on the quantum characteristic e (Corollary 2.4.19).

2.2 The ungraded algebras

We recall the standard definitions for the Hecke algebras of the complex reflection groups of the infinite series. Let $n, r \in \mathbb{N}^*$.

2.2.1 The Hecke algebra of type $G(r, 1, n)$

Let $\boldsymbol{u} = (u_1, \dots, u_r)$ be an r-tuple of elements of $F^\times$. We recall here the definition of the Ariki–Koike algebra $\mathrm{H}_n(q, \boldsymbol{u})$.

Definition 2.2.1 ([BrMa, ArKo])**.** The algebra $\mathrm{H}_n(q, \boldsymbol{u})$ is the unitary associative F-algebra generated by the elements $S, T_1, \dots, T_{n-1}$, subject to the following relations:

$$\prod_{k=1}^{r} (S - u_k) = 0, \tag{2.2.2a}$$

$$(T_a + 1)(T_a - q) = 0, \tag{2.2.2b}$$

$$ST_1ST_1 = T_1ST_1S, \tag{2.2.2c}$$

$$ST_a = T_aS, \qquad \text{if } a > 1, \tag{2.2.2d}$$

$$T_aT_{a'} = T_{a'}T_a, \qquad \text{if } |a - a'| > 1, \tag{2.2.2e}$$

$$T_bT_{b+1}T_b = T_{b+1}T_bT_{b+1}, \tag{2.2.2f}$$

for all $a, a' \in \{1, \dots, n\}$ and $b \in \{1, \dots, n-1\}$.

Using the terminology of [BrMa], we say that the algebra $\mathrm{H}_n(q, \boldsymbol{u})$ is a *Hecke algebra of type* $G(r, 1, n)$. Note that if $q = 1$ and $\boldsymbol{u}$ is such that (2.2.2a) is $S^r = 1$ then $\mathrm{H}_n(q, \boldsymbol{u})$ is isomorphic to the group algebra of $G(r, 1, n)$. In general, we say that $\mathrm{H}_n(q, \boldsymbol{u})$ is a *deformation* of $F[G(r, 1, n)]$. Let $X_1 \coloneqq S$ and define for any $a \in \{1, \dots, n-1\}$ the elements $X_{a+1} \in \mathrm{H}_n(q, \boldsymbol{u})$ by

$$qX_{a+1} \coloneqq T_aX_aT_a. \tag{2.2.3}$$

These elements $X_1, \dots, X_n$ pairwise commute ([ArKo, Lemma 3.3.(2)]). Moreover, Matsumoto's theorem ensures that (2.2.2e) and (2.2.2f) allow us to define $T_w \coloneqq T_{a_1} \cdots T_{a_m}$ for any reduced expression $w = s_{a_1} \cdots s_{a_m} \in \mathfrak{S}_n$, where $s_a \in \mathfrak{S}_n$ is the transposition $(a, a+1)$.

Theorem 2.2.4 ([ArKo, Theorem 3.10])**.** *The elements*

$$X_1^{m_1} \cdots X_n^{m_n} T_w \tag{2.2.5}$$

for all $m_1, \dots, m_n \in \{0, \dots, r-1\}$ *and* $w \in \mathfrak{S}_n$ *form a basis of the* F*-vector space* $\mathrm{H}_n(q, \boldsymbol{u})$.

Let $\boldsymbol{\Lambda} = (\Lambda_{i,j}) \in \mathbb{N}^{(I\times J)}$ be a weight. We assume that $\ell(\boldsymbol{\Lambda}) = r$, and we choose the parameters $u_1, \dots, u_r$ such that the relation (2.2.2a) in $\mathrm{H}_n(q, \boldsymbol{u})$ becomes

$$\prod_{i\in I}\prod_{j\in J}\left(S - \zeta^j q^i\right)^{\Lambda_{i,j}} = 0. \tag{2.2.6}$$

Definition 2.2.7. In the above setting, we define $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q, \zeta) \coloneqq \mathrm{H}_n(q, \boldsymbol{u})$.

We will often need the following condition on $\boldsymbol{\Lambda}$:

$$\Lambda_{i,j} = \Lambda_{i,j'} =: \Lambda_i, \qquad \text{for all } i \in I \text{ and } j, j' \in J. \tag{2.2.8}$$

In this case, the weight $(\Lambda_i)_{i\in I}$ has level $d = \frac{r}{p}$. Moreover, we can write (2.2.6) as

$$\prod_{i\in I}\prod_{j\in J}\left(S - \zeta^j q^i\right)^{\Lambda_i} = \prod_{i\in I}\left(S^p - q^{pi}\right)^{\Lambda_i} = 0.$$

Thus, we get the following result.

Proposition 2.2.9. *Suppose that* $\boldsymbol{\Lambda}$ *satisfies* (2.2.8)*. There is a well-defined algebra homomorphism* $\sigma : \mathrm{H}_n^{\boldsymbol{\Lambda}}(q, \zeta) \to \mathrm{H}_n^{\boldsymbol{\Lambda}}(q, \zeta)$ *given by:*

$$\begin{aligned} \sigma(S) &\coloneqq \zeta S, \\ \sigma(T_a) &\coloneqq T_a, \qquad \text{for all } a \in \{1, \dots, n-1\}. \end{aligned}$$

The homomorphism σ has order p, in particular σ is bijective. We will refer to σ as the *shift automorphism* of $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q, \zeta)$. In the remaining part of this section, we assume that (2.2.8) is satisfied, so that the shift automorphism is defined. The following lemma is an easy induction.

Lemma 2.2.10. *For every* $a \in \{1, \dots, n\}$ *we have* $\sigma(X_a) = \zeta X_a$.

Proposition 2.2.11. *The elements of* $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q, \zeta)$ *fixed by* σ *are exactly the elements in the* F*-span of* $X_1^{m_1} \cdots X_n^{m_n} T_w$ *for all* $m_1, \dots, m_n \in \{0, \dots, r-1\}$ *and* $w \in \mathfrak{S}_n$*, with the additional following condition:*

$$m_1 + \cdots + m_n = 0 \pmod p.$$

Proof. Let h be an arbitrary element of $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q, \zeta)$. By Theorem 2.2.4, we can write, with $\boldsymbol{m} = (m_a)_a$,

$$h = \sum_{\substack{\boldsymbol{m}\in\mathbb{N}^n, w\in\mathfrak{S}_n \\ 0\le m_a<r}} h_{\boldsymbol{m},w} X_1^{m_1} \cdots X_n^{m_n} T_w,$$

for some $h_{\boldsymbol{m},w} \in F$. Applying Lemma 2.2.10, we have

$$\sigma(h) = \sum_{\substack{\boldsymbol{m}\in\mathbb{N}^n, w\in\mathfrak{S}_n \\ 0\le m_a<r}} h_{\boldsymbol{m},w} \zeta^{m_1+\cdots+m_n} X_1^{m_1} \cdots X_n^{m_n} T_w,$$

thus $\sigma(h) = h$ if and only if $\zeta^{m_1+\cdots+m_n} = 1$ when $h_{\boldsymbol{m},w} \neq 0$. We conclude since ζ is a primitive pth root of unity. □

Note that the family in Proposition 2.2.11 is free (by Theorem 2.2.4), this is a basis of $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q, \zeta)^\sigma$, the fixed point subalgebra of $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q, \zeta)$ under the shift automorphism σ.

2.2.2 The Hecke algebra of type $G(r,p,n)$

Assume $\boldsymbol{\Lambda}$ satisfies the condition (2.2.8). In particular, for any $i \in I$ and $j \in J$ we have $\Lambda_{i,j} = \Lambda_i$. We will first define the algebra that Ariki [Ar95] associated with $G(r,p,n)$, and then relate this algebra to $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q,\zeta)$.

Definition 2.2.12 ([BrMa, Ar95])**.** We denote by $\mathrm{H}_{p,n}^{\boldsymbol{\Lambda}}(q)$ the unitary associative F-algebra generated by $s, t_1', t_1, \dots, t_{n-1}$, subject to the following relations:

$$\prod_{i\in I}\left(s - q^{pi}\right)^{\Lambda_i} = 0, \tag{2.2.13a}$$

$$(t_1'+1)(t_1'-q) = (t_a+1)(t_a-q) = 0, \tag{2.2.13b}$$

$$t_1't_2t_1' = t_2t_1't_2, \tag{2.2.13c}$$

$$t_at_{a+1}t_a = t_{a+1}t_at_{a+1}, \tag{2.2.13d}$$

$$(t_1't_1t_2)^2 = (t_2t_1't_1)^2, \tag{2.2.13e}$$

$$t_1't_a = t_at_1', \qquad \text{if } a \in \{3,\dots,n-1\}, \tag{2.2.13f}$$

$$t_at_b = t_bt_a, \qquad \text{if } |a-b|>1, \tag{2.2.13g}$$

$$st_a = t_as, \qquad \text{if } a \in \{2,\dots,n-1\}, \tag{2.2.13h}$$

$$st_1't_1 = t_1't_1s, \tag{2.2.13i}$$

$$\underbrace{st_1't_1t_1't_1\dots}_{p+1\text{ factors}} = \underbrace{t_1st_1't_1t_1'\dots}_{p+1\text{ factors}}, \tag{2.2.13j}$$

for all $a, b \in \{1,\dots,n-1\}$.

Using the terminology of [BrMa], we say that the algebra $\mathrm{H}_{p,n}^{\boldsymbol{\Lambda}}(q)$ is a Hecke algebra of type $G(r,p,n)$.

Remark 2.2.14. Assume that $p \geq 2$. The reader may have noticed that the above presentation is not the one given by Ariki [Ar95]. Instead of (2.2.13j) Ariki gives the following relation:

$$st_1't_1 = \left(q^{-1}t_1't_1\right)^{2-p} t_1st_1' + (q-1)\sum_{k=1}^{p-2}\left(q^{-1}t_1't_1\right)^{1-k} st_1'. \tag{2.2.13j'}$$

We claim that these two presentations define isomorphic algebras (we refer to §2.2.3 for more details). We conclude this remark by mentioning that the generators of [Ar95] are given by $a_0 = s, a_1 = t_1'$ and $a_k = t_{k-1}$ for all $k \in \{2,\dots,n\}$.

The next proposition proves that we recover the Ariki–Koike algebra $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q) := \mathrm{H}_n^{\boldsymbol{\Lambda}}(q,1)$ when $p = 1$.

Proposition 2.2.15. *The algebra homomorphisms $\phi : \mathrm{H}_{1,n}^{\boldsymbol{\Lambda}}(q) \to \mathrm{H}_n^{\boldsymbol{\Lambda}}(q)$ and $\psi : \mathrm{H}_n^{\boldsymbol{\Lambda}}(q) \to \mathrm{H}_{1,n}^{\boldsymbol{\Lambda}}(q)$ given by*

$$\begin{aligned} \phi(s) &:= S, \\ \phi(t_1') &:= S^{-1}T_1S, \\ \phi(t_a) &:= T_a, && \textit{for all } a \in \{1,\dots,n-1\}, \end{aligned}$$

and

$$\begin{aligned} \psi(S) &:= s, \\ \psi(T_a) &:= t_a, && \textit{for all } a \in \{1,\dots,n-1\}, \end{aligned}$$

are well-defined and inverse to each other. In particular, the algebras $\mathrm{H}_{1,n}^{\boldsymbol{\Lambda}}(q)$ and $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q)$ are isomorphic.

Proof. Note that since $p = 1$, the relation (2.2.13j) becomes $st'_1 = t_1 s$, thus we have

$$t'_1 = s^{-1} t_1 s. \tag{2.2.16}$$

Note by (2.2.13a) that s is indeed invertible, since $q \neq 0$ and $\ell(\Lambda) = r > 0$. We now check that ψ is an algebra homomorphism: all relations are straightforward except (2.2.2c), but this one follows from (2.2.13i) and (2.2.16). Concerning ϕ, again all relations are straightforward, except (2.2.13e) (if $n \geq 3$). Note the following consequence of (2.2.2c):

$$S^{-1} T_1 S T_1 = T_1 S T_1 S^{-1}. \tag{2.2.17}$$

In the following calculation, we adopt the following conventions:

- we use color when a quantity simplifies;
- we use underbrace when we will use a relation;
- we use parenthesis when we did use a relation.

We have:

$$\begin{aligned}
&\phi(t'_1)\phi(t_1)\phi(t_2)\phi(t'_1)\phi(t_1)\phi(t_2) = \phi(t_2)\phi(t'_1)\phi(t_1)\phi(t_2)\phi(t'_1)\phi(t_1)\\
\iff &\left[S^{-1}T_1S\right]T_1T_2\underbrace{\left[S^{-1}T_1S\right]T_1}_{(2.2.17)}T_2 = \underbrace{T_2\left[S^{-1}\right.}_{(2.2.2d)}\left.T_1S\right]T_1T_2\underbrace{\left[S^{-1}T_1S\right]T_1}_{(2.2.17)}.\\
\iff &S^{-1}T_1S\underbrace{T_1T_2\left(T_1\right.}_{(2.2.2f)}\left.ST_1\underbrace{S^{-1}\right)T_2}_{} = \left(S^{-1}T_2\right)T_1S\underbrace{T_1T_2\left(T_1\right.}_{(2.2.2f)}\left.ST_1S^{-1}\right)\\
\iff &T_1\underbrace{S\left(T_2\right.}_{}T_1\underbrace{\left.T_2\right)S}_{}T_1\left(T_2S^{-1}\right) = T_2T_1\underbrace{S\left(T_2\right.}_{}T_1\underbrace{\left.T_2\right)S}_{}T_1S^{-1}\\
\iff &T_1\left(T_2S\right)T_1\underbrace{\left(ST_2\right)T_1}_{}T_2 = T_2\underbrace{T_1\left(T_2S\right)}_{}T_1\left(ST_2\right)T_1\\
\iff &T_1T_2ST_1S\left(T_1T_2T_1\right) = \left(T_1T_2T_1\right)ST_1ST_2T_1\\
\iff &ST_1ST_1 = T_1ST_1S,
\end{aligned}$$

which allows us to conclude. Finally, the composition $\phi \circ \psi$ is the identity on the set of generators $\{S, T_1, \ldots, T_{n-1}\}$, and using (2.2.16) we find that $\psi \circ \phi$ is the identity on the set of generators $\{s, t'_1, t_1, \ldots, t_{n-1}\}$. Hence, the algebras homomorphisms ϕ and ψ are inverse isomorphisms and this concludes the proof. □

We now state the main result of this section.

Theorem 2.2.18. *The algebra homomorphism* $\phi : \mathrm{H}^{\Lambda}_{p,n}(q) \to \mathrm{H}^{\Lambda}_n(q, \zeta)$ *given by:*

$$\begin{aligned}
\phi(s) &:= S^p,\\
\phi(t'_1) &:= S^{-1}T_1S,\\
\phi(t_a) &:= T_a, && \text{for all } a \in \{1, \ldots, n-1\}.
\end{aligned}$$

is well-defined and one-to-one. Moreover, the elements $X_1^{m_1} \cdots X_n^{m_n} T_w \in \mathrm{H}^{\Lambda}_n(q, \zeta)$ *for all* $m_1, \ldots, m_n \in \{0, \ldots, r-1\}$ *and* $w \in \mathfrak{S}_n$ *such that* $m_1 + \cdots + m_n = 0 \pmod p$ *form an* F*-basis of* $\phi(\mathrm{H}^{\Lambda}_{p,n}(q))$.

Proof. If $p = 1$ we deduce the result from Theorem 2.2.4 and Proposition 2.2.15. If $p \geq 2$, by Remark 2.2.14 this is exactly [Ar95, Proposition 1.6]. □

In particular, using Proposition 2.2.11 we get the following one.

Corollary 2.2.19. *The algebra* $\mathrm{H}^{\wedge}_{p,n}(q)$ *is isomorphic via* ϕ *to* $\mathrm{H}^{\wedge}_{n}(q,\zeta)^{\sigma}$.

2.2.3 Two isomorphic presentations

We assume that $p \geq 2$. We prove here the statement of Remark 2.2.14: in the algebra $\mathrm{H}^{\wedge}_{p,n}(q)$, the relations (2.2.13j) and (2.2.13j') are equivalent. We will even prove a slightly more general statement, cf. Proposition 2.2.22. Let A be a unitary ring and $q \in A^{\times}$ an invertible element. Let s, t'_1, t_1 some symbols that satisfy

$$(t'_1 + 1)(t'_1 - q) = (t_1 + 1)(t_1 - q) = 0. \tag{2.2.20}$$

Lemma 2.2.21. *We have:*

$$\left(q^{-1}t'_1t_1\right)^{2-p}t_1st'_1 + (q-1)\sum_{k=1}^{p-2}\left(q^{-1}t'_1t_1\right)^{1-k}st'_1 = (\underbrace{t_1^{-1}t_1'^{-1}t_1^{-1}t_1'^{-1}\dots}_{p-2\ \textit{factors}})(\underbrace{\dots t_1t'_1t_1t'_1}_{p-2\ \textit{factors}})t_1st'_1.$$

Proof. For $p = 2$ we obtain

$$t_1st'_1 = t_1st'_1,$$

which is obviously true. If the equality is satisfied for $p - 1 \geq 2$, we obtain

$$\begin{aligned}
&\left(q^{-1}t'_1t_1\right)^{2-p}t_1st'_1 + (q-1)\sum_{k=1}^{p-2}\left(q^{-1}t'_1t_1\right)^{1-k}st'_1 \\
&= (q^{-1}t'_1t_1)^{2-p}t_1st'_1 + (q-1)\sum_{k=2}^{p-2}(q^{-1}t'_1t_1)^{1-k}st'_1 + (q-1)st'_1 \\
&= (q^{-1}t'_1t_1)^{-1}(q^{-1}t'_1t_1)^{3-p}t_1st'_1 + (q-1)(q^{-1}t'_1t_1)^{-1}\sum_{k=1}^{p-3}(q^{-1}t'_1t_1)^{1-k}st'_1 + (q-1)st'_1 \\
&= (q^{-1}t'_1t_1)^{-1}\left[(q^{-1}t'_1t_1)^{2-(p-1)}t_1st'_1 + (q-1)\sum_{k=1}^{(p-1)-2}(q^{-1}t'_1t_1)^{1-k}st'_1\right] + (q-1)st'_1 \\
&= (q^{-1}t'_1t_1)^{-1}(\underbrace{t_1^{-1}t_1'^{-1}\dots}_{p-3})(\underbrace{\dots t_1t'_1}_{p-3})t_1st'_1 + (q-1)st'_1 \\
&= q(\underbrace{t_1^{-1}t_1'^{-1}\dots}_{p-1})(\underbrace{\dots t_1t'_1}_{p-3})t_1st'_1 + (q-1)st'_1.
\end{aligned}$$

Let us now distinguish between two cases. If p is even, using $qt_1^{-1} = t_1 - (q-1)$ we obtain

$$\begin{aligned}
&(q^{-1}t_1't_1)^{2-p}t_1st_1' + (q-1)\sum_{k=1}^{p-2}(q^{-1}t_1't_1)^{1-k}st_1' \\
&= q(\underbrace{t_1^{-1}t_1'^{-1}\dots t_1'^{-1}t_1^{-1}}_{p-1})(\underbrace{t_1't_1\dots t_1t_1'}_{p-3})t_1st_1' + (q-1)st_1' \\
&= (\underbrace{t_1^{-1}t_1'^{-1}\dots t_1'^{-1}}_{p-2})[t_1 - (q-1)](\underbrace{t_1't_1\dots t_1t_1'}_{p-3})t_1st_1' + (q-1)st_1' \\
&= (\underbrace{t_1^{-1}t_1'^{-1}\dots t_1'^{-1}}_{p-2})(\underbrace{t_1t_1't_1\dots t_1t_1'}_{p-2})t_1st_1' - (q-1)(\underbrace{t_1^{-1}t_1'^{-1}\dots t_1'^{-1}}_{p-2})(\underbrace{t_1't_1\dots t_1t_1'}_{p-3})t_1st_1' + (q-1)st_1' \\
&= (\underbrace{t_1^{-1}t_1'^{-1}\dots t_1'^{-1}}_{p-2})(\underbrace{t_1t_1't_1\dots t_1t_1'}_{p-2})t_1st_1' - (q-1)t_1^{-1}t_1st_1' + (q-1)st_1' \\
&= (\underbrace{t_1^{-1}t_1'^{-1}\dots t_1'^{-1}}_{p-2})(\underbrace{t_1t_1't_1\dots t_1t_1'}_{p-2})t_1st_1',
\end{aligned}$$

thus we are done. If p is odd, similarly we obtain, now using $qt_1'^{-1} = t_1' - (q-1)$,

$$\begin{aligned}
&(q^{-1}t_1't_1)^{2-p}t_1st_1' + (q-1)\sum_{k=1}^{p-2}(q^{-1}t_1't_1)^{1-k}st_1' \\
&= q(\underbrace{t_1^{-1}t_1'^{-1}\dots t_1^{-1}t_1'^{-1}}_{p-1})(\underbrace{t_1t_1'\dots t_1t_1'}_{p-3})t_1st_1' + (q-1)st_1' \\
&= (\underbrace{t_1^{-1}t_1'^{-1}\dots t_1^{-1}}_{p-2})[t_1' - (q-1)](\underbrace{t_1t_1'\dots t_1t_1'}_{p-3})t_1st_1' + (q-1)st_1' \\
&= (\underbrace{t_1^{-1}t_1'^{-1}\dots t_1^{-1}}_{p-2})(\underbrace{t_1't_1t_1'\dots t_1t_1'}_{p-2})t_1st_1' - (q-1)(\underbrace{t_1^{-1}t_1'^{-1}\dots t_1^{-1}}_{p-2})(\underbrace{t_1t_1'\dots t_1t_1'}_{p-3})t_1st_1' + (q-1)st_1' \\
&= (\underbrace{t_1^{-1}t_1'^{-1}\dots t_1^{-1}}_{p-2})(\underbrace{t_1't_1t_1'\dots t_1t_1'}_{p-2})t_1st_1' - (q-1)t_1^{-1}t_1st_1' + (q-1)st_1' \\
&= (\underbrace{t_1^{-1}t_1'^{-1}\dots t_1^{-1}}_{p-2})(\underbrace{t_1't_1t_1'\dots t_1t_1'}_{p-2})t_1st_1',
\end{aligned}$$

thus we are done. ∎

Proposition 2.2.22. *We assume that s, t_1', t_1 satisfy, in addition to (2.2.20), the following relation:*

$$st_1't_1 = t_1't_1s. \tag{2.2.23}$$

The relations

$$st_1't_1 = (q^{-1}t_1't_1)^{2-p}t_1st_1' + (q-1)\sum_{k=1}^{p-2}(q^{-1}t_1't_1)^{1-k}st_1', \tag{Ar}$$

and

$$\underbrace{st_1't_1t_1't_1\dots}_{p+1} = \underbrace{t_1st_1't_1t_1'\dots}_{p+1}, \tag{BM}$$

are equivalent.

Proof. By Lemma 2.2.21, relation (Ar) is equivalent to

$$st'_1t_1 = (\underbrace{t_1^{-1}{t'_1}^{-1}t_1^{-1}{t'_1}^{-1}\dots}_{p-2 \text{ factors}})(\underbrace{\dots t_1t'_1t_1t'_1}_{p-2 \text{ factors}})t_1st'_1. \tag{2.2.24}$$

If p is even, this reads

$$st'_1t_1 = (\underbrace{t_1^{-1}{t'_1}^{-1}\dots t_1^{-1}{t'_1}^{-1}}_{p-2})(\underbrace{t_1t'_1\dots t_1t'_1}_{p-2})t_1st'_1,$$

whence we obtain

$$\underbrace{t'_1t_1\dots t'_1t_1}_{p-2}\,st'_1t_1 = \underbrace{t_1t'_1\dots t_1t'_1}_{p-2}\,t_1st'_1.$$

Thus, using (2.2.23) to bring s to the left on both sides, we obtain

$$\underbrace{st'_1t_1\dots t'_1t_1}_{p+1} = \underbrace{t_1st'_1t_1\dots t_1t'_1}_{p+1},$$

which is the desired result: the relations (Ar) and (BM) are equivalent. Now if p is odd, relation (2.2.24) reads

$$st'_1t_1 = (\underbrace{t_1^{-1}{t'_1}^{-1}\dots {t'_1}^{-1}t_1^{-1}}_{p-2})(\underbrace{t'_1t_1\dots t_1t'_1}_{p-2})t_1st'_1,$$

whence we obtain

$$\underbrace{t_1t'_1\dots t'_1t_1}_{p-2}\,st'_1t_1 = \underbrace{t'_1t_1\dots t_1t'_1}_{p-2}\,t_1st'_1.$$

Thus, using (2.2.23) to bring s to the left on both sides, we obtain

$$\underbrace{t_1st'_1\dots t'_1t_1}_{p+1} = \underbrace{st'_1t_1\dots t_1t'_1}_{p+1},$$

which is the desired result: the relations (Ar) and (BM) are thus equivalent. □

Using Proposition 2.2.22, it is now clear that the algebra $\mathrm{H}^{\mathbf{\Lambda}}_{p,n}(q)$ is isomorphic to the one defined by Ariki in [Ar95], as stated in Remark 2.2.14.

2.2.4 Removing repetitions

The following map:

$$\begin{array}{|ccc} I\times J & \longrightarrow & F^\times \\ (i,j) & \longmapsto & \zeta^j q^i \end{array},$$

is not one-to-one. The first aim of this subsection is to find a subset $J' \subseteq \{0,\dots,p-1\} \simeq J$ such that the restriction of the previous map to $I\times J'$ has the same image and is one-to-one. Moreover, for our purposes, we would like relation (2.2.6) to be of the form

$$\prod_{i\in I}\prod_{j\in J'}\left(S-\zeta^j q^i\right)^{\Lambda_{i,j}} = 0, \tag{2.2.25}$$

where $\Lambda = (\Lambda_{i,j})_{i\in I, j\in J'} \in \mathbb{N}^{(I\times J')}$ is a weight of level r. The second aim of this subsection is to know for which tuples $\Lambda \in \mathbb{N}^{(I\times J')}$ of level r there is some $\mathbf{\Lambda} \in \mathbb{N}^{(I\times J)}$ such that the relation (2.2.6) in $\mathrm{H}_n^{\mathbf{\Lambda}}(q,\zeta)$ is exactly (2.2.25). We will be particularly interested in the case where $\mathbf{\Lambda}$ satisfies (2.2.8). This will require some quite long but easy computations.

Let us define the following integer:

$$p' := \min\{m \in \mathbb{N}^* : \zeta^m \in \langle q \rangle\} \in \{1, \dots, p\}, \tag{2.2.26}$$

together with the following set:

$$J' := \{0, \dots, p' - 1\}.$$

Lemma 2.2.27. *We have:*

$$p' = \begin{cases} p, & \text{if } e = \infty, \\ \frac{p}{\gcd(p,e)}, & \text{if } e < \infty. \end{cases}$$

In particular, the integer p' divides p and depends only on p and e.

Proof. The statement for $e = \infty$ is obvious since each element of $\langle q \rangle \setminus \{1\}$ has infinite order. Thus, we now assume that $e < \infty$. For any $m \in \mathbb{N}^*$, the order of ζ^m in $F^\times$ is $\frac{p}{\gcd(p,m)}$. Since q is a primitive eth root of unity, the set $\langle q \rangle$ is precisely the set of elements of $F^\times$ of order dividing e. Hence:

$$\begin{aligned} \zeta^m \in \langle q \rangle &\iff \frac{p}{\gcd(p,m)} \text{ divides } e \\ &\iff \frac{p}{\gcd(p,m)} \text{ divides } \gcd(p,e) \\ &\iff \frac{p}{\gcd(p,e)} \text{ divides } \gcd(p,m). \end{aligned}$$

We conclude that the minimal $m \in \mathbb{N}^*$ such that $\zeta^m \in \langle q \rangle$ is $p' = \frac{p}{\gcd(p,e)}$. □

The first aim of this subsection is achieved thanks to the next lemma, which is a immediate consequence of the minimality of p'.

Lemma 2.2.28. *The elements $\zeta^j q^i$ for all $i \in I$ and $j \in J'$ are pairwise distinct.*

Let us denote by η the (unique) element of I such that:

$$\zeta^{p'} = q^\eta. \tag{2.2.29}$$

Note that $p' = p \iff \eta = 0 \iff \langle q \rangle \cap \langle \zeta \rangle = \{1\}$. In particular, if $\eta \neq 0$ then $e < \infty$. In that case, we are not necessarily in the setting of [HuMa12] (see [Lemma 2.6.(*a*), *loc. cit.*]). We now consider the following map:

$$\left| \begin{array}{ccc} J' \times \mathbb{Z}/\omega\mathbb{Z} & \longrightarrow & J \\ (j,a) & \longmapsto & j + p'a \end{array} \right. , \tag{2.2.30}$$

where $\omega := \frac{p}{p'}$. It is well-defined and surjective, hence bijective by a counting argument. Equation (2.2.6) becomes

$$\begin{aligned} \prod_{i\in I}\prod_{j\in J}\left(S - \zeta^j q^i\right)^{\Lambda_{i,j}} &= \prod_{i\in I}\prod_{j\in J'}\prod_{a\in\mathbb{Z}/\omega\mathbb{Z}}\left(S - \zeta^j (\zeta^{p'})^a q^i\right)^{\Lambda_{i,j+p'a}} \\ &= \prod_{i\in I}\prod_{j\in J'}\prod_{a\in\mathbb{Z}/\omega\mathbb{Z}}\left(S - \zeta^j q^{i+\eta a}\right)^{\Lambda_{i,j+p'a}}. \end{aligned} \tag{2.2.31}$$

For each $(i,j) \in I \times J'$, we define

$$\Lambda_{i,j} \coloneqq \sum_{i' \in I} \sum_{\substack{a \in \mathbb{Z}/\omega\mathbb{Z} \\ i'+\eta a = i}} \mathsf{\Lambda}_{i',j+p'a}, \tag{2.2.32}$$

so that, by (2.2.31),

$$\prod_{i \in I} \prod_{j \in J} \left(S - \zeta^j q^i\right)^{\mathsf{\Lambda}_{i,j}} = \prod_{i \in I} \prod_{j \in J'} \left(S - \zeta^j q^i\right)^{\Lambda_{i,j}}.$$

Hence, relation (2.2.6) transforms to the desired one (2.2.25). Conversely, it is clear that each weight $\boldsymbol{\Lambda} \in \mathbb{N}^{(I \times J')}$ comes from some $\boldsymbol{\mathsf{\Lambda}} \in \mathbb{N}^{(I \times J)}$ through (2.2.32), that is, for each $\boldsymbol{\Lambda} \in \mathbb{N}^{(I \times J')}$ there is some $\boldsymbol{\mathsf{\Lambda}} \in \mathbb{N}^{(I \times J)}$ such that (2.2.32) is satisfied. Indeed, given any $\boldsymbol{\Lambda} \in \mathbb{N}^{(I \times J')}$ it suffices to set

$$\mathsf{\Lambda}_{i,j} \coloneqq \begin{cases} \Lambda_{i,\jmath} & \text{if } j \text{ is the image of } (\jmath, 0) \text{ by the bijection of (2.2.30)}, \\ 0 & \text{otherwise}, \end{cases}$$

for all $(i,j) \in I \times J$.

Definition 2.2.33. Let $\boldsymbol{\Lambda} \in \mathbb{N}^{(I \times J')}$ be a weight of level r. We consider $\boldsymbol{\mathsf{\Lambda}} \in \mathbb{N}^{(I \times J)}$ a weight of level r which gives $\boldsymbol{\Lambda}$ through (2.2.32). We write $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q,\zeta) \coloneqq \mathrm{H}_n^{\boldsymbol{\mathsf{\Lambda}}}(q,\zeta)$. In particular, the relation (2.2.6) becomes (2.2.25).

We now assume that the weight $\boldsymbol{\mathsf{\Lambda}} \in \mathbb{N}^{(I \times J)}$ satisfies the condition (2.2.8), that is, factors to $\boldsymbol{\mathsf{\Lambda}} \in \mathbb{N}^{(I)}$. We want to know which condition we recover on $\boldsymbol{\Lambda}$. The defining equality (2.2.32) becomes

$$\Lambda_{i,j} = \sum_{i' \in I} \sum_{\substack{a \in \mathbb{Z}/\omega\mathbb{Z} \\ i'+\eta a = i}} \mathsf{\Lambda}_{i'},$$

for any $i \in I$ and $j \in J'$. In particular, for any $i \in I$ and $j, j' \in J'$ we have $\Lambda_{i,j} = \Lambda_{i,j'} =: \Lambda_i$, so that $\boldsymbol{\Lambda} \in \mathbb{N}^{(I)}$ is a weight of level ωd, and

$$\Lambda_i = \sum_{i' \in I} \sum_{\substack{a \in \mathbb{Z}/\omega\mathbb{Z} \\ i'+\eta a = i}} \mathsf{\Lambda}_{i'}, \tag{2.2.34}$$

for all $i \in I$.

Lemma 2.2.35. *For any $i \in I$ we have*

$$\#\{a \in \mathbb{Z}/\omega\mathbb{Z} : \eta a = i\} = \begin{cases} 0 & \textit{if } i \notin \eta I, \quad (2.2.36) \\ 1 & \textit{if } i \in \eta I. \quad (2.2.37) \end{cases}$$

Proof. The result is straightforward if $\eta = 0$, in particular in that case we have $\omega = 1$. Thus we assume $\eta \neq 0$, in particular $e < \infty$ and $I = \mathbb{Z}/e\mathbb{Z}$. Let us compute the cardinality of the fibre of i under the following group homomorphism:

$$\phi : \left| \begin{array}{ccc} \mathbb{Z} & \longrightarrow & I \\ a & \longmapsto & \eta a. \end{array} \right.$$

First, the image of ϕ is ηI, which proves (2.2.36). The element ω lies in $\ker \phi$. Indeed, we have $\mathrm{order}(\zeta^{p'}) = \mathrm{order}(q^\eta)$, hence

$$\omega = \frac{e}{\gcd(e,\eta)}, \tag{2.2.38}$$

thus $e = \gcd(e,\eta)\omega \mid \eta\omega$. As a consequence, we have a well-defined *surjective* map

$$\overline{\phi} : \left| \begin{array}{rcl} \mathbb{Z}/\omega\mathbb{Z} & \longrightarrow & \eta I \\ a & \longmapsto & \eta a \end{array} \right. .$$

We have, using (2.2.38),

$$\eta I = (\eta\mathbb{Z} + e\mathbb{Z})/e\mathbb{Z} \simeq \mathbb{Z}/\tfrac{e}{\gcd(e,\eta)}\mathbb{Z} = \mathbb{Z}/\omega\mathbb{Z},$$

thus, by a counting argument we get that the map $\overline{\phi}$ is bijective. This concludes the proof. □

The second aim of this subsection is achieved thanks to the following proposition.

Proposition 2.2.39. *A weight $\mathbf{\Lambda} \in \mathbb{N}^{(I)}$ of level ωd comes from a weight $\mathbf{\Lambda} \in \mathbb{N}^{(I)}$ of level d through* (2.2.34) *if and only if for all $i \in I$,*

$$\Lambda_i = \Lambda_{i+\eta},$$

that is, if and only if the weight $\mathbf{\Lambda}$ factors to a weight $\mathbf{\Lambda} \in \mathbb{N}^{(I/\eta I)}$ of level d.

Proof. First, by applying Lemma 2.2.35 to (2.2.34), we obtain the equivalent equality

$$\Lambda_i = \sum_{\substack{i' \in I \\ i' - i \in \eta I}} \Lambda_{i'} = \sum_{i' \in i + \eta I} \Lambda_{i'}, \tag{2.2.40}$$

for all $i \in I$. The necessary condition is hence straightforward. We now suppose that $\mathbf{\Lambda} \in \mathbb{N}^{(I)}$ factors to a weight $\mathbf{\Lambda} \in \mathbb{N}^{(I/\eta I)}$ of level d. For any $\gamma \in I/\eta I$, we choose any ω non-negative integers Λ_i for $i \in \gamma$ such that $\sum_{i \in \gamma} \Lambda_i = \Lambda_\gamma$. We conclude that (2.2.40) and thus (2.2.32) hold since $\Lambda_i = \Lambda_\gamma$ if $i \in \gamma$. □

The reader may have noticed the similarity of the equation of Proposition 2.2.39 with the condition (1.4.25). In §2.4.2 we will explicitly make the link between these two conditions.

Definition 2.2.41. We write $\mathrm{H}_n^{\mathbf{\Lambda}}(q,\zeta) := \mathrm{H}_n^{\mathbf{\Lambda}}(q,\zeta)$ and $\mathrm{H}_{p,n}^{\mathbf{\Lambda}}(q) := \mathrm{H}_{p,n}^{\mathbf{\Lambda}}(q)$ if $\mathbf{\Lambda} \in \mathbb{N}^{(I)}$ of level ωd and $\mathbf{\Lambda} \in \mathbb{N}^{(I)}$ of level d are as in Proposition 2.2.39. In particular, the cyclotomic relation (2.2.6) in $\mathrm{H}_n^{\mathbf{\Lambda}}(q,\zeta)$ is exactly (2.2.25).

2.3 The graded isomorphism of Brundan and Kleshchev

In this section, we generalise an isomorphism of Brundan and Kleshchev [BrKl-a] involving $\mathrm{H}_n^{\mathbf{\Lambda}}(q,1)$ to the case of the algebra $\mathrm{H}_n^{\mathbf{\Lambda}}(q,\zeta)$.

2.3.1 Statement

We consider the quiver Γ_e defined as follows:

- the vertex set is $\{q^i\}_{i \in I}$;
- there is a directed edge from v to qv for each vertex v of Γ_e.

We will often identify the vertex set with I in the canonical way. In particular, if i is a vertex then there is a directed arrow from i to $i+1$. For any $i, i' \in I$, with the notation of §1.2.2 we have

$$\begin{aligned}
i \to i' &\iff [i' = i+1 \text{ and } i \neq i'+1],\\
i \leftarrow i' &\iff [i = i'+1 \text{ and } i' \neq i+1],\\
i \leftrightarrows i' &\iff [i = i'+1 \text{ and } i' = i+1],\\
i \not\!\!- i' &\iff i \neq i', i' \pm 1.
\end{aligned}$$

The quiver Γ_e is the cyclic quiver with e vertices if $e < \infty$, and a two-sided infinite line if $e = \infty$.

Example 2.3.1. We give some examples of quivers Γ_e, where we use the identification between the vertex set of Γ_e and I.

Quiver Γ_2 $\qquad$ $0 \leftrightarrows 1$

Quiver Γ_4 $\qquad$ $\begin{array}{ccc} 0 & \longrightarrow & 1 \\ \uparrow & & \downarrow \\ 3 & \longleftarrow & 2 \end{array}$

Quiver Γ_∞ $\qquad$ $\cdots \longrightarrow -2 \longrightarrow -1 \longrightarrow 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots$

We recall the notation p' and J' introduced at §2.2.4. We set $K := I \times J'$. Let us consider p' non-zero elements $v_0, \dots, v_{p'-1}$ of F which lie in distinct orbits under the action of $\langle q \rangle$ on $F^\times$, that is,

$$\frac{v_k}{v_l} \notin \langle q \rangle, \qquad \text{for any } k \neq l. \tag{2.3.2}$$

We then consider the quiver Γ defined as follows:

- the vertex set is $V := \{v_j q^i\}_{i \in I, j \in J'}$;
- there is a directed edge from v to qv for each vertex v of Γ.

Since the elements v_k lie in different q-orbits, the vertex set V of Γ can be identified with $K = I \times J'$. More precisely, we have the following decomposition:

$$V = \bigsqcup_{j \in J'} \{v_j q^i\}_{i \in I}. \tag{2.3.3}$$

Since:

- the subquiver of Γ with vertex set $\{v_j q^i\}_{i \in I}$ is a copy of Γ_e;
- for any $j, j' \in J'$ with $j \neq j'$, there is no arrows between any element of $\{v_j q^i\}_{i \in I}$ and $\{v_{j'} q^i\}_{i \in I}$;
- the set J' has cardinality p';

we conclude from (2.3.3) that Γ is exactly p' disjoint copies of Γ_e. Thus, the quiver Γ depends only on e and p' and we write $\Gamma =: \Gamma_{e,p'}$. This quiver is loop-free and has no multiple edges.

As a consequence, we will often write $(i, j) \in I \times J'$ for the vertex $v_j q^i \in V$ of $\Gamma_{e,p'}$. For any $i, i' \in I$ and $j, j' \in J'$, what precedes ensures that the vertices (i, j) and (i', j') are in a same copy of Γ_e if and only if $j = j'$. Further, there is a directed edge from (i, j) to (i', j') if and only if $j = j'$ and there is a directed edge in Γ_e from i to i'.

Example 2.3.4. We give three examples of quivers $\Gamma_{e,p'}$. For aesthetic reasons, we write i_j instead of $v_j q^i$. We also recall from Lemma 2.2.27 that $p' = \frac{p}{\gcd(p,e)}$ if $e < \infty$ and $p' = p$ if $e = \infty$.

Case $(e,p) = (2,3)$ $\qquad 0_1 \leftrightarrows 1_1 \qquad 0_2 \leftrightarrows 1_2 \qquad 0_3 \leftrightarrows 1_3$

Case $(e,p) = (2,6)$ $\qquad 0_1 \leftrightarrows 1_1 \qquad 0_2 \leftrightarrows 1_2 \qquad 0_3 \leftrightarrows 1_3$

Case $(e,p) = (\infty,2)$

$$\cdots \longrightarrow -2_1 \longrightarrow -1_1 \longrightarrow 0_1 \longrightarrow 1_1 \longrightarrow 2_1 \longrightarrow \cdots$$

$$\cdots \longrightarrow -2_2 \longrightarrow -1_2 \longrightarrow 0_2 \longrightarrow 1_2 \longrightarrow 2_2 \longrightarrow \cdots$$

Now let $\boldsymbol{\Lambda} = (\Lambda_k)_{k \in K} \in \mathbb{N}^{(K)}$ be a weight of level r. Mimicking the definition of $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q,\zeta)$, let us choose a tuple $\boldsymbol{u} \in (F^\times)^r$ which is given by exactly $\Lambda_{i,j}$ copies of $v_j q^i$ for each $(i,j) \in I \times J'$ and set $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q,\boldsymbol{v}) \coloneqq \mathrm{H}_n(q,\boldsymbol{u})$. As a result, the relation (2.2.2a) in $\mathrm{H}_n(q,\boldsymbol{u})$ is

$$\prod_{i \in I} \prod_{j \in J'} \left(S - v_j q^i\right)^{\Lambda_{i,j}} = 0. \tag{2.3.5}$$

The remaining part of this section is devoted to the proof of the following theorem. Recall the definition of the cyclotomic quiver Hecke algebra $\mathrm{R}_n^{\boldsymbol{\Lambda}}(\Gamma_{e,p'})$ from Section 1.2.

Theorem 2.3.6. *There is an explicit F-algebra isomorphism*

$$\mathrm{H}_n^{\boldsymbol{\Lambda}}(q,\boldsymbol{v}) \simeq \mathrm{R}_n^{\boldsymbol{\Lambda}}(\Gamma_{e,p'}).$$

Brundan and Kleshchev [BrKl-a] proved Theorem 2.3.6 for $p = 1$. In that case, we have $p' = 1$, the tuple $\boldsymbol{v}$ has only one component (that can be taken equal to 1) and $\Gamma_{e,p'} = \Gamma_e$. Moreover, we say that the algebra $\mathrm{R}_n^{\boldsymbol{\Lambda}}(\Gamma_e)$ is the *cyclotomic quiver Hecke algebra of type A*. We will see that the same argument as in [BrKl-a] proves the general case. Such an isomorphism, for $e < \infty$, was already obtained by Rouquier [Rou, Corollary 3.20].

2.3.2 Candidate homomorphisms

Recall the definitions $X_1 \coloneqq S$ and $qX_{a+1} \coloneqq T_a X_a T_a$ from (2.2.3). To prove Theorem 2.3.6, it suffices to give an isomorphism between $\mathrm{H}_\alpha^{\boldsymbol{\Lambda}}(q,\boldsymbol{v})$ (defined at (2.3.9)) and $\mathrm{R}_\alpha^{\boldsymbol{\Lambda}}(\Gamma_{e,p'})$ for any $\alpha \vDash_K n$. Let M be a finite-dimensional $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q,\boldsymbol{v})$-module.

Lemma 2.3.7. *For any $a \in \{1,\dots,n\}$, the eigenvalues of X_a on M are of the form $v_j q^i$ for $i \in I$ and $j \in J'$.*

Proof. The statement is of course true for $a = 1$ by (2.2.6). By induction, using [ArKo] or [Gr, Lemma 4.7] we know that any eigenvalue of X_{a+1} differs from an eigenvalues of X_a by a power of q. □

Hence, as the elements $X_1,\dots,X_n$ pairwise commute, we can write M as a direct sum of generalised simultaneous eigenspaces

$$M = \bigoplus_{\boldsymbol{k} \in K^n} M(\boldsymbol{k}),$$

where $M(\boldsymbol{k}) = M(\boldsymbol{i},\boldsymbol{j})$ is defined by, for any $\boldsymbol{k} = (\boldsymbol{i},\boldsymbol{j}) \in K^n \simeq I^n \times J'^n$,

$$M(\boldsymbol{i},\boldsymbol{j}) \coloneqq \left\{ m \in M : \left(X_a - v_{j_a} q^{i_a}\right)^N m = 0 \text{ for all } 1 \leq a \leq n \right\},$$

where $N \gg 0$. Note that all but finitely many $M(\boldsymbol{k})$ are reduced to $\{0\}$. We now consider the family $\{e(\boldsymbol{k})\}_{\boldsymbol{k} \in K^n}$ of projections associated with the decomposition above. In particular:

- we have $e(\boldsymbol{k})e(\boldsymbol{k}') = \delta_{\boldsymbol{k},\boldsymbol{k}'}e(\boldsymbol{k})$;
- we have $\sum_{\boldsymbol{k}\in K^n} e(\boldsymbol{k}) = \mathrm{id}$ (this is a finite sum since all but finitely many $e(\boldsymbol{k})$ are zero);
- we have $e(\boldsymbol{k})M = M(\boldsymbol{k})$.

Remark 2.3.8. We already used the notation $e(\boldsymbol{k})$ for some generators of $\mathrm{R}_n^{\boldsymbol{\Lambda}}(\Gamma_{e,p'})$. This abuse of notation will be justified by the proof of Theorem 2.3.6, where we prove that these elements can be identified.

Since $e(\boldsymbol{k})$ is a polynomial in $X_1, \ldots, X_n$ we have $e(\boldsymbol{k}) \in \mathrm{H}_n^{\boldsymbol{\Lambda}}(q, \boldsymbol{v})$. Now if $\alpha \models_K n$ is a K-composition of n, the following element:

$$e(\alpha) := \sum_{\boldsymbol{k}\in K^\alpha} e(\boldsymbol{k}) \in \mathrm{H}_n^{\boldsymbol{\Lambda}}(q, \boldsymbol{v}),$$

is a central idempotent (the reader should compare this definition to (1.2.10)). We thus get a subalgebra

$$\mathrm{H}_\alpha^{\boldsymbol{\Lambda}}(q, \boldsymbol{v}) := e(\alpha)\mathrm{H}_n^{\boldsymbol{\Lambda}}(q, \boldsymbol{v}). \tag{2.3.9}$$

Remark 2.3.10. The subalgebra $\mathrm{H}_\alpha^{\boldsymbol{\Lambda}}(q, \boldsymbol{v})$ is either $\{0\}$ or a *block* of $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q, \boldsymbol{v})$ (see [LyMa]). This block has unit $e(\alpha)$.

Recall that the elements $y_a \in \mathrm{R}_\alpha^{\boldsymbol{\Lambda}}(\Gamma_{e,p'})$ for all $a \in \{1, \ldots, n\}$ are nilpotent (Lemma 1.2.23). Hence, each power series $f(y_1, \ldots, y_n) \in F[[y_1, \ldots, y_n]]$ in these elements is a well-defined element of $\mathrm{R}_\alpha^{\boldsymbol{\Lambda}}(\Gamma_{e,p'})$. In particular, for any $a \in \{1, \ldots, n-1\}$ and $\boldsymbol{k} \in K^\alpha$ the following power series is well-defined in $\mathrm{R}_\alpha^{\boldsymbol{\Lambda}}(\Gamma_{e,p'})$:

$$P_a(\boldsymbol{k}) := \begin{cases} 1, & \text{if } k_a = k_{a+1}, \\ (1-q)\left(1 - y_a(\boldsymbol{k})y_{a+1}(\boldsymbol{k})^{-1}\right)^{-1}, & \text{if } k_a \neq k_{a+1}, \end{cases} \in F[[y_a, y_{a+1}]], \tag{2.3.11}$$

where

$$y_a(\boldsymbol{k}) := v_{j_a} q^{i_a}(1 - y_a), \tag{2.3.12}$$

if $\boldsymbol{k} = (\boldsymbol{i}, \boldsymbol{j})$. For any $k = (i, j) \in K = I \times J'$, we define

$$\begin{aligned} q^k &:= q^i, & q^{-k} &:= q^{-i}, \\ v_k &:= v_j, & v_{-k} &:= v_j^{-1}, \end{aligned} \tag{2.3.13}$$

in particular we obtain $y_a(\boldsymbol{k}) = v_{k_a} q^{k_a}(1 - y_a)$ for any $\boldsymbol{k} \in K^n$. Note that

$$\begin{aligned} 1 - y_a(\boldsymbol{k})y_{a+1}(\boldsymbol{k})^{-1} &= \frac{y_{a+1}(\boldsymbol{k}) - y_a(\boldsymbol{k})}{y_{a+1}(\boldsymbol{k})} \\ &= \frac{(v_{k_{a+1}} q^{k_{a+1}} - v_{k_a} q^{k_a}) + v_{k_a} q^{k_a} y_a - v_{k_{a+1}} q^{k_{a+1}} y_{a+1}}{y_{a+1}(\boldsymbol{k})}. \end{aligned}$$

Thus, by (2.3.2) we know that this expression is indeed invertible when $k_a \neq k_{a+1}$.

For any $(w, f) \in \mathfrak{S}_n \times F[[y_1, \ldots, y_n]]$, we denote by $f^w \in F[[y_1, \ldots, y_n]]$ the usual right action of w on f. For instance, if $w = \tau$ is a transposition then $f^\tau(y_1, \ldots, y_n) = f(y_{\tau(1)}, \ldots, y_{\tau(n)})$. Let us give a lemma involving this action (see, for instance, [BrKl-a, (2.6)]).

Lemma 2.3.14. *For any $f \in F[[y_1, \ldots, y_n]], a \in \{1, \ldots, n-1\}$ and $\boldsymbol{k} \in K^\alpha$ we have*

$$f\psi_a e(\boldsymbol{k}) = \begin{cases} \psi_a f^{s_a} e(\boldsymbol{k}) + \partial_a(f)e(\boldsymbol{k}), & \text{if } k_a = k_{a+1}, \\ \psi_a f^{s_a} e(\boldsymbol{k}), & \text{if } k_a \neq k_{a+1}, \end{cases}$$

where $\partial_a(f) := \frac{f^{s_a} - f}{y_a - y_{a+1}} \in F[[y_1, \ldots, y_n]]$.

Proof. This is a consequence of (1.2.3f), (1.2.3h) and (1.2.3i). □

We say that a family $\{Q_a(\boldsymbol{k})\}_{a\in\{1,\dots,n-1\},\boldsymbol{k}\in K^\alpha}$ of elements of $F[[y_1,\dots,y_n]]$ satisfies the property (BK) if

$$Q_a(\boldsymbol{k}) \text{ is an invertible element of } F[[y_a, y_{a+1}]], \tag{2.3.15a}$$

$$Q_a(\boldsymbol{k}) = 1 - q + qy_{a+1} - y_a, \qquad \text{if } k_a = k_{a+1}, \tag{2.3.15b}$$

$$Q_a(\boldsymbol{k})Q_a(s_a\cdot\boldsymbol{k})^{s_a} = \begin{cases} (1-P_a(\boldsymbol{k}))(q+P_a(\boldsymbol{k})), & \text{if } k_a \not\to k_{a+1}, \quad (2.3.15\text{c-i})\\ \dfrac{(1-P_a(\boldsymbol{k}))(q+P_a(\boldsymbol{k}))}{y_{a+1}-y_a}, & \text{if } k_a \to k_{a+1}, \quad (2.3.15\text{c-ii})\\ \dfrac{(1-P_a(\boldsymbol{k}))(q+P_a(\boldsymbol{k}))}{y_a-y_{a+1}}, & \text{if } k_a \leftarrow k_{a+1}, \quad (2.3.15\text{c-iii})\\ \dfrac{(1-P_a(\boldsymbol{k}))(q+P_a(\boldsymbol{k}))}{(y_{a+1}-y_a)(y_a-y_{a+1})}, & \text{if } k_a \leftrightarrows k_{a+1}, \quad (2.3.15\text{c-iv}) \end{cases}$$

$$Q_{a+1}(s_{a+1}s_a\cdot\boldsymbol{k})^{s_a} = Q_a(s_as_{a+1}\cdot\boldsymbol{k})^{s_{a+1}}. \tag{2.3.15d}$$

We can now give the key of Theorem 2.3.6.

Theorem 2.3.16. *Let $\{Q_a(\boldsymbol{k})\}_{a\in\{1,\dots,n-1\},\boldsymbol{k}\in K^\alpha}$ be a family of elements of $F[[y_1,\dots,y_n]]$ which satisfies* (BK)*. There exist unique F-algebra homomorphisms $f : \mathrm{H}_\alpha^{\boldsymbol{\Lambda}}(q,\boldsymbol{v}) \to \mathrm{R}_\alpha^{\boldsymbol{\Lambda}}(\Gamma_{e,p'})$ and $g : \mathrm{R}_\alpha^{\boldsymbol{\Lambda}}(\Gamma_{e,p'}) \to \mathrm{H}_\alpha^{\boldsymbol{\Lambda}}(q,\boldsymbol{v})$ such that*

$$f(X_a) := \sum_{\boldsymbol{k}\in K^\alpha} y_a(\boldsymbol{k})e(\boldsymbol{k}),$$

$$f(T_b) := \sum_{\boldsymbol{k}\in K^\alpha} (\psi_b Q_b(\boldsymbol{k}) - P_b(\boldsymbol{k}))e(\boldsymbol{k}),$$

and, recalling the notation (2.3.13),

$$g(e(\boldsymbol{k})) := e(\boldsymbol{k}),$$

$$g(y_a) := \sum_{\boldsymbol{k}\in K^\alpha} (1 - v_{-k_a}q^{-k_a}X_a)e(\boldsymbol{k}),$$

$$g(\psi_b) := \sum_{\boldsymbol{k}\in K^\alpha} (T_b + P_b(\boldsymbol{k}))Q_b(\boldsymbol{k})^{-1}e(\boldsymbol{k}),$$

for all $a \in \{1,\dots,n\}$ and $b \in \{1,\dots,n-1\}$. Moreover, these homomorphisms are inverse to each other, hence $\mathrm{H}_\alpha^{\boldsymbol{\Lambda}}(q,\boldsymbol{v}) \simeq \mathrm{R}_\alpha^{\boldsymbol{\Lambda}}(\Gamma_{e,p'})$.

We will explain at the beginning of §2.3.3.2 how the elements $P_a(\boldsymbol{k})$ and $Q_a(\boldsymbol{k})$ are considered as elements of $\mathrm{H}_\alpha^{\boldsymbol{\Lambda}}(q,\boldsymbol{v})$. We note that there exist such families $\{Q_a(\boldsymbol{k})\}_{a,\boldsymbol{k}}$, see §2.4.1 for further details.

2.3.3 Proof of Theorems 2.3.6 and 2.3.16

In this subsection, we first check that the maps of Theorem 2.3.16 indeed define algebras homomorphisms: we check that the different defining relations (2.2.2b)–(2.2.2f), (2.2.6) (for f) and (1.2.3), (1.2.16), (1.2.19) (for g) are satisfied. The proof is exactly as in [BrKl-a, Section 4]: we will only give some details when the elements v_j are involved. The remaining parts of the argument require only notational changes from [BrKl-a].

2.3.3.1 The map f is a homomorphism

We prove that the images of the generators of $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q, \boldsymbol{v})$ by f satisfy the defining relations.

The proof of the quadratic relation (2.2.2b) is exactly the same as the one for [BrKl-a, Theorem 4.3]. Namely, it suffices to check that for any $\boldsymbol{k} \in K^\alpha$ we have $f(T_a)^2 e(\boldsymbol{k}) = (q - 1)f(T_a)e(\boldsymbol{k}) + qe(\boldsymbol{k})$, and the result follows since $\sum_{\boldsymbol{k} \in K^\alpha} e(\boldsymbol{k}) = 1$ in $\mathrm{R}_\alpha^{\boldsymbol{\Lambda}}(\Gamma_{e,p'})$.

The equality $f(X_1)f(X_2) = f(X_2)f(X_1)$ is clear. Hence, to check the length 4-braid relation (2.2.2c) it suffices to prove that $qf(X_2) = f(T_1)f(X_1)f(T_1)$. We will in fact prove that for any $a \in \{1, \dots, n-1\}$,

$$qf(X_{a+1}) = f(T_a)f(X_a)f(T_a). \tag{2.3.17}$$

Since we have just checked the relation (2.2.2b) for $f(T_a)$, it suffices to prove that for any $\boldsymbol{k} \in K^\alpha$,

$$f(X_a)f(T_a)e(\boldsymbol{k}) = (f(T_a) + 1 - q)f(X_{a+1})e(\boldsymbol{k}).$$

Once again, the rest of the proof is exactly the same as in the corresponding part of the proof of [BrKl-a, Theorem 4.3]. We write down here some of the details since we have to add some v_j in the calculations. We have

$$\begin{aligned} X_a T_a e(\boldsymbol{k}) &= (y_a(s_a \cdot \boldsymbol{k})\psi_a Q_a(\boldsymbol{k}) - y_a(\boldsymbol{k})P_a(\boldsymbol{k}))\, e(\boldsymbol{k}) \\ &= \Big(\psi_a y_{a+1}(\boldsymbol{k})Q_a(\boldsymbol{k}) + \delta_{k_a,k_{a+1}} v_{k_a} q^{k_a} Q_a(\boldsymbol{k}) - y_a(\boldsymbol{k})P_a(\boldsymbol{k})\Big)\, e(\boldsymbol{k}), \end{aligned}$$

and:

$$(T_a + 1 - q)X_{a+1}e(\boldsymbol{k}) = (\psi_a Q_a(\boldsymbol{k}) - P_a(\boldsymbol{k}) + 1 - q)\, y_{a+1}(\boldsymbol{k})e(\boldsymbol{k}).$$

Considering the two cases $k_a \neq k_{a+1}$ and $k_a = k_{a+1}$ separately and using (2.3.11), (2.3.12) and (2.3.15b), we can easily prove that the two above quantities are equal.

The commutation relations (2.2.2d) and (2.2.2e) are straightforward from the defining relations in $\mathrm{R}_\alpha^{\boldsymbol{\Lambda}}(\Gamma_{e,p'})$, and for (2.2.2f) we can reproduce the corresponding part of the proof of [BrKl-a, Theorem 4.3]

Finally, let us prove that the cyclotomic relation (2.2.6) is satisfied, that is,

$$\prod_{k \in K} \Big(f(X_1) - v_k q^k\Big)^{\Lambda_k} = 0.$$

We have, using (1.2.3a) and (1.2.3b),

$$\begin{aligned} \prod_{k \in K} \Big(f(X_1) - v_k q^k\Big)^{\Lambda_k} &= \prod_{k \in K} \left[\sum_{\boldsymbol{k} \in K^\alpha} \Big(v_{k_1} q^{k_1}(1 - y_1) - v_k q^k\Big)\, e(\boldsymbol{k})\right]^{\Lambda_k} \\ &= \prod_{k \in K} \left[\sum_{\boldsymbol{k} \in K^\alpha} \Big(v_{k_1} q^{k_1}(1 - y_1) - v_k q^k\Big)^{\Lambda_k} e(\boldsymbol{k})\right] \\ &= \sum_{\boldsymbol{k} \in K^\alpha} \prod_{k \in K} \left[\Big(v_{k_1} q^{k_1}(1 - y_1) - v_k q^k\Big)^{\Lambda_k} e(\boldsymbol{k})\right]. \end{aligned}$$

By (1.2.19), for any $\boldsymbol{k} \in K^\alpha$ the term for $k = k_1$ vanishes, hence we get the result.

To conclude, the map $f : \mathrm{H}_n^{\boldsymbol{\Lambda}}(q, \boldsymbol{v}) \to \mathrm{R}_\alpha^{\boldsymbol{\Lambda}}(\Gamma_{e,p'})$ defined on the generators $X_1, T_1, \dots, T_{n-1}$ yields a homomorphism of algebra. By restriction, we get an algebra homomorphism $f : \mathrm{H}_\alpha^{\boldsymbol{\Lambda}}(q, \boldsymbol{v}) \to \mathrm{R}_\alpha^{\boldsymbol{\Lambda}}(\Gamma_{e,p'})$. In particular, the image of X_a for $a > 1$ is the one given in Theorem 2.3.16, thanks to (2.2.3) and (2.3.17).

2.3.3.2 The map g is a homomorphism

In this paragraph, for any $m \in \mathrm{R}^{\mathbf{\Lambda}}_{\alpha}(\Gamma_{e,p'})$ we also write $m := g(m) \in \mathrm{H}^{\mathbf{\Lambda}}_{\alpha}(q, \boldsymbol{v})$. In particular, we have

$$y_a = \sum_{\boldsymbol{k}\in K^\alpha} (1 - v_{-k_a} q^{-k_a} X_a) e(\boldsymbol{k}) \in \mathrm{H}^{\mathbf{\Lambda}}_{\alpha}(q, \boldsymbol{v}),$$

thus we can consider the power series $P_a(\boldsymbol{k})$ and $Q_a(\boldsymbol{k})$ as elements of $\mathrm{H}^{\mathbf{\Lambda}}_{\alpha}(q, \boldsymbol{v})$, namely

$$\psi_a = \sum_{\boldsymbol{k}\in K^\alpha} (T_a + P_a(\boldsymbol{k})) Q_a(\boldsymbol{k})^{-1} e(\boldsymbol{k}) \in \mathrm{H}^{\mathbf{\Lambda}}_{\alpha}(q, \boldsymbol{v}).$$

Following Lusztig, define the following "intertwining element" in $\mathrm{H}^{\mathbf{\Lambda}}_{\alpha}(q, \boldsymbol{v})$ for any $a \in \{1, \dots, n-1\}$ by

$$\Phi_a := T_a + (1-q) \sum_{\substack{\boldsymbol{k}\in K^\alpha \\ k_a \neq k_{a+1}}} \left(1 - X_a X_{a+1}^{-1}\right)^{-1} e(\boldsymbol{k}) + \sum_{\substack{\boldsymbol{k}\in K^\alpha \\ k_a = k_{a+1}}} e(\boldsymbol{k}),$$

where $\left(1 - X_a X_{a+1}^{-1}\right)^{-1} e(\boldsymbol{k})$ denotes the inverse of $(1 - X_a X_{a+1}^{-1}) e(\boldsymbol{k})$ in $e(\boldsymbol{k}) \mathrm{H}^{\mathbf{\Lambda}}_{\alpha}(q, \boldsymbol{v}) e(\boldsymbol{k})$. Noticing that $y_a(\boldsymbol{k}) e(\boldsymbol{k}) = X_a e(\boldsymbol{k})$, we can check the following equality:

$$\Phi_a = \sum_{\boldsymbol{k}\in K^\alpha} (T_a + P_a(\boldsymbol{k})) e(\boldsymbol{k}).$$

We can give an analogue of [BrKl-a, Lemma 4.1]. Once again, we just have to write a (respectively $\boldsymbol{k}, k$) instead of their r (resp. $\boldsymbol{i}, i$), both in the statements and the proofs. Among all the relations in the lemma, we will make here an explicit use of the following one:

$$X_{a+1} \Phi_a e(\boldsymbol{k}) = \begin{cases} \Phi_a X_a e(\boldsymbol{k}), & \text{if } k_a \neq k_{a+1}, \\ \Phi_a X_a e(\boldsymbol{k}) + (q X_{a+1} - X_a) e(\boldsymbol{k}), & \text{if } k_a = k_{a+1}. \end{cases} \tag{2.3.18}$$

We now check the different relations of $\mathrm{R}^{\mathbf{\Lambda}}_{\alpha}(\Gamma_{e,p'})$. Relations (1.2.3a)–(1.2.3g), (1.2.16) and (1.2.19) follow as in the corresponding part of the proof of [BrKl-a, Theorem 4.2]. To check (1.2.3i), again we just follow the corresponding part of the proof of [BrKl-a, Theorem 4.2], but we need to add some v_j's. We have

$$y_{a+1} \psi_a e(\boldsymbol{k}) = (1 - v_{-k_a} q^{-k_a} X_{a+1}) \Phi_a Q_a(\boldsymbol{k})^{-1} e(\boldsymbol{k}).$$

If $k_a \neq k_{a+1}$, using (2.3.18) we get

$$y_{a+1} \psi_a e(\boldsymbol{k}) = \Phi_a Q_a(\boldsymbol{k})^{-1} (1 - v_{-k_a} q^{-k_a} X_a) e(\boldsymbol{k}) = \psi_a y_a e(\boldsymbol{k}),$$

whereas if $k_a = k_{a+1}$ we obtain

$$\begin{aligned} y_{a+1} \psi_a e(\boldsymbol{k}) &= (1 - v_{-k_a} q^{-k_a} X_{a+1})(T_a + 1) Q_a(\boldsymbol{k})^{-1} e(\boldsymbol{k}) \\ &= \Big((T_a + 1)(1 - v_{-k_a} q^{-k_a} X_a) + v_{-k_a} q^{-k_a} X_a - v_{-k_a} q^{1-k_a} X_{a+1} \Big) Q_a(\boldsymbol{k})^{-1} e(\boldsymbol{k}) \\ &= (\psi_a y_a + 1) e(\boldsymbol{k}), \end{aligned}$$

since $(v_{-k_a} q^{-k_a} X_a - v_{-k_a} q^{1-k_a} X_{a+1}) e(\boldsymbol{k}) = Q_a(\boldsymbol{k}) e(\boldsymbol{k})$. The proof of (1.2.3h) is similar.

2.3.3.3 Conclusion

As in [BrKl-a, Lemma 3.4], we have:

$$f(e(\boldsymbol{k})) = e(\boldsymbol{k}) \in \mathrm{R}_\alpha^{\boldsymbol{\Lambda}}(\Gamma_{e,p'}),$$

for all $\boldsymbol{k} \in K^\alpha$. It is now an easy exercise to show that $f \circ g$ is the identity of $\mathrm{R}_\alpha^{\boldsymbol{\Lambda}}(\Gamma_{e,p'})$, and then that $g \circ f$ is the identity of $\mathrm{H}_\alpha^{\boldsymbol{\Lambda}}(q, \boldsymbol{v})$. Hence, the homomorphisms f and g are inverse isomorphisms and Theorem 2.3.16 is proved. Summing the F-isomorphism $\mathrm{H}_\alpha^{\boldsymbol{\Lambda}}(q, \boldsymbol{v}) \simeq \mathrm{R}_\alpha^{\boldsymbol{\Lambda}}(\Gamma_{e,p'})$ over all $\alpha \models_K n$, we thus get the statement of Theorem 2.3.6. Note that since $\mathrm{H}_\alpha^{\boldsymbol{\Lambda}}(q, \boldsymbol{v})$ is zero for all but finitely many α, the same thing happens for $\mathrm{R}_\alpha^{\boldsymbol{\Lambda}}(\Gamma_{e,p'})$. In particular, the direct sum:

$$\mathrm{R}_n^{\boldsymbol{\Lambda}}(\Gamma_{e,p'}) = \bigoplus_{\alpha \models_K n} \mathrm{R}_\alpha^{\boldsymbol{\Lambda}}(\Gamma_{e,p'}),$$

has a finite number of non-vanishing terms.

2.3.4 An unexpected corollary

For any $j \in J'$, let us write $\boldsymbol{\Lambda}^j$ for the restriction of $\boldsymbol{\Lambda}$ to $I \times \{j\} \simeq I$. Since $\Gamma_{e,p'}$ is given by p' disjoint copies of the quiver Γ_e, we know from Theorem 1.3.57 that there is an algebra isomorphism

$$\mathrm{R}_n^{\boldsymbol{\Lambda}}(\Gamma_{e,p'}) \simeq \bigoplus_{\lambda \models_{J'} n} \mathrm{Mat}_{m_\lambda} \left(\mathrm{R}_{\lambda_0}^{\boldsymbol{\Lambda}^0}(\Gamma_e) \otimes \cdots \otimes \mathrm{R}_{\lambda_{p'-1}}^{\boldsymbol{\Lambda}^{p'-1}}(\Gamma_e) \right),$$

where $m_\lambda = \frac{n!}{\lambda_0! \cdots \lambda_{p'-1}!}$. For any $j \in J'$, we set:

$$\mathrm{H}_{\lambda_j}^{\boldsymbol{\Lambda}^j}(q) := \mathrm{H}_{\lambda_j}^{\boldsymbol{\Lambda}^j}(q, \boldsymbol{v}_{\mathrm{triv}}),$$

where $\boldsymbol{v}_{\mathrm{triv}}$ has only one coordinate, equal to 1. In particular, we saw from Theorem 2.3.6 or [BrKl-a] that we have the F-isomorphism $\mathrm{H}_{\lambda_j}^{\boldsymbol{\Lambda}^j}(q) \simeq \mathrm{R}_{\lambda_j}^{\boldsymbol{\Lambda}^j}(\Gamma_e)$. We deduce the following result.

Theorem 2.3.19. *Let $\boldsymbol{v} \in (F^\times)^{p'}$ satisfying the distinct q-orbit condition* (2.3.2)*. We have an (explicit) F-algebra isomorphism*

$$\mathrm{H}_n^{\boldsymbol{\Lambda}}(q, \boldsymbol{v}) \simeq \bigoplus_{\lambda \models_{J'} n} \mathrm{Mat}_{m_\lambda} \left(\mathrm{H}_{\lambda_0}^{\boldsymbol{\Lambda}^0}(q) \otimes \cdots \otimes \mathrm{H}_{\lambda_{p'-1}}^{\boldsymbol{\Lambda}^{p'-1}}(q) \right).$$

In particular, the algebras $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q, \boldsymbol{v})$ and $\oplus_{\lambda \models_{J'} n} \mathrm{H}_{\lambda_0}^{\boldsymbol{\Lambda}^0}(q) \otimes \cdots \otimes \mathrm{H}_{\lambda_{p'-1}}^{\boldsymbol{\Lambda}^{p'-1}}(q)$ are Morita equivalent. Note that since the following condition is satisfied (recall (2.3.2)):

$$\prod_{j < j' \in J'} \prod_{i, i' \in I} \prod_{-n < a < n} \left(q^a (v_j q^i) - v_{j'} q^{i'} \right) \in F^\times,$$

the Morita equivalence is known by [DiMa, Theorem 1.1]. Therefore, Theorem 2.3.19 provides an explicit isomorphism from which the Morita equivalence of [DiMa] follows.

Remark 2.3.20. If $\boldsymbol{\Lambda}^0 = \cdots = \boldsymbol{\Lambda}^{p'-1}$, by [PA, Corollary 3.2] or Chapter 3 we know that the algebra of Theorem 2.3.19 is a cyclotomic Yokonuma–Hecke algebra of type A, as introduced in [ChPA15].

2.4 Restricting the grading

In this section, we prove our second main result, given in Corollary 2.4.17: we give a cyclotomic quiver Hecke-like presentation for $\mathrm{H}^{\mathbf{\Lambda}}_{p,n}(q)$. The key is to make a careful choice for the family $\{Q_a(\boldsymbol{k})\}_{a,\boldsymbol{k}}$.

2.4.1 A nice family

We consider the quiver $\Gamma_{e,p'}$ with vertex set $V = \{v_j q^i\}_{i\in I, j\in J'} \simeq K = I \times J'$ of §2.3.1, where $v_0, \dots, v_{p'-1} \in F^\times$ satisfy (2.3.2). We give here a particular choice for the family $\{Q_a(\boldsymbol{k})\}_{a,\boldsymbol{k}}$. We recall the definition of the family $\{P_a(\boldsymbol{k})\}_{a,\boldsymbol{k}}$ of (2.3.11) and also the property (BK), defined by the conditions (2.3.15).

Lemma 2.4.1 ([StWe, (5.4)]). *The family $\{Q_a(\boldsymbol{k})\}_{1\leq a<n, \boldsymbol{k}\in K^n}$ given by:*

$$Q_a(\boldsymbol{k}) := \begin{cases} 1 - q + qy_{a+1} - y_a, & \text{if } k_a = k_{a+1}, \\ \frac{1-P_a(\boldsymbol{k})}{y_{a+1}-y_a}, & \text{if } k_a \leftarrow k_{a+1} \text{ or } k_a \leftrightarrows k_{a+1}, \\ 1 - P_a(\boldsymbol{k}), & \text{otherwise}, \end{cases}$$

satisfies the property (BK).

Remark 2.4.2. The condition "$k_a \leftarrow k_{a+1}$ or $k_a \leftrightarrows k_{a+1}$" is equivalent to "$v_{k_a} q^{k_a} = qv_{k_{a+1}} q^{k_{a+1}}$". With $\boldsymbol{k} = (\boldsymbol{i}, \boldsymbol{j})$, that means "$i_a = i_{a+1} + 1$ and $j_a = j_{a+1}$".

The family given in [BrKl-a, (4.36)] would be the following one:

$$Q^{\mathrm{BK}}_a(\boldsymbol{k}) := \begin{cases} 1 - q + qy_{a+1} - y_a, & \text{if } k_a = k_{a+1}, \\ (y_a(\boldsymbol{k}) - qy_{a+1}(\boldsymbol{k}))/(y_a(\boldsymbol{k}) - y_{a+1}(\boldsymbol{k})), & \text{if } k_a \not- k_{a+1}, \\ (y_a(\boldsymbol{k}) - qy_{a+1}(\boldsymbol{k}))/(y_a(\boldsymbol{k}) - y_{a+1}(\boldsymbol{k}))^2, & \text{if } k_a \rightarrow k_{a+1}, \\ v_{k_a} q^{k_a}, & \text{if } k_a \leftarrow k_{a+1}, \\ v_{k_a} q^{k_a}/(y_a(\boldsymbol{k}) - y_{a+1}(\boldsymbol{k})), & \text{if } k_a \leftrightarrows k_{a+1}. \end{cases} \tag{2.4.3}$$

We will see in Remark 2.4.16 why the choice of Lemma 2.4.1 is more adapted to our problem. For the convenience of the reader, we will now give a proof of Lemma 2.4.1.

Proof of Lemma 2.4.1. First, let us prove that $Q_a(\boldsymbol{k})$ is well-defined. If $k_a \neq k_{a+1}$ we have $P_a(\boldsymbol{k}) = \frac{(1-q)y_{a+1}(\boldsymbol{k})}{y_{a+1}(\boldsymbol{k})-y_a(\boldsymbol{k})}$, thus:

$$1 - P_a(\boldsymbol{k}) = \frac{qy_{a+1}(\boldsymbol{k}) - y_a(\boldsymbol{k})}{y_{a+1}(\boldsymbol{k}) - y_a(\boldsymbol{k})} = \frac{qv_{k_{a+1}} q^{k_{a+1}}(1 - y_{a+1}) - v_{k_a} q^{k_a}(1 - y_a)}{y_{a+1}(\boldsymbol{k}) - y_a(\boldsymbol{k})}.$$

In particular, if $k_a \leftarrow k_{a+1}$ or $k_a \leftrightarrows k_{a+1}$ we get (recall Remark 2.4.2):

$$1 - P_a(\boldsymbol{k}) = \frac{v_{k_a} q^{k_a}(y_a - y_{a+1})}{y_{a+1}(\boldsymbol{k}) - y_a(\boldsymbol{k})},$$

thus:

$$\frac{1 - P_a(\boldsymbol{k})}{y_{a+1} - y_a} = \frac{v_{k_a} q^{k_a}}{y_a(\boldsymbol{k}) - y_{a+1}(\boldsymbol{k})},$$

which is well-defined.

As suggested in [StWe], we now notice that, if $k_a \neq k_{a+1}$:

$$(1 - P_a(s_a \cdot \boldsymbol{k}))^{s_a} = q + P_a(\boldsymbol{k}). \tag{2.4.4}$$

This is a straightforward consequence of the equality $P_a(\boldsymbol{k}) + P_a(s_a \cdot \boldsymbol{k})^{s_a} = 1 - q$ (see [BrKl-a, (4.28)]). Let us now check that (BK) is satisfied. First, the element $Q_a(\boldsymbol{k})$ is of course invertible when $k_a = k_{a+1}$ (since $1 - q \neq 0$), and the invertibility in the remaining cases follows from the above calculations so (2.3.15a) holds. Moreover, equation (2.3.15b) is true by definition.

We now check the different relations (2.3.15c) involving $Q_a(\boldsymbol{k})Q_a(s_a \cdot \boldsymbol{k})^{s_a}$. If $k_a \not\to k_{a+1}$ (in particular, $k_a \neq k_{a+1}$) then $Q_a(\boldsymbol{k}) = 1 - P_a(\boldsymbol{k})$ and we immediately deduce (2.3.15c-i) from (2.4.4). If $k_a \to k_{a+1}$ then $Q_a(\boldsymbol{k}) = 1 - P_a(\boldsymbol{k})$ and $Q_a(s_a \cdot \boldsymbol{k}) = \frac{1-P_a(s_a \cdot \boldsymbol{k})}{y_{a+1}-y_a}$. Thus:

$$Q_a(\boldsymbol{k})Q_a(s_a \cdot \boldsymbol{k})^{s_a} = (1 - P_a(\boldsymbol{k}))\frac{q + P_a(\boldsymbol{k})}{y_a - y_{a+1}},$$

so (2.3.15c-ii) holds. The proof of (2.3.15c-iii) is similar. Now if $k_a \leftrightarrows k_{a+1}$ then $Q_a(\boldsymbol{k}) = \frac{1-P_a(\boldsymbol{k})}{y_{a+1}-y_a}$ and $Q_a(s_a \cdot \boldsymbol{k}) = \frac{1-P_a(s_a \cdot \boldsymbol{k})}{y_{a+1}-y_a}$, thus:

$$Q_a(\boldsymbol{k})Q_a(s_a \cdot \boldsymbol{k})^{s_a} = \frac{1 - P_a(\boldsymbol{k})}{y_{a+1} - y_a} \cdot \frac{q + P_a(\boldsymbol{k})}{y_a - y_{a+1}},$$

so (2.3.15c-iv) holds.

Finally, to prove equation (2.3.15d) it suffices to see that $P_{a+1}(s_{a+1}s_a \cdot \boldsymbol{k})^{s_a} = P_a(s_a s_{a+1} \cdot \boldsymbol{k})^{s_{a+1}}$. This equality follows from [BrKl-a, (4.29)] and the braid relation $s_a s_{a+1} s_a = s_{a+1} s_a s_{a+1}$. □

Remark 2.4.5. We deduce from the calculations made at the beginning of the proof of Lemma 2.4.1 that $Q_a(\boldsymbol{k}) = Q_a^{\mathrm{BK}}(\boldsymbol{k})$ if $k_a = k_{a+1}$, $k_a \not\to k_{a+1}$ or $k_a \leftrightarrows k_{a+1}$.

2.4.2 Intertwining

In this subsection, we show how our previous works allow us to prove the main result of this chapter, Corollary 2.4.17. For any $j \in J'$, let us set $v_j := \zeta^j$. It follows from the definition of p' that $v_0, \dots, v_{p'-1}$ satisfy the distinct q-orbit condition (2.3.2). In particular, the vertex set of $\Gamma_{e,p'}$ can be identified with $V = \{\zeta^j q^i\}_{i\in I, j\in J'}$. Let us consider a weight $\boldsymbol{\Lambda} = (\Lambda_k)_{k\in K}$ of level r, such that

$$\Lambda_{i,j} = \Lambda_{i,j'} =: \Lambda_i, \qquad \text{for all } i \in I \text{ and } j, j' \in J'. \tag{2.4.6}$$

We suppose that the associated tuple $\boldsymbol{\Lambda} = (\Lambda_i)_{i\in I}$, of level ωd, satisfies the condition of Proposition 2.2.39, that is (recall the notation η of (2.2.29)),

$$\Lambda_i = \Lambda_{i+\eta}, \qquad \text{for all } i \in I, \tag{2.4.7}$$

so that the algebras $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q,\zeta), \mathrm{H}_{p,n}^{\boldsymbol{\Lambda}}(q)$ (recall Definition 2.2.41) and the shift automorphism of $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q,\zeta)$ (recall Proposition 2.2.9) are well-defined. We will use the above condition (2.4.7) and the results of Section 1.4 to define a particular automorphism σ of $\mathrm{R}_n^{\boldsymbol{\Lambda}}(\Gamma_{e,p'})$.

Let us define $\sigma : V \to V$ by:

$$\sigma(v) := \zeta v, \tag{2.4.8}$$

for all $v \in V$. Note that σ is well-defined since V is also given by $\{\zeta^j q^i\}_{i\in I, j\in J}$. Moreover, the reader may have noticed the similarity with the map of Proposition 2.2.9.

Lemma 2.4.9. *The map $\sigma : V \to V$ induces an automorphism of the quiver $\Gamma_{e,p'}$, that is:*

- *the map $\sigma : V \to V$ is a bijection;*
- *if (v, v') is an edge of $\Gamma_{e,p'}$ then $(\sigma(v), \sigma(v'))$ is also an edge of $\Gamma_{e,p'}$.*

In particular, the map σ satisfies the assumptions of Section 1.4.

Proof. The first point is satisfied since ζ is a root of unity. Now let (v, v') be an edge of $\Gamma_{e,p'}$. By definition, we have $v' = qv$, thus $\zeta v' = \zeta(qv) = q(\zeta v)$. Hence, we have $\sigma(v') = q\sigma(v)$, thus we have proved that $(\sigma(v), \sigma(v'))$ is an edge of $\Gamma_{e,p'}$. We conclude recalling Remark 1.4.2. □

The action of σ on V is algebraically easy. Let us now describe how σ acts "graphically" on the set V of the vertices of $\Gamma_{e,p'}$, that is, on $K = I \times J'$. Let $i \in I, j \in J'$ and set $v \coloneqq \zeta^j q^i$. We have:

$$\sigma(\zeta^j q^i) = \zeta^{j+1} q^i. \tag{2.4.10}$$

Hence, if $j < p' - 1$ then σ just translates the vertex v to the copy of Γ_e directly on its right. If $j = p' - 1$, we have $j + 1 = p' \notin J'$ thus we write:

$$\sigma(\zeta^{p'-1} q^i) = \zeta\zeta^{p'-1} q^i = \zeta^{p'} q^i = q^{i+\eta}. \tag{2.4.11}$$

It means that v is translated to the first copy of Γ_e *and* rotated by η. Note that depending on e and p', there may not be any translation or rotation.

Example 2.4.12. Recall the quivers of Example 2.3.4.

Case $(e, p) = (2, 3)$**.** We have $p' = 3, \eta = 0$ and the map σ is given by the product of 3-cycles $(0_1, 0_2, 0_3)(1_1, 1_2, 1_3)$.

Case $(e, p) = (2, 6)$**.** we have $p' = 3, \eta = 1$ and the map σ is given by the 6-cycle $(0_1, 0_2, 0_3, 1_1, 1_2, 1_3)$.

Case $(e, p) = (\infty, 2)$**.** we have $p' = 2, \eta = 0$ and the map σ is given by the product of transpositions $\prod_{i \in I}(i_1, i_2)$.

In particular, note that σ has indeed order p.

By Theorem 1.4.5 and Lemma 1.4.24, the permutation σ of the vertices of V induces an isomorphism $\mathrm{R}^{\mathbf{\Lambda}}_{\alpha}(\Gamma_{e,p'}) \to \mathrm{R}^{\sigma \cdot \mathbf{\Lambda}}_{\sigma \cdot \alpha}(\Gamma_{e,p'})$ for any $\alpha \models_K n$. Let us now check that the weight $\mathbf{\Lambda}$ satisfies the σ-stability condition (1.4.25).

Proposition 2.4.13. *For any $k \in K = I \times J'$ we have $\Lambda_k = \Lambda_{\sigma(k)}$.*

Proof. We have seen above that for $(i, j) \in I \times J'$:

- if $j < p' - 1$ then $\sigma(i, j) = (i, j + 1)$;
- if $j = p' - 1$ then $\sigma(i, j) = (i + \eta, 0)$.

Thus, we deduce the result from (2.4.6) and (2.4.7). □

By Lemma 1.4.28, we know that the map σ induces an automorphism of the cyclotomic quiver Hecke algebra $\mathrm{R}^{\mathbf{\Lambda}}_n(\Gamma_{e,p'})$. We will refer to it as the *shift automorphism* of $\mathrm{R}^{\mathbf{\Lambda}}_n(\Gamma_{e,p'})$.

Lemma 2.4.14. *The power series $y_a(\boldsymbol{k}), P_a(\boldsymbol{k})$ and $Q_a(\boldsymbol{k})$ of $\mathrm{R}^{\mathbf{\Lambda}}_n(\Gamma_{e,p'})$ are shift-invariant. Moreover:*

$$y_a(\sigma(\boldsymbol{k})) = \zeta y_a(\boldsymbol{k}),$$
$$P_a(\sigma(\boldsymbol{k})) = P_a(\boldsymbol{k}),$$
$$Q_a(\sigma(\boldsymbol{k})) = Q_a(\boldsymbol{k}).$$

Proof. The first statement is clear since y_a and y_{a+1} are shift-invariant (by definition, just recall Theorem 1.4.5). Recall that $V = \{\zeta^j q^i\}_{i\in I, j\in J'} \simeq K$ is the vertex set of $\Gamma_{e,p'}$. The image of $\boldsymbol{k} \in K^n$ in V^n is $(\zeta^{k_1}q^{k_1}, \dots, \zeta^{k_n}q^{k_n})$, where $\zeta^k q^k = \zeta^j q^i$ if $k = (i,j)$ (recall (2.3.13)). In particular, the image of $\sigma(\boldsymbol{k})$ in V^n is $(\zeta\zeta^{k_1}q^{k_1}, \dots, \zeta\zeta^{k_n}q^{k_n})$. Thus, we have

$$y_a(\sigma(\boldsymbol{k})) = \zeta\zeta^{k_a}q^{k_a}(1-y_a) = \zeta y_a(\boldsymbol{k}).$$

Hence, if $k_a \neq k_{a+1}$ we have

$$\begin{aligned}
P_a(\sigma(\boldsymbol{k})) &= (1-q)\Big(1 - y_a(\sigma(\boldsymbol{k}))y_{a+1}(\sigma(\boldsymbol{k}))^{-1}\Big)^{-1} \\
&= (1-q)\Big((1 - \zeta\zeta^{-1}y_a(\boldsymbol{k})y_{a+1}(\boldsymbol{k})^{-1}\Big)^{-1} \\
&= P_a(\boldsymbol{k}),
\end{aligned}$$

and this equality is obvious is $k_a = k_{a+1}$. The last equality $Q_a(\sigma(\boldsymbol{k})) = Q_a(\boldsymbol{k})$ is now immediate. □

Let us now denote by $\widetilde{\sigma} : \mathrm{H}_n^{\boldsymbol{\Lambda}}(q,\zeta) \to \mathrm{H}_n^{\boldsymbol{\Lambda}}(q,\zeta)$ the shift automorphism of $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q,\zeta)$ (defined in Proposition 2.2.9). Recalling the choice for $\boldsymbol{v}$ that we made at the beginning of §2.4.2, we consider the F-algebra isomorphism $f : \mathrm{H}_n^{\boldsymbol{\Lambda}}(q,\zeta) \to \mathrm{R}_n^{\boldsymbol{\Lambda}}(\Gamma_{e,p'})$ from Theorems 2.3.6 and 2.3.16, defined with the family $\{Q_a(\boldsymbol{k})\}_{a,\boldsymbol{k}}$ of Lemma 2.4.1.

Theorem 2.4.15. *We have $\sigma^{-1} \circ f = f \circ \widetilde{\sigma}$.*

Proof. Since we deal with algebra homomorphisms, it suffices to check the equality on the generators $S, T_1, \dots, T_{n-1}$ of $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q,\zeta)$. We successively have, using Lemma 2.4.14 (recall that, by definition, $S = X_1$):

$$\begin{aligned}
\sigma^{-1} \circ f(S) &= \sum_{\boldsymbol{k}\in K^n} \sigma^{-1}(y_1(\boldsymbol{k})e(\boldsymbol{k})) \\
&= \sum_{\boldsymbol{k}\in K^n} y_1(\boldsymbol{k})e(\sigma^{-1}(\boldsymbol{k})) \\
&= \sum_{\boldsymbol{k}\in K^n} y_1(\sigma(\boldsymbol{k}))e(\boldsymbol{k}) \\
&= \zeta f(S) \\
&= f(\zeta S) \\
&= f \circ \widetilde{\sigma}(S),
\end{aligned}$$

and:

$$\begin{aligned}
\sigma^{-1} \circ f(T_a) &= \sum_{\boldsymbol{k}\in K^n} \sigma^{-1}\Big([\psi_a Q_a(\boldsymbol{k}) - P_a(\boldsymbol{k})]e(\boldsymbol{k})\Big) \\
&= \sum_{\boldsymbol{k}\in K^n} [\psi_a Q_a(\boldsymbol{k}) - P_a(\boldsymbol{k})]e(\sigma^{-1}(\boldsymbol{k})) \\
&= \sum_{\boldsymbol{k}\in K^n} [\psi_a Q_a(\sigma(\boldsymbol{k})) - P_a(\sigma(\boldsymbol{k}))]e(\boldsymbol{k}) \\
&= \sum_{\boldsymbol{k}\in K^n} [\psi_a Q_a(\boldsymbol{k}) - P_a(\boldsymbol{k})]e(\boldsymbol{k}) \\
&= f(T_a) \\
&= f \circ \widetilde{\sigma}(T_a).
\end{aligned}$$

Note that the above sums over K^n are in fact finite, since all but finitely many elements $e(\boldsymbol{k}) \in \mathrm{R}_n^{\boldsymbol{\Lambda}}(\Gamma_{e,p'})$ are zero (recall, for instance, §2.3.3.3). □

Remark 2.4.16. Theorem 2.4.15 fails if we consider the homomorphism f built from the family $\{Q_a^{\mathrm{BK}}(\boldsymbol{k})\}_{a,\boldsymbol{k}}$ of (2.4.3). For instance, Lemma 2.4.14 is no longer valid with $Q_a^{\mathrm{BK}}(\boldsymbol{k})$, since if $k_a \leftarrow k_{a+1}$:

$$Q_a^{\mathrm{BK}}(\sigma(\boldsymbol{k})) = \zeta\zeta^{k_a}q^{k_a} = \zeta Q_a^{\mathrm{BK}}(\boldsymbol{k}),$$

and the same result holds if $k_a \to k_{a+1}$.

We now recall some notation and facts from Section 1.4. If $\alpha \vDash_K n$, we denote by $[\alpha]$ its orbit under the action of $\langle\sigma\rangle$ (this action is defined in Lemma 1.4.4). We have an associated subset $K^{[\alpha]} = \sqcup_{\beta\in[\alpha]} K^\beta$ of K^n (see (1.4.7)). The quotient set of $K^{[\alpha]}$ by the equivalence relation $\sim$ generated by $\boldsymbol{k} \sim \sigma(\boldsymbol{k})$ for all $\boldsymbol{k} \in K^{[\alpha]}$ is $K_\sigma^{[\alpha]}$ (Definition 1.4.12). Here, each equivalence class $\gamma \in K_\sigma^{[\alpha]}$ has cardinality $o_\gamma = p$, and is given by $\gamma = \{\boldsymbol{k}, \sigma(\boldsymbol{k}), \dots, \sigma^{p-1}(\boldsymbol{k})\}$ for any $\boldsymbol{k} \in \gamma$. Finally, thanks to the canonical map $K^n/\!\sim\, \to (K/\!\sim)^n$ and Lemma 2.4.9, for any $\gamma \in K_\sigma^{[\alpha]}$ and $a \in \{1, \dots, n\}$ we have well-defined statements $\gamma_a = \gamma_{a+1}, \gamma_a \to \gamma_{a+1}$, etc. (see Remark 1.4.2 and before Remark 1.4.17). Moreover, since $\boldsymbol{\Lambda}$ is σ-stable (Proposition 2.4.13) the integer Λ_{γ_a} is well-defined.

Corollary 2.4.17. *The F-algebra isomorphism $f : \mathrm{H}_n^{\boldsymbol{\Lambda}}(q,\zeta) \to \mathrm{R}_n^{\boldsymbol{\Lambda}}(\Gamma_{e,p'})$ induces an isomorphism between $\mathrm{H}_{p,n}^{\boldsymbol{\Lambda}}(q)$ and $\mathrm{R}_n^{\boldsymbol{\Lambda}}(\Gamma_{e,p'})^\sigma$. Hence, we have the following F-algebra isomorphism:*

$$\mathrm{H}_{p,n}^{\boldsymbol{\Lambda}}(q) \simeq \bigoplus_{[\alpha]} \mathrm{R}_{[\alpha]}^{\boldsymbol{\Lambda}}(\Gamma_{e,p'})^\sigma,$$

where $[\alpha]$ runs over the orbits of the K-compositions of n under the action of $\langle\sigma\rangle$, and the subalgebra $\mathrm{H}_{p,[\alpha]}^{\boldsymbol{\Lambda}}(q)$ has a presentation given by the generators

$$\{e(\gamma)\}_{\gamma\in K_\sigma^{[\alpha]}} \cup \{y_1, \dots, y_n\} \cup \{\psi_1, \dots, \psi_{n-1}\},$$

and the relations (1.4.20), (1.4.22) and

$$y_1^{\gamma_1} e(\gamma) = 0,$$

for all $\gamma \in K_\sigma^{[\alpha]}$.

Proof. Using Theorem 2.4.15, for any $h \in \mathrm{H}_n^{\boldsymbol{\Lambda}}(q,\zeta)$ we have:

$$\widetilde{\sigma}(h) = h \iff f\circ\widetilde{\sigma}(h) = f(h) \iff \sigma^{-1}\circ f(h) = f(h) \iff f(h) = \sigma\circ f(h),$$

hence:

$$h \text{ is fixed under } \widetilde{\sigma} \iff f(h) \text{ is fixed under } \sigma.$$

Using Corollary 2.2.19, we get:

$$\mathrm{H}_{p,n}^{\boldsymbol{\Lambda}}(q) \simeq \mathrm{H}_n^{\boldsymbol{\Lambda}}(q,\zeta)^{\widetilde{\sigma}} \simeq \mathrm{R}_n^{\boldsymbol{\Lambda}}(\Gamma_{e,p'})^\sigma,$$

as desired. We deduce the second statement from the equality $\mathrm{R}_n^{\boldsymbol{\Lambda}}(\Gamma_{e,p'})^\sigma = \oplus_{[\alpha]}\mathrm{R}_{[\alpha]}^{\boldsymbol{\Lambda}}(\Gamma_{e,p'})^\sigma$ (note that this direct sum is finite by Theorem 2.3.16) and Theorem 1.4.36, where we gave a presentation for $\mathrm{R}_{[\alpha]}^{\boldsymbol{\Lambda}}(\Gamma_{e,p'})^\sigma$. ∎

Recall from Remark 1.4.37 that $\mathrm{R}_n^{\boldsymbol{\Lambda}}(\Gamma_{e,p'})$ is naturally $\mathbb{Z}$-graded. From this grading, Theorem 2.3.6 and the isomorphism f, we can endow $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q,\zeta)$ with a (non-trivial) $\mathbb{Z}$-grading.

Corollary 2.4.18. *The shift automorphism $\widetilde{\sigma} : \mathrm{H}_n^{\boldsymbol{\Lambda}}(q,\zeta) \to \mathrm{H}_n^{\boldsymbol{\Lambda}}(q,\zeta)$ is homogeneous with respect to the previous grading and the subalgebra $\mathrm{H}_{p,n}^{\boldsymbol{\Lambda}}(q)$ is a graded subalgebra of $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q,\zeta)$.*

Proof. Recall from Remark 1.4.37 that $\sigma : \mathrm{R}_n^{\mathbf{\Lambda}}(\Gamma_{e,p'}) \to \mathrm{R}_n^{\mathbf{\Lambda}}(\Gamma_{e,p'})$ is homegeneous and that $\mathrm{R}_n^{\mathbf{\Lambda}}(\Gamma_{e,p'})^\sigma$ is a graded subalgebra. We thus deduce the first assertion from Theorem 2.4.15 and the second one from Corollary 2.4.17. □

We now give an analogue of a classical corollary of [BrKl-a, Theorem 1.1].

Corollary 2.4.19. *If $\tilde{q} \in F \setminus \{0, 1\}$ has the same order $e \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ as q then*

$$\mathrm{H}_{p,n}^{\mathbf{\Lambda}}(\tilde{q}) \simeq \mathrm{H}_{p,n}^{\mathbf{\Lambda}}(q),$$

as F-algebras.

Proof. We know from Lemma 2.2.27 and Theorem 2.3.6 that the algebras $\mathrm{H}_n^{\mathbf{\Lambda}}(q)$ and $\mathrm{H}_n^{\mathbf{\Lambda}}(\tilde{q})$ are isomorphic to the same cyclotomic quiver Hecke algebra $\mathrm{R}_n^{\mathbf{\Lambda}}(\Gamma_{e,p'})$. Moreover, we have the following isomorphism:

$$\mathrm{H}_{p,n}^{\mathbf{\Lambda}}(q) \simeq \mathrm{R}_n^{\mathbf{\Lambda}}(\Gamma_{e,p'})^\sigma,$$

where σ is uniquely determined by the quiver $\Gamma_{e,p'}$ and the element $\eta \in I$ such that $q^\eta = \zeta^{p'}$. To prove that $\mathrm{H}_{p,n}^{\mathbf{\Lambda}}(\tilde{q}) \simeq \mathrm{H}_{p,n}^{\mathbf{\Lambda}}(q)$, it thus suffices to prove that there is a primitive pth root of unity $\tilde{\zeta} \in F^\times$ such that:

$$\tilde{q}^\eta = \tilde{\zeta}^{p'}$$

(recall from Lemma 2.2.27 that p' does not depend on the chosen primitive pth root of unity). To deal with the case $e = \infty$, it suffices to set $\tilde{\zeta} \coloneqq \zeta$. Recall that, in that case, we have $\eta = 0$ and $p' = p$. Hence, we now assume that $e < \infty$. Since q and $\tilde{q}$ are both primitive eth roots of unity, there is some $a \in \mathbb{Z}$, invertible modulo e, such that $\tilde{q} = q^a$. In particular, for any $k \in \mathbb{Z}$ we have $\tilde{q} = q^{a+ke}$. Since $q^\eta = \zeta^{p'}$, we get:

$$\tilde{q}^\eta = (\zeta^{a+ke})^{p'}.$$

Therefore, it suffices to prove that there is some $k \in \mathbb{Z}$ such that $a + ke$ is invertible modulo p, that is, such that $\tilde{\zeta} \coloneqq \zeta^{a+ke}$ is a primitive pth root of unity. A quick (but very powerful) argument is to use Dirichlet's theorem about arithmetic progression (see also [Hu07, Lemma 3.5]): since a and e are coprime, the set $\{a + ke\}_{k \in \mathbb{N}}$ contains infinitely many prime numbers. In particular, it contains a prime $\wp$ which does not divide p, hence which is coprime to p. It now suffices to choose $k \in \mathbb{N}$ such that $\wp = a + ke$. □

Chapter 3

Cyclotomic Yokonuma–Hecke algebras

This chapter is adapted from [Ro17-a].

3.1 Overview

In this chapter, we prove that cyclotomic Yokonuma–Hecke algebras of type A are cyclotomic quiver Hecke algebras and we give an explicit isomorphism together with its inverse. As in Chapter 2, we use the isomorphism of Brundan and Kleshchev [BrKl-a] and the quiver is given by disjoint copies of the cyclic quiver Γ_e. However, the generalisation is now not straightforward. Finally, we relate this work to an isomorphism of Lusztig involving Yokonuma–Hecke and tensor products of Iwahori–Hecke algebras of type A.

We now give a brief overview of the chapter. Given $d \in \mathbb{N}^*$, we first define in Section 3.2 the cyclotomic Yokonuma–Hecke algebra $\mathrm{Y}^{\mathbf{\Lambda}}_{d,n}(q)$ where $q \in F \setminus \{0, 1\}$ has order $e \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ in $F^\times$ and $\mathbf{\Lambda}$ is a weight. We begin Section 3.3 by considering in $\mathrm{Y}^{\mathbf{\Lambda}}_{d,n}(q)$ a natural system $\{e(\alpha)\}_{\alpha \models_{ed} n}$ of pairwise orthogonal central idempotents. Then, as in [BrKl-a] (and Chapter 2) we define some "quiver Hecke generators", now for $\mathrm{Y}^{\mathbf{\Lambda}}_\alpha(q) := e(\alpha)\mathrm{Y}^{\mathbf{\Lambda}}_{d,n}(q)$, and we check that they satisfy the defining relations of the quiver Hecke algebra $\mathrm{R}^{\mathbf{\Lambda}}_\alpha(\Gamma_{e,d})$ (see Theorem 3.3.1), where $\Gamma_{e,d}$ is the quiver given by d disjoint copies of the cyclic quiver Γ_e with e vertices defined in §2.3.1. In Section 3.4 we define the "Yokonuma–Hecke generators" of $\mathrm{R}^{\mathbf{\Lambda}}_\alpha(\Gamma_{e,d})$ and again check the corresponding defining relations, see Theorem 3.4.1. We can now prove in Section 3.5 the main theorem of the chapter, Theorem 3.5.1: we have an F-isomorphism of algebras

$$\mathrm{Y}^{\mathbf{\Lambda}}_\alpha(q) \simeq \mathrm{R}^{\mathbf{\Lambda}}_\alpha(\Gamma_{e,d}).$$

Indeed, we prove that we have defined inverse algebra homomorphisms. Note that we also have an F-isomorphism $\mathrm{Y}^{\mathbf{\Lambda}}_{d,n}(q) \simeq \mathrm{R}^{\mathbf{\Lambda}}_n(\Gamma_{e,d})$, see (3.5.2). We justify in Section 3.6 that the isomorphism of Theorem 3.5.1 remains true for the *degenerate* cyclotomic Yokonuma–Hecke algebra $\mathrm{Y}^{\mathbf{\Lambda}}_{d,n}(1)$ that we define in §3.6.1, see Theorem 3.6.20. We end the section with Corollary 3.6.22, which states that, under certain conditions, the algebras $\mathrm{Y}^{\mathbf{\Lambda}}_{d,n}(q)$ and $\mathrm{Y}^{\mathbf{\Lambda}}_{d,n}(1)$ are isomorphic. We end the chapter with Theorem 3.7.3, where we show that we recover the isomorphism $\mathrm{Y}^{\mathbf{\Lambda}}_{d,n}(q) \simeq \oplus_{\lambda \models_d n} \mathrm{Mat}_{m_\lambda} \mathrm{H}^{\mathbf{\Lambda}}_\lambda(q)$ of [Lu], where $m_\lambda := \frac{n!}{\lambda_1! \cdots \lambda_d!}$ and the algebra $\mathrm{H}^{\mathbf{\Lambda}}_\lambda(q)$ is a tensor product of Ariki–Koike algebras. More precisely, we recover the explicit form given in [JacPA, PA], see Theorem 3.7.10.

3.2 Setting

Let $d \in \mathbb{N}^*$ and assume that the field F contains a primitive dth root of unity ξ. In particular, the characteristic of F does not divide d. Recall that $q \in F^\times$ and $e \in \mathbb{N}^* \cup \{\infty\}$ is minimal such that $1 + q + \cdots + q^{e-1} = 0$. Except in Section 3.6, the element q will always be taken different from 1. Recall that $I = \mathbb{Z}/e\mathbb{Z}$ (with $I = \mathbb{Z}$ if $e = \infty$) and set $J := \mathbb{Z}/d\mathbb{Z} \simeq \{1, \dots, d\}$. Unless mentioned otherwise, we have $K := I \times J$. We will use the *quantum characteristic* of F, given by

$$\operatorname{char}_q(F) := \begin{cases} e, & \text{if } e < \infty, \\ 0, & \text{if } e = \infty. \end{cases}$$

In particular, we have $I = \mathbb{Z}/\operatorname{char}_q(F)\mathbb{Z}$ and $\operatorname{char}_1(F)$ is exactly the usual characteristic of F.

3.2.1 Cyclotomic Yokonuma–Hecke algebras

Let $\mathbf{\Lambda} = (\Lambda_i)_{i \in I} \in \mathbb{N}^{(I)}$ be a weight with $\ell(\mathbf{\Lambda}) = \sum_{i \in I} \Lambda_i > 0$. The *cyclotomic Yokonuma–Hecke algebra of type A*, denoted by $\mathrm{Y}^{\mathbf{\Lambda}}_{d,n}(q)$, is the unitary associative F-algebra generated by the elements

$$g_1, \dots, g_{n-1}, t_1, \dots, t_n, X_1, \tag{3.2.1}$$

subject to the following relations:

$$\begin{aligned}
t_a^d &= 1, && && (3.2.2a)\\
t_a t_{a'} &= t_{a'} t_a, && && (3.2.2b)\\
t_a g_b &= g_b t_{s_b(a)}, && && (3.2.2c)\\
g_b^2 &= q + (q-1) g_b e_b, && && (3.2.2d)\\
g_b g_{b'} &= g_{b'} g_b, && \text{if } |b - b'| > 1, && (3.2.2e)\\
g_{c+1} g_c g_{c+1} &= g_c g_{c+1} g_c, && && (3.2.2f)\\
X_1 g_1 X_1 g_1 &= g_1 X_1 g_1 X_1, && && (3.2.2g)\\
X_1 g_b &= g_b X_1, && \text{if } b > 1, && (3.2.2h)\\
X_1 t_a &= t_a X_1, && && (3.2.2i)\\
\prod_{i \in I} (X_1 - q^i)^{\Lambda_i} &= 0. && && (3.2.2j)
\end{aligned}$$

for all $a, a' \in \{1, \dots, n\}$, $b, b' \in \{1, \dots, n-1\}$ and $c \in \{1, \dots, n-2\}$, where $e_b := \frac{1}{d} \sum_{j \in J} t_b^j t_{b+1}^{-j}$. Note that the presentation comes from [ChPA15], excepting the normalisation in (3.2.2d) which was used in [ChPou]. In particular, it comes from (3.2.2j) that X_1 is invertible in $\mathrm{Y}^{\mathbf{\Lambda}}_{d,n}(q)$. When $d = 1$, the algebra $\mathrm{Y}^{\mathbf{\Lambda}}_{1,n}(q)$ is the Ariki–Koike algebra $\mathrm{H}^{\mathbf{\Lambda}}_n(q) := \mathrm{H}^{\mathbf{\Lambda}}_n(q, 1)$, defined in Chapter 2 and used in [BrKl-a], hence is a deformation of $F[G(r, 1, n)]$ where $r := \ell(\mathbf{\Lambda})$. In this case, the element e_a becomes 1. We write $g_a^{(1)}$ (respectively $X_1^{(1)}$) for the element g_a (resp. X_1) when $d = 1$, that is, considered in $\mathrm{H}^{\mathbf{\Lambda}}_n(q)$. Finally, note that when $r = 1$ then $\mathrm{Y}_{d,n}(q) := \mathrm{Y}^{\mathbf{\Lambda}}_{d,n}(q)$ is a deformation of $F[G(d, 1, n)]$.

Following [ChPA15], we define inductively X_{a+1} for any $a \in \{1, \dots, n-1\}$ by

$$qX_{a+1} := g_a X_a g_a \tag{3.2.3}$$

(note that the q comes from our different normalisation in (3.2.2d)). As for X_1, we introduce the notation $X_a^{(1)}$ to denote X_a in the case $d = 1$. The family $\{t_1, \dots, t_n, X_1, \dots, X_n\}$ is commutative and we have the following equalities:

$$
\begin{aligned}
g_b X_a &= X_a g_b, && \text{if } a \neq b, b+1, && (3.2.4a)\\
g_b X_{b+1} &= X_b g_b + (q-1) X_{b+1} e_b, && && (3.2.4b)\\
X_{b+1} g_b &= g_b X_b + (q-1) X_{b+1} e_b, && && (3.2.4c)
\end{aligned}
$$

for all $a \in \{1, \dots, n\}$ and $b \in \{1, \dots, n-1\}$.

The proof of the following result is the same as in [ChPA15, Proposition 4.7], where we write $g_w \coloneqq g_{a_1} \cdots g_{a_r}$ for a reduced expression $w = s_{a_1} \cdots s_{a_r} \in \mathfrak{S}_n$. By Matsumoto's theorem (see for instance [GePf, Theorem 1.2.2]) the value of g_w does not depend on the choice of the reduced expression, since the generators g_a satisfy the same braid relations as the elements $s_a \in \mathfrak{S}_n$.

Proposition 3.2.5. *The algebra $\mathrm{Y}^{\mathbf{\Lambda}}_{d,n}(q)$ is a finite-dimensional F-vector space and a generating family is given by the elements $g_w X_1^{u_1} \cdots X_n^{u_n} t_1^{v_1} \cdots t_n^{v_n}$ for all $w \in \mathfrak{S}_n$, $u_a \in \{0, \dots, \ell(\mathbf{\Lambda}) - 1\}$ and $v_a \in J$.*

Note that the above family is actually an F-basis of $\mathrm{Y}^{\mathbf{\Lambda}}_{d,n}(q)$, although here we will never make use of this fact (see [ChPA15, Theorem 4.15]).

3.2.2 The quiver

Recall from §2.3.1 that the quiver $\Gamma_{e,d}$ is given by d disjoint copies of the cyclic quiver Γ_e with e vertices. The quiver $\Gamma_{e,d}$ is described in the following way:

- the vertices are the elements of $K \coloneqq I \times J$;
- for each $(i, j) \in K$ there is a directed edge from (i, j) to $(i + 1, j)$.

In particular, there is an arrow between (i, j) and (i', j') in $\Gamma_{e,d}$ if and only if there is an arrow between i and i' in Γ_e and $j = j'$. Moreover, the set K is finite if and only if e is finite. We consider the diagonal action of $\mathfrak{S}_n$ on $K^n \simeq I^n \times J^n$, that is, $\sigma \cdot (\boldsymbol{i}, \boldsymbol{j}) \coloneqq (\sigma \cdot \boldsymbol{i}, \sigma \cdot \boldsymbol{j})$. We will need the following notation:

$$
\begin{aligned}
I^\alpha &\coloneqq \{\boldsymbol{i} \in I^n : \text{there exists } \boldsymbol{j} \in J^n \text{ such that } (\boldsymbol{i}, \boldsymbol{j}) \in K^\alpha\},\\
J^\alpha &\coloneqq \{\boldsymbol{j} \in J^n : \text{there exists } \boldsymbol{i} \in I^n \text{ such that } (\boldsymbol{i}, \boldsymbol{j}) \in K^\alpha\}.
\end{aligned}
$$

The sets I^α and J^α are finite and stable under the action of $\mathfrak{S}_n$. Note that K^α is included in $I^\alpha \times J^\alpha$ (we don't have the equality in general).

Finally, we extend the weight $\mathbf{\Lambda} \in \mathbb{N}^{(I)}$ of §3.2.1 to an element of $\mathbb{N}^{(K)}$, which we still denote by $\mathbf{\Lambda}$, by defining

$$
\Lambda_{i,j} \coloneqq \Lambda_i,
$$

for all $(i, j) \in K = I \times J$. We will consider the cyclotomic quiver Hecke algebra $\mathrm{R}^{\mathbf{\Lambda}}_\alpha(\Gamma_{e,d})$, as defined in Chapter 1.

Remark 3.2.6. The *cyclotomic Khovanov–Lauda algebra* of [BrKl-a] is the quiver Hecke algebra $\mathrm{R}^{\mathbf{\Lambda}}_\alpha(\Gamma_e)$, that is, the algebra $\mathrm{R}^{\mathbf{\Lambda}}_\alpha(\Gamma_{e,1})$. We write $e^{(1)}(\boldsymbol{i})$, $y_a^{(1)}$ and $\psi_a^{(1)}$ for the generators of $\mathrm{R}^{\mathbf{\Lambda}}_\alpha(\Gamma_e)$. The reason for this notation will appear in §3.3.1.

3.3 Quiver Hecke generators of $\mathrm{Y}^{\mathbf{\Lambda}}_{\alpha}(q)$

In this section, our first task is to define some central idempotents $e(\alpha) \in \mathrm{Y}^{\mathbf{\Lambda}}_{d,n}(q)$ for any $\alpha \models_K n$, with $\sum_\alpha e(\alpha) = 1$. We will then prove the following theorem.

Theorem 3.3.1. *For any $\alpha \models_K n$, we can construct an explicit algebra homomorphism*

$$\rho_{\mathrm{RY}} : \mathrm{R}^{\mathbf{\Lambda}}_{\alpha}(\Gamma_{e,d}) \to \mathrm{Y}^{\mathbf{\Lambda}}_{\alpha}(q),$$

where $\mathrm{Y}^{\mathbf{\Lambda}}_{\alpha}(q) \coloneqq e(\alpha)\mathrm{Y}^{\mathbf{\Lambda}}_{d,n}(q)$.

Note that $\mathrm{Y}^{\mathbf{\Lambda}}_{\alpha}(q)$ is a unitary algebra (if not reduced to $\{0\}$), with unit $e(\alpha)$. To define this algebra homomorphism, it suffices to define the images of the generators (1.2.2) and check that they satisfy the defining relations of the cyclotomic quiver Hecke algebra: the same strategy was used by Brundan and Kleshchev in [BrKl-a] for $d = 1$.

For a generator g of $\mathrm{R}^{\mathbf{\Lambda}}_{\alpha}(\Gamma_{e,d})$, we will use as well the notation g for the corresponding element that we will define in $\mathrm{Y}^{\mathbf{\Lambda}}_{\alpha}(q)$. There will be no possible confusion since we will work with elements of $\mathrm{Y}^{\mathbf{\Lambda}}_{\alpha}(q)$.

3.3.1 Definition of the images of the generators

We define now our different "quiver Hecke generators".

3.3.1.1 Image of $e(\boldsymbol{i}, \boldsymbol{j})$

Let M be a finite-dimensional $\mathrm{Y}^{\mathbf{\Lambda}}_{d,n}(q)$-module. Each X_a acts on M as an endomorphism of the finite-dimensional F-vector space (see Proposition 3.2.5). In particular, by (3.2.2j) the eigenvalues of X_1 can be written q^i for $i \in I$. Hence, applying [CuWa, Lemma 5.2] we know that the eigenvalues of each X_a are of the form q^i for $i \in I$. Concerning the t_a, by (3.2.2a) (they are diagonalisable and) their eigenvalues are dth roots of unity.

As the elements of the family $\{X_a, t_a\}_{1 \leq a \leq n}$ pairwise commute, using Cayley–Hamilton theorem we can write M as the direct sum of its *weight spaces* (simultaneous generalised eigenspaces)

$$M(\boldsymbol{i}, \boldsymbol{j}) \coloneqq \left\{ v \in M : \left(X_a - q^{i_a}\right)^N v = \left(t_a - \xi^{j_a}\right) v = 0 \text{ for all } 1 \leq a \leq n \right\} \tag{3.3.2}$$

for $(\boldsymbol{i}, \boldsymbol{j}) \in I^n \times J^n$, where $N \gg 0$ and ξ is the given primitive dth root of unity in F that we considered at the very beginning of Section 3.2. Observe that some $M(\boldsymbol{i}, \boldsymbol{j})$ may be reduced to zero; in fact, only a finite number of them are non-zero.

Remark 3.3.3. The element e_a acts on $M(\boldsymbol{i}, \boldsymbol{j})$ as 0 if $j_a \neq j_{a+1}$ and as 1 if $j_a = j_{a+1}$.

We can now consider the family of projections $\{e(\boldsymbol{k})\}_{\boldsymbol{k} \in K^n}$ associated with the decomposition $M = \oplus_{\boldsymbol{k} \in K^n} M(\boldsymbol{k})$. The element $e(\boldsymbol{k})$ is the projection onto $M(\boldsymbol{k})$ along $\oplus_{\boldsymbol{k}' \neq \boldsymbol{k}} M(\boldsymbol{k}')$, in particular $e(\boldsymbol{k})^2 = 0$ and if $\boldsymbol{k} \neq \boldsymbol{k}'$ then $e(\boldsymbol{k})e(\boldsymbol{k}') = 0$. Moreover, only a finite number of projections $e(\boldsymbol{k})$ are non-zero.

As the $e(\boldsymbol{k})$ are polynomials in $X_1, \dots, X_n, t_1, \dots t_n$ (in fact $e(\boldsymbol{k})$ is the product of commuting projections onto the corresponding generalised eigenspaces of X_a and t_a), they belong to $\mathrm{Y}^{\mathbf{\Lambda}}_{d,n}(q)$.

Remark 3.3.4. The above polynomials do not depend on the finite-dimensional $\mathrm{Y}^{\mathbf{\Lambda}}_{d,n}(q)$-module M.

We are now able to define our central idempotents. We set, for any $\alpha \models_K n$,

$$e(\alpha) := \sum_{\boldsymbol{k}\in K^\alpha} e(\boldsymbol{k}).$$

Since K^α is a $\mathfrak{S}_n$-orbit, the element $e(\alpha)$ is indeed central. Though we will not use this fact, we can notice that according to [PA, Corollary 3.2] and [LyMa], the subalgebras $\mathrm{Y}^{\boldsymbol{\Lambda}}_\alpha(q) := e(\alpha)\mathrm{Y}^{\boldsymbol{\Lambda}}_{d,n}(q)$ which are not reduced to zero are the *blocks* of the Yokonuma–Hecke algebra $\mathrm{Y}^{\boldsymbol{\Lambda}}_{d,n}(q)$; see also [CuWa, §6.3]. For $d = 1$, the element $e(\alpha)$ is the element e_α of [BrKl-a, (1.3)].

We will sometimes need the following elements:

$$\begin{aligned} e(\alpha)(\boldsymbol{i}) &:= \sum_{\boldsymbol{j}\in J^\alpha} e(\alpha)e(\boldsymbol{i},\boldsymbol{j}) = \sum_{\substack{\boldsymbol{j}\in J^\alpha\\(\boldsymbol{i},\boldsymbol{j})\in K^\alpha}} e(\boldsymbol{i},\boldsymbol{j}),\\ e(\alpha)(\boldsymbol{j}) &:= \sum_{\boldsymbol{i}\in I^\alpha} e(\alpha)e(\boldsymbol{i},\boldsymbol{j}) = \sum_{\substack{\boldsymbol{i}\in I^\alpha\\(\boldsymbol{i},\boldsymbol{j})\in K^\alpha}} e(\boldsymbol{i},\boldsymbol{j}). \end{aligned} \tag{3.3.5}$$

For $d = 1$, we recover with $e(\alpha)(\boldsymbol{i})$ the element $e(\boldsymbol{i})$ of [BrKl-a, §4.1]; we denote it by $e^{(1)}(\boldsymbol{i})$. Finally, note that:

$$e(\alpha)(\boldsymbol{i}) \cdot e(\alpha)(\boldsymbol{j}) = e(\alpha)(\boldsymbol{j}) \cdot e(\alpha)(\boldsymbol{i}) = e(\boldsymbol{i},\boldsymbol{j}).$$

From now, unless mentioned otherwise, we always work in $\mathrm{Y}^{\boldsymbol{\Lambda}}_\alpha(q)$. Every relation should be multiplied by $e(\alpha)$ and we write $e(\boldsymbol{i})$ (respectively $e(\boldsymbol{j})$) for $e(\alpha)(\boldsymbol{i})$ (resp. $e(\alpha)(\boldsymbol{j})$).

We give now a few useful lemmas.

Lemma 3.3.6. *Let $a \in \{1,\dots,n-1\}$ and $\boldsymbol{j} \in J^\alpha$. If $j_a \neq j_{a+1}$ then*

$$\begin{aligned} g_a^2 e(\boldsymbol{j}) &= q e(\boldsymbol{j}),\\ g_a X_{a+1} e(\boldsymbol{j}) &= X_a g_a e(\boldsymbol{j}),\\ X_{a+1} g_a e(\boldsymbol{j}) &= g_a X_a e(\boldsymbol{j}). \end{aligned}$$

Proof. We deduce it the from relations (3.2.2d), (3.2.4b), (3.2.4c) and from $e_a e(\boldsymbol{j}) = 0$ (since $j_a \neq j_{a+1}$, see Remark 3.3.3). □

For the next lemma, we should compare with [JacPA, (15)].

Lemma 3.3.7. *For any $a \in \{1,\dots,n-1\}$ and $\boldsymbol{j} \in J^\alpha$ we have $g_a e(\boldsymbol{j}) = e(s_a \cdot \boldsymbol{j})g_a$.*

Proof. Let $M := \mathrm{Y}^{\boldsymbol{\Lambda}}_\alpha(q)$. Given the relation (3.2.2c), we see that g_a maps $M(\boldsymbol{j})$ to $M(s_a \cdot \boldsymbol{j})$. Fix now $\boldsymbol{j} \in J^\alpha$ and let $\boldsymbol{j}' \in J^\alpha$ and $v \in M(\boldsymbol{j}')$. If $\boldsymbol{j}' = \boldsymbol{j}$ then we obtain

$$g_a e(\boldsymbol{j})v = e(s_a \cdot \boldsymbol{j})g_a v \ (= g_a v),$$

and if $\boldsymbol{j}' \neq \boldsymbol{j}$, since $g_a v \in M(s_a \cdot \boldsymbol{j}')$ we have

$$g_a e(\boldsymbol{j})v = e(s_a \cdot \boldsymbol{j})g_a v \ (= 0).$$

Hence, $e(s_a \cdot \boldsymbol{j})g_a$ and $g_a e(\boldsymbol{j})$ coincide on each $M(\boldsymbol{j}')$ for $\boldsymbol{j}' \in J^\alpha$ thus coincide on $M = \oplus_{\boldsymbol{j}'} M(\boldsymbol{j}')$ and we conclude since $M = \mathrm{Y}^{\boldsymbol{\Lambda}}_\alpha(q)$ is a unitary algebra. □

Corollary 3.3.8. *Let $a \in \{1,\dots,n\}$ and $\boldsymbol{j} \in J^\alpha$. If $j_a = j_{a+1}$ then g_a and $e(\boldsymbol{j})$ commute.*

Remark 3.3.9 (About Brundan and Kleshchev's proof - I). Let $a \in \{1, \dots, n-1\}$. If $\boldsymbol{j} \in J^\alpha$ satisfies $j_a = j_{a+1}$, the following relations are satisfied in the unitary algebra $e(\boldsymbol{j})\mathrm{Y}_\alpha^{\boldsymbol{\Lambda}}(q)e(\boldsymbol{j})$ between $g_a e(\boldsymbol{j})$ and the elements $X_b e(\boldsymbol{j})$ for any $b \in \{1, \dots, n\}$ with $b \neq a, a+1$:

$$\begin{aligned}
g_a^2 &= q + (q-1)g_a, \\
qX_{a+1} &= g_a X_a g_a, \\
g_a X_b &= X_b g_a, \\
g_a X_{a+1} &= X_a g_a + (q-1)X_{a+1}, \\
X_{a+1} g_a &= g_a X_a + (q-1)X_{a+1}.
\end{aligned}$$

These are exactly the relations (3.2.2d), (3.2.3) and (3.2.4) for $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q) = \mathrm{Y}_{1,n}^{\boldsymbol{\Lambda}}(q)$. Hence, when these only elements, together with $e^{(1)}(\boldsymbol{i})$ for any $\boldsymbol{i} \in I^\alpha$, and these only relations, together with those involving the elements $e^{(1)}(\boldsymbol{i})$, are used to prove any relation $(*)$ in [BrKl-a, §4] (in $\mathrm{H}_\alpha^{\boldsymbol{\Lambda}}(q)$), we claim that the *same proof* in $\mathrm{Y}_\alpha^{\boldsymbol{\Lambda}}(q)$ holds for $(*)$, involving $g_a e(\boldsymbol{j})$ instead of $g_a^{(1)}$, the elements $X_b^{\pm 1} e(\boldsymbol{j})$ instead of $X_b^{(1)^{\pm 1}}$ and $e(\boldsymbol{i}, \boldsymbol{j})$ instead of $e^{(1)}(\boldsymbol{i})$. If $\boldsymbol{j}$ satisfies in addition $j_{a+1} = j_{a+2}$, we can add to the previous list the element $g_{a+1} e(\boldsymbol{j})$, which stands for $g_{a+1}^{(1)}$.

Lemma 3.3.10. *For any $a \in \{1, \dots, n-1\}$ and $(\boldsymbol{i}, \boldsymbol{j}) \in K^\alpha$ such that $j_a \neq j_{a+1}$ we have $g_a e(\boldsymbol{i}) e(\boldsymbol{j}) = e(s_a \cdot \boldsymbol{i}) g_a e(\boldsymbol{j})$, that is, $g_a e(\boldsymbol{i}, \boldsymbol{j}) = e(s_a \cdot (\boldsymbol{i}, \boldsymbol{j})) g_a$.*

Proof. Given Lemma 3.3.6, we show as in Lemma 3.3.7 that $g_a e(\boldsymbol{i}) e(\boldsymbol{j}) = e(s_a \cdot \boldsymbol{i}) g_a e(\boldsymbol{j})$. We obtain the final result applying Lemma 3.3.7. □

3.3.1.2 Image of y_a

We are now able to define the elements $y_a \in \mathrm{Y}_\alpha^{\boldsymbol{\Lambda}}(q)$ for all $a \in \{1, \dots, n\}$. We define the following elements of $\mathrm{Y}_\alpha^{\boldsymbol{\Lambda}}(q)$ for any $a \in \{1, \dots, n\}$:

$$y_a := \sum_{\boldsymbol{i} \in I^\alpha} \left(1 - q^{-i_a} X_a\right) e(\boldsymbol{i}) \in \mathrm{Y}_\alpha^{\boldsymbol{\Lambda}}(q).$$

We can notice that $\sum_{\boldsymbol{i}} (q^{i_a} - X_a) e(\boldsymbol{i})$ is the nilpotent part of the Jordan–Chevalley decomposition of X_a. In particular, the element y_a is nilpotent. As a consequence, we will be able to make calculations in the ring $F[[y_1, \dots, y_n]]$ of power series in the commuting variables $y_1, \dots, y_n$. We will sometimes also need the following element:

$$y_a(\boldsymbol{i}) := q^{i_a}(1 - y_a),$$

which satisfies

$$y_a(\boldsymbol{i}) e(\boldsymbol{i}) = X_a e(\boldsymbol{i}). \tag{3.3.11}$$

We end this paragraph with a lemma.

Lemma 3.3.12. *For any $\boldsymbol{j} \in J^\alpha$ such that $j_a \neq j_{a+1}$ we have*

$$\begin{aligned}
g_a y_{a+1} e(\boldsymbol{j}) &= y_a g_a e(\boldsymbol{j}), \\
y_{a+1} g_a e(\boldsymbol{j}) &= g_a y_a e(\boldsymbol{j}).
\end{aligned}$$

Proof. Indeed, we have $g_a y_{a+1} e(\boldsymbol{j}) = \sum_{\boldsymbol{i}} (g_a - q^{-i_{a+1}} g_a X_{a+1}) e(\boldsymbol{i}, \boldsymbol{j})$ and applying Lemmas 3.3.6 and 3.3.10 we obtain

$$\begin{aligned}
g_a y_{a+1} e(\boldsymbol{j}) &= \sum_{\boldsymbol{i} \in I^\alpha} (1 - q^{-i_{a+1}} X_a) e(s_a \cdot \boldsymbol{i}) g_a e(\boldsymbol{j}) \\
&= \sum_{\boldsymbol{i} \in I^\alpha} (1 - q^{-i_a} X_a) e(\boldsymbol{i}) g_a e(\boldsymbol{j}) \\
&= y_a g_a e(\boldsymbol{j}).
\end{aligned}$$

□

3.3.1.3 Image of ψ_a

We first define some invertible elements $Q_a(\boldsymbol{i},\boldsymbol{j}) \in F[[y_a, y_{a+1}]]^\times$ for any $a \in \{1,\dots,n-1\}$ and $(\boldsymbol{i},\boldsymbol{j}) \in K^\alpha$ by

$$Q_a(\boldsymbol{i},\boldsymbol{j}) \coloneqq \begin{cases} \begin{cases} 1-q+qy_{a+1}-y_a, & \text{if } i_a = i_{a+1}, \\ (y_a(\boldsymbol{i}) - qy_{a+1}(\boldsymbol{i}))/(y_a(\boldsymbol{i}) - y_{a+1}(\boldsymbol{i})), & \text{if } i_a \not\to i_{a+1}, \\ (y_a(\boldsymbol{i}) - qy_{a+1}(\boldsymbol{i}))/(y_a(\boldsymbol{i}) - y_{a+1}(\boldsymbol{i}))^2, & \text{if } i_a \to i_{a+1}, \\ q^{i_a}, & \text{if } i_a \leftarrow i_{a+1}, \\ q^{i_a}/(y_a(\boldsymbol{i}) - y_{a+1}(\boldsymbol{i})), & \text{if } i_a \leftrightarrows i_{a+1}, \end{cases} & \text{if } j_a = j_{a+1}, \\ f_{a,\boldsymbol{j}}, & \text{if } j_a \neq j_{a+1}, \end{cases}$$

where $f_{a,\boldsymbol{j}} \in \{1,q\}$ is given for $j_a \neq j_{a+1}$ by

$$f_{a,\boldsymbol{j}} \coloneqq \begin{cases} q, & \text{if } j_a < j_{a+1}, \\ 1, & \text{if } j_a > j_{a+1}, \end{cases}$$

with $<$ being any total ordering on $J = \mathbb{Z}/d\mathbb{Z} \simeq \{1,\dots,d\}$. For $d = 1$ or for $j_a = j_{a+1}$, the power series $Q_a(\boldsymbol{i},\boldsymbol{j})$ coincides with the definition given at [BrKl-a, (4.36)].

Remark 3.3.13. The elements $Q_a(\boldsymbol{i},\boldsymbol{j})$ depend only on (i_a, i_{a+1}) and (j_a, j_{a+1}). Moreover, as in [BrKl-a] the explicit expression of $Q_a(\boldsymbol{i},\boldsymbol{j})$ for $j_a = j_{a+1}$ does not really matter. Only its properties are essential, namely, those in Lemma 3.3.16.

Remark 3.3.14. The scalar $f_{a,\boldsymbol{j}}$ is only an artefact: if q admits a square root $q^{1/2}$ in F, we can simply set $f_{a,\boldsymbol{j}} \coloneqq q^{1/2}$.

Finally, we give an easy lemma about the elements $f_{a,\boldsymbol{j}}$.

Lemma 3.3.15. *Let $a \in \{1,\dots,n-1\}$ and $\boldsymbol{j} \in J^\alpha$. If $j_a \neq j_{a+1}$ then $f_{a,\boldsymbol{j}} f_{a,s_a\cdot\boldsymbol{j}} = q$.*

As in §2.3.2, for any $Q \in F[[y_1,\dots,y_n]]$ and $\sigma \in \mathfrak{S}_n$ we have an element $Q^\sigma \in F[[y_1,\dots,y_n]]$ obtained by place permutation of the variables. We will use later the following properties satisfied by $Q_a(\boldsymbol{i},\boldsymbol{j})$, where $Q^\sigma(\boldsymbol{i},\boldsymbol{j}) \coloneqq Q(\boldsymbol{i},\boldsymbol{j})^\sigma$ (see [BrKl-a, (4.35)]).

Lemma 3.3.16. *Let $a \in \{1,\dots,n-1\}$ and $(\boldsymbol{i},\boldsymbol{j}) \in K^\alpha$. We have*

$$Q_{a+1}^{s_a}(\boldsymbol{i},\boldsymbol{j}) = Q_a^{s_{a+1}}(s_{a+1}s_a \cdot (\boldsymbol{i},\boldsymbol{j})),$$
$$Q_a^{s_{a+1}}(\boldsymbol{i},\boldsymbol{j}) = Q_{a+1}^{s_a}(s_a s_{a+1} \cdot (\boldsymbol{i},\boldsymbol{j})).$$

Proof. We check only the first equality, the second being straightforward considering $(\boldsymbol{i}',\boldsymbol{j}') \coloneqq s_a s_{a+1} \cdot (\boldsymbol{i},\boldsymbol{j})$. For any $i \in I$, let us write $y_a(i) \coloneqq q^i(1-y_a)$, so that $y_a(\boldsymbol{i}) = y_a(i_a)$. Noticing that $s_{a+1}s_a = (a, a+2, a+1)$, we obtain, if $j_{a+1} = j_{a+2}$,

$$Q_a(s_{a+1}s_a \cdot (\boldsymbol{i},\boldsymbol{j})) = \begin{cases} 1-q+qy_{a+1}-y_a, & \text{if } i_{a+1} = i_{a+2}, \\ (y_a(i_{a+1}) - qy_{a+1}(i_{a+2}))/(y_a(i_{a+1}) - y_{a+1}(i_{a+2})), & \text{if } i_{a+1} \not\to i_{a+2}, \\ (y_a(i_{a+1}) - qy_{a+1}(i_{a+2}))/(y_a(i_{a+1}) - y_{a+1}(i_{a+2}))^2, & \text{if } i_{a+1} \to i_{a+2}, \\ q^{i_{a+1}}, & \text{if } i_{a+1} \leftarrow i_{a+2}, \\ q^{i_{a+1}}/(y_a(i_{a+1}) - y_{a+1}(i_{a+2})), & \text{if } i_{a+1} \leftrightarrows i_{a+2}, \end{cases}$$

and we conclude using $y_a(i_{a+1})^{s_{a+1}} = y_a(i_{a+1})$ and $y_{a+1}(i_{a+2})^{s_{a+1}} = y_{a+2}(i_{a+2})$. If $j_{a+1} \neq j_{a+2}$, we have $Q_a(s_{a+1}s_a \cdot (\boldsymbol{i},\boldsymbol{j})) = f_{a,(a,a+2,a+1)\cdot\boldsymbol{j}} = \begin{cases} q, & \text{if } j_{a+1} < j_{a+2}, \\ 1, & \text{if } j_{a+1} > j_{a+2}, \end{cases}$ and this is exactly $f_{a+1,\boldsymbol{j}}$. □

Let $a \in \{1, \dots, n-1\}$. We now introduce the following element of $\mathrm{Y}_\alpha^{\boldsymbol{\Lambda}}(q)$:

$$\Phi_a := g_a + (1-q) \sum_{\substack{(\boldsymbol{i},\boldsymbol{j}) \in K^\alpha \\ i_a \neq i_{a+1} \\ j_a = j_{a+1}}} \left(1 - X_a X_{a+1}^{-1}\right)^{-1} e(\boldsymbol{i},\boldsymbol{j}) + \sum_{\substack{(\boldsymbol{i},\boldsymbol{j}) \in K^\alpha \\ i_a = i_{a+1} \\ j_a = j_{a+1}}} e(\boldsymbol{i},\boldsymbol{j}),$$

where $\left(1 - X_a X_{a+1}^{-1}\right)^{-1} e(\boldsymbol{k})$ denotes the inverse of $(1 - X_a X_{a+1}^{-1}) e(\boldsymbol{k})$ in $e(\boldsymbol{k}) \mathrm{Y}_\alpha^{\boldsymbol{\Lambda}}(q) e(\boldsymbol{k})$. Note that this element is indeed invertible, since for any $\boldsymbol{k} = (\boldsymbol{i},\boldsymbol{j})$ with $i_a \neq i_{a+1}$ its only eigenvalue $1 - q^{i_a - i_{a+1}}$ is non-zero, thanks to the definition of I. In particular, we have

$$\Phi_a e(\boldsymbol{j}) = g_a e(\boldsymbol{j}),$$

if $j_a \neq j_{a+1}$. For $d = 1$ we get the "intertwining element" defined in [BrKl-a, §4.2], and we write it $\Phi_a^{(1)}$.

Though we will not need this until Section 3.4, we define now the power series $P_a(\boldsymbol{i},\boldsymbol{j}) \in F[[y_a, y_{a+1}]]$ for any $a \in \{1, \dots, n-1\}$ and $(\boldsymbol{i},\boldsymbol{j}) \in K^\alpha$ by

$$P_a(\boldsymbol{i},\boldsymbol{j}) := \begin{cases} \begin{cases} 1, & \text{if } i_a = i_{a+1}, \\ (1-q)(1 - y_a(\boldsymbol{i}) y_{a+1}(\boldsymbol{i})^{-1})^{-1}, & \text{if } i_a \neq i_{a+1}, \end{cases} & \text{if } j_a = j_{a+1}, \\ 0, & \text{if } j_a \neq j_{a+1}. \end{cases}$$

For $d = 1$ or for $j_a = j_{a+1}$ we recover the definition given at [BrKl-a, (4.27)]. Moreover, we have the following equality:

$$\Phi_a = \sum_{\boldsymbol{k} \in K^\alpha} (g_a + P_a(\boldsymbol{k})) e(\boldsymbol{k}). \tag{3.3.17}$$

Indeed, the only non-obvious fact to check is $P_a(\boldsymbol{i},\boldsymbol{j}) e(\boldsymbol{i},\boldsymbol{j}) = (1-q)\left(1 - X_a X_{a+1}^{-1}\right)^{-1} e(\boldsymbol{i},\boldsymbol{j})$ if $i_a \neq i_{a+1}$ and $j_a = j_{a+1}$, but this is clear by (3.3.11). We will also use the following equality (the same one as in Lemma 3.3.16):

$$P_{a+1}^{s_a}(\boldsymbol{i},\boldsymbol{j}) = P_a^{s_{a+1}}(s_{a+1} s_a \cdot (\boldsymbol{i},\boldsymbol{j})). \tag{3.3.18}$$

Lemma 3.3.19. *Let $a, a' \in \{1, \dots, n-1\}$ and $b \in \{1, \dots, n\}$. We have the following properties:*

$$\begin{aligned}
\Phi_a e(\boldsymbol{j}) &= e(s_a \cdot \boldsymbol{j}) \Phi_a, && && (3.3.20a) \\
\Phi_a e(\boldsymbol{i},\boldsymbol{j}) &= e(s_a \cdot (\boldsymbol{i},\boldsymbol{j})) \Phi_a, && && (3.3.20b) \\
\Phi_a X_b &= X_b \Phi_a, && \text{if } b \neq a, a+1, && (3.3.20c) \\
\Phi_a y_b &= y_b \Phi_a, && \text{if } b \neq a, a+1, && (3.3.20d) \\
\Phi_a Q_{a'}(\boldsymbol{k}) &= Q_{a'}(\boldsymbol{k}) \Phi_a, && \text{if } |a - a'| > 1, && (3.3.20e) \\
\Phi_a \Phi_{a'} &= \Phi_{a'} \Phi_a, && \text{if } |a - a'| > 1. && (3.3.20f)
\end{aligned}$$

Proof. We will use results from [BrKl-a, Lemma 4.1] (which is this lemma for $d = 1$).

(3.3.20a) Using Lemma 3.3.7, it is clear if $j_a \neq j_{a+1}$ since then $\Phi_a e(\boldsymbol{j}) = g_a e(\boldsymbol{j}) = e(s_a \cdot \boldsymbol{j}) g_a = e(s_a \cdot \boldsymbol{j}) \Phi_a$. Using Corollary 3.3.8 it is clear if $j_a = j_{a+1}$ since $e(\boldsymbol{j})$ commutes with every term in the definition of Φ_a.

(3.3.20b) If $j_a = j_{a+1}$, we claim that the relation comes applying (3.3.20a) and Remark 3.3.9 on the equality $\Phi_a^{(1)} e^{(1)}(\boldsymbol{i}) = e^{(1)}(s_a \cdot \boldsymbol{i}) \Phi_a^{(1)}$. If $j_a \neq j_{a+1}$ it follows directly from Lemma 3.3.10.

(3.3.20c) Straightforward using (3.2.4a) and since $e(\boldsymbol{i},\boldsymbol{j})$ are polynomials in $X_1, \dots, X_n$.

(3.3.20d) Using (3.3.5), (3.3.20b) and (3.3.20c) we get:

$$\begin{aligned}\Phi_a y_b &= \sum_{\boldsymbol{i},\boldsymbol{j}} (1 - q^{-i_b} X_b) e(s_a \cdot (\boldsymbol{i},\boldsymbol{j})) \Phi_a \\ &= \sum_{\boldsymbol{i}} (1 - q^{-i_b} X_b) e(s_a \cdot \boldsymbol{i}) \Phi_a = y_b \Phi_a,\end{aligned}$$

since $(s_a \cdot \boldsymbol{i})_b = i_b$.

(3.3.20e) Since $Q_{a'}(\boldsymbol{k}) \in F[[y_{a'}, y_{a'+1}]]$ and $a' \neq a, a+1$ and $a'+1 \neq a, a+1$ it follows from (3.3.20d).

(3.3.20f) Let us write $\widetilde{\Phi}_a \coloneqq \Phi_a - g_a$. Using (3.2.4a) and Lemma 3.3.10 we obtain

$$\begin{aligned}\widetilde{\Phi}_a g_{a'} = g_{a'} \Bigg(&(1-q) \sum_{\substack{i_a \neq i_{a+1} \\ j_a = j_{a+1}}} (1 - X_a X_{a+1}^{-1})^{-1} e(s_{a'} \cdot (\boldsymbol{i},\boldsymbol{j})) \\ &+ \sum_{\substack{i_a = i_{a+1} \\ j_a = j_{a+1}}} e(s_{a'} \cdot (\boldsymbol{i},\boldsymbol{j})) \Bigg) = g_{a'} \widetilde{\Phi}_a,\end{aligned}$$

and exchanging a and a' we get $g_a \widetilde{\Phi}_{a'} = \widetilde{\Phi}_{a'} g_a$. Noticing that $\widetilde{\Phi}_a \widetilde{\Phi}_{a'} = \widetilde{\Phi}_{a'} \widetilde{\Phi}_a$ (we do not use here $|a - a'| > 1$) and using (3.2.2e), we obtain

$$\begin{aligned}\Phi_a \Phi_{a'} &= (g_a + \widetilde{\Phi}_a)(g_{a'} + \widetilde{\Phi}_{a'}) \\ &= g_a g_{a'} + \widetilde{\Phi}_a g_{a'} + g_a \widetilde{\Phi}_{a'} + \widetilde{\Phi}_a \widetilde{\Phi}_{a'} \\ &= g_{a'} g_a + g_{a'} \widetilde{\Phi}_a + \widetilde{\Phi}_{a'} g_a + \widetilde{\Phi}_{a'} \widetilde{\Phi}_a \\ &= (g_{a'} + \widetilde{\Phi}_{a'})(g_a + \widetilde{\Phi}_a) \\ &= \Phi_{a'} \Phi_a.\end{aligned}$$

□

We are now ready to define our elements ψ_a for any $a \in \{1, \ldots, n-1\}$:

$$\psi_a \coloneqq \sum_{\boldsymbol{k} \in K^\alpha} \Phi_a Q_a(\boldsymbol{k})^{-1} e(\boldsymbol{k}) \in \mathrm{Y}_\alpha^{\boldsymbol{\Lambda}}(q).$$

As usual, we write $\psi_a^{(1)}$ for ψ_a when $d = 1$, and this element $\psi_a^{(1)}$ corresponds with the ψ_a of [BrKl-a, §4.3]. Note finally that for any $\boldsymbol{j} \in J^\alpha$ we have

$$\psi_a e(\boldsymbol{j}) = f_{a,\boldsymbol{j}}^{-1} g_a e(\boldsymbol{j}),$$

if $j_a \neq j_{a+1}$.

3.3.2 Check of the defining relations

We now check the defining relations (1.2.3), (1.2.16) and (1.2.19) for the elements we have just defined. The idea is the following: when an element $e(\boldsymbol{i},\boldsymbol{j})$ lies in a relation to check, if $j_a = j_{a+1}$ then we get immediately the result by Remark 3.3.9 rewriting the same proof as [BrKl-a, Theorem 4.2], and if $j_a \neq j_{a+1}$ then it will be easy (at least, easier than in [BrKl-a]) to prove the relation. Recall that we always work in $\mathrm{Y}_\alpha^{\boldsymbol{\Lambda}}(q)$. In particular every relation should be multiplied by $e(\alpha)$ and we write $e(\boldsymbol{i})$ (respectively $e(\boldsymbol{j})$) instead of $e(\alpha)(\boldsymbol{i})$ (resp. $e(\alpha)(\boldsymbol{j})$).

(**1.2.19**) We do exactly the same proof as in $\mathrm{H}_\alpha^{\boldsymbol{\Lambda}}(q)$. Let $(\boldsymbol{i}, \boldsymbol{j}) \in K^\alpha$ and set $M := \mathrm{Y}_\alpha^{\boldsymbol{\Lambda}}(q)$. Recall that the action of X_1 on M is given by the action of $e(\alpha)X_1$. By (3.2.2j) we have $\prod_{i\in I}(X_1 - q^i)^{\Lambda_i} = 0$, hence

$$\prod_{i\in I}\left[\left(X_1 - q^i\right)^{\Lambda_i} e(\boldsymbol{i})\right] = 0. \tag{3.3.21}$$

As an endomorphism of $M(\boldsymbol{i})$, the element $(X_1 - q^i)^{\Lambda_i}$ is invertible if $i \neq i_1$ since its only eigenvalue $(q^{i_1} - q^i)^{\Lambda_i}$ is non-zero (note that Λ_i may be equal to 0). This means that there exist elements $(X_1 - q^i)^{-\Lambda_i} e(\boldsymbol{i})$ such that $(X_1 - q^i)^{\Lambda_i} e(\boldsymbol{i}) \cdot (X_1 - q^i)^{-\Lambda_i} e(\boldsymbol{i}) = e(\boldsymbol{i})$. Hence, multiplying by all these inverses, the equation (3.3.21) becomes

$$(X_1 - q^{i_1})^{\Lambda_{i_1}} e(\boldsymbol{i}) = 0.$$

Finally, since $y_1^{\Lambda_{i_1}} = \sum_{\boldsymbol{i}'\in I^\alpha}(1 - q^{-i'_1}X_1)^{\Lambda_{i_1}} e(\boldsymbol{i}')$ we obtain

$$y_1^{\Lambda_{i_1}} e(\boldsymbol{i}, \boldsymbol{j}) = (1 - q^{-i_1}X_1)^{\Lambda_{i_1}} e(\boldsymbol{i}, \boldsymbol{j}) = 0.$$

(**1.2.3a**) Straightforward from the definition of $e(\boldsymbol{k})$ for all $\boldsymbol{k} \in K^\alpha$.

(**1.2.3b**) Idem.

(**1.2.3c**) Straightforward since y_a and $e(\boldsymbol{k})$ both lie in the commutative subalgebra generated by $t_1, \dots, t_n$ and $X_1, \dots, X_n$.

(**1.2.3d**) Straightforward by (3.3.20b) and since $Q_a(\boldsymbol{k}')$ and $e(\boldsymbol{k}')$ commute with $e(\boldsymbol{k})$.

(**1.2.3e**) True since $\{X_a\}_a$ is commutative.

(**1.2.3f**) True by (3.3.20d).

(**1.2.3g**) Let $|a - b| > 1$. We have, using (1.2.3d), (3.3.20e) and (3.3.20f),

$$\begin{aligned}
\psi_a\psi_b &= \sum_{\boldsymbol{k}} \Phi_a Q_a(\boldsymbol{k})^{-1} e(\boldsymbol{k})\psi_b \\
&= \sum_{\boldsymbol{k}} \Phi_a Q_a(\boldsymbol{k})^{-1} \psi_b e(s_b \cdot \boldsymbol{k}) \\
&= \sum_{\boldsymbol{k}} \Phi_a Q_a(\boldsymbol{k})^{-1} \Phi_b Q_b(s_b \cdot \boldsymbol{k})^{-1} e(s_b \cdot \boldsymbol{k}) \\
&= \sum_{\boldsymbol{k}} \Phi_b Q_b(s_b \cdot \boldsymbol{k})^{-1} \Phi_a Q_a(\boldsymbol{k})^{-1} e(s_b \cdot \boldsymbol{k}).
\end{aligned}$$

Hence, noticing that $Q_a(\boldsymbol{k}) = Q_a(s_b \cdot \boldsymbol{k})$ and $Q_b(s_b \cdot \boldsymbol{k}) = Q_b(s_a s_b \cdot \boldsymbol{k})$ (see Remark 3.3.13) we obtain

$$\begin{aligned}
\psi_a\psi_b &= \sum_{\boldsymbol{k}} \Phi_b Q_b(s_b \cdot \boldsymbol{k})^{-1} \psi_a e(s_b \cdot \boldsymbol{k}) \\
&= \sum_{\boldsymbol{k}} \Phi_b Q_b(s_b \cdot \boldsymbol{k})^{-1} e(s_a s_b \cdot \boldsymbol{k}) \psi_a \\
&= \psi_b\psi_a.
\end{aligned}$$

(1.2.3h) First, if $\boldsymbol{k} = (\boldsymbol{i}, \boldsymbol{j})$ satisfies $j_a = j_{a+1}$ then by Remark 3.3.9 we obtain from

$$\psi_a^{(1)} y_{a+1}^{(1)} e^{(1)}(\boldsymbol{i}) = \begin{cases} (y_a^{(1)} \psi_a^{(1)} + 1) e^{(1)}(\boldsymbol{i}), & \text{if } i_a = i_{a+1}, \\ y_a^{(1)} \psi_a^{(1)} e^{(1)}(\boldsymbol{i}), & \text{if } i_a \neq i_{a+1}, \end{cases}$$

the following equality:

$$\psi_a y_{a+1} e(\boldsymbol{i}, \boldsymbol{j}) = \begin{cases} (y_a \psi_a + 1) e(\boldsymbol{i}, \boldsymbol{j}), & \text{if } i_a = i_{a+1} \text{ and } j_a = j_{a+1}, \\ y_a \psi_a e(\boldsymbol{i}, \boldsymbol{j}), & \text{if } i_a \neq i_{a+1} \text{ and } j_a = j_{a+1}. \end{cases}$$

Hence it remains to deal with the case $j_a \neq j_{a+1}$ (and no condition on $\boldsymbol{i}$). Using Lemma 3.3.12 we obtain

$$\begin{aligned} \psi_a y_{a+1} e(\boldsymbol{i}, \boldsymbol{j}) &= \Phi_a Q_a(\boldsymbol{i}, \boldsymbol{j})^{-1} y_{a+1} e(\boldsymbol{i}, \boldsymbol{j}) \\ &= f_{a,\boldsymbol{j}}^{-1} g_a y_{a+1} e(\boldsymbol{i}, \boldsymbol{j}) \\ &= f_{a,\boldsymbol{j}}^{-1} y_a g_a e(\boldsymbol{i}, \boldsymbol{j}) \\ &= y_a \Phi_a Q_a(\boldsymbol{i}, \boldsymbol{j})^{-1} e(\boldsymbol{i}, \boldsymbol{j}) \\ &= y_a \psi_a e(\boldsymbol{i}, \boldsymbol{j}). \end{aligned}$$

Finally, we have proved

$$\psi_a y_{a+1} e(\boldsymbol{k}) = \begin{cases} (y_a \psi_a + 1) e(\boldsymbol{k}), & \text{if } k_a = k_{a+1}, \\ y_a \psi_a e(\boldsymbol{k}), & \text{if } k_a \neq k_{a+1}, \end{cases}$$

which is exactly (1.2.3h).

(1.2.3i) Similar.

Remark 3.3.22. Thanks to relations (1.2.3f), (1.2.3h) and (1.2.3i), given $f \in F[[y_1, \dots, y_n]]$ and $\boldsymbol{k} \in K^\alpha$ such that $k_a \neq k_{a+1}$ we have $f \psi_a e(\boldsymbol{k}) = \psi_a f^{s_a} e(\boldsymbol{k})$. In particular, this holds if $j_a \neq j_{a+1}$ with $\boldsymbol{k} = (\boldsymbol{i}, \boldsymbol{j})$.

(1.2.16a) Once again, the result is straightforward if $j_a = j_{a+1}$ using Remark 3.3.9. Let us then suppose $j_a \neq j_{a+1}$. Hence, necessarily we have $k_a \not\leftrightarrow k_{a+1}$ so we have to prove $\psi_a^2 e(\boldsymbol{k}) = e(\boldsymbol{k})$. We have

$$\begin{aligned} \psi_a^2 e(\boldsymbol{k}) &= \psi_a e(s_a \cdot \boldsymbol{k}) \psi_a \\ &= \Phi_a Q_a(s_a \cdot \boldsymbol{k})^{-1} e(s_a \cdot \boldsymbol{k}) \psi_a \\ &= f_{a, s_a \cdot \boldsymbol{j}}^{-1} g_a \Phi_a Q_a(\boldsymbol{k})^{-1} e(\boldsymbol{k}) \\ &= (f_{a, s_a \cdot \boldsymbol{j}} f_{a,\boldsymbol{j}})^{-1} g_a^2 e(\boldsymbol{k}). \end{aligned}$$

Applying Lemmas 3.3.6 and 3.3.15 we find $\psi_a^2 e(\boldsymbol{k}) = e(\boldsymbol{k})$ (recall $e(\boldsymbol{k}) = e(\boldsymbol{i}) e(\boldsymbol{j}) = e(\boldsymbol{j}) e(\boldsymbol{i})$) thus we are done.

(1.2.16b) If $j_a = j_{a+1} = j_{a+2}$, we get the result using Remark 3.3.9. Let us then suppose that we are not in that case: we have to prove $\psi_{a+1}\psi_a\psi_{a+1}e(\boldsymbol{k}) = \psi_a\psi_{a+1}\psi_a e(\boldsymbol{k})$. We will intensively use (1.2.3d). Note also that

$$\Phi_a e(\boldsymbol{i},\boldsymbol{j}) = \begin{cases} \left(g_a + (1-q)(1-X_aX_{a+1}^{-1})^{-1}\right)e(\boldsymbol{i},\boldsymbol{j}), & \text{if } i_a \neq i_{a+1} \text{ and } j_a = j_{a+1}, \\ (g_a+1)e(\boldsymbol{i},\boldsymbol{j}), & \text{if } i_a = i_{a+1} \text{ and } j_a = j_{a+1}, \\ g_a e(\boldsymbol{i},\boldsymbol{j}), & \text{otherwise } (j_a \neq j_{a+1}), \end{cases}$$

and

$$\psi_a e(\boldsymbol{i},\boldsymbol{j}) = \begin{cases} \Phi_a Q_a(\boldsymbol{i},\boldsymbol{j})^{-1}e(\boldsymbol{i},\boldsymbol{j}), & \text{if } j_a = j_{a+1}, \\ f_{a,\boldsymbol{j}}^{-1} g_a e(\boldsymbol{i},\boldsymbol{j}), & \text{if } j_a \neq j_{a+1}. \end{cases}$$

It is convenient to introduce some notation. The couple $(\boldsymbol{i},\boldsymbol{j})$ shall only be modified by the action of s_a or s_{a+1}, hence we only write $((i_a, i_{a+1}, i_{a+2}),(j_a, j_{a+1}, j_{a+2}))$ for $(\boldsymbol{i},\boldsymbol{j})$. Moreover, for clarity we forget comas and only write the indexation, substituting 0 to a. Thus, $((i_a, i_{a+1}, i_{a+2}),(j_a, j_{a+1}, j_{a+2}))$ becomes $((012),(012))$. Finally, as $\mathfrak{S}_n$ acts diagonally on $I \times J$ we can write (012) instead of $((012),(012))$. Because an example beats lines of explanation, here is one: $\psi_0 e(102)$ stands for $\psi_a e(s_a \cdot \boldsymbol{k})$.

Case $j_0 = j_1 \neq j_2$. Let us first compute $\psi_1\psi_0\psi_1 e(012)$ and $\psi_0\psi_1\psi_0 e(012)$. We have

$$\begin{aligned} \psi_1\psi_0\psi_1 e(012) &= \psi_1\psi_0 e(021)\psi_1 \\ &= \psi_1 e(201)\psi_0\psi_1 \\ &= \Phi_1 Q_1(201)^{-1}e(201)\psi_0\psi_1. \end{aligned}$$

Since $Q_1(201)^{-1} \in F[[y_1, y_2]]$ and recalling Remark 3.3.22 we obtain

$$\psi_1\psi_0\psi_1 e(012) = \Phi_1 e(201)\psi_0 Q_1^{s_0}(201)^{-1}\psi_1 = \Phi_1 e(201)\psi_0\psi_1 Q_1^{s_0s_1}(201)^{-1}.$$

By Lemma 3.3.16 we have $Q_1^{s_0s_1}(201) = Q_1(201)^{s_0s_1} = (Q_1(201)^{s_0})^{s_1} = (Q_0(s_1s_0\cdot(201))^{s_1})^{s_1} = Q_0(012)$. Hence,

$$\psi_1\psi_0\psi_1 e(012) = \Phi_1 e(201)\psi_0\psi_1 Q_0(012)^{-1}.$$

As

$$\psi_0\psi_1\psi_0 e(012) = \psi_0\psi_1\Phi_0 e(012)Q_0(012)^{-1},$$

to have (1.2.16b) it suffices to prove

$$\Phi_1 e(201)\psi_0\psi_1 = \psi_0\psi_1\Phi_0 e(012). \tag{3.3.23}$$

We now distinguish two subcases. If $i_0 \neq i_1$ then

$$\Phi_1 e(201)\psi_0\psi_1 = \left(g_1 + (1-q)(1-X_1X_2^{-1})^{-1}\right)\psi_0 e(021)\psi_1.$$

Recalling (3.2.4a) and Lemma 3.3.6 we obtain

$$\begin{aligned} \Phi_1 e(201)\psi_0\psi_1 &= f_{0,(021)}^{-1}\left(g_1g_0 + (1-q)g_0(1-X_0X_2^{-1})^{-1}\right)\psi_1 e(012) \\ &= f_{0,(021)}^{-1}f_{1,(012)}^{-1}\left(g_1g_0g_1 + (1-q)g_0g_1(1-X_0X_1^{-1})^{-1}\right)e(012). \end{aligned}$$

Using the braid relation (3.2.2f) this becomes, recalling (3.3.20b),

$$\begin{aligned}\Phi_1 e(201)\psi_0\psi_1 &= f_{0,(021)}^{-1} f_{1,(012)}^{-1}\left(g_0g_1g_0 + (1-q)g_0g_1(1-X_0X_1^{-1})^{-1}\right)e(012)\\ &= f_{0,(021)}^{-1} f_{1,(012)}^{-1} g_0g_1\left(g_0 + (1-q)(1-X_0X_1^{-1})^{-1}\right)e(012)\\ &= f_{0,(021)}^{-1} f_{1,(012)}^{-1} g_0g_1\Phi_0 e(012)\\ &= f_{0,(021)}^{-1} f_{1,(012)}^{-1} g_0g_1 e(102)\Phi_0,\end{aligned}$$

and then, noticing that $f_{1,(012)} = f_{1,(102)}$ and $f_{0,(021)} = f_{0,(120)}$, we obtain

$$\begin{aligned}\Phi_1 e(201)\psi_0\psi_1 &= f_{0,(120)}^{-1} f_{1,(012)}^{-1} g_0g_1 e(102)\Phi_0\\ &= f_{0,(120)}^{-1} g_0 e(120)\psi_1\Phi_0\\ &= \psi_0\psi_1\Phi_0 e(012),\end{aligned}$$

thus (3.3.23) is proved in the case $i_0 \neq i_1$. Now if $i_0 = i_1$ then

$$\begin{aligned}\Phi_1 e(201) &= (g_1+1)e(201),\\ \Phi_0 e(012) &= (g_0+1)e(012),\end{aligned}$$

thus with the same calculation as above (even easier) we obtain

$$\begin{aligned}\Phi_1 e(201)\psi_0\psi_1 &= f_{0,(021)}^{-1} f_{1,(012)}^{-1}(g_1g_0g_1 + g_0g_1)e(012)\\ &= f_{0,(120)}^{-1} f_{1,(102)}^{-1}(g_0g_1g_0 + g_0g_1)e(012)\\ &= \psi_0\psi_1\Phi_0 e(012),\end{aligned}$$

so we got (3.3.23). Until the end of the proof we use the same arguments as here, arguments which we will thus not recall.

Case $j_0 \neq j_1 = j_2$. Similar.

Case $j_0 = j_2 \neq j_1$. Once again we begin with the computation of $\psi_1\psi_0\psi_1 e(012)$ and $\psi_0\psi_1\psi_0 e(012)$. We have

$$\psi_1\psi_0\psi_1 e(012) = \psi_1\Phi_0 Q_0(021)^{-1} e(021)\psi_1 = \psi_1\Phi_0 e(021)\psi_1 Q_0^{s_1}(021)^{-1},$$

and

$$\psi_0\psi_1\psi_0 e(012) = \psi_0\Phi_1 Q_1(102)^{-1} e(102)\psi_0 = \psi_0\Phi_1 e(102)\psi_0 Q_1^{s_0}(102)^{-1}.$$

Since $Q_0^{s_1}(021)^{-1} = Q_1^{s_0}(102)^{-1}$, it suffices to prove

$$\psi_1\Phi_0 e(021)\psi_1 = \psi_0\Phi_1 e(102)\psi_0.$$

Once again we distinguish two subcases. If $i_0 \neq i_2$ then

$$\begin{aligned}\psi_1\Phi_0 e(021)\psi_1 &= \psi_1 e(201)\Phi_0\psi_1\\ &= f_{1,(201)}^{-1} g_1\left(g_0 + (1-q)(1-X_0X_1^{-1})^{-1}\right)\psi_1 e(012)\\ &= f_{1,(201)}^{-1} f_{1,(012)}^{-1}\left(g_1g_0g_1 + (1-q)g_1^2(1-X_0X_2^{-1})^{-1}\right)e(012)\\ &= f_{1,(201)}^{-1} f_{1,(012)}^{-1}\left(g_1g_0g_1 + (1-q)q(1-X_0X_2^{-1})^{-1}\right)e(012).\end{aligned}$$

Similarly, we find

$$\begin{aligned}\psi_0\Phi_1 e(102)\psi_0 &= \psi_0 e(120)\Phi_1\psi_0\\ &= f_{0,(120)}^{-1} g_0 \left(g_1 + (1-q)(1-X_1X_2^{-1})^{-1}\right)\psi_0 e(012)\\ &= f_{0,(120)}^{-1} f_{0,(012)}^{-1}\left(g_0g_1g_0 + (1-q)g_0^2(1-X_0X_2^{-1})^{-1}\right)e(012)\\ &= f_{0,(120)}^{-1} f_{0,(012)}^{-1}\left(g_0g_1g_0 + (1-q)q(1-X_0X_2^{-1})^{-1}\right)e(012),\end{aligned}$$

thus we conclude since $f_{1,(201)} = f_{0,(012)}$ and $f_{1,(012)} = f_{0,(120)}$ (we see it on this particular case or we can use Lemma 3.3.16). Now if $i_0 = i_2$, as above we obtain, with $\alpha := f_{1,(201)}^{-1} f_{1,(012)}^{-1} = f_{0,(120)}^{-1} f_{0,(012)}^{-1}$,

$$\begin{aligned}\psi_1\Phi_0 e(021)\psi_1 &= \alpha(g_1g_0g_1 + g_1^2)e(012)\\ &= \alpha(g_1g_0g_1 + q)e(012)\\ &= \alpha(g_0g_1g_0 + q)e(012)\\ &= \alpha(g_0g_1g_0 + g_0^2)e(012)\\ &= \psi_0\Phi_1 e(102)\psi_0 e(012).\end{aligned}$$

Case $\#\{j_a, j_{a+1}, j_{a+2}\} = 3$. We have $j_a \neq j_{a+1}$ and $j_a \neq j_{a+2}$ and $j_{a+1} \neq j_{a+2}$ thus we immediately obtain

$$\begin{aligned}\psi_1\psi_0\psi_1 e(012) &= f_{1,(201)}^{-1} f_{0,(021)}^{-1} f_{1,(012)}^{-1} g_1g_0g_1 e(012)\\ &= f_{0,(120)}^{-1} f_{1,(102)}^{-1} f_{0,(012)}^{-1} g_0g_1g_0 e(012)\\ &= \psi_0\psi_1\psi_0 e(012),\end{aligned}$$

since $f_{1,(201)} = f_{0,(012)}$, $f_{0,(021)} = f_{1,(102)}$ and $f_{1,(012)} = f_{0,(120)}$.

3.4 Yokonuma–Hecke generators of $\mathrm{R}_\alpha^{\mathbf{\Lambda}}(\Gamma)$

Let $\mathbf{\Lambda}$ be a weight as in Section 3.3. The aim of this section is to prove the following theorem.

Theorem 3.4.1. *For any $\alpha \models_K n$, we can construct an explicit algebra homomorphism*

$$\rho_{\mathrm{YR}} : \mathrm{Y}_{d,n}^{\mathbf{\Lambda}}(q) \to \mathrm{R}_\alpha^{\mathbf{\Lambda}}(\Gamma_{e,d}).$$

Note that we do not consider yet $\mathrm{Y}_\alpha^{\mathbf{\Lambda}}(q)$. In particular, it suffices to define the images of the generators (3.2.1) and check if they satisfy the defining relations (3.2.2) of the cyclotomic Yokonuma–Hecke algebra. As in Section 3.3, we use the same notation for a generator and its image.

3.4.1 Definition of the images of the generators

It is easier this time to define these images. First, since the elements $y_1, \dots, y_n$ are nilpotent (Lemma 1.2.23), we can consider power series in these variables. Hence, the quantities $P_a(\boldsymbol{k})$, $Q_a(\boldsymbol{k})$ and $y_a(\boldsymbol{i})$ that we defined in §3.3.1 are also well-defined as elements of $\mathrm{R}_\alpha^{\mathbf{\Lambda}}(\Gamma_{e,d})$. We define finally as in (3.3.5) the elements $e(\boldsymbol{i})$ and $e(\boldsymbol{j})$ of $\mathrm{R}_\alpha^{\mathbf{\Lambda}}(\Gamma_{e,d})$ for any $\boldsymbol{i} \in I^\alpha$ and $\boldsymbol{j} \in J^\alpha$.

We recall that ξ is a primitive dth root of unity in F. Our "Yokonuma–Hecke generators" of $\mathrm{R}^{\mathbf{\Lambda}}_\alpha(\Gamma_{e,d})$ are given below.

$$\begin{aligned}
g_a &:= \sum_{\boldsymbol{k}\in K^\alpha} \left(\psi_a Q_a(\boldsymbol{k}) - P_a(\boldsymbol{k})\right) e(\boldsymbol{k}), && \text{for any } a \in \{1,\dots,n-1\},\\
t_a &:= \sum_{\boldsymbol{j}\in J^\alpha} \xi^{j_a} e(\boldsymbol{j}), && \text{for any } a \in \{1,\dots,n\},\\
X_a &:= \sum_{\boldsymbol{i}\in I^\alpha} y_a(\boldsymbol{i}) e(\boldsymbol{i}), && \text{for any } a \in \{1,\dots,n\}.
\end{aligned}$$

As usual, we write $g_a^{(1)}$ and $X_a^{(1)}$ for the corresponding elements when $d = 1$, thus recovering the elements of [BrKl-a, §4.4].

Remark 3.4.2 (About Brundan and Kleshchev's proof - II). This remark is similar to Remark 3.3.9. Let $\boldsymbol{j} \in J^\alpha$ such that $j_a = j_{a+1}$ and consider a relation in [BrKl-a, §4] that involves only $\psi_a^{(1)}$, $e^{(1)}(\boldsymbol{i})$ and $y_b^{(1)}$ for any $\boldsymbol{i} \in I^\alpha$ and $b \in \{1,\dots,n\}$ and that proof does not require any cyclotomic relation (1.2.19). Then by the same proof, the same relation holds between $\psi_a e(\boldsymbol{j})$, $e(\boldsymbol{i},\boldsymbol{j})$ and $y_b e(\boldsymbol{j})$ in the unitary algebra $e(\boldsymbol{j})\mathrm{R}^{\mathbf{\Lambda}}_\alpha(\Gamma_{e,d})e(\boldsymbol{j})$. If $\boldsymbol{j}$ satisfies in addition $j_{a+1} = j_{a+2}$, we will be able to add relations with $\psi_{a+1}^{(1)}$, which we substitute by $\psi_{a+1}e(\boldsymbol{j})$.

3.4.2 Check of the defining relations

As in §3.3.2, we will use Remark 3.4.2 when $j_a = j_{a+1}$ to get the result from the same corresponding proof of [BrKl-a, Theorem 4.3], and when $j_a \neq j_{a+1}$ we will need a few calculations.

(3.2.2a) Straightforward since $e(\boldsymbol{j})e(\boldsymbol{j}') = \delta_{\boldsymbol{j},\boldsymbol{j}'}e(\boldsymbol{j})$ and $\xi^d = 1$.

(3.2.2b) Straightforward since $e(\boldsymbol{j})e(\boldsymbol{j}') = e(\boldsymbol{j}')e(\boldsymbol{j})$.

(3.2.2c) According to (1.2.3a), it suffices to prove $t_b g_a e(\boldsymbol{i},\boldsymbol{j}) = g_a t_{s_a(b)} e(\boldsymbol{i},\boldsymbol{j})$ for every $(\boldsymbol{i},\boldsymbol{j}) \in K^\alpha$. For $(\boldsymbol{i},\boldsymbol{j}) \in K^\alpha$, we have, using (1.2.3d),

$$\begin{aligned}
t_b g_a e(\boldsymbol{i},\boldsymbol{j}) &= t_b(\psi_a Q_a(\boldsymbol{i},\boldsymbol{j}) - P_a(\boldsymbol{i},\boldsymbol{j}))e(\boldsymbol{i},\boldsymbol{j})\\
&= t_b\left[e(s_a\cdot(\boldsymbol{i},\boldsymbol{j}))\psi_a Q_a(\boldsymbol{i},\boldsymbol{j}) - e(\boldsymbol{i},\boldsymbol{j})P_a(\boldsymbol{i},\boldsymbol{j})\right]\\
&= \xi^{(s_a\cdot\boldsymbol{j})_b}\psi_a Q_a(\boldsymbol{i},\boldsymbol{j})e(\boldsymbol{i},\boldsymbol{j}) - \xi^{j_b}P_a(\boldsymbol{i},\boldsymbol{j})e(\boldsymbol{i},\boldsymbol{j})\\
&= \psi_a Q_a(\boldsymbol{i},\boldsymbol{j})\xi^{(s_a\cdot\boldsymbol{j})_b}e(\boldsymbol{i},\boldsymbol{j}) - P_a(\boldsymbol{i},\boldsymbol{j})\xi^{j_b}e(\boldsymbol{i},\boldsymbol{j}),
\end{aligned}$$

and

$$g_a t_{s_a(b)} e(\boldsymbol{i},\boldsymbol{j}) = g_a \xi^{j_{s_a(b)}} e(\boldsymbol{i},\boldsymbol{j}) = \psi_a Q_a(\boldsymbol{i},\boldsymbol{j})\xi^{j_{s_a(b)}}e(\boldsymbol{i},\boldsymbol{j}) - P_a(\boldsymbol{i},\boldsymbol{j})\xi^{j_{s_a(b)}}e(\boldsymbol{i},\boldsymbol{j}).$$

As $(s_a\cdot\boldsymbol{j})_b = j_{s_a(b)}$ (by definition of the action of $\mathfrak{S}_n$ on J^n), it suffices to prove the following:

$$P_a(\boldsymbol{i},\boldsymbol{j})\xi^{j_b} = P_a(\boldsymbol{i},\boldsymbol{j})\xi^{j_{s_a(b)}}.$$

But this is clear if $b \notin \{a, a+1\}$ since $b = s_a(b)$, and if $b \in \{a, a+1\}$ it is clear if $j_a = j_{a+1}$ and obvious if $j_a \neq j_{a+1}$ since then $P_a(\boldsymbol{i},\boldsymbol{j}) = 0$.

(3.2.2d). Let $(\boldsymbol{i},\boldsymbol{j}) \in K^\alpha$ and let us prove $g_a^2 e(\boldsymbol{i},\boldsymbol{j}) = (q + (q-1)g_a e_a)e(\boldsymbol{i},\boldsymbol{j})$. Summing over all $(\boldsymbol{i},\boldsymbol{j}) \in K^\alpha$ will conclude. If $j_a = j_{a+1}$ then it is immediate applying Remark 3.4.2 on $(g_a^{(1)})^2 = q + (q-1)g_a^{(1)}$ and left-multiplying by $e(\boldsymbol{i})$, recalling $e_a e(\boldsymbol{j}) = e(\boldsymbol{j})$ and Corollary 3.3.8. If now $j_a \neq j_{a+1}$, since $e_a e(\boldsymbol{j}) = 0$ it suffices to prove $g_a^2 e(\boldsymbol{i},\boldsymbol{j}) = qe(\boldsymbol{i},\boldsymbol{j})$. But, recalling $Q_a(\boldsymbol{i},\boldsymbol{j}) = f_{a,\boldsymbol{j}}$ and $P_a(\boldsymbol{i},\boldsymbol{j}) = 0$,

$$\begin{aligned}
g_a^2 e(\boldsymbol{i},\boldsymbol{j}) &= g_a(\psi_a Q_a(\boldsymbol{i},\boldsymbol{j}) - P_a(\boldsymbol{i},\boldsymbol{j}))e(\boldsymbol{i},\boldsymbol{j}) \\
&= f_{a,\boldsymbol{j}} g_a \psi_a e(\boldsymbol{i},\boldsymbol{j}) \\
&= f_{a,\boldsymbol{j}} g_a e(s_a \cdot (\boldsymbol{i},\boldsymbol{j}))\psi_a \\
&= f_{a,\boldsymbol{j}} \Big(\psi_a Q_a(s_a \cdot (\boldsymbol{i},\boldsymbol{j})) - P_a(s_a \cdot (\boldsymbol{i},\boldsymbol{j}))\Big)\psi_a e(\boldsymbol{i},\boldsymbol{j}) \\
&= f_{a,\boldsymbol{j}} f_{a,s_a\cdot \boldsymbol{j}} \psi_a^2 e(\boldsymbol{i},\boldsymbol{j}),
\end{aligned}$$

hence we conclude using Lemma 3.3.15 and (1.2.16a), since $j_a \neq j_{a+1}$ implies $(i_a, j_a) \not\rightarrow (i_{a+1}, j_{a+1})$.

(3.2.2e). Let us prove $g_a g_b e(\boldsymbol{k}) = g_b g_a e(\boldsymbol{k})$ for every $\boldsymbol{k} \in K^\alpha$. By (1.2.3f) the element ψ_b commutes with the elements $P_a(\boldsymbol{k})$ and $Q_a(\boldsymbol{k})$ of $F[[y_a, y_{a+1}]]$. Moreover, $Q_a(s_b \cdot \boldsymbol{k}) = Q_a(\boldsymbol{k})$ and $P_a(s_b \cdot \boldsymbol{k}) = P_a(\boldsymbol{k})$, hence

$$\begin{aligned}
g_a g_b e(\boldsymbol{k}) &= g_a(\psi_b Q_b(\boldsymbol{k}) - P_b(\boldsymbol{k}))e(\boldsymbol{k}) \\
&= g_a e(s_b \cdot \boldsymbol{k})\psi_b Q_b(\boldsymbol{k}) - g_a e(\boldsymbol{k}) P_b(\boldsymbol{k}) \\
&= (\psi_a Q_a(\boldsymbol{k}) - P_a(\boldsymbol{k}))\psi_b Q_b(\boldsymbol{k}) e(\boldsymbol{k}) - (\psi_a Q_a(\boldsymbol{k}) - P_a(\boldsymbol{k}))P_b(\boldsymbol{k})e(\boldsymbol{k}) \\
&= \psi_a\psi_b Q_a(\boldsymbol{k})Q_b(\boldsymbol{k})e(\boldsymbol{k}) - \psi_b Q_b(\boldsymbol{k})P_a(\boldsymbol{k})e(\boldsymbol{k}) - \psi_a Q_a(\boldsymbol{k})P_b(\boldsymbol{k})e(\boldsymbol{k}) \\
&\qquad + P_a(\boldsymbol{k})P_b(\boldsymbol{k})e(\boldsymbol{k}),
\end{aligned}$$

and we conclude since that expression is symmetric in a and b (recalling (1.2.3g)).

(3.2.2f). Again it suffices to prove $g_{a+1}g_a g_{a+1} e(\boldsymbol{i},\boldsymbol{j}) = g_a g_{a+1} g_a e(\boldsymbol{i},\boldsymbol{j})$ for all $(\boldsymbol{i},\boldsymbol{j}) \in K^\alpha$. If $j_a = j_{a+1} = j_{a+2}$ we get the result using Remark 3.4.2. Let us then suppose that we are not in that case. We will intensively use (1.2.3d). Recall the following fact:

$$g_a e(\boldsymbol{i},\boldsymbol{j}) = \begin{cases} (\psi_a Q_a(\boldsymbol{i},\boldsymbol{j}) - P_a(\boldsymbol{i},\boldsymbol{j}))e(\boldsymbol{i},\boldsymbol{j}), & \text{if } j_a = j_{a+1}, \\ f_{a,\boldsymbol{j}}\psi_a e(\boldsymbol{i},\boldsymbol{j}), & \text{if } j_a \neq j_{a+1}. \end{cases}$$

Finally, as during the proof of (1.2.16b) in §3.3.2, we write for example $g_0 e(102)$ instead of $g_a e(s_a \cdot \boldsymbol{k})$. Thus, given our hypothesis on j_0, j_1 and j_2 we have:

$$\psi_1\psi_0\psi_1 e(012) = \psi_0\psi_1\psi_0 e(012). \tag{3.4.3}$$

Case $j_0 = j_1 \neq j_2$. Let us first compute $g_1 g_0 g_1 e(012)$ and $g_0 g_1 g_0 e(012)$. We set $\alpha := f_{1,(012)} f_{0,(021)}$. We have:

$$\begin{aligned}
g_1 g_0 g_1 e(012) &= f_{1,(012)} g_1 g_0 e(021)\psi_1 \\
&= f_{1,(012)} f_{0,(021)} g_1 e(201)\psi_0\psi_1 \\
&= \alpha\psi_1 Q_1(201)\psi_0\psi_1 e(012) - \alpha P_1(201)\psi_0\psi_1 e(012) \\
&= \alpha\psi_1\psi_0\psi_1 e(012) Q_1^{s_0 s_1}(201) - \alpha\psi_0\psi_1 e(012) P_1^{s_0 s_1}(201).
\end{aligned}$$

We have already seen that $Q_1^{s_0s_1}(201) = Q_0(012)$ and similarly we have $P_1^{s_0s_1}(201) = P_0(012)$ (see (3.3.18)). Hence we obtain, using (3.4.3) and noticing $f_{1,(012)} = f_{0,(120)}$ and $f_{0,(021)} = f_{1,(102)}$,

$$\begin{aligned}
g_1g_0g_1e(012) &= \alpha\psi_0\psi_1\psi_0e(012)Q_0(012) - \alpha\psi_0\psi_1e(012)P_0(012)\\
&= \alpha\psi_0\psi_1e(102)\psi_0Q_0(012) - f_{1,(012)}f_{0,(021)}\psi_0e(021)\psi_1P_0(012)\\
&= f_{1,(102)}f_{0,(120)}\psi_0e(120)\psi_1\psi_0Q_0(012) - f_{1,(012)}g_0\psi_1e(012)P_0(012)\\
&= f_{1,(102)}g_0\psi_1e(102)\psi_0Q_0(012) - g_0g_1P_0(012)e(012)\\
&= g_0g_1(\psi_0Q_0(012) - P_0(012))e(012)\\
&= g_0g_1g_0e(012),
\end{aligned}$$

so we are done.

Case $j_0 \neq j_1 = j_2$. Similar.

Case $j_0 = j_2 \neq j_1$. Given these assumptions we have

$$\psi_0^2e(012) = \psi_1^2e(012) = e(012). \tag{3.4.4}$$

Hence, using (3.4.4), with $\alpha \coloneqq f_{1,(012)}f_{1,(201)}$,

$$\begin{aligned}
g_1g_0g_1e(012) &= f_{1,(012)}g_1(\psi_0Q_0(021) - P_0(021))e(021)\psi_1\\
&= f_{1,(012)}g_1e(201)\psi_0Q_0(021)\psi_1 - f_{1,(012)}g_1e(021)P_0(021)\psi_1\\
&= \alpha\psi_1\psi_0\psi_1e(012)Q_0^{s_1}(021) - \alpha\psi_1^2e(012)P_0^{s_1}(021)\\
&= \alpha\psi_0\psi_1\psi_0e(012)Q_1^{s_0}(102) - \alpha\psi_0^2e(012)P_1^{s_0}(102)\\
&= \alpha\psi_0e(120)\psi_1Q_1(102)\psi_0 - \alpha\psi_0e(102)P_1(102)\psi_0.
\end{aligned}$$

Noticing $f_{1,(012)} = f_{0,(120)} = f_{0,(102)}$ and $f_{1,(201)} = f_{0,(012)}$ we finally obtain

$$\begin{aligned}
g_1g_0g_1e(012) &= f_{0,(012)}g_0(\psi_1Q_1(102) - P_1(102))e(102)\psi_0\\
&= f_{0,(012)}g_0g_1\psi_0e(012)\\
&= g_0g_1g_0e(012).
\end{aligned}$$

Case $\#\{j_0, j_1, j_2\} = 3$. We immediately obtain

$$\begin{aligned}
g_1g_0g_1e(012) &= f_{1,(201)}f_{0,(021)}f_{1,(012)}\psi_1\psi_0\psi_1e(012)\\
&= f_{0,(120)}f_{1,(102)}f_{0,(012)}\psi_0\psi_1\psi_0e(012)\\
&= g_0g_1g_0e(012),
\end{aligned}$$

since $f_{1,(201)} = f_{0,(012)}$ and $f_{0,(021)} = f_{1,(102)}$ and $f_{1,(012)} = f_{0,(120)}$.

(3.2.2g). Since for $a \in \{1, \dots, n-1\}$ it is clear that $X_{a+1}X_a = X_aX_{a+1}$, it remains to prove that $qX_{a+1} = g_aX_ag_a$ and we will conclude taking $a = 1$. As we proved (3.2.2d), it suffices to prove (3.2.4b). Let $(\boldsymbol{i}, \boldsymbol{j}) \in K^\alpha$ and let us prove

$$g_aX_{a+1}e(\boldsymbol{i}, \boldsymbol{j}) = \begin{cases} X_ag_ae(\boldsymbol{i}, \boldsymbol{j}) + (q-1)X_{a+1}e(\boldsymbol{i}, \boldsymbol{j}), & \text{if } j_a = j_{a+1},\\ X_ag_ae(\boldsymbol{i}, \boldsymbol{j}), & \text{if } j_a \neq j_{a+1}.\end{cases}$$

Again, we deduce the case $j_a = j_{a+1}$ from Remark 3.4.2. If $j_a \neq j_{a+1}$ we have, using (1.2.3d) and (1.2.3h),

$$\begin{aligned} g_a X_{a+1} e(\boldsymbol{i}, \boldsymbol{j}) &= q^{i_{a+1}} g_a e(\boldsymbol{i}, \boldsymbol{j})(1 - y_{a+1}) \\ &= q^{i_{a+1}} f_{a,\boldsymbol{j}} \psi_a (1 - y_{a+1}) e(\boldsymbol{i}, \boldsymbol{j}) \\ &= q^{i_{a+1}} f_{a,\boldsymbol{j}} (1 - y_a) e(s_a \cdot (\boldsymbol{i}, \boldsymbol{j})) \psi_a \\ &= f_{a,\boldsymbol{j}} X_a \psi_a e(\boldsymbol{i}, \boldsymbol{j}) \\ &= X_a g_a e(\boldsymbol{i}, \boldsymbol{j}). \end{aligned}$$

(3.2.2h). We prove in fact (3.2.4a), that is, $g_a X_b = X_b g_a$ for $b \neq a, a+1$. As y_b commutes with ψ_a by (1.2.3f) we have, for any $\boldsymbol{k} \in K^\alpha$ (where $y_a(\boldsymbol{k}) := y_a(\boldsymbol{i})$ with $\boldsymbol{k} = (\boldsymbol{i}, \boldsymbol{j})$),

$$\begin{aligned} g_a X_b e(\boldsymbol{k}) &= g_a e(\boldsymbol{k}) y_b(\boldsymbol{k}) \\ &= y_b(\boldsymbol{k})(\psi_a Q_a(\boldsymbol{k}) - P_a(\boldsymbol{k})) e(\boldsymbol{k}) \\ &= y_b(\boldsymbol{k}) e(s_a \cdot \boldsymbol{k}) \psi_a Q_a(\boldsymbol{k}) - y_b(\boldsymbol{k}) e(\boldsymbol{k}) P_a(\boldsymbol{k}) \\ &= X_b g_a e(\boldsymbol{k}), \end{aligned}$$

since $y_b(\boldsymbol{k}) e(s_a \cdot \boldsymbol{k}) = q^{(s_a \cdot \boldsymbol{i})_b}(1 - y_b) e(s_a \cdot \boldsymbol{k}) = q^{i_b}(1 - y_b) e(s_a \cdot \boldsymbol{k}) = X_b e(s_a \cdot \boldsymbol{k})$.

(3.2.2i). We prove in fact $X_a t_b = t_b X_a$ for every a, b. That is straightforward from (1.2.3c).

(3.2.2j). We have, using (1.2.3a)–(1.2.3c),

$$\begin{aligned} \prod_{i \in I} (X_1 - q^i)^{\Lambda_i} &= \prod_{i \in I} \left[\sum_{\boldsymbol{i} \in I^\alpha} \left(q^{i_1}(1 - y_1) - q^i \right) e(\boldsymbol{i}) \right]^{\Lambda_i} \\ &= \prod_{i \in I} \left[\sum_{\boldsymbol{i} \in I^\alpha} \left(q^{i_1}(1 - y_1) - q^i \right)^{\Lambda_i} e(\boldsymbol{i}) \right] \\ &= \sum_{\boldsymbol{i} \in I^\alpha} \prod_{i \in I} \left[\left(q^{i_1}(1 - y_1) - q^i \right)^{\Lambda_i} e(\boldsymbol{i}) \right]. \end{aligned}$$

Noticing that for each $\boldsymbol{i} \in I^\alpha$ the term for $i = i_1$ vanishes by (1.2.19), we get the result.

3.5 Isomorphism theorem

We give now the main result of this chapter. Let $\boldsymbol{\Lambda}$ be a weight as in Sections 3.3 and 3.4.

3.5.1 Statement

Theorem 3.5.1. *There is a presentation of the algebra $\mathrm{Y}_\alpha^{\boldsymbol{\Lambda}}(q)$ given by the generators (1.2.2) and the relations (1.2.3), (1.2.16) and (1.2.19), that is, we have an F-algebra isomorphism*

$$\mathrm{R}_\alpha^{\boldsymbol{\Lambda}}(\Gamma_{e,d}) \xrightarrow{\sim} \mathrm{Y}_\alpha^{\boldsymbol{\Lambda}}(q).$$

Since $\mathrm{Y}_{d,n}^{\boldsymbol{\Lambda}}(q) = \oplus_{\alpha \models_K n} \mathrm{Y}_\alpha^{\boldsymbol{\Lambda}}(q)$, we deduce the following algebra F-isomorphism:

$$\mathrm{R}_n^{\boldsymbol{\Lambda}}(\Gamma_{e,d}) \simeq \mathrm{Y}_{d,n}^{\boldsymbol{\Lambda}}(q). \tag{3.5.2}$$

Note that we already saw in §2.3.3.3 that there are only finitely many non-zero terms in the decomposition $\mathrm{R}_n^{\boldsymbol{\Lambda}}(\Gamma_{e,d}) = \bigoplus_{\alpha \models_K n} \mathrm{R}_\alpha^{\boldsymbol{\Lambda}}(\Gamma_{e,d})$. Now recalling that the cyclotomic quiver Hecke algebra is naturally graded (Proposition 1.2.17), we obtain the following corollary.

Corollary 3.5.3. *The cyclotomic Yokonuma–Hecke algebra inherits the grading of the cyclotomic quiver Hecke algebra.*

Moreover, as we obtain a presentation of $\mathrm{Y}^{\mathbf{\Lambda}}_{d,n}(q)$ which does not depend on q, we also get another one (see Corollary 3.6.21 for a slight improvement).

Corollary 3.5.4. *Let $\tilde{q} \in F \setminus \{0,1\}$. If $\mathrm{char}_{\tilde{q}}(F) = \mathrm{char}_q(F)$ then the algebras $\mathrm{Y}^{\mathbf{\Lambda}}_{d,n}(q)$ and $\mathrm{Y}^{\mathbf{\Lambda}}_{d,n}(\tilde{q})$ are isomorphic.*

Let us now prove Theorem 3.5.1. First, as we have a (non-unitary) algebra homomorphism $\mathrm{Y}^{\mathbf{\Lambda}}_{\alpha}(q) \to \mathrm{Y}^{\mathbf{\Lambda}}_{d,n}(q)$, by Theorem 3.4.1 we get an algebra homomorphism $\mathrm{Y}^{\mathbf{\Lambda}}_{\alpha}(q) \to \mathrm{R}^{\mathbf{\Lambda}}_{\alpha}(\Gamma_{e,d})$, that we still call ρ_{YR}. We will prove that $\rho_{\mathrm{YR}} : \mathrm{Y}^{\mathbf{\Lambda}}_{\alpha}(q) \to \mathrm{R}^{\mathbf{\Lambda}}_{\alpha}(\Gamma_{e,d})$ and $\rho_{\mathrm{RY}} : \mathrm{R}^{\mathbf{\Lambda}}_{\alpha}(\Gamma_{e,d}) \to \mathrm{Y}^{\mathbf{\Lambda}}_{\alpha}(q)$ (from Theorem 3.3.1) satisfy $\rho_{\mathrm{YR}} \circ \rho_{\mathrm{RY}} = \mathrm{id}_{\mathrm{R}^{\mathbf{\Lambda}}_{\alpha}(\Gamma_{e,d})}$ and $\rho_{\mathrm{RY}} \circ \rho_{\mathrm{YR}} = \mathrm{id}_{\mathrm{Y}^{\mathbf{\Lambda}}_{\alpha}(q)}$. Since these are algebra homomorphisms, it suffices to prove that they are identity on generators. To clarify the proof, let us add a $^{\mathrm{Y}}$ on the quiver Hecke generators of $\mathrm{Y}^{\mathbf{\Lambda}}_{\alpha}(q)$ and a $^{\mathrm{R}}$ on the Yokonuma–Hecke generators of $\mathrm{R}^{\mathbf{\Lambda}}_{\alpha}(\Gamma_{e,d})$.

3.5.2 Proof of $\rho_{\mathrm{YR}} \circ \rho_{\mathrm{RY}} = \mathrm{id}_{\mathrm{R}^{\mathbf{\Lambda}}_{\alpha}(\Gamma)}$

We have to check that $\rho_{\mathrm{YR}}(\rho_{\mathrm{RY}}(e(\boldsymbol{k}))) = e(\boldsymbol{k})$ for all $\boldsymbol{k} \in K^{\alpha}$, that $\rho_{\mathrm{YR}}(\rho_{\mathrm{RY}}(y_a)) = y_a$ for all $1 \leq a \leq n$ and that $\rho_{\mathrm{YR}}(\rho_{\mathrm{RY}}(\psi_a)) = \psi_a$ for all $1 \leq a < n$.

Let us start by finding the image of $e(\boldsymbol{k})$ by $\rho_{\mathrm{YR}} \circ \rho_{\mathrm{RY}}$. By definition of ρ_{RY} we have $\rho_{\mathrm{RY}}(e(\boldsymbol{k})) = e^{\mathrm{Y}}(\boldsymbol{k})$, so we have to prove $\rho_{\mathrm{YR}}(e^{\mathrm{Y}}(\boldsymbol{k})) = e(\boldsymbol{k})$. Let $M := \mathrm{R}^{\mathbf{\Lambda}}_{\alpha}(\Gamma_{e,d})$. The algebra homomorphism ρ_{YR} gives M a structure of $\mathrm{Y}^{\mathbf{\Lambda}}_{\alpha}(q)$-module, finite-dimensional thanks to Theorem 1.2.24. If $M(\boldsymbol{k})$ denotes the weight space as in (3.3.2), by Remark 3.3.4 we know that the projection onto $M(\boldsymbol{k})$ along $\oplus_{\boldsymbol{k}' \neq \boldsymbol{k}} M(\boldsymbol{k}')$ is given by $\rho_{\mathrm{YR}}(e^{\mathrm{Y}}(\boldsymbol{k}))$. We prove that $e(\boldsymbol{k})$ is this projection too.

Let $(\boldsymbol{i}, \boldsymbol{j}) \in K^{\alpha}$. For any $1 \leq a \leq n$, we have $\rho_{\mathrm{YR}}(X_a) = X^{\mathrm{R}}_a = \sum_{\boldsymbol{i}'} (q^{i'_a} - q^{i'_a} y_a) e(\boldsymbol{i}')$ so:

$$X^{\mathrm{R}}_a - q^{i_a} = \sum_{\boldsymbol{i}' \in I^{\alpha}} \left[\left(q^{i'_a} - q^{i_a} \right) - q^{i'_a} y_a \right] e(\boldsymbol{i}').$$

Since y_a is nilpotent, thanks to (1.2.3a)–(1.2.3b) we have

$$\left\{ v \in M : \left(X^{\mathrm{R}}_a - q^{i_a} \right)^N v = 0 \right\} = \left(\sum_{\substack{\boldsymbol{i}' \in I^{\alpha} \\ i'_a = i_a}} e(\boldsymbol{i}') \right) M,$$

hence, for $N \gg 0$ we have

$$M(\boldsymbol{i}) := \left\{ v \in M : \left(X^{\mathrm{R}}_a - q^{i_a} \right)^N v = 0 \text{ for all } a \right\} = e(\boldsymbol{i}) M.$$

In a similar way we have $M(\boldsymbol{j}) = e(\boldsymbol{j}) M$ where $M(\boldsymbol{j}) := \{ v \in M : (t^{\mathrm{R}}_a - \xi^{j_a}) v = 0 \text{ for all } a \}$, thus

$$M(\boldsymbol{k}) = e(\boldsymbol{k}) M.$$

Hence, as $\oplus_k M(\boldsymbol{k}) = M$ we conclude that $e(\boldsymbol{k})$ is the desired projection and finally $e(\boldsymbol{k}) = \rho_{\mathrm{YR}}(e^{\mathrm{Y}}(\boldsymbol{k}))$.

The end of the proof is without any difficulty. We have:

$$\begin{aligned}
\rho_{\mathrm{YR}}(\rho_{\mathrm{RY}}(y_a)) = \rho_{\mathrm{YR}}(y_a^{\mathrm{Y}}) &= \sum_{\boldsymbol{i}\in I^\alpha} \left[1 - q^{-i_a}\rho_{\mathrm{YR}}(X_a)\right] \rho_{\mathrm{YR}}(e^{\mathrm{Y}}(\boldsymbol{i}))\\
&= \sum_{\boldsymbol{i}\in I^\alpha} \left[1 - q^{-i_a} X_a^{\mathrm{R}}\right] e(\boldsymbol{i})\\
&= \sum_{\boldsymbol{i}\in I^\alpha} \left[1 - q^{-i_a} \sum_{\boldsymbol{i}'\in I^\alpha} y_a(\boldsymbol{i}')e(\boldsymbol{i}')\right] e(\boldsymbol{i})\\
&= \sum_{\boldsymbol{i}\in I^\alpha} \left[1 - q^{-i_a} y_a(\boldsymbol{i})\right] e(\boldsymbol{i})\\
&= \sum_{\boldsymbol{i}\in I^\alpha} \left[1 - q^{-i_a} q^{i_a}(1-y_a)\right] e(\boldsymbol{i})\\
&= y_a.
\end{aligned}$$

Thus, we have $\rho_{\mathrm{YR}}(Q_a^{\mathrm{Y}}(\boldsymbol{k})) = Q_a(\boldsymbol{k})$ and $\rho_{\mathrm{YR}}(P_a^{\mathrm{Y}}(\boldsymbol{k})) = P_a(\boldsymbol{k})$. Hence, recalling (3.3.17),

$$\begin{aligned}
\rho_{\mathrm{YR}}(\rho_{\mathrm{RY}}(\psi_a)) = \rho_{\mathrm{YR}}(\psi_a^{\mathrm{Y}}) &= \sum_{\boldsymbol{k}\in K^\alpha} \rho_{\mathrm{YR}}(\Phi_a)\rho_{\mathrm{YR}}(Q_a^{\mathrm{Y}}(\boldsymbol{k}))^{-1}\rho_{\mathrm{YR}}(e^{\mathrm{Y}}(\boldsymbol{k}))\\
&= \sum_{\boldsymbol{k}\in K^\alpha}\left(\sum_{\boldsymbol{k}'\in K^\alpha}\left[\rho_{\mathrm{YR}}(g_a) + \rho_{\mathrm{YR}}(P_a^{\mathrm{Y}}(\boldsymbol{k}'))\right] e(\boldsymbol{k}')\right) Q_a(\boldsymbol{k})^{-1} e(\boldsymbol{k})\\
&= \sum_{\boldsymbol{k}\in K^\alpha} (g_a^{\mathrm{R}} + P_a(\boldsymbol{k}))Q_a(\boldsymbol{k})^{-1}e(\boldsymbol{k})\\
&= \sum_{\boldsymbol{k}\in K^\alpha}\left(\left[\sum_{\boldsymbol{k}'\in K^\alpha}(\psi_a Q_a(\boldsymbol{k}') - P_a(\boldsymbol{k}'))\, e(\boldsymbol{k}')\right] + P_a(\boldsymbol{k})\right) Q_a(\boldsymbol{k})^{-1}e(\boldsymbol{k})\\
&= \sum_{\boldsymbol{k}\in K^\alpha}\left[(\psi_a Q_a(\boldsymbol{k}) - P_a(\boldsymbol{k})) + P_a(\boldsymbol{k})\right] Q_a(\boldsymbol{k})^{-1}e(\boldsymbol{k})\\
&= \psi_a.
\end{aligned}$$

3.5.3 Proof of $\rho_{\mathrm{RY}} \circ \rho_{\mathrm{YR}} = \mathrm{id}_{\mathrm{Y}_\alpha^\Lambda(q)}$

This is even easier: we have to check that $\rho_{\mathrm{RY}}(g_a^{\mathrm{R}}) = g_a$ for any $1 \leq a < n$ and that $\rho_{\mathrm{RY}}(X_a^{\mathrm{R}}) = X_a$ and $\rho_{\mathrm{RY}}(t_a^{\mathrm{R}}) = t_a$ for any $1 \leq a \leq n$. We have

$$\begin{aligned}
\rho_{\mathrm{RY}}(g_a^{\mathrm{R}}) &= \sum_{\boldsymbol{k}\in K^\alpha}\left[\psi_a^{\mathrm{Y}} Q_a^{\mathrm{Y}}(\boldsymbol{k}) - P_a^{\mathrm{Y}}(\boldsymbol{k})\right] e^{\mathrm{Y}}(\boldsymbol{k})\\
&= \sum_{\boldsymbol{k}\in K^\alpha}\left[\Phi_a Q_a^{\mathrm{Y}}(\boldsymbol{k})^{-1} Q_a^{\mathrm{Y}}(\boldsymbol{k}) - P_a^{\mathrm{Y}}(\boldsymbol{k})\right] e^{\mathrm{Y}}(\boldsymbol{k})\\
&= \sum_{\boldsymbol{k}\in K^\alpha}\left[\Phi_a - P_a^{\mathrm{Y}}(\boldsymbol{k})\right] e^{\mathrm{Y}}(\boldsymbol{k})\\
&= g_a.
\end{aligned}$$

Recalling (3.3.11), we have

$$\rho_{\mathrm{RY}}(X_a^{\mathrm{R}}) = \sum_{\boldsymbol{i}\in I^\alpha} y_a^{\mathrm{Y}}(\boldsymbol{i}) e^{\mathrm{Y}}(\boldsymbol{i}) = \sum_{\boldsymbol{i}\in I^\alpha} X_a e^{\mathrm{Y}}(\boldsymbol{i}) = X_a.$$

Finally,

$$\rho_{\mathrm{RY}}(t_a^{\mathrm{R}}) = \sum_{\boldsymbol{j}\in J^\alpha} \xi^{j_a} e^{\mathrm{Y}}(\boldsymbol{j}) = \sum_{\boldsymbol{j}\in J^\alpha} t_a e^{\mathrm{Y}}(\boldsymbol{j}) = t_a.$$

The proof of Theorem 3.5.1 is now over.

3.6 Degenerate case

In this section, we extend the previous results to the case $q = 1$. In particular, we need to define a new "degenerate" cyclotomic Yokonuma–Hecke algebra. Many calculations are not written, since they are entirely similar to the non-degenerate case. Note the following thing: since the cyclotomic quiver Hecke algebra has no q in its presentation, we do not need to define some new cyclotomic quiver Hecke algebra.

Let $\mathbf{\Lambda} = (\Lambda_k)_{k\in K} \in \mathbb{N}^{(K)}$ be a weight; we assume that $\ell(\mathbf{\Lambda}) = \sum_{k\in K} \Lambda_k$ satisfies $\ell(\mathbf{\Lambda}) > 0$. Moreover, as in Section 3.3 we suppose that for any $i \in I$ and $j, j' \in J$, we have

$$\Lambda_{i,j} = \Lambda_{i,j'} =: \Lambda_i.$$

In particular, we will write $\mathbf{\Lambda}$ as well for the weight $(\Lambda_i)_{i\in I}$.

3.6.1 Degenerate cyclotomic Yokonuma–Hecke algebras

We introduce here the degenerate cyclotomic Yokonuma–Hecke algebra: this algebra can be seen as the rational degeneration of the cyclotomic Yokonuma–Hecke algebra $\mathrm{Y}^{\mathbf{\Lambda}}_{d,n}(q)$.

The *degenerate cyclotomic Yokonuma–Hecke algebra of type A*, denoted by $\mathrm{Y}^{\mathbf{\Lambda}}_{d,n}(1)$, is the unitary associative F-algebra generated by the elements

$$f_1, \dots, f_{n-1}, t_1, \dots, t_n, x_1, \dots, x_n \tag{3.6.1}$$

subject to the following relations:

$$\begin{aligned}
t_a^d &= 1, && && (3.6.2a)\\
t_a t_{a'} &= t_{a'} t_a, && && (3.6.2b)\\
t_a f_b &= f_b t_{s_b(a)}, && && (3.6.2c)\\
f_b^2 &= 1, && && (3.6.2d)\\
f_b f_{b'} &= f_{b'} f_b, && \text{if } |b - b'| > 1, && (3.6.2e)\\
f_{c+1} f_c f_{c+1} &= f_c f_{c+1} f_c, && && (3.6.2f)\\
x_a x_{a'} &= x_{a'} x_a, && && (3.6.2g)\\
f_b x_{b+1} &= x_b f_b + e_b, && && (3.6.2h)\\
f_b x_a &= x_a f_b, && \text{if } a \neq b, b+1, && (3.6.2i)\\
x_a t_{a'} &= t_{a'} x_a, && && (3.6.2j)\\
\prod_{i\in I} (x_1 - i)^{\Lambda_i} &= 0, && && (3.6.2k)
\end{aligned}$$

for all $a, a' \in \{1, \dots, n\}$, $b, b' \in \{1, \dots, n-1\}$ and $c \in \{1, \dots, n-2\}$, with $e_b := \frac{1}{d}\sum_{j\in J} t_b^j t_{b+1}^{-j}$.

We obtained this presentation by setting $X_a = 1 + (q-1)x_a$ in $\mathrm{Y}^{\mathbf{\Lambda}}_{d,n}(q)$, simplifying by $(1-q)$ as much as we can and then setting $q = 1$ (according to the transformation made by Drinfeld [Dr] to define degenerate Hecke algebras). As in the non-degenerate case, the element e_a satisfies $e_a^2 = e_a$ and commutes with f_a. Finally, note that by (3.6.2d) and (3.6.2h) we have

$$\begin{aligned}
x_{a+1} &= f_a x_a f_a + f_a e_a, && (3.6.3a)\\
x_{a+1} f_a &= f_a x_a + e_a, && (3.6.3b)
\end{aligned}$$

for all $a \in \{1, \dots, n-1\}$. When $d = 1$, the algebra $\mathrm{H}_n^{\mathbf{\Lambda}}(1) := \mathrm{Y}_{1,n}^{\mathbf{\Lambda}}(1)$ is the *cyclotomic Hecke algebra* of [BrKl-a]. In particular, the element e_a becomes 1, and f_a (respectively x_b) is the element s_a (resp. x_b) of [BrKl-a, §3].

We will use the following lemma (see [ChPA15, Lemma 2.15] for the non-degenerate case).

Lemma 3.6.4. *For any $u, v \in \mathbb{N}$ and $a \in \{1, \dots, n-1\}$ we have the following equalities:*

$$f_a x_a x_{a+1} = x_a x_{a+1} f_a, \tag{3.6.5a}$$

$$f_a x_{a+1}^v = x_a^v f_a + e_a \sum_{m=0}^{v-1} x_a^m x_{a+1}^{v-1-m}, \tag{3.6.5b}$$

$$f_a x_a^u = x_{a+1}^u f_a - e_a \sum_{m=0}^{u-1} x_a^m x_{a+1}^{u-1-m}, \tag{3.6.5c}$$

$$f_a x_a^u x_{a+1}^v = \begin{cases} x_a^v x_{a+1}^u f_a + e_a \displaystyle\sum_{m=0}^{v-u-1} x_a^{u+m} x_{a+1}^{v-1-m}, & \text{if } u \leq v, \\ x_a^v x_{a+1}^u f_a - e_a \displaystyle\sum_{m=0}^{u-v-1} x_a^{u-1+m} x_{a+1}^{v-m}, & \text{if } u \geq v. \end{cases} \tag{3.6.5d}$$

Proof. We deduce (3.6.5a) from different previous relations. The relations (3.6.5b) and (3.6.5c) can be proved by an easy induction. The equality (3.6.5d) follows finally from these previous equalities. □

As the elements g_a for all $a \in \{1, \dots, n-1\}$ satisfy the same braid relations as the transpositions $s_a \in \mathfrak{S}_n$, for each $w \in \mathfrak{S}_n$ there is a well-defined element $g_w := g_{a_1} \cdots g_{a_r} \in \mathrm{Y}_{d,n}^{\mathbf{\Lambda}}(1)$ which does not depend on the reduced expression $w = s_{a_1} \cdots s_{a_r}$.

Proposition 3.6.6. *The algebra $\mathrm{Y}_{d,n}^{\mathbf{\Lambda}}(1)$ is a finite-dimensional F-vector space and a family of generators is given by the elements $f_w x_1^{u_1} \cdots x_n^{u_n} t_1^{v_1} \cdots t_n^{v_n}$ for all $w \in \mathfrak{S}_n, u_a \in \{0, \dots, \ell(\mathbf{\Lambda}) - 1\}$ and $v_a \in J$.*

Proof. We use a similar method to [ArKo, OgPA]. As the unit element belongs to the above family, it suffices to prove that the F-vector space V spanned by these elements is stable under (right-)multiplication by the generators of $\mathrm{Y}_{d,n}^{\mathbf{\Lambda}}(1)$.

Let us consider $\alpha := f_w x_1^{u_1} \cdots x_n^{u_n} t_1^{v_1} \cdots t_n^{v_n}$ as in the proposition. By (3.6.2a) and (3.6.2b) the element αt_a remains in V. Moreover, writing (by (3.6.2c) and (3.6.2i))

$$\alpha f_a = f_w x_1^{u_1} \cdots x_{a-1}^{u_{a-1}} \left(x_a^{u_a} x_{a+1}^{u_{a+1}} f_a\right) x_{a+2}^{u_{a+2}} \cdots x_n^{u_n} t_1^{v_1} \cdots t_n^{v_n},$$

and using (3.6.5d) we conclude that $\alpha f_a \in V$, noticing that the element

$$x_1^{u_1} \cdots x_{a-1}^{u_{a-1}} \left(e_a x_a^{u'_a} x_{a+1}^{u'_{a+1}}\right) x_{a+2}^{u_{a+2}} \cdots x_n^{u_n} t_1^{v_1} \cdots t_n^{v_n}$$

belongs to V for every $u'_a, u'_{a+1} \in \{0, \dots, \ell(\mathbf{\Lambda}) - 1\}$. Finally, according to (3.6.3a), to prove that αx_a remains in V it suffices now to prove that $\alpha x_1 \in V$, but this is clear by (3.6.2g), (3.6.2j) and (3.6.2k). □

Now let M be a finite-dimensional $\mathrm{Y}_{d,n}^{\mathbf{\Lambda}}(1)$-module. It is a finite-dimensional F-vector space thanks to Proposition 3.6.6. By (3.6.2k), the eigenvalues of x_1 on M belong to I. We prove in Lemma 3.6.8 that all the x_a have in fact their eigenvalues in I. This is the degenerate analogue of [CuWa, Lemma 5.2], which we used in §3.3.1.

Lemma 3.6.7. *We have*

$$x_a\phi_a = \phi_a x_{a+1},$$
$$\phi_a^2 = (x_{a+1} - x_a - e_a)(x_a - x_{a+1} - e_a),$$

where ϕ_a is the "intertwining operator" defined by

$$\phi_a \coloneqq f_a(x_a - x_{a+1}) + e_a.$$

Proof. These are straightforward calculations. We have, using (3.6.2h),

$$\begin{aligned} x_a\phi_a &= (f_a x_{a+1} - e_a)(x_a - x_{a+1}) + x_a e_a \\ &= f_a(x_a - x_{a+1})x_{a+1} + x_{a+1}e_a \\ &= \phi_a x_{a+1}, \end{aligned}$$

and

$$\begin{aligned} \phi_a^2 &= f_a(x_a - x_{a+1})f_a(x_a - x_{a+1}) + 2f_a(x_a - x_{a+1})e_a + e_a \\ &= f_a(f_a x_{a+1} - e_a - f_a x_a - e_a)(x_a - x_{a+1}) + 2f_a(x_a - x_{a+1})e_a + e_a \\ &= (x_{a+1} - x_a)(x_a - x_{a+1}) + e_a \\ &= (x_{a+1} - x_a - e_a)(x_a - x_{a+1} - e_a). \end{aligned}$$

□

Lemma 3.6.8. *The eigenvalues of x_a belong to I for every $a \in \{1, \dots, n\}$.*

Proof. We proceed by induction on a. The proposition is true for $a = 1$ and we suppose that it is true for some $a \in \{1, \dots, n-1\}$. Let λ be an eigenvalue of x_{a+1} (in a algebraic closure of F). As the family $\{x_a, x_{a+1}, e_a\}$ is commutative, we can find a common eigenvector v in the eigenspace of x_{a+1} associated with λ. We have $x_a v = iv$ and $e_a v = \delta v$ for some $i \in I$ (by induction hypothesis) and $\delta \in \{0, 1\}$ (since $e_a^2 = e_a$). We distinguish now whether $\phi_a v$ vanishes or not:

- if $\phi_a v \neq 0$, we get by Lemma 3.6.7

  $$x_a(\phi_a v) = \phi_a(x_{a+1}v) = \lambda\phi_a v,$$

  hence λ is an eigenvalue for x_a and by induction hypothesis we get $\lambda \in I$;

- if $\phi_a v = 0$, by the same lemma we have

  $$\phi_a^2 v = (\lambda - i - \delta)(i - \lambda - \delta)v = 0,$$

  hence $\lambda = i \pm \delta \in I$.

□

3.6.2 Quiver Hecke generators of $\mathrm{Y}_{d,n}^{\mathbf{\Lambda}}(1)$

We proceed as in Section 3.3: we define some central idempotents, then some "quiver Hecke generators" on which we check the defining relations of $\mathrm{R}_\alpha^{\mathbf{\Lambda}}(\Gamma_{e,d})$. The proofs are entirely similar to the non-degenerate case (even easier; note that once again the "hard work" has been made in [BrKl-a]), hence we won't write them down. However, we will still define the different involved elements.

3.6.2.1 Image of $e(\boldsymbol{i}, \boldsymbol{j})$

Let M be a finite-dimensional $\mathrm{Y}_{d,n}^{\boldsymbol{\Lambda}}(1)$-module. We know that the t_a are diagonalisable with eigenvalues in J. Hence, recalling Lemma 3.6.8, we can write (recall that the family $\{X_a, t_a\}_{1\leq a\leq n}$ is commutative)

$$M = \bigoplus_{(\boldsymbol{i},\boldsymbol{j})\in I^n\times J^n} M(\boldsymbol{i}, \boldsymbol{j}),$$

with:

$$M(\boldsymbol{i}, \boldsymbol{j}) \coloneqq \left\{ v \in M : (x_a - i_a)^N v = (t_a - \xi^{j_a})v = 0 \text{ for all } 1 \leq a \leq n \right\},$$

with $N \gg 0$. Since M is a finite-dimensional F-vector space, only finitely many $M(\boldsymbol{i}, \boldsymbol{j})$ are non-zero. Considering once again the family of projections $\{e(\boldsymbol{k})\}_{\boldsymbol{k}\in K^n}$ associated with $M = \oplus_{\boldsymbol{k}\in K^n} M(\boldsymbol{k})$, we define for any $\alpha \models_K n$

$$e(\alpha) \coloneqq \sum_{\boldsymbol{k}\in K^\alpha} e(\boldsymbol{k}),$$

and we set $\mathrm{Y}_\alpha^{\boldsymbol{\Lambda}}(1) \coloneqq e(\alpha)\mathrm{Y}_{d,n}^{\boldsymbol{\Lambda}}(1)$. We can now define, for any $\boldsymbol{i} \in I^\alpha$ and $\boldsymbol{j} \in J^\alpha$,

$$\begin{aligned} e(\alpha)(\boldsymbol{i}) &\coloneqq \sum_{\boldsymbol{j}\in J^\alpha} e(\alpha)e(\boldsymbol{i}, \boldsymbol{j}), \\ e(\alpha)(\boldsymbol{j}) &\coloneqq \sum_{\boldsymbol{i}\in I^\alpha} e(\alpha)e(\boldsymbol{i}, \boldsymbol{j}). \end{aligned} \tag{3.6.9}$$

In particular, with $e(\alpha)(\boldsymbol{i})$ for $d = 1$ we recover the element $e(\boldsymbol{i})$ of [BrKl-a, §3.1].

From now on, unless mentioned otherwise we always work in $\mathrm{Y}_\alpha^{\boldsymbol{\Lambda}}(1)$. Every relation should be multiplied by $e(\alpha)$ and we write $e(\boldsymbol{i})$ and $e(\boldsymbol{j})$ without any (α).

Lemma 3.6.10. *If $1 \leq a < n$ and $\boldsymbol{j} \in J^\alpha$ is such that $j_a \neq j_{a+1}$ then we have:*

$$\begin{aligned} f_a x_{a+1} e(\boldsymbol{j}) &= x_a f_a e(\boldsymbol{j}), \\ x_{a+1} f_a e(\boldsymbol{j}) &= f_a x_a e(\boldsymbol{j}). \end{aligned}$$

Lemma 3.6.11. *For $1 \leq a < n$ and $\boldsymbol{j} \in J^\alpha$ we have $f_a e(\boldsymbol{j}) = e(s_a \cdot \boldsymbol{j}) f_a$. In particular, if $j_a = j_{a+1}$ then f_a and $e(\boldsymbol{j})$ commute. Moreover, if $j_a \neq j_{a+1}$ then $f_a e(\boldsymbol{i}, \boldsymbol{j}) = e(s_a \cdot (\boldsymbol{i}, \boldsymbol{j})) f_a$.*

Remark 3.6.12 (About Brundan and Kleshchev's proof - III). Let $a \in \{1, \dots, n-1\}$. If $\boldsymbol{j} \in J^\alpha$ satisfies $j_a = j_{a+1}$, when a proof in [BrKl-a, §3.3] needs only the elements f_a, x_b, $e(\boldsymbol{i})$ and the corresponding relations in $\mathrm{H}_\alpha^{\boldsymbol{\Lambda}}(1)$, we claim that the same proof holds in $e(\boldsymbol{j})\mathrm{Y}_\alpha^{\boldsymbol{\Lambda}}(1)e(\boldsymbol{j})$. We extend this claim to the case $j_a = j_{a+1} = j_{a+2}$.

3.6.2.2 Image of y_a

We define the following elements of $\mathrm{Y}_\alpha^{\boldsymbol{\Lambda}}(1)$ for $1 \leq a \leq n$:

$$y_a \coloneqq \sum_{\boldsymbol{i}\in I^\alpha} (x_a - i_a) e(\boldsymbol{i}) \in \mathrm{Y}_\alpha^{\boldsymbol{\Lambda}}(1).$$

When $d = 1$ we recover the elements defined in [BrKl-a, §3.3]. These elements are nilpotent: we will be able to make calculations in the ring $F[[y_1, \dots, y_n]]$.

Lemma 3.6.13. *For $\boldsymbol{j} \in J^\alpha$ such that $j_a \neq j_{a+1}$ we have:*

$$\begin{aligned} f_a y_{a+1} e(\boldsymbol{j}) &= y_a f_a e(\boldsymbol{j}), \\ y_{a+1} f_a e(\boldsymbol{j}) &= f_a y_a e(\boldsymbol{j}). \end{aligned}$$

3.6.2.3 Image of ψ_a

We first define some elements $p_a(\boldsymbol{i}, \boldsymbol{j}) \in F[[y_a, y_{a+1}]]$ for $1 \leq a < n$ and $(\boldsymbol{i}, \boldsymbol{j}) \in K^\alpha$ by:

$$p_a(\boldsymbol{i}, \boldsymbol{j}) := \begin{cases} \begin{cases} 1 & \text{if } i_a = i_{a+1}, \\ (i_a - i_{a+1} + y_a - y_{a+1})^{-1} & \text{if } i_a \neq i_{a+1}, \end{cases} & \text{if } j_a = j_{a+1}, \\ 0 & \text{if } j_a \neq j_{a+1}, \end{cases}$$

and then some invertible elements $q_a(\boldsymbol{i}, \boldsymbol{j}) \in F[[y_a, y_{a+1}]]^\times$ for $1 \leq a < n$ and $(\boldsymbol{i}, \boldsymbol{j}) \in K^\alpha$ by:

$$q_a(\boldsymbol{i}, \boldsymbol{j}) := \begin{cases} \begin{cases} 1 + y_{a+1} - y_a & \text{if } i_a = i_{a+1}, \\ 1 - p_a(\boldsymbol{i}, \boldsymbol{j}) & \text{if } i_a \not- i_{a+1}, \\ \left(1 - p_a(\boldsymbol{i}, \boldsymbol{j})^2\right)/(y_{a+1} - y_a) & \text{if } i_a \to i_{a+1}, \\ 1 & \text{if } i_a \leftarrow i_{a+1}, \\ (1 - p_a(\boldsymbol{i}, \boldsymbol{j}))/(y_{a+1} - y_a) & \text{if } i_a \leftrightarrows i_{a+1}, \end{cases} & \text{if } j_a = j_{a+1}, \\ 1 & \text{if } j_a \neq j_{a+1}. \end{cases}$$

Remark 3.6.14. As in [BrKl-a], the explicit expression of $q_a(\boldsymbol{i}, \boldsymbol{j})$ does not really matter; we only need some properties satisfied by these power series.

Lemma 3.6.15. *We have:*

$$\begin{aligned} p_{a+1}^{s_a}(\boldsymbol{i}, \boldsymbol{j}) &= p_a^{s_{a+1}}(s_{a+1}s_a \cdot (\boldsymbol{i}, \boldsymbol{j})), \\ q_{a+1}^{s_a}(\boldsymbol{i}, \boldsymbol{j}) &= q_a^{s_{a+1}}(s_{a+1}s_a \cdot (\boldsymbol{i}, \boldsymbol{j})). \end{aligned}$$

We now introduce the following element of $\mathrm{Y}_\alpha^{\boldsymbol{\Lambda}}(1)$:

$$\varphi_a := f_a + \sum_{\substack{(\boldsymbol{i}, \boldsymbol{j}) \in K^\alpha \\ i_a \neq i_{a+1} \\ j_a = j_{a+1}}} (x_a - x_{a+1})^{-1} e(\boldsymbol{i}, \boldsymbol{j}) + \sum_{\substack{(\boldsymbol{i}, \boldsymbol{j}) \in K^\alpha \\ i_a = i_{a+1} \\ j_a = j_{a+1}}} e(\boldsymbol{i}),$$

where $(x_a - x_{a+1})^{-1} e(\boldsymbol{k})$ denotes the inverse of $(x_a - x_{a+1}) e(\boldsymbol{k})$ in $e(\boldsymbol{k}) \mathrm{Y}_\alpha^{\boldsymbol{\Lambda}}(1) e(\boldsymbol{k})$. In particular, we have:

$$\begin{aligned} \varphi_a e(\boldsymbol{j}) &= f_a e(\boldsymbol{j}) \qquad \text{if } j_a \neq j_{a+1}, \\ \varphi_a &= \sum_{\boldsymbol{k} \in K^\alpha} (f_a + p_a(\boldsymbol{k})) e(\boldsymbol{k}). \end{aligned}$$

Moreover, for $d = 1$ the element φ_a is the "intertwining element" defined in [BrKl-a, §3.2].

Lemma 3.6.16. *We have the following properties:*

$$\begin{aligned} \varphi_b e(\boldsymbol{j}) &= e(s_b \cdot \boldsymbol{j}) \varphi_b, \\ \varphi_b e(\boldsymbol{i}, \boldsymbol{j}) &= e(s_b \cdot (\boldsymbol{i}, \boldsymbol{j})) \varphi_b, \\ \varphi_b x_a &= x_a \varphi_b, && \textit{if } a \neq b, b+1, \\ \varphi_b y_a &= y_a \varphi_b, && \textit{if } a \neq b, b+1, \\ \varphi_b q_{b'}(\boldsymbol{k}) &= q_{b'}(\boldsymbol{k}) \varphi_b, && \textit{if } |b - b'| > 1, \\ \varphi_b \varphi_{b'} &= \varphi_{b'} \varphi_b, && \textit{if } |b - b'| > 1, \end{aligned}$$

for all $a \in \{1, \ldots, n\}$ and $b, b' \in \{1, \ldots, n-1\}$.

Our element ψ_a is defined for any $a \in \{1, \dots, n-1\}$ by

$$\psi_a := \sum_{\boldsymbol{k} \in K^\alpha} \phi_a q_a(\boldsymbol{k})^{-1} e(\boldsymbol{k}) \in \mathrm{Y}_\alpha^{\boldsymbol{\Lambda}}(1).$$

When $d = 1$ this element ψ_a corresponds to the ψ_a of [BrKl-a, §3.3]. Note finally that for $\boldsymbol{j} \in J^\alpha$ we have:

$$\psi_a e(\boldsymbol{j}) = f_a e(\boldsymbol{j}) \qquad \text{if } j_a \neq j_{a+1}.$$

3.6.2.4 Check of the defining relations

Theorem 3.6.17. *The elements $y_1, \dots, y_n, \psi_1, \dots, \psi_{n-1}$ and $e(\boldsymbol{k})$ for all $\boldsymbol{k} \in K^\alpha$ satisfy the defining relations (1.2.3), (1.2.16) and (1.2.19) of $\mathrm{R}_\alpha^{\boldsymbol{\Lambda}}(\Gamma_{e,d})$.*

The painstaking verification is exactly the same as in §3.3.2: we apply Remark 3.6.12 on the proof of [BrKl-a, Theorem 3.2] for the cases $j_a = j_{a+1}$, and when $j_a \neq j_{a+1}$ then entirely similar (even the same) relations as in §3.3.2 are satisfied. Note two small differences with the proof in §3.3.2:

- we write $(x_a - x_b)$ instead of $(1-q)(1 - X_a X_b^{-1})$;
- the elements $f_{a,\boldsymbol{j}}$ are equal to 1.

3.6.3 Degenerate Yokonuma–Hecke generators of $\mathrm{R}_\alpha^{\boldsymbol{\Lambda}}(\Gamma)$

We proceed as in Section 3.4. Once again, the proofs are entirely similar to the non-degenerate case, hence we do not write them down.

First of all, since the elements $y_1, \dots, y_n \in \mathrm{R}_\alpha^{\boldsymbol{\Lambda}}(\Gamma_{e,d})$ are nilpotent we can consider power series in these variables. Hence, the quantities $p_a(\boldsymbol{k})$, $q_a(\boldsymbol{k})$ that we defined in §3.6.2.3 are also well-defined as elements of $\mathrm{R}_\alpha^{\boldsymbol{\Lambda}}(\Gamma_{e,d})$. We define finally as in (3.6.9) the elements $e(\boldsymbol{i})$ and $e(\boldsymbol{j})$ of $\mathrm{R}_\alpha^{\boldsymbol{\Lambda}}(\Gamma_{e,d})$ for $\boldsymbol{i} \in I^\alpha$ and $\boldsymbol{j} \in J^\alpha$.

We recall that ξ is a primitive dth root of unity in F. Our "degenerate Yokonuma–Hecke generators" of $\mathrm{R}_\alpha^{\boldsymbol{\Lambda}}(\Gamma_{e,d})$ are given by

$$\begin{aligned}
f_b &:= \sum_{\boldsymbol{k} \in K^\alpha} \left(\psi_b q_b(\boldsymbol{k}) - p_b(\boldsymbol{k})\right) e(\boldsymbol{k}), \\
t_a &:= \sum_{\boldsymbol{j} \in J^\alpha} \xi^{j_a} e(\boldsymbol{j}), \\
x_a &:= \sum_{\boldsymbol{i} \in I^\alpha} (y_a + i_a) e(\boldsymbol{i}),
\end{aligned}$$

for all $a \in \{1, \dots, n\}$ and $b \in \{1, \dots, n-1\}$. When $d = 1$, the element f_a (respectively x_a) is the element s_a (resp. x_a) of [BrKl-a, §3.4].

Remark 3.6.18 (About Brundan and Kleshchev's proof - IV). Let $a \in \{1, \dots, n-1\}$. If $\boldsymbol{j} \in J^\alpha$ satisfies $j_a = j_{a+1}$, when a proof in [BrKl-a, §3.4] needs only the elements $\psi_a e(\boldsymbol{j}), y_b e(\boldsymbol{j}), e(\boldsymbol{i}, \boldsymbol{j})$ and the corresponding relations in $\mathrm{R}_\alpha^{\boldsymbol{\Lambda}}(\Gamma_e)$, we claim that the same proof holds in $e(\boldsymbol{j})\mathrm{R}_\alpha^{\boldsymbol{\Lambda}}(\Gamma_{e,d})e(\boldsymbol{j})$. We extend this claim to the case $j_a = j_{a+1} = j_{a+2}$.

Finally, similarly to §3.6.2.4 we have the following theorem. Once again the check of the various relations is exactly the same as in §3.4.2.

Theorem 3.6.19. *The elements $f_1, \dots, f_{n-1}, t_1, \dots, t_n, x_1, \dots, x_n$ satisfy the defining relations (3.6.2) of $\mathrm{Y}_{d,n}^{\boldsymbol{\Lambda}}(1)$.*

3.6.4 Isomorphism theorem

We give now the degenerate version of Theorem 3.5.1.

Theorem 3.6.20. *There is a presentation of the degenerate cyclotomic Yokonuma–Hecke algebra* $\mathrm{Y}^{\mathbf{\Lambda}}_{\alpha}(1)$ *given by the generators* (1.2.2) *and the relations* (1.2.3), (1.2.16) *and* (1.2.19), *that is, we have an algebra isomorphism:*

$$\mathrm{R}^{\mathbf{\Lambda}}_{n}(\Gamma_{e,d}) \xrightarrow{\sim} \mathrm{Y}^{\mathbf{\Lambda}}_{d,n}(1).$$

The proof of this theorem is entirely similar to the one of Theorem 3.5.1. In particular, by Theorem 3.6.17 we can define an algebra homomorphism $\rho_{\mathrm{RY}} : \mathrm{R}^{\mathbf{\Lambda}}_{\alpha}(\Gamma_{e,d}) \to \mathrm{Y}^{\mathbf{\Lambda}}_{\alpha}(1)$ and by Theorem 3.6.19 we can define another algebra homomorphism $\rho_{\mathrm{YR}} : \mathrm{Y}^{\mathbf{\Lambda}}_{d,n}(1) \to \mathrm{R}^{\mathbf{\Lambda}}_{\alpha}(\Gamma_{e,d})$. From the inclusion $\mathrm{Y}^{\mathbf{\Lambda}}_{\alpha}(1) \subseteq \mathrm{Y}^{\mathbf{\Lambda}}_{d,n}(1)$ we deduce an algebra homomorphism $\rho_{\mathrm{YR}} : \mathrm{Y}^{\mathbf{\Lambda}}_{\alpha}(1) \to \mathrm{R}^{\mathbf{\Lambda}}_{\alpha}(\Gamma_{e,d})$. We prove then that ρ_{RY} and ρ_{YR} are inverse homomorphisms, taking the images of the different defining generators.

Together with Theorem 3.5.1 we get the following corollaries (cf. [BrKl-a, Corollary 1.3]).

Corollary 3.6.21. *If* q *and* $\tilde{q}$ *are two arbitrary elements of* $F^{\times}$ *with* $\operatorname{char}_q(F) = \operatorname{char}_{\tilde{q}}(F)$ *then* $\mathrm{Y}^{\mathbf{\Lambda}}_{d,n}(q)$ *and* $\mathrm{Y}^{\mathbf{\Lambda}}_{d,n}(\tilde{q})$ *are isomorphic algebras.*

Corollary 3.6.22. *If* F *has characteristic* $\operatorname{char}_q(F)$ *then the cyclotomic Yokonuma–Hecke algebra* $\mathrm{Y}^{\mathbf{\Lambda}}_{d,n}(q)$ *is isomorphic to its rational degeneration* $\mathrm{Y}^{\mathbf{\Lambda}}_{d,n}(1)$. *This applies in particular when* F *has characteristic* 0 *and* q *is generic.*

3.7 A commutative diagram

We assume here that $F = \mathbb{C}$. Let $q \in \mathbb{C}^{\times}$ be a primitive eth root of unity and write

$$\mathrm{BK} : \mathrm{H}^{\mathbf{\Lambda}}_{n}(q) \xrightarrow{\sim} \mathrm{R}^{\mathbf{\Lambda}}_{n}(\Gamma_e)$$

for the $\mathbb{C}$-algebra isomorphism of [BrKl-a]. For any $\lambda \vDash_d n$, define $\mathrm{H}^{\mathbf{\Lambda}}_{\lambda}(q) \coloneqq \mathrm{H}^{\mathbf{\Lambda}}_{\lambda_1}(q) \otimes \cdots \otimes \mathrm{H}^{\mathbf{\Lambda}}_{\lambda_d}(q)$ and recall from (1.3.2) the definition of the integer m_λ. We have an algebra isomorphism

$$\mathrm{JPA} : \mathrm{Y}^{\mathbf{\Lambda}}_{d,n}(q) \xrightarrow{\sim} \bigoplus_{\lambda \vDash_d n} \mathrm{Mat}_{m_\lambda} \mathrm{H}^{\mathbf{\Lambda}}_{\lambda}(q),$$

proved by Lusztig [Lu] when $\ell(\mathbf{\Lambda}) = 1$, and then explicitly constructed by Jacon–Poulain d'Andecy [JacPA] when $\ell(\mathbf{\Lambda}) = 1$ and Poulain d'Andecy [PA] in the general case. This isomorphism is defined on the generators as follows:

$$\mathrm{JPA}(t_a) = \sum_{\mathfrak{t} \in J^n} \xi^{\mathfrak{t}_a} E_{\mathfrak{t},\mathfrak{t}}, \tag{3.7.1a}$$

$$\mathrm{JPA}(X_a) = \sum_{\mathfrak{t} \in J^n} X_{\pi_{\mathfrak{t}}(a)} E_{\mathfrak{t},\mathfrak{t}}, \tag{3.7.1b}$$

$$\mathrm{JPA}(g_a) = \sum_{\substack{\mathfrak{t} \in J^n \\ \mathfrak{t}_a = \mathfrak{t}_{a+1}}} g_{\pi_{\mathfrak{t}}(a)} E_{\mathfrak{t},\mathfrak{t}} + \sum_{\substack{\mathfrak{t} \in J^n \\ \mathfrak{t}_a \neq \mathfrak{t}_{a+1}}} \sqrt{q} E_{\mathfrak{t}, s_a \cdot \mathfrak{t}}, \tag{3.7.1c}$$

where we recall from §1.3.3.2 the notation $E_{\mathfrak{t},\mathfrak{t}'}$ and $\sqrt{q} \in \mathbb{C}^{\times}$ is a square root of q.

Remark 3.7.2. Note two slight differences with [PA]:

- our elements $E_{\mathfrak{t}',\mathfrak{t}}$ are written E_{χ}, where χ is a character of $(\mathbb{Z}/d\mathbb{Z})^n = J^n$;

- Poulain d'Andecy considers left cosets instead of our right ones, in particular his minimal length representatives π_χ satisfy $\pi_\chi = \pi_{\mathfrak{t}}^{-1}$.

Recall from §2.3.1 that the quiver $\Gamma_{e,d}$ is the disjoint union of d copies of the cyclic quiver Γ_e with e vertices. In particular, the vertex set of $\Gamma_{e,d}$ is exactly $K \simeq K = I \times J$. The two previous results, together with our Theorem 1.3.57, gives straightforwardly the following theorem.

Theorem 3.7.3. *We have an algebra isomorphism*

$$\Phi_n^{\mathbf{\Lambda}} \circ \mathrm{BK} \circ \mathrm{JPA} : \mathrm{Y}_{d,n}^{\mathbf{\Lambda}}(q) \simeq \mathrm{R}_n^{\mathbf{\Lambda}}(\Gamma_{e,d}),$$

where:

- *the homomorphism* $\mathrm{BK} : \oplus_\lambda \mathrm{Mat}_{m_\lambda} \mathrm{H}_\lambda^{\mathbf{\Lambda}}(q) \to \oplus_\lambda \mathrm{Mat}_{m_\lambda} \mathrm{R}_\lambda^{\mathbf{\Lambda}}(\Gamma_{e,d})$ *is naturally induced by* $\mathrm{BK} : \mathrm{H}_n^{\mathbf{\Lambda}}(q) \to \mathrm{R}_n^{\mathbf{\Lambda}}(\Gamma_e)$*;*
- *the homomorphism* $\Phi_n^{\mathbf{\Lambda}} : \oplus_\lambda \mathrm{Mat}_{m_\lambda} \mathrm{R}_\lambda^{\mathbf{\Lambda}}(\Gamma_{e,d}) \to \mathrm{R}_n^{\mathbf{\Lambda}}(\Gamma_{e,d})$ *is the isomorphism of Theorem 1.3.57.*

An algebra isomorphism $\mathrm{Y}_{d,n}^{\mathbf{\Lambda}}(q) \to \mathrm{R}_n^{\mathbf{\Lambda}}(\Gamma_{e,d})$ was already constructed in Theorem 3.5.1; we denote it by $\widetilde{\mathrm{BK}}$. An interesting question is to know whether we recover the same isomorphism as above. In other words, does the diagram of Figure 3.1 commute?

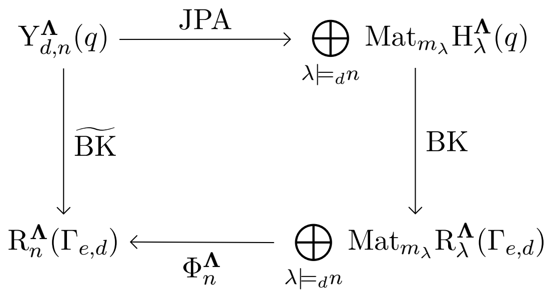


Figure 3.1: A commutative diagram?

As we deal with algebra homomorphisms, it suffices to check that the images of the generators of $\mathrm{Y}_{d,n}^{\mathbf{\Lambda}}(q)$ are the same. We will use the following notation: for $\mathfrak{t} \in J^n$ we set $\mathfrak{t}^* := \pi_{\mathfrak{t}} \cdot \mathfrak{t}$. With $\lambda := [\mathfrak{t}]$, we have of course $\mathfrak{t}^* = \mathfrak{t}^\lambda$. Moreover, we will keep on using the notation $\mathfrak{t}$ of Section 1.3 for the elements of J^n, which we denoted by $\boldsymbol{j}$ from Section 3.2 to Section 3.6.

Image of t_a. Let $a \in \{1, \dots, n\}$. Recall from §3.4.1 that

$$\widetilde{\mathrm{BK}}(t_a) = \sum_{\boldsymbol{j} \in J^n} e(\boldsymbol{j}) \xi^{j_a} = \sum_{\mathfrak{t} \in J^n} e(\mathfrak{t}) \xi^{\mathfrak{t}_a} \in \mathrm{R}_n^{\mathbf{\Lambda}}(\Gamma_{e,d}). \tag{3.7.4}$$

Recalling (3.7.1a), we obtain

$$\mathrm{BK} \circ \mathrm{JPA}(t_a) = \sum_{\mathfrak{t} \in J^n} \sum_{\boldsymbol{k} \in K^{\mathfrak{t}^*}} \xi^{\mathfrak{t}_a} e(\boldsymbol{k}) E_{\mathfrak{t},\mathfrak{t}} \in \bigoplus_{\lambda \vDash_d n} \mathrm{Mat}_{m_\lambda} \mathrm{R}_\lambda^{\mathbf{\Lambda}}(\Gamma_{e,d}).$$

Hence, with the usual manipulations, we obtain

$$\begin{aligned}
\Phi_n^{\boldsymbol{\Lambda}} \circ \mathrm{BK} \circ \mathrm{JPA}(t_a) &= \sum_{\mathfrak{t}\in J^n} \sum_{\boldsymbol{k}\in K^{\mathfrak{t}*}} \xi^{\mathfrak{t}_a} \psi_{\pi_{\mathfrak{t}}^{-1}} \psi_{\pi_{\mathfrak{t}}} e(\pi_{\mathfrak{t}}^{-1} \cdot \boldsymbol{k}) \\
&= \sum_{\mathfrak{t}\in J^n} \sum_{\boldsymbol{k}\in K^{\mathfrak{t}}} \xi^{\mathfrak{t}_a} e(\boldsymbol{k}) \\
\Phi_n^{\boldsymbol{\Lambda}} \circ \mathrm{BK} \circ \mathrm{JPA}(t_a) &= \sum_{\mathfrak{t}\in J^n} \xi^{\mathfrak{t}_a} e(\mathfrak{t}) \in \mathrm{R}_n^{\boldsymbol{\Lambda}}(\Gamma_{e,d}),
\end{aligned}$$

thus it coincides with (3.7.4).

Image of X_a. Let $a \in \{1, \dots, n\}$ (it is in fact enough to study the case $a = 1$). Recall from §3.4.1 that

$$\widetilde{\mathrm{BK}}(X_a) = \sum_{\boldsymbol{i}\in I^n} q^{i_a}(1 - y_a)e(\boldsymbol{i}) = \sum_{\boldsymbol{k}\in K^n} q^{k_a}(1 - y_a)e(\boldsymbol{k}) \in \mathrm{R}_n^{\boldsymbol{\Lambda}}(\Gamma_{e,d}), \tag{3.7.5}$$

where we write $q^k := q^i$ for $k = (i, j) \in K = I \times J$. Recalling (3.7.1b), we obtain

$$\mathrm{BK} \circ \mathrm{JPA}(X_a) = \sum_{\mathfrak{t}\in J^n} X_{\pi_{\mathfrak{t}}(a)} E_{\mathfrak{t},\mathfrak{t}} \in \bigoplus_{\lambda \models_d n} \mathrm{Mat}_{m_\lambda} \mathrm{R}_\lambda^{\boldsymbol{\Lambda}}(\Gamma_{e,d}).$$

We write, where $j := \mathfrak{t}_a$ and $\lambda := [\mathfrak{t}]$,

$$X_{\pi_{\mathfrak{t}}(a)} = \sum_{\boldsymbol{k}^j \in K_j^{\lambda_j}} q^{k^j_{\pi_{\mathfrak{t}}(a)}} (1 - y_{\pi_{\mathfrak{t}}(a)}) e(\boldsymbol{k}^j) \in \mathrm{R}_{\lambda_j}^{\boldsymbol{\Lambda}}(\Gamma_e) \subseteq \mathrm{R}_\lambda^{\boldsymbol{\Lambda}}(\Gamma_{e,d}),$$

where $\boldsymbol{k}^j \in K_j^{\lambda_j}$ is indexed by $(\boldsymbol{\lambda}_{j-1} + 1, \dots, \boldsymbol{\lambda}_j)$. Hence, we have

$$X_{\pi_{\mathfrak{t}}(a)} = \sum_{\boldsymbol{k}\in K^{\mathfrak{t}^\lambda}} q^{k_{\pi_{\mathfrak{t}}(a)}} (1 - y_{\pi_{\mathfrak{t}}(a)}) e(\boldsymbol{k}) \in \mathrm{R}_\lambda^{\boldsymbol{\Lambda}}(\Gamma_{e,d}),$$

and thus

$$\begin{aligned}
\Phi_n^{\boldsymbol{\Lambda}} \circ \mathrm{BK} \circ \mathrm{JPA}(X_a) &= \sum_{\mathfrak{t}\in J^n} \sum_{\boldsymbol{k}\in K^{\mathfrak{t}*}} q^{k_{\pi_{\mathfrak{t}}(a)}} \Phi_n^{\boldsymbol{\Lambda}}\big((1 - y_{\pi_{\mathfrak{t}}(a)}) e(\boldsymbol{k}) E_{\mathfrak{t},\mathfrak{t}}\big) \\
&= \sum_{\mathfrak{t}\in J^n} \sum_{\boldsymbol{k}\in K^{\mathfrak{t}*}} q^{k_{\pi_{\mathfrak{t}}(a)}} \psi_{\pi_{\mathfrak{t}}^{-1}} (1 - y_{\pi_{\mathfrak{t}}(a)}) e(\boldsymbol{k}) \psi_{\pi_{\mathfrak{t}}} \\
&= \sum_{\mathfrak{t}\in J^n} \sum_{\boldsymbol{k}\in K^{\mathfrak{t}*}} q^{k_{\pi_{\mathfrak{t}}(a)}} \psi_{\pi_{\mathfrak{t}}^{-1}} (1 - y_{\pi_{\mathfrak{t}}(a)}) \psi_{\pi_{\mathfrak{t}}} e(\boldsymbol{k}^{\mathfrak{t}}) \\
\Phi_n^{\boldsymbol{\Lambda}} \circ \mathrm{BK} \circ \mathrm{JPA}(X_a) &= \sum_{\mathfrak{t}\in J^n} \sum_{\boldsymbol{k}^{\mathfrak{t}}\in K^{\mathfrak{t}}} q^{k^{\mathfrak{t}}_a} \psi_{\pi_{\mathfrak{t}}^{-1}} \psi_{\pi_{\mathfrak{t}}} (1 - y_a) e(\boldsymbol{k}^{\mathfrak{t}}),
\end{aligned}$$

where we have used Lemma 1.3.39. Hence, using Proposition 1.3.35 we finally obtain

$$\Phi_n^{\boldsymbol{\Lambda}} \circ \mathrm{BK} \circ \mathrm{JPA}(X_a) = \sum_{\mathfrak{t}\in J^n} \sum_{\boldsymbol{k}\in K^{\mathfrak{t}}} q^{k_a}(1 - y_a)e(\boldsymbol{k}) = \sum_{\boldsymbol{k}\in K^n} q^{k_a}(1 - y_a)e(\boldsymbol{k}),$$

which is (3.7.5).

Image of g_a. Let $a \in \{1, \dots, n-1\}$. Recall from §3.4.1 that

$$\widetilde{\mathrm{BK}}(g_a) = \sum_{\boldsymbol{k} \in K^n} (\psi_a Q_a(\boldsymbol{k}) - P_a(\boldsymbol{k}))e(\boldsymbol{k}) \in \mathrm{R}_n^{\boldsymbol{\Lambda}}(\Gamma_{e,d}), \tag{3.7.6}$$

where $Q_a(\boldsymbol{k}), P_a(\boldsymbol{k}) \in \mathbb{C}[[y_a, y_{a+1}]]$ are some power series. For convenience, we write $\widehat{Q}_a(\boldsymbol{k}), \widehat{P}_a(\boldsymbol{k}) \in \mathbb{C}[[Y_a, Y_{a+1}]]$ the underlying power series, which satisfy $\widehat{Q}_a(\boldsymbol{k})(y_a, y_{a+1}) = Q_a(\boldsymbol{k})$ and $\widehat{P}_a(\boldsymbol{k})(y_a, y_{a+1}) = P_a(\boldsymbol{k})$. These power series depend only on k_a and k_{a+1}, that is, $\widehat{Q}_a(\boldsymbol{k})(Y, Y') = \widehat{Q}_{a'}(\boldsymbol{k}')(Y, Y')$ and $\widehat{P}_a(\boldsymbol{k})(Y, Y') = \widehat{P}_{a'}(\boldsymbol{k}')(Y, Y')$ if $k_a = k'_{a'}$ and $k_{a+1} = k'_{a'+1}$. Moreover, recall that if $\boldsymbol{k}$ and $s_a \cdot \boldsymbol{k}$ are labellings of two different $\mathfrak{t} \in J^n$, we have $Q_a(\boldsymbol{k}) = \sqrt{q}$ and $P_a(\boldsymbol{k}) = 0$ (cf. Remark 3.3.14).

Recalling (3.7.1c), we obtain

$$\mathrm{BK} \circ \mathrm{JPA}(g_a) = \sum_{\substack{\mathfrak{t} \in J^n \\ \mathfrak{t}_a = \mathfrak{t}_{a+1}}} g_{\pi_\mathfrak{t}(a)} E_{\mathfrak{t},\mathfrak{t}} + \sum_{\substack{\mathfrak{t} \in J^n \\ \mathfrak{t}_a \neq \mathfrak{t}_{a+1}}} \sqrt{q} E_{\mathfrak{t}, s_a \cdot \mathfrak{t}} \in \bigoplus_{\lambda \models_d n} \mathrm{Mat}_{m_\lambda} \mathrm{R}_\lambda^{\boldsymbol{\Lambda}}(\Gamma_{e,d}).$$

With $j := \mathfrak{t}_a$ and $\lambda := [\mathfrak{t}]$, we have

$$g_{\pi_\mathfrak{t}(a)} = \sum_{\boldsymbol{k}^j \in K_j^{\lambda_j}} (\psi_{\pi_\mathfrak{t}(a)} Q_{\pi_\mathfrak{t}(a)}(\boldsymbol{k}^j) - P_{\pi_\mathfrak{t}(a)}(\boldsymbol{k}^j)) e(\boldsymbol{k}^j) \in \mathrm{R}_{\lambda_j}^{\boldsymbol{\Lambda}}(\Gamma_e) \subseteq \mathrm{R}_\lambda^{\boldsymbol{\Lambda}}(\Gamma_{e,d})$$

(recall that $\boldsymbol{k}^j \in K_j^{\lambda_j}$ is indexed by $(\boldsymbol{\lambda}_{j-1} + 1, \dots, \boldsymbol{\lambda}_j)$), hence

$$g_{\pi_\mathfrak{t}(a)} = \sum_{\boldsymbol{k} \in K^{\mathfrak{t}^\lambda}} (\psi_{\pi_\mathfrak{t}(a)} Q_{\pi_\mathfrak{t}(a)}(\boldsymbol{k}) - P_{\pi_\mathfrak{t}(a)}(\boldsymbol{k})) e(\boldsymbol{k}) \in \mathrm{R}_\lambda^{\boldsymbol{\Lambda}}(\Gamma_{e,d}).$$

We obtain

$$\Phi_n^{\boldsymbol{\Lambda}} \circ \mathrm{BK} \circ \mathrm{JPA}(g_a) = \underbrace{\sum_{\substack{\mathfrak{t} \in J^n \\ \mathfrak{t}_a = \mathfrak{t}_{a+1}}} \sum_{\boldsymbol{k} \in K^{\mathfrak{t}^*}} \psi_{\pi_\mathfrak{t}^{-1}} (\psi_{\pi_\mathfrak{t}(a)} Q_{\pi_\mathfrak{t}(a)}(\boldsymbol{k}) - P_{\pi_\mathfrak{t}(a)}(\boldsymbol{k})) e(\boldsymbol{k}) \psi_{\pi_\mathfrak{t}}}_{=:S_1} + \underbrace{\sum_{\substack{\mathfrak{t} \in J^n \\ \mathfrak{t}_a \neq \mathfrak{t}_{a+1}}} \sum_{\boldsymbol{k} \in K^{\mathfrak{t}^*}} \sqrt{q} \psi_{\pi_\mathfrak{t}^{-1}} e(\boldsymbol{k}) \psi_{\pi_{s_a \cdot \mathfrak{t}}}}_{=:S_2}. \tag{3.7.7}$$

We first focus on the first sum S_1. Let $\mathfrak{t} \in J^n$ such that $\mathfrak{t}_a = \mathfrak{t}_{a+1}$ and $\boldsymbol{k} \in K^{\mathfrak{t}^*}$. We can notice that thanks to Proposition 1.3.12 we have $\pi_\mathfrak{t}(a+1) = \pi_\mathfrak{t}(a) + 1$. Using Lemma 1.3.39 and the properties of $\widehat{Q}$ we have (recalling the notation $\boldsymbol{k}^\mathfrak{t}$ introduced at (1.3.23)):

$$\begin{aligned} Q_{\pi_\mathfrak{t}(a)}(\boldsymbol{k}) e(\boldsymbol{k}) \psi_{\pi_\mathfrak{t}} &= \widehat{Q}_{\pi_\mathfrak{t}(a)}(\boldsymbol{k})(y_{\pi_\mathfrak{t}(a)}, y_{\pi_\mathfrak{t}(a)+1}) \psi_{\pi_\mathfrak{t}} e(\boldsymbol{k}^\mathfrak{t}) \\ &= \psi_{\pi_\mathfrak{t}} \widehat{Q}_{\pi_\mathfrak{t}(a)}(\boldsymbol{k})(y_a, y_{a+1}) e(\boldsymbol{k}^\mathfrak{t}) \\ &= \psi_{\pi_\mathfrak{t}} \widehat{Q}_a(\boldsymbol{k}^\mathfrak{t})(y_a, y_{a+1}) e(\boldsymbol{k}^\mathfrak{t}) \\ &= \psi_{\pi_\mathfrak{t}} Q_a(\boldsymbol{k}^\mathfrak{t}) e(\boldsymbol{k}^\mathfrak{t}). \end{aligned}$$

The same proof gives $P_{\pi_\mathfrak{t}(a)}(\boldsymbol{k}) e(\boldsymbol{k}) \psi_{\pi_\mathfrak{t}} = \psi_{\pi_\mathfrak{t}} P_a(\boldsymbol{k}^\mathfrak{t}) e(\boldsymbol{k}^\mathfrak{t})$. We thus have

$$\psi_{\pi_\mathfrak{t}^{-1}} (\psi_{\pi_\mathfrak{t}(a)} Q_{\pi_\mathfrak{t}(a)}(\boldsymbol{k}) - P_{\pi_\mathfrak{t}(a)}(\boldsymbol{k})) e(\boldsymbol{k}) \psi_{\pi_\mathfrak{t}} = \psi_{\pi_\mathfrak{t}^{-1}} (\psi_{\pi_\mathfrak{t}(a)} \psi_{\pi_\mathfrak{t}} Q_a(\boldsymbol{k}^\mathfrak{t}) - \psi_{\pi_\mathfrak{t}} P_a(\boldsymbol{k}^\mathfrak{t})) e(\boldsymbol{k}^\mathfrak{t}).$$

Using (1.2.3c) and Lemma 1.3.41, we obtain

$$\psi_{\pi_\mathfrak{t}^{-1}} (\psi_{\pi_\mathfrak{t}(a)} Q_{\pi_\mathfrak{t}(a)}(\boldsymbol{k}) - P_{\pi_\mathfrak{t}(a)}(\boldsymbol{k})) e(\boldsymbol{k}) \psi_{\pi_\mathfrak{t}} = \psi_{\pi_\mathfrak{t}^{-1}} \psi_{\pi_\mathfrak{t}} (\psi_a Q_a(\boldsymbol{k}^\mathfrak{t}) - P_a(\boldsymbol{k}^\mathfrak{t})) e(\boldsymbol{k}^\mathfrak{t}).$$

Finally, since $s_a \cdot \mathfrak{t} = \mathfrak{t}$, we obtain by Proposition 1.3.35

$$\psi_{\pi_{\mathfrak{t}}^{-1}}(\psi_{\pi_{\mathfrak{t}}(a)} Q_{\pi_{\mathfrak{t}}(a)}(\boldsymbol{k}) - P_{\pi_{\mathfrak{t}}(a)}(\boldsymbol{k}))e(\boldsymbol{k})\psi_{\pi_{\mathfrak{t}}} = (\psi_a Q_a(\boldsymbol{k}^{\mathfrak{t}}) - P_a(\boldsymbol{k}^{\mathfrak{t}}))e(\boldsymbol{k}^{\mathfrak{t}}),$$

so the first sum becomes

$$\begin{aligned} &\sum_{\substack{\mathfrak{t}\in J^n \\ \mathfrak{t}_a = \mathfrak{t}_{a+1}}} \sum_{\boldsymbol{k}\in K^{\mathfrak{t}*}} \psi_{\pi_{\mathfrak{t}}^{-1}}(\psi_{\pi_{\mathfrak{t}}(a)} Q_{\pi_{\mathfrak{t}}(a)}(\boldsymbol{k}) - P_{\pi_{\mathfrak{t}}(a)}(\boldsymbol{k}))e(\boldsymbol{k})\psi_{\pi_{\mathfrak{t}}} \\ &\qquad = \sum_{\substack{\mathfrak{t}\in J^n \\ \mathfrak{t}_a = \mathfrak{t}_{a+1}}} \sum_{\boldsymbol{k}^{\mathfrak{t}}\in K^{\mathfrak{t}}} (\psi_a Q_a(\boldsymbol{k}^{\mathfrak{t}}) - P_a(\boldsymbol{k}^{\mathfrak{t}}))e(\boldsymbol{k}^{\mathfrak{t}}). \end{aligned} \tag{3.7.8}$$

We now focus on the second sum S_2. Let $\mathfrak{t} \in J^n$ with $\mathfrak{t}_a \neq \mathfrak{t}_{a+1}$ and let $\boldsymbol{k} \in K^{\mathfrak{t}^*}$. By Lemma 1.3.40 we directly have

$$\psi_{\pi_{\mathfrak{t}}^{-1}} e(\boldsymbol{k})\psi_{\pi_{s_a\cdot\mathfrak{t}}} = e(\boldsymbol{k}^{\mathfrak{t}})\psi_{\pi_{\mathfrak{t}}^{-1}}\psi_{\pi_{s_a\cdot\mathfrak{t}}} = e(\boldsymbol{k}^{\mathfrak{t}})\psi_a,$$

thus

$$\begin{aligned} \sum_{\substack{\mathfrak{t}\in J^n \\ \mathfrak{t}_a \neq \mathfrak{t}_{a+1}}} \sum_{\boldsymbol{k}\in K^{\mathfrak{t}*}} \sqrt{q}\psi_{\pi_{\mathfrak{t}}^{-1}} e(\boldsymbol{k})\psi_{\pi_{s_a\cdot\mathfrak{t}}} &= \sum_{\substack{\mathfrak{t}\in J^n \\ \mathfrak{t}_a \neq \mathfrak{t}_{a+1}}} \sum_{\boldsymbol{k}^{\mathfrak{t}}\in K^{\mathfrak{t}}} \sqrt{q} e(\boldsymbol{k}^{\mathfrak{t}})\psi_a \\ &= \sum_{\substack{\mathfrak{t}\in J^n \\ \mathfrak{t}_a \neq \mathfrak{t}_{a+1}}} \sum_{\boldsymbol{k}^{\mathfrak{t}}\in K^{\mathfrak{t}}} \sqrt{q}\psi_a e(s_a \cdot \boldsymbol{k}^{\mathfrak{t}}) \\ &= \sum_{\substack{\mathfrak{t}\in J^n \\ \mathfrak{t}_a \neq \mathfrak{t}_{a+1}}} \sum_{\boldsymbol{k}^{s_a\cdot\mathfrak{t}}\in K^{s_a\cdot\mathfrak{t}}} \sqrt{q}\psi_a e(\boldsymbol{k}^{s_a\cdot\mathfrak{t}}) \\ &= \sum_{\substack{\mathfrak{t}\in J^n \\ \mathfrak{t}_a \neq \mathfrak{t}_{a+1}}} \sum_{\boldsymbol{k}^{\mathfrak{t}}\in K^{\mathfrak{t}}} \sqrt{q}\psi_a e(\boldsymbol{k}^{\mathfrak{t}}) \\ &= \sum_{\substack{\mathfrak{t}\in J^n \\ \mathfrak{t}_a \neq \mathfrak{t}_{a+1}}} \sum_{\boldsymbol{k}^{\mathfrak{t}}\in K^{\mathfrak{t}}} (Q_a(\boldsymbol{k}^{\mathfrak{t}})\psi_a - P_a(\boldsymbol{k}^{\mathfrak{t}}))e(\boldsymbol{k}^{\mathfrak{t}}). \end{aligned} \tag{3.7.9}$$

Finally, by (3.7.7)–(3.7.9) we obtain

$$\begin{aligned} \Phi_n^{\boldsymbol{\Lambda}} \circ \mathrm{BK} \circ \mathrm{JPA}(g_a) &= \sum_{\substack{\mathfrak{t}\in J^n \\ \mathfrak{t}_a = \mathfrak{t}_{a+1}}} \sum_{\boldsymbol{k}\in K^{\mathfrak{t}}} (\psi_a Q_a(\boldsymbol{k}) - P_a(\boldsymbol{k}))e(\boldsymbol{k}) + \sum_{\substack{\mathfrak{t}\in J^n \\ \mathfrak{t}_a \neq \mathfrak{t}_{a+1}}} \sum_{\boldsymbol{k}\in K^{\mathfrak{t}}} (\psi_a Q_a(\boldsymbol{k}) - P_a(\boldsymbol{k}))e(\boldsymbol{k}) \\ &= \sum_{\mathfrak{t}\in J^n} \sum_{\boldsymbol{k}\in K^{\mathfrak{t}}} (\psi_a Q_a(\boldsymbol{k}) - P_a(\boldsymbol{k}))e(\boldsymbol{k}) \\ &= \sum_{\boldsymbol{k}\in K^n} (\psi_a Q_a(\boldsymbol{k}) - P_a(\boldsymbol{k}))e(\boldsymbol{k}), \end{aligned}$$

which is (3.7.6).

To conclude, we have checked that the algebra homomorphisms $\widetilde{\mathrm{BK}}$ and $\Phi_n^{\boldsymbol{\Lambda}} \circ \mathrm{BK} \circ \mathrm{JPA}$ coincide on every generator of $\mathrm{Y}_{d,n}^{\boldsymbol{\Lambda}}(q)$, hence we have the following theorem.

Theorem 3.7.10. *We have*

$$\widetilde{\mathrm{BK}} = \Phi_n^{\boldsymbol{\Lambda}} \circ \mathrm{BK} \circ \mathrm{JPA},$$

and the diagram of Figure 3.1 commutes.

Chapter 4

Stuttering blocks of Ariki–Koike algebras

This chapter is a slightly modified version of [Ro17-b].

4.1 Introduction

Let $r, p, n \in \mathbb{N}^*$ with p dividing r. Let $\mathbf{\Lambda}$ be a weight of level r associated to a multicharge $\kappa \in (\mathbb{Z}/e\mathbb{Z})^r$ such that the subalgebra $\mathrm{H}^{\mathbf{\Lambda}}_{p,n}(q)$ of the Ariki–Koike algebra $\mathrm{H}^{\mathbf{\Lambda}}_n(q)$ is well-defined (see Chapter 2). Recall from the introduction that, if $\boldsymbol{\lambda}$ is a Kleshchev r-partition of n, the restriction $\mathcal{D}^{\boldsymbol{\lambda}}\Big\downarrow^{\mathrm{H}^{\mathbf{\Lambda}}_n(q)}_{\mathrm{H}^{\mathbf{\Lambda}}_{p,n}(q)}$ of the irreducible $\mathrm{H}^{\mathbf{\Lambda}}_n(q)$-module $\mathcal{D}^{\boldsymbol{\lambda}}$ is isomorphic to a sum of irreducible $\mathrm{H}^{\mathbf{\Lambda}}_{p,n}(q)$-modules, whose number depends on the cardinality of the orbit $[\boldsymbol{\lambda}]$ of $\boldsymbol{\lambda}$ under the shift action. A natural question is then to determine the extreme cardinalities of the orbits under this action, and thus the extremal number of irreducible $\mathrm{H}^{\mathbf{\Lambda}}_{p,n}(q)$-module that appear during the restriction process. The answer is an easy exercise when considering all r-partitions of n.

Proposition 4.1.1. *Let $\mathcal{C} \coloneqq \{\#[\boldsymbol{\lambda}] : \boldsymbol{\lambda}$ is an r-partition of $n\} \subseteq \mathbb{N}^*$. We have $\max \mathcal{C} = p$ and $\min \mathcal{C} = \frac{p}{\gcd(p,n)}$.*

Already with this Proposition 4.1.1, we can give some results about the representation theory of $\mathrm{H}^{\mathbf{\Lambda}}_{p,n}(q)$, such as the number of "Specht modules" that appear in the restriction of Specht modules of $\mathrm{H}^{\mathbf{\Lambda}}_n(q)$ to $\mathrm{H}^{\mathbf{\Lambda}}_{p,n}(q)$ (as defined in [HuMa10]). We can also prove that a natural basis of $\mathrm{H}^{\mathbf{\Lambda}}_{p,n}(q)$ is not "adapted" cellular (cf. §4.5.2.5). In order to give block-analogue answers, we introduce a shift action on $Q^+ = \mathbb{N}^{\mathbb{Z}/e\mathbb{Z}}$ where $e \in \mathbb{N}_{\geq 2}$ is a multiple of p. More precisely, for any $\alpha \in Q^+$ we define $\sigma \cdot \alpha$ by shifting coordinates by $\eta \coloneqq \frac{e}{p}$ and we write $[\alpha]$ for the orbit of α. Recalling that $I = \mathbb{Z}/e\mathbb{Z}$, note that $\alpha = (\alpha_i)_{i \in I} \in Q^+$ satisfies $\sum_{i \in I} \alpha_i = n$ if and only if $\alpha \models_I n$. Hence, we can use the notation from Chapter 2: the subalgebra $\mathrm{H}^{\mathbf{\Lambda}}_{[\alpha]}(q) = \oplus_{\beta \in [\alpha]} \mathrm{H}^{\mathbf{\Lambda}}_\beta(q)$ of $\mathrm{H}^{\mathbf{\Lambda}}_n(q)$ is stable under the shift automorphism $\sigma : \mathrm{H}^{\mathbf{\Lambda}}_n(q) \to \mathrm{H}^{\mathbf{\Lambda}}_n(q)$, and we denote by $\mathrm{H}^{\mathbf{\Lambda}}_{p,[\alpha]}(q)$ the subalgebra of fixed points . For any r-partition $\boldsymbol{\lambda}$, we have an associated element $\alpha_\kappa(\boldsymbol{\lambda}) \in Q^+$. The two shift actions that we have defined are compatible in the following way: if $\boldsymbol{\lambda}$ is an r-partition then

$$\alpha_\kappa({}^\sigma\boldsymbol{\lambda}) = \sigma \cdot \alpha_\kappa(\boldsymbol{\lambda})$$

(see Lemma 4.2.29). Hence, if $\alpha \coloneqq \alpha_\kappa(\boldsymbol{\lambda})$ we always have $\#[\boldsymbol{\lambda}] \geq \#[\alpha]$. It is easy to see in small examples that we may have a strict inequality. However, the main results of this chapter, Theorem 4.2.31 and Corollary 4.2.34, prove that equality holds if we allow us to choose among all r-partitions $\boldsymbol{\mu}$ with $\alpha_\kappa(\boldsymbol{\mu}) = \alpha_\kappa(\boldsymbol{\lambda})$. It leads to a more precise version of the "min part" of Proposition 4.1.1.

Theorem 4.1.2. *Let $\boldsymbol{\lambda}$ be an r-partition and let $\alpha := \alpha_\kappa(\boldsymbol{\lambda})$. There exists an r-partition $\boldsymbol{\mu}$ with $\alpha_\kappa(\boldsymbol{\mu}) = \alpha$ and $\#[\boldsymbol{\mu}] = \#[\alpha]$.*

Wada [Wa] proved a more precise version of the "max part" of Proposition 4.1.1. In order to classify the blocks of $\mathrm{H}^{\boldsymbol{\Lambda}}_{p,n}(q)$, Wada proved that there (almost) always exists an r-partition $\boldsymbol{\mu}$ with $\alpha_\kappa(\boldsymbol{\mu}) = \alpha$ and $\#[\boldsymbol{\mu}] = p$. His proof uses the classification result of [LyMa] and is very short. In contrast, the proof of Theorem 4.1.2 that we present here is quite long and we did not find a way to use [LyMa]. At least, as in [LyMa], we use the *abacus representation* of partitions.

Theorem 4.1.2 allows us to give the block-analogues of the results for $\mathrm{H}^{\boldsymbol{\Lambda}}_{p,n}(q)$ that we deduced from Proposition 4.1.1, that is, the same results but for $\mathrm{H}^{\boldsymbol{\Lambda}}_{p,[\alpha]}(q)$ instead of $\mathrm{H}^{\boldsymbol{\Lambda}}_{p,n}(q)$. We can also deduce from Theorem 4.1.2 some consequences about the blocks of $\mathrm{H}^{\boldsymbol{\Lambda}}_{n}(q)$. We say that an r-partition $\boldsymbol{\lambda}$ (respectively an element $\alpha \in Q^+$) is *stuttering* if $\#[\boldsymbol{\lambda}] < p$ (resp. $\#[\alpha] < p$). By Theorem 4.1.2, we know that the block indexed by a stuttering $\alpha \in Q^+$ always contains a stuttering r-partition.

The chapter is organised as follows. Section 4.2 is devoted to combinatorics. More specifically, in §4.2.1 we define partitions of integers and to each partition λ we associate an element $\alpha(\lambda) \in Q^+ = \mathbb{N}^{\mathbb{Z}/e\mathbb{Z}}$. In §4.2.2 we recall the abacus representation of partitions. In §4.2.3, to an e-core λ we associate the *e-abacus variable* $x = (x_0, \dots, x_{e-1}) \in \mathbb{Z}^e$. The main fact of this subsection is the equality

$$\alpha(\lambda)_0 = \frac{1}{2}\sum_{i=0}^{e-1} x_i^2$$

(cf. Theorem 4.2.13). We deduce this equality from [GKS], and we show how to obtain it using abacus manipulations. In §4.2.4 we extend the previous definitions to multipartitions, so we can in §4.2.5 define the two shift maps $\boldsymbol{\lambda} \mapsto {}^\sigma\boldsymbol{\lambda}$ and $\alpha \mapsto \sigma \cdot \alpha$ involved in the statement of our main results, Theorem 4.2.31 and Corollary 4.2.34. Theorem 4.2.31 is the case $\#[\alpha] = 1$ of Theorem 4.1.2 and Corollary 4.2.34 is the general case.

Section 4.3 is devoted to technical tools that we need to prove Theorem 4.2.31. The reader who wants to focus on the proof of Theorem 4.2.31 may, in the first instance, skip this section. In §4.3.1, we study the existence of a chain of interchanges $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ in a family of binary matrices (Corollary 4.3.8). In §4.3.2, we recall a special case of a general theorem of Gale [Ga] and Ryser [Ry] about the existence of a binary matrix with prescribed row and column sums. We apply the results of §4.3.1 to impose extra conditions on block sums (Proposition 4.3.14). Finally, we gathered in §4.3.3 some inequalities; in particular, Lemma 4.3.25 is a special case of a Jensen's inequality for strongly convex functions and Lemma 4.3.27 is an application to an inequality involving the fractional part map.

In Section 4.4, we prove the main result, Theorem 4.2.31. After a preliminary step in §4.4.1, we give in §4.4.2 a key lemma (Lemma 4.4.5), which reduces the proof of Theorem 4.2.31 to a (strongly) convex optimisation problem over the integers with linear constraints. We find in §4.4.3 a partial solution, in §4.4.4 we use Proposition 4.3.14 to find a solution and eventually in §4.4.5 we prove Theorem 4.2.31.

Finally, we give in Section 4.5 two applications of Corollary 4.2.34. The general idea is that we will have more precise results with Corollary 4.2.34 than with Proposition 4.1.1. We quickly recall in §4.5.1 the theory of cellular algebras of Graham and Lehrer [GrLe], the Ariki–Koike algebra $\mathrm{H}^{\boldsymbol{\Lambda}}_{n}(q)$ and its blocks being particular cases. In §4.5.2 we use the map $\mu := \sum_{j=0}^{p-1} \sigma^j$ to construct a family of bases for $\mathrm{H}^{\boldsymbol{\Lambda}}_{p,[\alpha]}(q) = \mu(\mathrm{H}^{\boldsymbol{\Lambda}}_{[\alpha]}(q))$ (Proposition 4.5.18). We deduce in §4.5.2.4 that $\mathrm{H}^{\boldsymbol{\Lambda}}_{p,[\alpha]}(q)$ is a cellular algebra if $\#[\alpha] = p$, and $\mathrm{H}^{\boldsymbol{\Lambda}}_{p,n}(q)$ is cellular if p and n are coprime.

Then, using Corollary 4.2.34, we show that if $\#[\alpha] < p$ and p is odd then the bases that we constructed for $\mathrm{H}^{\mathbf{\Lambda}}_{p,[\alpha]}(q)$ are not *adapted cellular* (see §4.5.2.5). Finally, in §4.5.3, we study the maximal number of "Specht modules of $\mathrm{H}^{\mathbf{\Lambda}}_{p,[\alpha]}(q)$" (see [HuMa10]) that appear when restricting the Specht modules of $\mathrm{H}^{\mathbf{\Lambda}}_{[\alpha]}(q)$ to $\mathrm{H}^{\mathbf{\Lambda}}_{p,[\alpha]}(q)$.

4.2 Combinatorics

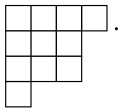
In this section, we recall standard definitions of combinatorics such as (multi)partitions and their associated abaci. We also introduce two *shift* actions and then state our main result, Theorem 4.2.31. We identify $\mathbb{Z}/e\mathbb{Z}$ with the set $\{0, \ldots, e-1\}$.

4.2.1 Partitions

A *partition* of n is a non-increasing sequence of positive integers $\lambda = (\lambda_0, \ldots, \lambda_{h-1})$ of sum n. We will write $|\lambda| := n$ and $h(\lambda) := h$. If λ is a partition, we denote by $\mathcal{Y}(\lambda)$ its Young diagram, defined by:

$$\mathcal{Y}(\lambda) := \left\{(a,b) \in \mathbb{N}^2 : 0 \leq a \leq h(\lambda) - 1 \text{ and } 0 \leq b \leq \lambda_a - 1\right\}.$$

Example 4.2.1. We represent the Young diagram associated with the partition $(4, 3, 3, 1)$ by

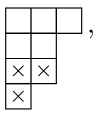


We refer to the elements of $\mathcal{Y}(\lambda)$ as *nodes*. A node $\gamma \in \mathcal{Y}(\lambda)$ is *removable* (respectively *addable*) if $\mathcal{Y}(\lambda) \setminus \{\gamma\}$ (resp. $\mathcal{Y}(\lambda) \cup \{\gamma\}$) is the Young diagram of a partition. A *rim hook* of λ is a subset of $\mathcal{Y}(\lambda)$ of the following form:

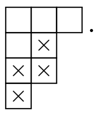
$$r^{\lambda}_{(a,b)} := \{(a', b') \in \mathcal{Y}(\lambda) : a' \geq a,\ b' \geq b \text{ and } (a'+1, b'+1) \notin \mathcal{Y}(\lambda)\},$$

where $(a, b) \in \mathcal{Y}(\lambda)$. We say that $r^{\lambda}_{(a,b)}$ is an *h-rim hook* if it has cardinality h. Note that 1-rim hooks are exactly removable nodes. The *hand* of a rim hook $r^{\lambda}_{(a,b)}$ is the node $(a, b') \in r^{\lambda}_{(a,b)}$ with maximal b'. The set $\mathcal{Y}(\lambda) \setminus r^{\lambda}_{(a,b)}$ is the Young diagram of a certain partition μ, obtained by *unwrapping* or *removing* the rim hook $r^{\lambda}_{(a,b)}$ from λ. Conversely, we say that λ is obtained from μ by *wrapping on* or *adding* the rim hook $r^{\lambda}_{(a,b)}$. We say that a partition λ is an *e-core* if λ has no e-rim hooks.

Example 4.2.2. We consider the partition $\lambda := (3, 2, 2, 1)$. An example of a 3-rim hook is

$r^{\lambda}_{(2,0)} =$ 

and a 4-rim hook is for instance

$r^{\lambda}_{(1,0)} =$ 

We can check that λ has no 5-rim hook so it is a 5-core. We will see in §4.2.3 how to use abaci to easily know whether a partition is an e-core.

Let λ be a partition. The *residue* of a node $\gamma = (a, b) \in \mathcal{Y}(\lambda)$ is $\operatorname{res}(\gamma) := b - a \pmod e$. For any $i \in \mathbb{Z}/e\mathbb{Z}$, an i-node is a node with residue i. We denote by $n^i(\lambda)$ the multiplicity of i in the multiset of residues of all elements of $\mathcal{Y}(\lambda)$. Note that $\sum_{i=0}^{e-1} n^i(\lambda) = |\lambda|$.

Let Q be a free $\mathbb{Z}$-module of rank e and let $\{\alpha_i\}_{i\in\mathbb{Z}/e\mathbb{Z}}$ be a basis. We have $Q = \oplus_{i=0}^{e-1}\mathbb{Z}\alpha_i$ and we define $Q^+ := \oplus_{i=0}^{e-1}\mathbb{N}\alpha_i$. For any $\alpha \in Q$, we denote by $|\alpha| \in \mathbb{Z}$ the sum of its coordinates in the basis $\{\alpha_i\}_{i\in\mathbb{Z}/e\mathbb{Z}}$. If λ is a partition we define

$$\alpha(\lambda) := \sum_{\gamma\in\mathcal{Y}(\lambda)} \alpha_{\operatorname{res}(\gamma)} = \sum_{i=0}^{e-1} n^i(\lambda)\alpha_i \in Q^+.$$

Note that $|\alpha(\lambda)| = |\lambda|$. More generally, if Γ is any finite subset of $\mathbb{N}^2$ we will write $\alpha(\Gamma) := \sum_{\gamma\in\Gamma} \alpha_{\operatorname{res}(\gamma)}$.

Remark 4.2.3. If r^λ is an h-rim hook then $\alpha(r^\lambda) = \sum_{i=0}^{h-1} \alpha_{i_0+i}$ for some $i_0 \in \mathbb{Z}/e\mathbb{Z}$. In particular, if r^λ is an e-rim hook then $\alpha(r^\lambda) = \sum_{i=0}^{e-1} \alpha_i$.

Finally, if for $\alpha \in Q^+$ there exists a partition λ such that $\alpha = \alpha(\lambda)$, we say that $\alpha \in Q^+$ is *associated* with λ. For an arbitrary $\alpha \in Q^+$, there can exist two different partitions $\lambda \neq \mu$ such that $\alpha = \alpha(\lambda) = \alpha(\mu)$. However, if we restrict to e-cores then the map $\lambda \mapsto \alpha(\lambda)$ is one-to-one (see [JamKe, 2.7.41 Theorem] or [LyMa]). Hence, the following subset of Q^+:

$$Q^* := \left\{\alpha \in Q^+ : \alpha \text{ is associated with some } e\text{-core}\right\},$$

is in bijection with the set of e-cores. The aim of §4.2.3 is to explicit a bijection between Q^* and $\mathbb{Z}^{e-1}$.

4.2.2 Abaci

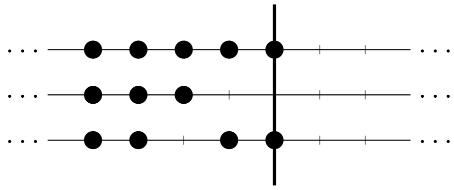
The abacus representation of a partition has been first introduced by James [Jam]. Here, we follow the construction of [LyMa]. To a partition $\lambda = (\lambda_0, \dots, \lambda_{h-1})$ we associate the *β-number* $\beta(\lambda)$ defined as the strictly decreasing sequence $(\lambda_{a-1} - a)_{a\geq 1}$, where $\lambda_{a-1} := 0$ for any $a > h$. Note that $\beta(\lambda)_a = -a$ if $a > h$. This construction can be reverted: if $\beta = (\beta_a)_{a\geq 1}$ is a strictly decreasing sequence of integers with $\beta_a = -a$ for any $a > h$ then $\beta = \beta(\lambda)$ where $\lambda = (\lambda_0, \dots, \lambda_{h-1})$ is the partition given by $\lambda_a := \beta_{a+1} + a + 1$ for all $a \in \{0, \dots, h-1\}$. The following result is well-known (see for instance [JamKe, 2.7.13 Lemma]).

Lemma 4.2.4. *Let $h \in \mathbb{N}^*$. A partition λ has an h-rim hook if and only if there is an element $b \in \beta(\lambda)$ such that $b - h \notin \beta(\lambda)$. In that case, if μ is the partition that we obtain by removing this h-rim hook, then $\beta(\mu)$ is obtained by replacing b by $b - h$ in $\beta(\lambda)$ and then sorting in decreasing order.*

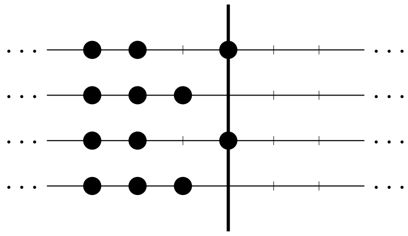
In particular, if μ is a partition and if $b \in \beta(\mu)$ and $h \in \mathbb{N}^*$ are such that $b + h \notin \beta(\mu)$, then replacing b by $b+h$ in $\beta(\mu)$ and sorting in decreasing order is equivalent to adding an h-rim hook to μ. Indeed, the sequence that we obtain from $\beta(\mu)$ is strictly decreasing thus is the β-number of a certain partition λ. By Lemma 4.2.4, the partition μ is obtained from λ by unwrapping an h-rim hook, that is, the partition λ is obtained from μ by wrapping on an h-rim hook (see also [Ma15, Lemma 5.26]).

Lemma 4.2.4 ensures that for any partition λ, there is a unique e-core $\overline{\lambda}$ that can be obtained by successively removing e-rim hooks. We say that $\overline{\lambda}$ is the *e-core of* λ. We now consider an abacus with e-runners, each runner being a horizontal copy of $\mathbb{Z}$ and disposed in the following way: the 0th runner is on top and the origins of each copy of $\mathbb{Z}$ are aligned with respect to a vertical line. We record the elements of $\beta(\lambda)$ on this abacus according to the following rule: there is a bead at position $j \in \mathbb{Z}$ on the runner $i \in \{0, \dots, e-1\}$ if and only if there exists $a \geq 1$ such that $\beta(\lambda)_a = i + je$. We say that this abacus is the *e-abacus associated with* λ.

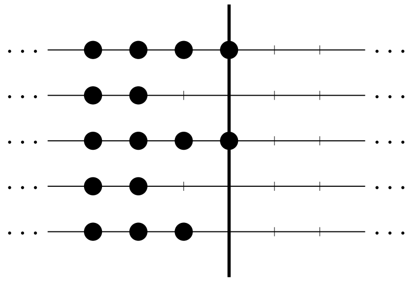
Example 4.2.5. We consider the partition $\lambda = (3, 2, 2, 1)$ from Example 4.2.2. Its β-number is $\beta(\lambda) = (2, 0, -1, -3, -5, \dots)$. The associated 3-abacus is



the associated 4-abacus is



and the associated 5-abacus is

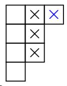
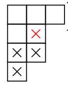
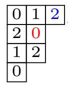


Recall that counting the number of gaps up each bead (continuing counting on the left starting from the $(e-1)$th runner when reaching the 0th one) recovers the underlying partition.

Let λ be a partition and let us consider its associated e-abacus. We give the abacus interpretation of Lemma 4.2.4 in the two particular cases $h = 1$ and $h = e$.

Corollary 4.2.6.
- *We can move a bead on position $j \in \mathbb{Z}$ on runner $i \in \{0, \dots, e-1\}$ to the previously free position j on runner $i-1$ (to the previously free position $j-1$ on runner $e-1$ if $i = 0$) if and only if λ has a removable i-node.*
- *We can move a bead on position j on runner i to the previously free position j on runner $i+1$ (to the previously free position $j+1$ on runner 0 if $i = e-1$) if and only if λ has an addable $(i+1)$-node.*

Corollary 4.2.7.
- *We can slide a bead on position j on runner i to the previously free position $j-1$ on the same runner if and only if λ has an e-rim hook of hand residue i. Hence, the partition λ is an e-core if and only if its associated e-abacus has no gap, that is, no bead has a gap on its left.*
- *We can slide a bead on position j on runner i to the previously free position $j+1$ on the same runner if and only if λ has an addable e-rim hook of hand residue i. Hence, we can always add an e-rim hook of hand residue i to λ.*

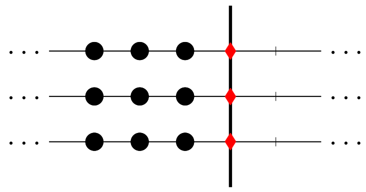
Example 4.2.8. We consider the partition $\lambda = (3, 2, 2, 1)$ of Example 4.2.2. Recall that we gave in Example 4.2.5 the e-abaci for $e \in \{3, 4, 5\}$. The 3-abacus of λ has only one gap thus $r^{\lambda}_{(2,0)}$ is the only 3-rim hook that we can remove. The 4-abacus of λ has two gaps, corresponding to the two lonely beads on runners 0 and 2. Sliding left the bead on runner 2 (respectively 0) corresponds to removing the 4-rim hook $r^{\lambda}_{(0,1)} =$  (resp. $r^{\lambda}_{(1,0)} =$ ). The hand residue, in blue (resp. red), matches since the multiset of residues is given by 

. The 5-abacus of λ has no gap thus λ is a 5-core, as we saw in Example 4.2.2.

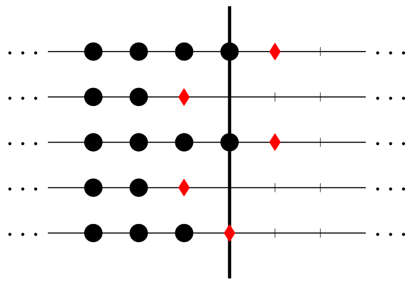
4.2.3 Parametrisation of Q^*

In this subsection, we will parametrise by $\mathbb{Z}^{e-1}$ the set Q^* of all $\alpha \in Q^+$ that are associated with e-cores. Given an e-abacus associated to an e-core λ and $i \in \{0, \dots, e-1\}$, let us write $x_i(\lambda) \in \mathbb{Z}$ for the position of the first gap on the runner i. We say that $x_0(\lambda), \dots, x_{e-1}(\lambda)$ are the *parameters* of the e-abacus, or the *e-abacus variables* of λ. We will also use the notation $x(\lambda) = (x_0(\lambda), \dots, x_{e-1}(\lambda)) \in \mathbb{Z}^e$.

Example 4.2.9. We use $\blacklozenge$ to denote the position of each $x_i(\lambda)$. The 3-abacus associated with the empty partition (which is a 3-core indeed), of associated β-number $(-1, -2, \dots)$, is



thus the associated parameters are $x_0(\emptyset) = x_1(\emptyset) = x_2(\emptyset) = 0$. As we saw in Example 4.2.5, the 5-abacus associated with the 5-core $\lambda = (3, 2, 2, 1)$ is



thus the associated parameters are

$$x_0(\lambda) = x_2(\lambda) = 1, \qquad x_1(\lambda) = x_3(\lambda) = -1, \qquad x_4(\lambda) = 0.$$

We have the following consequences of Corollary 4.2.6.

Lemma 4.2.10. *Let λ be an e-core. For all $i \in \{0, \dots, e-1\}$ we have $x_i(\lambda) = n^i(\lambda) - n^{i+1}(\lambda)$.*

Corollary 4.2.11. *Let λ be an e-core. For all $i \in \{1, \dots, e-1\}$ we have $n^i(\lambda) = n^0(\lambda) - x_0(\lambda) - \cdots - x_{i-1}(\lambda)$.*

If λ is an e-core then Lemma 4.2.10 ensures that $x_0(\lambda) + \cdots + x_{e-1}(\lambda) = 0$. Using Corollaries 4.2.6 and 4.2.7, we can also prove the converse.

Proposition 4.2.12. *Let $x_0, \dots, x_{e-1} \in \mathbb{Z}$. Then $x_0 + \dots + x_{e-1} = 0$ if and only if there is an e-core λ such that $x_i = x_i(\lambda)$ for all $i \in \{0, \dots, e-1\}$.*

We thus have a bijection

$$\{e\text{-cores}\} \overset{1:1}{\longleftrightarrow} \{(x_0, \dots, x_{e-1}) \in \mathbb{Z}^e : x_0 + \dots + x_{e-1} = 0\} =: \mathbb{Z}^e_0.$$

The function n^0 defined on the set of e-cores is a symmetric polynomial in $x_0, \dots, x_{e-1}$. Indeed, exchanging the runners i and $i+1$ for any $i \in \{0, \dots, e-2\}$ only modifies the number of $(i+1)$-nodes (by Corollary 4.2.6) and we conclude since the symmetric group $\mathfrak{S}(\{0, \dots, e-1\})$ is generated by the transpositions $(i, i+1)$ for all $i \in \{0, \dots, e-2\}$. We will explicit this symmetric polynomial in Theorem 4.2.13 using [GKS, Bijection 2]. We will give in Theorem 4.2.18 an equivalent formula, obtained by an abacus manipulation (see also [Ol, top of page 24]). We denote by $\|\cdot\|$ the euclidean norm on tuples of integers.

Theorem 4.2.13. *Let λ be an e-core. We have:*

$$n^0(\lambda) = \frac{1}{2}\|x(\lambda)\|^2 = \frac{1}{2}\sum_{i=0}^{e-1} x_i(\lambda)^2.$$

Proof. For any $i \in \{0, \dots, e-1\}$, our integer $x_i(\lambda)$ is exactly the integer n_i of [GKS, §2]. Let us recall the construction of n_i. A node $\gamma = (a, b)$ of $\mathcal{Y}(\lambda)$ is *exposed* if it is at the end of a row, that is, if $(a, b+1) \notin \mathcal{Y}(\lambda)$. For any $j \in \mathbb{Z}$, we define the region R_j of $\mathcal{Y}(\lambda)$ as the set of nodes $(a, b) \in \mathcal{Y}(\lambda)$ such that $e(j-1) \leq b - a < ej$. The integer n_i is then defined as the greatest integer j such that R_j contains an exposed i-node (if such a node does not exist, we consider the nodes of the "(-1)th column" of $\mathcal{Y}(\lambda)$, which are all exposed). Considering the e-abacus associated with λ, it is now clear that $n_i = x_i(\lambda)$, since:

- the beads on runner i correspond to exposed i-nodes, by definition of the β-number (cf. [JamKe, 2.7.38 Lemma]);
- the rightmost bead on runner i corresponds to the i-node in the region R_j for the largest possible j (two different beads on the same runner correspond to exposed nodes in two different regions).

Thus, we can apply the result of [GKS, Bijection 2]: we have

$$|\lambda| = \frac{e}{2}\|x(\lambda)\|^2 + \langle b, x(\lambda)\rangle,$$

where $b := (0, 1, \dots, e-1) \in \mathbb{Z}^e$ and $\langle \cdot, \cdot \rangle$ is the scalar product associated with $\|\cdot\|$. Since $|\lambda| = \sum_{i=0}^{e-1} n^i(\lambda)$, using Corollary 4.2.11 and Proposition 4.2.12 we obtain

$$\begin{aligned}
n^0(\lambda) &= |\lambda| - \sum_{i=1}^{e-1} n^i(\lambda) \\
&= \frac{e}{2}\|x(\lambda)\|^2 + \langle b, x(\lambda)\rangle - (e-1)n^0(\lambda) + \sum_{i=1}^{e-1}\sum_{j=0}^{i-1} x_j(\lambda) \\
&= \frac{e}{2}\|x(\lambda)\|^2 + \sum_{j=0}^{e-1} j x_j(\lambda) - (e-1)n^0(\lambda) + \sum_{j=0}^{e-2}(e-j-1)x_j(\lambda) \\
&= \frac{e}{2}\|x(\lambda)\|^2 - (e-1)n^0(\lambda) - (e-1)\underbrace{\sum_{j=0}^{e-1} x_j(\lambda)}_{=0} \\
&= \frac{e}{2}\|x(\lambda)\|^2 - (e-1)n^0(\lambda),
\end{aligned}$$

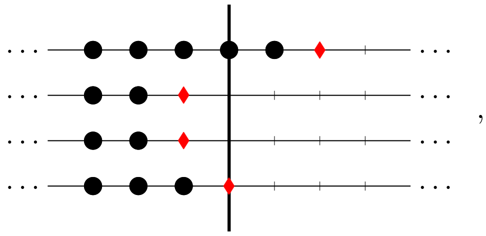
and we conclude. $\square$

Remark 4.2.14. Let λ be an e-core. Using Corollary 4.2.11 and Theorem 4.2.13 we obtain

$$n^i(\lambda) = \frac{1}{2}\|x(\lambda)\|^2 - x_0(\lambda) - \cdots - x_{i-1}(\lambda),$$

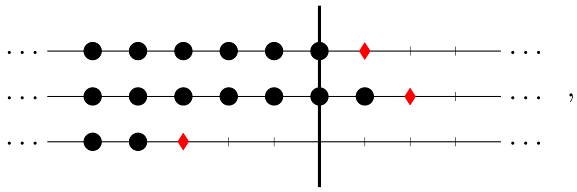
for all $i \in \{1, \ldots, e-1\}$.

Example 4.2.15. We take $p = 2$ and $e = 4$. We consider the parameter $x := (2, -1, -1, 0) \in \mathbb{Z}_0^4$. The corresponding 4-abacus is



the β-number is then $(4, 0, -1, -4, -5, \ldots)$ and this corresponds to the 4-core $\lambda = (5, 2, 2)$. The multiset of residues is $\begin{smallmatrix} 0&1&2&3&0 \\ 3&0 \\ 2&3 \end{smallmatrix}$ and the number of 0-nodes is $3 = \frac{1}{2}(2^2 + 1^2 + 1^2 + 0^2)$.

Example 4.2.16. We take $p = e = 3$. We consider the parameter $x := (1, 2, -3) \in \mathbb{Z}_0^3$. The corresponding 4-abacus is:



the β-number is then $(4, 1, 0, -2, -3, -5, -6, -8, -9, \ldots)$ and this corresponds to the 4-core $\lambda = (5, 3, 3, 2, 2, 1, 1)$. The multiset of residues is $\begin{smallmatrix} 0&1&2&0&1 \\ 2&0&1 \\ 1&2&0 \\ 0&1 \\ 2&0 \\ 1 \\ 0 \end{smallmatrix}$ and the number of 0-nodes is $7 = \frac{1}{2}(1^2 + 2^2 + 3^2)$.

We will now prove the formula of Theorem 4.2.13 using an abacus manipulation.

Lemma 4.2.17. *Let $0 \leq i < i' \leq e-1$ and $h \in \mathbb{Z}$. Let λ be an e-core and let μ be the e-core whose parameters satisfy*

$$\begin{aligned} x_i(\mu) &= x_i(\lambda) + h, \\ x_{i'}(\mu) &= x_{i'}(\lambda) - h, \\ x_j(\mu) &= x_j(\lambda), \qquad \textit{for all } j \neq i, i'. \end{aligned}$$

Then $n^0(\mu) = n^0(\lambda) + h[x_i(\lambda) - x_{i'}(\lambda)] + h^2$.

Proof. Note that the e-core μ is well-defined thanks to Proposition 4.2.12. By Corollary 4.2.6 and since $i < i'$, moving beads from runner i down to runner i' or from runner i' up to runner i only changes the number of j-nodes for $j \in \{i+1, \ldots, i'\}$. Since $0 \leq i < i' \leq e-1$, we deduce that these operations do not change the number of 0-nodes. Hence, to determine $n^0(\mu)$ it suffices to consider additions and deletions of e-rim hook, more specifically bead slides on runners i and i'.

Noticing that exchanging λ and μ changes the sign of h, an easy calculation shows that we can assume that $h \geq 0$. Moreover, by induction it suffices to prove the lemma for $h = 1$. Thus, we have

$$\begin{aligned} x_i(\mu) &= x_i(\lambda) + 1, \\ x_{i'}(\mu) &= x_{i'}(\lambda) - 1, \\ x_j(\mu) &= x_j(\lambda), \qquad \text{for all } j \neq i, i'. \end{aligned}$$

To get from the e-abacus of λ to the e-abacus of μ, we need to perform $|\delta|$ slides on runner i or i', where $\delta \coloneqq x_i(\lambda) - x_{i'}(\lambda) + 1$. More precisely:

- if $\delta \geq 0$ then we slide right δ times the rightmost bead on runner i' and then move it up to runner i (thus we added δ times a 0-node);
- if $\delta \leq 0$ then we move up the rightmost bead on runner i' to runner i and we slide it $-\delta$ times to the left (thus we removed $-\delta$ times a 0-node).

We conclude that $n^0(\mu) = n^0(\lambda) + \delta$, which is exactly the desired formula for $h = 1$. □

Write σ_1 (respectively σ_2) for the homogeneous elementary symmetric polynomial of degree 1 (resp. of degree 2) in $e - 1$ indeterminates. We have:

$$\begin{aligned} \sigma_1(x_0, \dots, x_{e-2}) &= \sum_{i=0}^{e-2} x_i, \\ \sigma_2(x_0, \dots, x_{e-2}) &= \sum_{0 \leq i < j \leq e-2} x_i x_j. \end{aligned}$$

Theorem 4.2.18. *Let $\check{x} = (x_0, \dots, x_{e-2}) \in \mathbb{Z}^{e-1}$. The number of 0-nodes in the e-core λ of parameter $x \coloneqq (x_0, \dots, x_{e-2}, -x_0 - \cdots - x_{e-2}) \in \mathbb{Z}_0^e$ is $n^0(\lambda) = \sigma_1(\check{x})^2 - \sigma_2(\check{x})$.*

Proof. We start with the e-abacus of the empty partition $\lambda_{(-1)} \coloneqq \emptyset$. We use the runner $e - 1$ as a buffer. For i from 0 to $e - 2$, we apply Lemma 4.2.17 to $\lambda_{(i-1)}$ and runners i and $e - 1$ with $h \coloneqq x_i$. The e-core $\lambda_{(i)}$ that we obtain satisfies

$$\begin{aligned} x_j(\lambda_{(i)}) &= x_j, \qquad \text{for all } j \in \{0, \dots, i\}, \\ x_j(\lambda_{(i)}) &= 0, \qquad \text{for all } j \in \{i+1, \dots, e-2\}, \\ x_{e-1}(\lambda_{(i)}) &= -x_0 - \cdots - x_i, \end{aligned}$$

and

$$n^0(\lambda_{(i)}) = n^0(\lambda_{(i-1)}) + x_i\big[x_i(\lambda_{(i-1)}) - x_{e-1}(\lambda_{(i-1)})\big] + x_i^2.$$

If $i = 0$ the above formula just reads $n^0(\lambda_{(0)}) = x_0^2$, and for any $i \in \{1, \dots, e-2\}$ we obtain

$$\begin{aligned} n^0(\lambda_{(i)}) &= n^0(\lambda_{(i-1)}) + x_i\big[0 - (-x_0 - \cdots - x_{i-1})\big] + x_i^2 \\ &= n^0(\lambda_{(i-1)}) + x_i(x_0 + \cdots + x_{i-1}) + x_i^2. \end{aligned}$$

Since $\lambda = \lambda_{(e-2)}$, we have

$$\begin{aligned} n^0(\lambda) &= x_0^2 + \sum_{i=1}^{e-2} \left[x_i(x_0 + \cdots + x_{i-1}) + x_i^2\right] \\ &= \sum_{i=0}^{e-2} x_i^2 + \sigma_2(\check{x}) \\ &= \sigma_1(\check{x})^2 - \sigma_2(\check{x}), \end{aligned}$$

as desired. □

Remark 4.2.19. Since $n^0(\lambda) = \|\check{x}\|^2 + \sigma_2(\check{x})$ and $x_{e-1}(\lambda)^2 = (-x_0 - \cdots - x_{e-2})^2 = \|\check{x}\|^2 + 2\sigma_2(\check{x})$, we recover the formula of Theorem 4.2.13.

4.2.4 Multipartitions

Let $d, \eta, p \in \mathbb{N}^*$ and assume that $e = \eta p$. We define $r := dp$ and we identify $\mathbb{Z}/r\mathbb{Z}$ with the set $\{0, \dots, r-1\}$. Let $\kappa = (\kappa_0, \dots, \kappa_{r-1}) \in (\mathbb{Z}/e\mathbb{Z})^r$ be a multicharge. An *r-partition* (or *multipartition*) of n is an r-tuple $\boldsymbol{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(r-1)})$ of partitions such that $|\boldsymbol{\lambda}| := |\lambda^{(0)}| + \cdots + |\lambda^{(r-1)}| = n$. We write $\boldsymbol{\lambda} \in \mathcal{P}_n^\kappa$ if $\boldsymbol{\lambda}$ is an r-partition of n. We say that κ is *compatible* with (d, η, p) when

$$\kappa_{k+d} = \kappa_k + \eta, \qquad \text{for all } k \in \mathbb{Z}/r\mathbb{Z}. \tag{4.2.20}$$

Thus, the multicharge κ is compatible with (d, η, p) if and only if

$$\kappa = (\kappa_0, \dots, \kappa_{d-1}, \kappa_0 + \eta, \dots, \kappa_{d-1} + \eta, \dots\dots, \kappa_0 + (p-1)\eta, \dots, \kappa_{d-1} + (p-1)\eta). \tag{4.2.21}$$

Example 4.2.22. If $d = 1$ and $\eta = p = 2$ (thus $e = 4$ and $r = 2$), the multicharge $\kappa := (0, 2) \in (\mathbb{Z}/4\mathbb{Z})^2$ is compatible with (d, η, p).

The Young diagram of an r-partition $\boldsymbol{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(r-1)})$ is the subset of $\mathbb{N}^3$ defined by

$$\mathcal{Y}(\boldsymbol{\lambda}) := \bigcup_{c=0}^{r-1} \left(\mathcal{Y}(\lambda^{(c)}) \times \{c\}\right).$$

The *κ-residue* of a node $\gamma = (a, b, c) \in \mathcal{Y}(\boldsymbol{\lambda})$ is $\operatorname{res}_\kappa(\gamma) := b - a + \kappa_c \pmod{e}$. For any $i \in \mathbb{Z}/e\mathbb{Z}$, we denote by $n_\kappa^i(\boldsymbol{\lambda})$ its multiplicity in the multiset of κ-residues of all elements of $\mathcal{Y}(\boldsymbol{\lambda})$. We also define

$$\alpha_\kappa(\boldsymbol{\lambda}) := \sum_{\gamma \in \mathcal{Y}(\boldsymbol{\lambda})} \alpha_{\operatorname{res}_\kappa(\gamma)} = \sum_{i=0}^{e-1} n_\kappa^i(\boldsymbol{\lambda}) \alpha_i \in Q^+.$$

We have $|\alpha_\kappa(\boldsymbol{\lambda})| = |\boldsymbol{\lambda}|$. By [LyMa], the blocks of $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q)$ partition the set of r-partitions of n via the map $\boldsymbol{\lambda} \mapsto \alpha_\kappa(\boldsymbol{\lambda})$. We say that two r-partitions $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ *belong* to the same block of $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q)$ if $\alpha_\kappa(\boldsymbol{\lambda}) = \alpha_\kappa(\boldsymbol{\mu})$. Finally, if $\boldsymbol{\lambda}$ is an e-multicore, for any $k \in \{0, \dots, r-1\}$ we write

$$x^{(k)}(\boldsymbol{\lambda}) := x(\lambda^{(k)}) \in \mathbb{Z}_0^e,$$

for the parameter of the e-abacus associated to the e-core $\lambda^{(k)}$.

Remark 4.2.23. For ordinary partitions, which are 1-partitions, we recover the definitions of §4.2.1 if $\kappa = 0$. In particular, if λ is a partition then $n^i(\lambda) = n_0^i(\lambda)$ for all $i \in \{0, \dots, e-1\}$ and $\alpha(\lambda) = \alpha_0(\lambda)$. Moreover, if λ is an e-core then $x^{(0)}(\lambda) = x(\lambda)$.

The next lemma is straightforward.

Lemma 4.2.24. *Let λ be a partition and $i, \delta, \kappa_* \in \mathbb{Z}/e\mathbb{Z}$. We have*

$$n^i_{\kappa_*+\delta}(\lambda) = n^{i-\delta}_{\kappa_*}(\lambda).$$

We now give a generalisation of Lemma 4.2.10 and Theorem 4.2.13 in the setting of multipartitions. Recall that we identify $\{0,\dots,e-1\}$ (respectively $\{0,\dots,r-1\}$) with $\mathbb{Z}/e\mathbb{Z}$ (resp. $\mathbb{Z}/r\mathbb{Z}$).

Lemma 4.2.25. *Let $\boldsymbol{\lambda}$ be an e-multicore. For all $i \in \{0,\dots,e-1\}$ we have*

$$n^i_\kappa(\boldsymbol{\lambda}) - n^{i+1}_\kappa(\boldsymbol{\lambda}) = \sum_{k=0}^{r-1} x^{(k)}_{i-\kappa_k}(\boldsymbol{\lambda}).$$

Proof. Write $\boldsymbol{\lambda} = (\lambda^{(0)},\dots,\lambda^{(r-1)})$ and let $i \in \{0,\dots,e-1\}$. By Lemmas 4.2.10 and 4.2.24 we have

$$\begin{aligned}
n^i_\kappa(\boldsymbol{\lambda}) - n^{i+1}_\kappa(\boldsymbol{\lambda}) &= \sum_{k=0}^{r-1}\left[n^i_{\kappa_k}(\lambda^{(k)}) - n^{i+1}_{\kappa_k}(\lambda^{(k)})\right] \\
&= \sum_{k=0}^{r-1}\left[n^{i-\kappa_k}(\lambda^{(k)}) - n^{i+1-\kappa_k}(\lambda^{(k)})\right] \\
&= \sum_{k=0}^{r-1} x_{i-\kappa_k}(\lambda^{(k)}). \\
&= \sum_{k=0}^{r-1} x^{(k)}_{i-\kappa_k}(\boldsymbol{\lambda}).
\end{aligned}$$

□

Finally, for any $i \in \{0,\dots,e-1\}$ define $L_i(x) := \sum_{i'=0}^{i-1} x_{i'}$ for all $x \in \mathbb{Z}^e$. By Corollary 4.2.11, if $\boldsymbol{\lambda} = (\lambda^{(0)},\dots,\lambda^{(r-1)})$ is an e-multicore we have

$$n^0_\kappa(\boldsymbol{\lambda}) = \sum_{k=0}^{r-1} n^0_{\kappa_k}(\lambda^{(k)}) = \sum_{k=0}^{r-1} n^{-\kappa_k}(\lambda^{(k)}) = \sum_{k=0}^{r-1}\left[n^0(\lambda^{(k)}) - L_{-\kappa_k}(x^{(k)}(\boldsymbol{\lambda}))\right].$$

Hence, by Theorem 4.2.13,

$$n^0_\kappa(\boldsymbol{\lambda}) = \sum_{k=0}^{r-1}\left[\frac{1}{2}\|x^{(k)}(\boldsymbol{\lambda})\|^2 - L_{-\kappa_k}(x^{(k)}(\boldsymbol{\lambda}))\right]. \tag{4.2.26}$$

4.2.5 Shifts

We are now ready to define our two shift maps.

Definition 4.2.27. Recall that e is determined by η and p. For any $i \in \mathbb{Z}/e\mathbb{Z}$ we define $\sigma_{\eta,p} \cdot \alpha_i := \alpha_{i+\eta} \in Q^+$, and we extend $\sigma_{\eta,p}$ to a $\mathbb{Z}$-linear map $Q \to Q$.

Definition 4.2.28. Recall that r is determined by d and p. If $\boldsymbol{\lambda} = (\lambda^{(0)},\dots,\lambda^{(r-1)})$ is an r-partition, we define

$${}^{\sigma_{d,p}}\boldsymbol{\lambda} := (\lambda^{(r-d)},\dots,\lambda^{(r-1)},\lambda^{(0)},\dots,\lambda^{(r-d-1)}).$$

For any $\alpha \in Q^+$, we denote by $\mathcal{P}_\alpha^\kappa$ the subset of $\mathcal{P}_n^\kappa$ given by r-partitions $\boldsymbol{\lambda}$ such that $\alpha_\kappa(\boldsymbol{\lambda}) = \alpha$. The two shifts of Definitions 4.2.27 and 4.2.28 are compatible in the following way.

Lemma 4.2.29. *Assume that the multicharge κ is compatible with (d, η, p). If $\boldsymbol{\lambda}$ is an r-partition then $\alpha_\kappa({}^{\sigma_{d,p}}\boldsymbol{\lambda}) = \sigma_{\eta,p} \cdot \alpha_\kappa(\boldsymbol{\lambda})$. In other words, the map $\sigma_{d,p}$ induces a bijection between $\mathcal{P}_\alpha^\kappa$ and $\mathcal{P}_{\sigma_{\eta,p}\cdot\alpha}^\kappa$.*

Proof. Recall that we are identifying $\mathbb{Z}/e\mathbb{Z}$ (respectively $\mathbb{Z}/r\mathbb{Z}$) with $\{0, \dots, e-1\}$ (resp. $\{0, \dots, r-1\}$). We write $\boldsymbol{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(r-1)})$. Using the compatibility equation (4.2.20) for the multicharge κ and Lemma 4.2.24 we have

$$\begin{aligned}
\alpha_\kappa({}^{\sigma_{d,p}}\boldsymbol{\lambda}) &= \alpha_\kappa(\lambda^{(r-d)}, \dots, \lambda^{(r-1)}, \lambda^{(0)}, \dots, \lambda^{(r-d-1)}) \\
&= \sum_{i=0}^{e-1} n_\kappa^i(\lambda^{(r-d)}, \dots, \lambda^{(r-1)}, \lambda^{(0)}, \dots, \lambda^{(r-d-1)})\alpha_i \\
&= \sum_{i=0}^{e-1}\sum_{k=0}^{r-1} n_{\kappa_k}^i(\lambda^{(r-d+k)})\alpha_i \\
&= \sum_{i=0}^{e-1}\sum_{k=0}^{r-1} n_{\kappa_k}^i(\lambda^{(k-d)})\alpha_i \\
&= \sum_{i=0}^{e-1}\sum_{k=0}^{r-1} n_{\kappa_{k+d}}^i(\lambda^{(k)})\alpha_i \\
&= \sum_{i=0}^{e-1}\sum_{k=0}^{r-1} n_{\kappa_k+\eta}^i(\lambda^{(k)})\alpha_i \\
&= \sum_{i=0}^{e-1}\sum_{k=0}^{r-1} n_{\kappa_k}^{i-\eta}(\lambda^{(k)})\alpha_i \\
&= \sum_{i=0}^{e-1}\sum_{k=0}^{r-1} n_{\kappa_k}^i(\lambda^{(k)})\alpha_{i+\eta} \\
&= \sigma_{\eta,p} \cdot \sum_{i=0}^{e-1}\sum_{k=0}^{r-1} n_{\kappa_k}^i(\lambda^{(k)})\alpha_i \\
&= \sigma_{\eta,p} \cdot \sum_{i=0}^{e-1} n_\kappa^i(\lambda^{(0)}, \dots, \lambda^{(r-1)})\alpha_i \\
&= \sigma_{\eta,p} \cdot \alpha(\boldsymbol{\lambda}),
\end{aligned}$$

as desired. The second statement follows. □

Remark 4.2.30. Let p' be an integer that divides p. We have $r = dp = (p'd)\frac{p}{p'}$ and $e = p\eta = \frac{p}{p'}(p'\eta)$. The multicharge κ, which is compatible with (d, η, p), is also compatible with $(p'd, p'\eta, \frac{p}{p'})$: we have $\kappa_{i+d} = \kappa_i + \eta$ thus $\kappa_{i+p'd} = \kappa_i + p'\eta$. Then, applying Definitions 4.2.27 and 4.2.28 we obtain

$$\begin{aligned}
\sigma_{\eta,p}^{p'} &= \sigma_{p'\eta,\frac{p}{p'}}, && \text{in } Q^+, \\
\sigma_{d,p}^{p'} &= \sigma_{p'd,\frac{p}{p'}}, && \text{on } r\text{-partitions}.
\end{aligned}$$

We can now state the main theorem of the chapter, which will be proved in Section 4.4.

Theorem 4.2.31. *Let $\boldsymbol{\lambda}$ be an r-partition and let $\alpha := \alpha_\kappa(\boldsymbol{\lambda}) \in Q^+$. Assume that κ is compatible with (d, η, p). If $\sigma_{\eta,p} \cdot \alpha = \alpha$ then there is an r-partition $\boldsymbol{\mu} \in \mathcal{P}_\alpha^\kappa$ with ${}^{\sigma_{d,p}}\boldsymbol{\mu} = \boldsymbol{\mu}$.*

We say that an r-partition $\boldsymbol{\mu}$ as in Theorem 4.2.31 is *stuttering*. We will often drop the subscripts and only write σ for $\sigma_{d,p}$ and $\sigma_{\eta,p}$ when the meaning is clear from the context.

Example 4.2.32. We consider the setting of Example 4.2.22 and the bipartition $\boldsymbol{\lambda} := ((5,2,1),(1,1))$. The multiset of κ-residues is

$$\begin{array}{|c|c|c|c|c|}\hline 0&1&2&3&0\\\hline 3&0\\\cline{1-2} 2\\\cline{1-1}\end{array}\qquad \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array},$$

thus $\alpha_\kappa(\boldsymbol{\lambda}) = 3(\alpha_0+\alpha_2)+2(\alpha_1+\alpha_3) =: \alpha$. Hence, we have $\sigma\cdot\alpha = \alpha$ but ${}^\sigma\boldsymbol{\lambda} = ((1,1),(5,2,1)) \neq \boldsymbol{\lambda}$. We now consider the partition $\mu := (3,1,1)$. The residue multiset of the bipartition (μ,μ) is

$$\begin{array}{|c|c|c|}\hline 0&1&2\\\hline 3\\\cline{1-1} 2\\\cline{1-1}\end{array}\qquad \begin{array}{|c|c|c|}\hline 2&3&0\\\hline 1\\\cline{1-1} 0\\\cline{1-1}\end{array},$$

thus $\alpha_\kappa(\mu,\mu) = 3(\alpha_0+\alpha_2)+2(\alpha_1+\alpha_3) = \alpha$. Hence, the stuttering bipartition (μ,μ) is as in Theorem 4.2.31.

Remark 4.2.33. Two particular cases of Theorem 4.2.31 easily follow from Lemma 4.2.29.

(*i*) If ${}^\sigma\boldsymbol{\lambda} = \boldsymbol{\lambda}$ then $\sigma\cdot\alpha = \alpha$ and there is nothing to prove.

(*ii*) If $\boldsymbol{\lambda}$ is the only r-partition in $\mathcal{P}^\kappa_\alpha$ (*e.g.* when the associated Ariki–Koike algebra is semi-simple, see [Ar94]) then ${}^\sigma\boldsymbol{\lambda} \in \mathcal{P}^\kappa_{\sigma\cdot\alpha} = \mathcal{P}^\kappa_\alpha$. Hence, if $\sigma\cdot\alpha = \alpha$ we conclude that ${}^\sigma\boldsymbol{\lambda} = \boldsymbol{\lambda}$.

Let us denote by $[\boldsymbol{\lambda}]$ (respectively by $[\alpha]$) the orbit of an r-partition $\boldsymbol{\lambda}$ (resp. of $\alpha \in Q^+$) under the action of σ. We now state Theorem 4.1.2 from the introduction.

Corollary 4.2.34. *Assume that κ is compatible with (d,η,p) and let $\alpha \in Q^+$ such that $\mathcal{P}^\kappa_\alpha$ is not empty. Then $\#[\alpha]$ is the smallest element of the set $\{\#[\boldsymbol{\lambda}] : \boldsymbol{\lambda} \in \mathcal{P}^\kappa_\alpha\}$. In other words, if $\boldsymbol{\lambda}$ is an r-partition and $\alpha := \alpha_\kappa(\boldsymbol{\lambda})$, if $\sigma^j\cdot\alpha = \alpha$ for some $j \in \{0,\dots,p-1\}$ then there exists an r-partition $\boldsymbol{\mu}$ such that $\alpha_\kappa(\boldsymbol{\mu}) = \alpha$ and ${}^{\sigma^j}\boldsymbol{\mu} = \boldsymbol{\mu}$.*

Proof. The second part of the statement is clear. Let $\mathcal{C}$ be the set $\{\#[\boldsymbol{\lambda}] : \boldsymbol{\lambda} \in \mathcal{P}^\kappa_\alpha\}$ and let us prove that $\#[\alpha]$ is the smallest element of $\mathcal{C}$. For each $\boldsymbol{\lambda} \in \mathcal{P}^\kappa_\alpha$, by Lemma 4.2.29 we have $\alpha_\kappa({}^\sigma\boldsymbol{\lambda}) = \sigma\cdot\alpha_\kappa(\boldsymbol{\lambda})$ thus $\#[\boldsymbol{\lambda}] \geq \#[\alpha]$, hence $\#[\alpha]$ is a lower bound of $\mathcal{C}$. To prove that it is the smallest element, it suffices to prove that there is an r-partition $\boldsymbol{\mu} \in \mathcal{P}^\kappa_\alpha$ such that $\#[\boldsymbol{\mu}] \leq \#[\alpha]$. Write $p' := \#[\alpha]$. The integer p' divides p since σ has order p. By Remark 4.2.30, we know that κ is compatible with $(p'd, p'\eta, \frac{p}{p'})$. Moreover, we have $\sigma^{p'}_{\eta,p}\cdot\alpha = \alpha$ thus Remark 4.2.30 also gives

$$\sigma_{p'\eta,\frac{p}{p'}}\cdot\alpha = \alpha.$$

Hence, by Theorem 4.2.31 applied with $(p'd, p'\eta, \frac{p}{p'})$ we know that there is an r-partition $\boldsymbol{\mu} \in \mathcal{P}^\kappa_\alpha$ such that

$${}^{\sigma_{dp',\frac{p}{p'}}}\boldsymbol{\mu} = \boldsymbol{\mu},$$

that is, by another application of Remark 4.2.30,

$${}^{\sigma^{p'}_{d,p}}\boldsymbol{\mu} = \boldsymbol{\mu}.$$

Hence, we have $\#[\boldsymbol{\mu}] \leq p'$ and we conclude since $p' = \#[\alpha]$. □

Remark 4.2.35. By [LyMa], two r-partitions are in a same $\mathcal{P}^\kappa_\alpha$ if and only if they belong to the same block of $\mathrm{H}^{\boldsymbol{\Lambda}}_n(q)$. Thus, Corollary 4.2.34 gives a little information about the r-partitions that belong to each block. As we mentionned in the introduction, Wada [Wa] proved that the maximum of the set $\{\#[\boldsymbol{\lambda}] : \boldsymbol{\lambda} \in \mathcal{P}^\kappa_\alpha\}$ of Corollary 4.2.34 is always p, provided that this set has at least two elements. His proof is very short but relies on the (non trivial) fact that if $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are in $\mathcal{P}^\kappa_\alpha$ then they are *Jantzen equivalent* (cf. [LyMa]). On the contrary, we did not find a way to use [LyMa] to prove Theorem 4.2.31.

We conclude this section by a reduction step for our main theorem. We assume that the multicharge κ is compatible with (d, η, p). For any $\ell \in \{0, \dots, d-1\}$, we define the multicharge $\kappa^{(\ell)} \in (\mathbb{Z}/e\mathbb{Z})^p$ by

$$\kappa^{(\ell)} := (\kappa_\ell, \kappa_{\ell+d}, \dots, \kappa_{\ell+(p-1)d}) = (\kappa_\ell, \kappa_\ell + \eta, \dots, \kappa_\ell + (p-1)\eta). \tag{4.2.36}$$

We first need the following lemma.

Lemma 4.2.37. *Let $\ell \in \{0, \dots, d-1\}$, let λ be a partition and let μ be a partition obtained from λ by wrapping on an η-rim hook. We define the two p-partitions λ^p and μ^p by $\lambda^p := (\lambda, \dots, \lambda)$ and $\mu^p := (\mu, \dots, \mu)$. If $\alpha := \alpha_{\kappa^{(\ell)}}(\lambda^p)$ and $\beta := \alpha_{\kappa^{(\ell)}}(\mu^p)$ then $\beta = \alpha + \alpha_0 + \dots + \alpha_{e-1}$.*

Proof. By Remark 4.2.3, we have $\alpha_{\kappa_\ell}(\mu) = \alpha_{\kappa_\ell}(\lambda) + \alpha_{i_0} + \dots + \alpha_{i_0+\eta-1}$ for some $i_0 \in \mathbb{Z}/e\mathbb{Z}$. Thus, for any $j \in \{0, \dots, p-1\}$ we have

$$\begin{aligned}
\alpha_{\kappa_\ell + j\eta}(\mu) &= \sigma^j \cdot \alpha_{\kappa_\ell}(\mu) \\
&= \sigma^j \cdot \alpha_{\kappa_\ell}(\lambda) + \sum_{i=0}^{\eta-1} \sigma^j \cdot \alpha_{i_0+i} \\
&= \alpha_{\kappa_\ell+j\eta}(\lambda) + \sum_{i=0}^{\eta-1} \alpha_{i_0+i+j\eta}.
\end{aligned}$$

We obtain

$$\begin{aligned}
\beta &= \alpha_{\kappa^{(\ell)}}(\mu^p) \\
&= \sum_{j=0}^{p-1} \alpha_{\kappa_\ell+j\eta}(\mu) \\
&= \sum_{j=0}^{p-1} \alpha_{\kappa_\ell+j\eta}(\lambda) + \sum_{j=0}^{p-1}\sum_{i=0}^{\eta-1} \alpha_{i_0+i+j\eta} \\
&= \alpha_{\kappa^{(\ell)}}(\lambda^p) + \alpha_0 + \dots + \alpha_{e-1} \\
&= \alpha + \alpha_0 + \dots + \alpha_{e-1}.
\end{aligned}$$

□

If $\boldsymbol{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(r-1)})$ is an r-partition, its *e-multicore* is the r-partition $\overline{\boldsymbol{\lambda}} := (\overline{\lambda^{(0)}}, \dots, \overline{\lambda^{(r-1)}})$. We say that $\boldsymbol{\lambda}$ is an *e-multicore* if $\boldsymbol{\lambda} = \overline{\boldsymbol{\lambda}}$, that is, if each $\lambda^{(k)}$ for $k \in \{0, \dots, r-1\}$ is an e-core.

Proposition 4.2.38. *It suffices to prove Theorem 4.2.31 for the e-multicores.*

Proof. Let $\boldsymbol{\lambda}$ be an r-partition such that $\sigma \cdot \alpha_\kappa(\boldsymbol{\lambda}) = \alpha_\kappa(\boldsymbol{\lambda})$ and let $\overline{\boldsymbol{\lambda}}$ be its e-multicore. By definition of the e-multicore and by Remark 4.2.3, we have $\alpha_\kappa(\boldsymbol{\lambda}) = \alpha_\kappa(\overline{\boldsymbol{\lambda}}) + w\sum_{i=0}^{e-1} \alpha_i$ where $w \in \mathbb{N}$ is the number of e-rim hooks that we need to wrap on to obtain $\boldsymbol{\lambda}$ from $\overline{\boldsymbol{\lambda}}$. Since $\alpha_\kappa(\boldsymbol{\lambda})$ and $\sum_{i=0}^{e-1} \alpha_i$ are both stable by σ, we have $\sigma \cdot \alpha_\kappa(\overline{\boldsymbol{\lambda}}) = \alpha_\kappa(\overline{\boldsymbol{\lambda}})$. If Theorem 4.2.31 is true for the e-multicore $\overline{\boldsymbol{\lambda}}$, we can find a stuttering r-partition $\widetilde{\boldsymbol{\mu}} = {}^\sigma\widetilde{\boldsymbol{\mu}}$ with $\alpha_\kappa(\widetilde{\boldsymbol{\mu}}) = \alpha_\kappa(\overline{\boldsymbol{\lambda}})$. Write $\widetilde{\boldsymbol{\mu}} = (\widetilde{\mu}^{(0)}, \dots, \widetilde{\mu}^{(r-1)})$ and let $\mu^{(0)}$ be a partition obtained by wrapping on w times an η-rim hook to $\widetilde{\mu}^{(0)}$. We define

$$\begin{aligned}
\mu^{(jd)} &:= \mu^{(0)}, &&\text{for all } j \in \{1, \dots, p-1\}, \\
\mu^{(k)} &:= \widetilde{\mu}^{(k)}, &&\text{for all } k \in \{0, \dots, r-1\} \setminus \{0, d, \dots, (p-1)d\}.
\end{aligned}$$

The r-partition $\boldsymbol{\mu} := (\mu^{(0)}, \dots, \mu^{(r-1)})$ satisfies $\boldsymbol{\mu} = {}^{\sigma}\boldsymbol{\mu}$. Moreover, since $\widetilde{\mu}^{(0)} = \widetilde{\mu}^{(jd)}$ for all $j \in \{1, \dots, p-1\}$, we can apply w times Lemma 4.2.37 with $\ell := 0$ starting from the p-partition $(\widetilde{\mu}^{(0)}, \dots, \widetilde{\mu}^{(0)})$. We obtain

$$\begin{aligned}
\alpha_\kappa(\boldsymbol{\mu}) &= \alpha_{\kappa^{(0)}}(\mu^{(0)}, \dots, \mu^{(0)}) + \sum_{\ell=1}^{d-1} \alpha_{\kappa^{(\ell)}}(\mu^{(\ell)}, \dots, \mu^{(\ell)}) \\
&= \alpha_{\kappa^{(0)}}(\widetilde{\mu}^{(0)}, \dots, \widetilde{\mu}^{(0)}) + w \sum_{i=0}^{e-1} \alpha_i + \sum_{\ell=1}^{d-1} \alpha_{\kappa^{(\ell)}}(\widetilde{\mu}^{(\ell)}, \dots, \widetilde{\mu}^{(\ell)}) \\
&= \alpha_\kappa(\widetilde{\boldsymbol{\mu}}) + w \sum_{i=0}^{e-1} \alpha_i \\
&= \alpha_\kappa(\overline{\boldsymbol{\lambda}}) + w \sum_{i=0}^{e-1} \alpha_i \\
&= \alpha_\kappa(\boldsymbol{\lambda}).
\end{aligned}$$

Hence, Theorem 4.2.31 is proved for $\boldsymbol{\lambda}$. □

Remark 4.2.39. Since the η-rim hooks that we wrap on are arbitrary, the stuttering r-partition in Theorem 4.2.31 is not unique in general. Moreover, using the same idea of wrapping on η-rim hooks we can easily prove Theorem 4.2.31 in the particular case $\eta = 1$ (that is, $p = e$). Finally, if $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are as in Theorem 4.2.31 and if $\boldsymbol{\lambda}$ is an e-multicore, then $\boldsymbol{\mu}$ is not necessary an e-multicore.

4.3 Binary tools and inequalities

In this section, we introduce two technical tools that we will need to prove Theorem 4.2.31. In §4.3.1, given a family of binary matrices satisfying some conditions, our aim is to prove that we can find a series of *compatible* submatrices $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. We will need to study some particular cases (Lemma 4.3.6 and Proposition 4.3.7) before stating the main result, Corollary 4.3.8. We use this result to prove in §4.3.2 the existence of a binary matrix with prescribed row, (partial) column and block sums. Finally, we will give §4.3.3 some inequalities. The first one will be reminiscent of the binary setting, and the others will use convexity.

We use $|\cdot| : \mathbb{R}^n \to \mathbb{R}^n$ to denote the sum of the coordinates (we warn the reader that we do not take the sum of the absolute values) and we write $\|\cdot\|$ for the euclidean norm.

4.3.1 Binary matrices

Given two matrices with entries in $\{0, 1\}$ whose row sums (respectively column sums) are pairwise equal, we can get from the one to the other by replacing submatrices $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ by $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ (cf. [Ry]). These interchanges do not change the row or column sums, however they may change block sums. The results of this section, particularly Corollary 4.3.5, will be used to prove Proposition 4.3.14 in §4.3.2, where we show the existence of a binary matrix with prescribed row, column and block sums. Note that Chernyak and Chernyak [ChCh] considered matrices with prescribed row, column and block sums, but they did not study the existence problem.

We call *binary matrix* a matrix with entries is $\{0, 1\}$. If M is an $m \times n$ binary matrix, we write $M_{\ell k}$ for its entry at $(\ell, k) \in \{1, \dots, m\} \times \{1, \dots, n\}$. We denote by $\gamma_{\ell,k}(M)$ the binary matrix that we obtain from M by changing the entry (ℓ, k) to $1 - M_{\ell k}$. We write $R_\ell(M)$ for the ℓth row of M. Note that if $|M|$ denotes the sum of the entries of M then $|M| = \sum_\ell |R_\ell(M)|$.

Finally, if the number of rows of M is even, we will systematically write $M = \begin{pmatrix} M^+ \\ M^- \end{pmatrix}$ where M^+ and M^- have the same size, and we define $\gamma^{\pm}_{\ell,k}(M) := \begin{pmatrix} \gamma_{\ell,k}(M^+) \\ \gamma_{\ell,k}(M^-) \end{pmatrix}$.

Definition 4.3.1. Let $A = \begin{pmatrix} A^+ \\ A^- \end{pmatrix}$ and $B = \begin{pmatrix} B^+ \\ B^- \end{pmatrix}$ be two binary matrices with the same even number of rows. We say that the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is a *compatible* submatrix of $\left(\begin{array}{c|c} A & B \end{array} \right)$ if there exist ℓ, k, k' such that

$$\begin{aligned} A^+_{\ell k} &= 1, & B^+_{\ell k'} &= 0, \\ A^-_{\ell k} &= 0, & B^-_{\ell k'} &= 1. \end{aligned}$$

In that case, we will write $A \models_{\ell,k,k'} B$. We denote by $\gamma_{\ell,k,k'}(A, B) := (\gamma^{\pm}_{\ell,k}(A), \gamma^{\pm}_{\ell,k'}(B))$ the pair of binary matrices that we obtain if we replace the submatrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Example 4.3.2. We consider the binary matrices $A = \begin{pmatrix} A^+ \\ A^- \end{pmatrix}$ and $B = \begin{pmatrix} B^+ \\ B^- \end{pmatrix}$ defined by

$$\begin{aligned} A^+ &:= \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, & B^+ &:= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\ A^- &:= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & B^- &:= \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}. \end{aligned}$$

The red entries prove that $A \models_{1,2,3} B$. With $(\widetilde{A}, \widetilde{B}) := \gamma_{1,2,3}(A, B)$, we have

$$\begin{aligned} \widetilde{A}^+ &:= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & \widetilde{B}^+ &:= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ \widetilde{A}^- &:= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & \widetilde{B}^- &:= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}. \end{aligned}$$

If A and B are two binary matrices with the same even number of rows, the set of all pairs $\gamma_{\ell,k,k'}(A, B)$ where ℓ, k, k' are such that $A \models_{\ell,k,k'} B$ is denoted by $\Gamma(A, B)$. Moreover, we will write $A \models B$ if the set $\Gamma(A, B)$ is non-empty, that is, if there exist ℓ, k, k' such that $A \models_{\ell,k,k'} B$.

We can generalise these notations to a family $(A_i)_{1 \le i \le n}$ of binary matrices with the same even number of rows. Let $((\ell_i, k_i, k'_i))_{1 \le i \le n-1}$ be a family of triples such that

$$A_i \models_{\ell_i,k_i,k'_i} A_{i+1},$$

for all $i \in \{1, \ldots, n-1\}$. For any $i \in \{2, \ldots, n-1\}$ we have

$$A_{i-1} \models_{\ell_{i-1},k_{i-1},k'_{i-1}} A_i \models_{\ell_i,k_i,k'_i} A_{i+1},$$

thus, according to Definition 4.3.1,

$$(\ell_{i-1}, k'_{i-1}) \neq (\ell_i, k_i).$$

Hence, for all $i \in \{2, \ldots, n-1\}$ we have

$$\begin{aligned} A_{i-1} &\models_{\ell_{i-1},k_{i-1},k'_{i-1}} \gamma^{\pm}_{\ell_i,k_i}(A_i), \\ \gamma^{\pm}_{\ell_{i-1},k'_{i-1}}(A_i) &\models_{\ell_i,k_i,k'_i} A_{i+1}, \end{aligned}$$

and

$$\gamma^{\pm}_{\ell_i,k_i}(\gamma^{\pm}_{\ell_{i-1},k'_{i-1}}(A_i)) = \gamma^{\pm}_{\ell_{i-1},k'_{i-1}}(\gamma^{\pm}_{\ell_i,k_i}(A_i)). \tag{4.3.3}$$

We denote by $\gamma_{((\ell_i,k_i,k'_i))_{1\le i\le n-1}}((A_i)_{1\le i\le n})$ the family $(\widetilde{A}_i)_{1\le i\le n}$ defined by

$$\begin{aligned}
\widetilde{A}_1 &\coloneqq \gamma^{\pm}_{\ell_1,k_1}(A_1),\\
\widetilde{A}_i &\coloneqq \gamma^{\pm}_{\ell_i,k_i}(\gamma^{\pm}_{\ell_{i-1},k'_{i-1}}(A_i)), \qquad \text{for all } i \in \{2,\dots,n-1\},\\
\widetilde{A}_n &\coloneqq \gamma^{\pm}_{\ell_{n-1},k'_{n-1}}(A_n).
\end{aligned}$$

By (4.3.3), no choice has been made to define $\widetilde{A}_i$ for $i \in \{2,\dots,n-1\}$. Finally, we denote by $\Gamma(A_1,\dots,A_n)$ the set of all families $\gamma_{((\ell_i,k_i,k'_i))_{1\le i\le n-1}}((A_i)_{1\le i\le n})$ where $((\ell_i,k_i,k'_i))_{1\le i\le n-1}$ is such that $A_1 \models_{\ell_1,k_1,k'_1} \cdots \models_{\ell_{n-1},k_{n-1},k'_{n-1}} A_n$, and we will write $A_1 \models \cdots \models A_n$ if $\Gamma(A_1,\dots,A_n)$ is non-empty.

The following properties are straightforward from the definition.

Proposition 4.3.4. *Let A and B be two binary matrices with the same even number of rows such that $A \models_{\ell,k,k'} B$. If $(\widetilde{A},\widetilde{B}) \coloneqq \gamma_{\ell,k,k'}(A,B)$ then*

$$\widetilde{A}^+_{\ell k} = A^+_{\ell k} - 1, \qquad \widetilde{B}^+_{\ell k'} = B^+_{\ell k'} + 1,$$
$$\widetilde{A}^-_{\ell k} = A^-_{\ell k} + 1, \qquad \widetilde{B}^-_{\ell k'} = B^-_{\ell k'} - 1,$$

the other entries being unchanged. Hence, the following equalities are satisfied:

$$\widetilde{A}^+_{\ell k} + \widetilde{A}^-_{\ell k} = A^+_{\ell k} + A^-_{\ell k}, \qquad \widetilde{B}^+_{\ell k'} + \widetilde{B}^-_{\ell k'} = B^+_{\ell k'} + B^-_{\ell k'},$$
$$R_\ell(\widetilde{A}^+) + R_\ell(\widetilde{B}^+) = R_\ell(A^+) + R_\ell(B^+), \qquad R_\ell(\widetilde{A}^-) + R_\ell(\widetilde{B}^-) = R_\ell(A^-) + R_\ell(B^-),$$

and

$$|\widetilde{A}^+| = |A^+| - 1, \qquad |\widetilde{B}^+| = |B^+| + 1,$$
$$|\widetilde{A}^-| = |A^-| + 1, \qquad |\widetilde{B}^-| = |B^-| - 1.$$

As a consequence, if $A \models B \models C$ and $(\widehat{A},\widehat{B},\widehat{C}) \in \Gamma(A,B,C)$ then $|\widehat{B}^+| = |B^+|$ and $|\widehat{B}^-| = |B^-|$.

Corollary 4.3.5. *Let $(A_i)_{1\le i\le n}$ be a family of binary matrices with the same even number of rows. Assume that $i_0,\dots,i_s$ are distinct integers such that $A_{i_0} \models \dots \models A_{i_s}$ and let $(\widetilde{A}_{i_0},\dots,\widetilde{A}_{i_s}) \in \Gamma(A_{i_0},\dots,A_{i_s})$. Then*

$$|\widetilde{A}^+_{i_0}| = |A^+_{i_0}| - 1, \qquad |\widetilde{A}^+_{i_s}| = |A^+_{i_s}| + 1,$$
$$|\widetilde{A}^-_{i_0}| = |A^-_{i_0}| + 1, \qquad |\widetilde{A}^-_{i_s}| = |A^-_{i_s}| - 1,$$

and for all $t \in \{1,\dots,s-1\}$ we have

$$|\widetilde{A}^+_{i_t}| = |A^+_{i_t}|,$$
$$|\widetilde{A}^-_{i_t}| = |A^-_{i_t}|.$$

The following, easy to prove, lemma is very important in the proof of Proposition 4.3.14.

Lemma 4.3.6. *Let A and B be two binary matrices with the same even number of rows. We assume that*

$$|R_\ell(A^+)| + |R_\ell(B^+)| = |R_\ell(A^-)| + |R_\ell(B^-)|, \qquad \text{for all } \ell,$$
$$|A^+| > |A^-|.$$

Then $A \models B$.

Proof. Since $|A^+| > |A^-|$, there is some ℓ such that $|R_\ell(A^+)| > |R_\ell(A^-)|$. Since the matrices have their entries in $\{0,1\}$, for all k we have

$$\begin{pmatrix} A^+_{\ell k} \\ A^-_{\ell k} \end{pmatrix} \in \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

Thus, there is some k such that $\begin{pmatrix} A^+_{\ell k} \\ A^-_{\ell k} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Moreover, we have

$$|R_\ell(B^+)| = |R_\ell(B^-)| + \big(|R_\ell(A^-)| - |R_\ell(A^+)|\big) < |R_\ell(B^-)|.$$

Again, we deduce that there is some k' such that $\begin{pmatrix} B^+_{\ell k'} \\ B^-_{\ell k'} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Finally, we have

$$\begin{aligned} A^+_{\ell k} &= 1, & B^+_{\ell k'} &= 0, \\ A^-_{\ell k} &= 0, & B^-_{\ell k'} &= 1, \end{aligned}$$

thus $A \models B$. □

Let us now give a generalisation of Lemma 4.3.6 to an arbitrary number of matrices.

Proposition 4.3.7. *Let $(A_i)_{1 \le i \le n}$ be a family of binary matrices with the same even number of rows. We assume that*

$$\begin{aligned} \sum_{i=1}^{n} |R_\ell(A_i^+)| &= \sum_{i=1}^{n} |R_\ell(A_i^-)|, && \textit{for all } \ell, \\ |A_1^+| &> |A_1^-|, \\ |A_i^+| &\ge |A_i^-|, && \textit{for all } i \in \{2, \dots, n-1\}. \end{aligned}$$

Then there exists a sequence $1 < i_1, \dots, i_{s-1} < n$ of distinct integers such that

$$A_1 \models A_{i_1} \models A_{i_2} \models \dots \models A_{i_{s-1}} \models A_n.$$

Proof. We consider the following binary matrices with an even number of rows:

$$B_1 := \begin{pmatrix} A_2 & A_3 & \cdots & A_{n-1} & A_n \end{pmatrix}.$$

For each ℓ we have $|R_\ell(B_1^+)| = \sum_{i=2}^{n} |R_\ell(A_i^+)|$ and $|R_\ell(B_1^-)| = \sum_{i=2}^{n} |R_\ell(A_i^-)|$. Thus,

$$\begin{aligned} |R_\ell(A_1^+)| + |R_\ell(B_1^+)| &= \sum_{i=1}^{n} |R_\ell(A_i^+)| \\ &= \sum_{i=1}^{n} |R_\ell(A_i^-)| \\ &= |R_\ell(A_1^-)| + \sum_{i=2}^{n} |R_\ell(A_i^-)| \\ |R_\ell(A_1^+)| + |R_\ell(B_1^+)| &= |R_\ell(A_1^-)| + |R_\ell(B_1^-)|. \end{aligned}$$

Since $|A_1^+| > |A_1^-|$, we can apply Lemma 4.3.6 to the matrices A and B_1. Hence, if we define I_1 as the set of integers $i \in \{2, \dots, n\}$ such that $A_1 \models A_i$, then I_1 is not empty. If $n \in I_1$ then the proof is over, and otherwise we start an induction. Assume that for some integer s we have some pairwise disjoint non-empty subsets $I_0 \coloneqq \{1\}, I_1, \ldots, I_{s-1}$ of $\{1, \ldots, n-1\}$ such that for all $t \in \{1, \ldots, s-1\}$ we have

$$\text{for all } i_t \in I_t, \text{ there exists } i_{t-1} \in I_{t-1} \text{ such that } A_{i_{t-1}} \models A_{i_t}.$$

In the following, we write $i \notin I_0 \cup \cdots \cup I_{s-1}$ to mean $i \in \{1, \ldots, n\} \setminus (I_0 \cup \cdots \cup I_{s-1})$. We define the two following binary matrices with the same even number of rows:

$$\widehat{A}_s^+ \coloneqq \Big(A_i\Big)_{i \in I_0 \cup \cdots \cup I_{s-1}},$$
$$B_s \coloneqq \Big(A_i\Big)_{i \notin I_0 \cup \cdots \cup I_{s-1}}.$$

Note that the matrix B_s is not empty since $n \in \{1, \ldots, n\} \setminus (I_0 \cup \cdots \cup I_{s-1})$. For all ℓ we have

$$|R_\ell(\widehat{A}_s^+)| = \sum_{i \in I_0 \cup \cdots \cup I_{s-1}} |R_\ell(A_i^+)|,$$
$$|R_\ell(\widehat{A}_s^-)| = \sum_{i \in I_0 \cup \cdots \cup I_{s-1}} |R_\ell(A_i^-)|,$$

and

$$|R_\ell(B_s^+)| = \sum_{i \notin I_0 \cup \cdots \cup I_{s-1}} |R_\ell(A_i^+)|,$$
$$|R_\ell(B_s^-)| = \sum_{i \notin I_0 \cup \cdots \cup I_{s-1}} |R_\ell(A_i^-)|.$$

Thus,

$$|R_\ell(\widehat{A}_s^+)| + |R_\ell(B_s^+)| = \sum_{i=1}^{n} |R_\ell(A_i^+)| = \sum_{i=1}^{n} |R_\ell(A_i^-)| = |R_\ell(\widehat{A}_s^-)| + |R_\ell(B_s^-)|.$$

Furthermore, since $|A_i^+| \geq |A_i^-|$ for all $i \in I_1 \cup \cdots \cup I_{s-1} \subseteq \{2, \ldots, n-1\}$ and $|A_1^+| > |A_1^-|$ we obtain

$$\begin{aligned}
|\widehat{A}_s^+| &= \sum_{i \in I_1 \cup \cdots \cup I_{s-1}} |A_i^+| + |A_1^+| \\
&\geq \sum_{i \in I_1 \cup \cdots \cup I_{s-1}} |A_i^-| + |A_1^+| \\
&\geq |\widehat{A}_s^-| - |A_1^-| + |A_1^+| \\
|\widehat{A}_s^+| &> |\widehat{A}_s^-|.
\end{aligned}$$

As a consequence, we can apply Lemma 4.3.6 to the matrices $\widehat{A}_s$ and B_s. Hence, the set I_s of integers $i \in \{1, \ldots, n\} \setminus (I_0 \cup \cdots \cup I_{s-1})$ such that $A_{\widehat{\imath}} \models A_i$ for some $\widehat{\imath} \in I_0 \cup \cdots \cup I_{s-1}$ is non-empty. Moreover, by construction such an integer $\widehat{\imath}$ is necessary in I_{s-1}. We stop here if $n \in I_s$, and otherwise we continue the induction with $I_0, I_1, \ldots, I_s$.

Since the sets that we construct are non-empty, pairwise disjoint and included in $\{1, \ldots, n\}$, there is some integer s such that $n \in I_s$. By construction, for any $t \in \{1, \ldots, s\}$ if $i_t \in I_t$ then there exists $i_{t-1} \in I_{t-1}$ such that $A_{i_{t-1}} \models A_{i_t}$. Hence, starting with $n \in I_s$, since the sets $(I_t)_{0 \leq t \leq s}$ are pairwise disjoint and $I_0 = \{1\}$, we can find a sequence $1 < i_1, \ldots, i_{s-1} < n$ of distinct integers such that $A_1 \models A_{i_1} \models \ldots \models A_{i_{s-1}} \models A_n$. □

Corollary 4.3.8. *Let $(A_i)_{1\leq i\leq n}$ be a family of matrices with the same even number of rows. We assume that*

$$\sum_{i=1}^{n}|R_\ell(A_i^+)| = \sum_{i=1}^{n}|R_\ell(A_i^-)|, \qquad \text{for all } \ell,$$
$$|A_{i_0}^+| > |A_{i_0}^-|, \qquad \text{for some } i_0 \in \{1,\dots,n\}.$$

Then there exists a sequence of distinct integers $i_1,\dots,i_s$ distinct from i_0 such that

$$A_{i_0} \models A_{i_1} \models A_{i_2} \models \dots \models A_{i_{s-1}} \models A_{i_s},$$

with $|A_{i_s}^+| < |A_{i_s}^-|$.

Proof. Let $m \in \{1,\dots,n-1\}$ be the number of $i \in \{1,\dots,n\}$ such that $|A_i^+| \geq |A_i^-|$. Let $(j_k)_{1\leq k\leq n}$ be a reordering of $\{1,\dots,n\}$ with $j_1 = i_0$ such that

$$|A_{j_k}^+| \geq |A_{j_k}^-|, \qquad \text{for all } k \in \{1,\dots,m\},$$
$$|A_{j_k}^+| < |A_{j_k}^-|, \qquad \text{for all } k \in \{m+1,\dots,n\}.$$

We define the following binary matrix with an even number of rows:

$$A \coloneqq \begin{pmatrix} A_{j_{m+1}} & \cdots & A_{j_n} \end{pmatrix}.$$

For all ℓ we have

$$\sum_{k=1}^{m}|R_\ell(A_{j_k}^+)| + |R_\ell(A^+)| = \sum_{k=1}^{m}|R_\ell(A_{j_k}^-)| + |R_\ell(A^-)|.$$

Hence, we can apply Proposition 4.3.7 to the family $(A_{j_1},\dots,A_{j_m},A)$. We find a sequence $i_1,\dots,i_{s-1}$ of distinct elements of $\{j_2,\dots,j_m\}$ such that

$$A_{j_1} = A_{i_0} \models A_{i_1} \models \dots \models A_{i_{s-1}} \models A.$$

We conclude since $A_{i_{s-1}} \models A$ implies that there exists $i_s \in \{j_{m+1},\dots,j_n\}$ such that $A_{i_{s-1}} \models A_{i_s}$. ☐

4.3.2 Application to binary averaging

The following result is well-known.

Lemma 4.3.9. *Let $w_0,\dots,w_{n-1} \in \{0,\dots,p\}$. For all $i \in \{0,\dots,n-1\}$ we define $v_i \coloneqq \frac{w_i}{p}$ and we set $v \coloneqq (v_0,\dots,v_{n-1}) \in [0,1]^n$. There exist some vectors $\epsilon^0,\dots,\epsilon^{p-1} \in \{0,1\}^n$ such that*

$$v = \frac{1}{p}\sum_{j=0}^{p-1}\epsilon^j.$$

In particular,

$$\frac{1}{p}\sum_{j=0}^{p-1}|\epsilon^j| = \frac{1}{p}\sum_{j=0}^{p-1}\|\epsilon^j\|^2 = |v|.$$

If in addition $|v| \in \mathbb{N}$ then for all $j \in \{0,\dots,p-1\}$ we can choose ϵ^j such that $|\epsilon^j| = \|\epsilon^j\|^2 = |v|$.

The last result is equivalent to the existence of a binary $p \times n$ matrix with row sums $(|v|, \dots, |v|)$ and column sums $(w_0, \dots, w_{n-1})$. By a general result of [Ga, Ry], we know that such a matrix exists, since the conjugate $(p, \dots, p)$ (with $|v|$ terms) of the partition $(|v|, \dots, |v|)$ dominates the partition $\widetilde{w}$ for the usual dominance order on partitions, where $\widetilde{w}$ is the partition obtained by rearranging the entries of w in decreasing order. However, for the convenience of the reader we give a simplified proof for the particular setting of Lemma 4.3.9.

Proof. For any $i \in \{0, \dots, n-1\}$, we define the set

$$W_i := \{w_0 + \cdots + w_{i-1} + 1, \dots, w_0 + \cdots + w_i\}.$$

For any $j \in \{0, \dots, p-1\}$, we consider the element $\epsilon^j := (\epsilon^j_0, \dots, \epsilon^j_{n-1}) \in \{0,1\}^n$ defined by

$$\epsilon^j_i := \begin{cases} 1 & \text{if } W_i \text{ contains an element of residue } j \text{ modulo } p, \\ 0 & \text{otherwise,} \end{cases}$$

for any $i \in \{0, \dots, n-1\}$. Since W_i has cardinality w_i and is given by at most p successive integers, the set of residues modulo p of the elements of W_i has also cardinality w_i. Hence, there are exactly w_i integers ϵ^j_i for all $j \in \{0, \dots, p-1\}$ that are equal to 1. The other are 0, thus

$$\sum_{j=0}^{p-1} \epsilon^j_i = w_i.$$

The ith component of $\frac{1}{p}\sum_{j=0}^{p-1} \epsilon^j$ is thus $\frac{w_i}{p} = v_i$ and we obtain

$$\frac{1}{p}\sum_{j=0}^{p-1} \epsilon^j = v.$$

Since $|\cdot|$ is additive, we deduce that $\frac{1}{p}\sum_{j=0}^{p-1} |\epsilon^j| = |v|$. Moreover, since $\epsilon^j \in \{0,1\}^n$ we have $|\epsilon^j| = \|\epsilon^j\|^2$ thus $\frac{1}{p}\sum_{j=0}^{p-1} \|\epsilon^j\|^2 = |v|$.

Now assume that $|v| \in \mathbb{N}$. There are in the set $\{1, \dots, |v|p = |w|\}$ exactly $|v|$ integers of residue j modulo p for each $j \in \{0, \dots, p-1\}$. Since $\{W_i\}_{i \in \{0, \dots, n-1\}}$ is a partition of $\{1, \dots, |w|\}$, we deduce that

$$\sum_{i=0}^{n-1} \epsilon^j_i = \#\Big\{\text{elements of } \{1, \dots, |w|\} \text{ of residue } j \text{ modulo } p\Big\} = |v|,$$

for all $j \in \{0, \dots, p-1\}$. Hence $|\epsilon^j| = |v|$ and we conclude. □

We will use Corollary 4.3.8 of §4.3.1 to generalise Lemma 4.3.9: see Proposition 4.3.14. Let us first give an easy lemma.

Lemma 4.3.10. *Let $a_0, \dots, a_{p-1}$ be integers of sum a multiple of p. The following integer:*

$$m := \max\{a_j - a_{j'} : j, j' \in \{0, \dots, p-1\}\} \in \mathbb{N},$$

satisfies $m = 0$ or $m \geq 2$.

Proof. Assume $m \leq 1$. Then, for all $j, j' \in \{0, \dots, p-1\}$ we have $|a_j - a_{j'}| \leq 1$. If $j_0 \in \{0, \dots, p-1\}$ is such that a_{j_0} is the minimum of $\{a_j\}_{j \in \{0, \dots, p-1\}}$ then for all $j \in \{0, \dots, p-1\}$, there exists $\epsilon_j \in \{0,1\}$ such that $a_j = a_{j_0} + \epsilon_j$. From the hypothesis, we know that $pa_{j_0} + \sum_{j=0}^{p-1} \epsilon_j$ is a multiple of p, thus $\sum_{j=0}^{p-1} \epsilon_j$ is a multiple of p. Since $\epsilon_{j_0} = 0$, we deduce that $\epsilon_j = 0$ for all j. We conclude that $a_{j_0} = a_j$ for all $j \in \{0, \dots, p-1\}$ thus $m = 0$. □

We need to introduce some notation in order to state Proposition 4.3.14. For any $\ell \in \{0, \ldots, d-1\}$ and $i \in \{0, \ldots, e-1\}$, let $w_i^{(\ell)} \in \{0, \ldots, p\}$ and set $v_i^{(\ell)} := \frac{w_i^{(\ell)}}{p}$. For each $\ell \in \{0, \ldots, d-1\}$ we define

$$v^{(\ell)} := (v_0^{(\ell)}, \ldots, v_{e-1}^{(\ell)}).$$

We obtain a $d \times e$ matrix

$$V := \begin{pmatrix} v^{(0)} \\ \vdots \\ v^{(d-1)} \end{pmatrix}.$$

We assume that for all $\ell \in \{0, \ldots, d-1\}$ we have $|v^{(\ell)}| \in \mathbb{N}$. Hence, for all $\ell \in \{0, \ldots, d-1\}$ we can apply Lemma 4.3.9 (with $n := e$). We obtain some vectors $\epsilon^{j(\ell)} \in \{0,1\}^e$ for all $j \in \{0, \ldots, p-1\}$, such that

$$v^{(\ell)} = \frac{1}{p} \sum_{j=0}^{p-1} \epsilon^{j(\ell)}, \tag{4.3.11}$$

and

$$|\epsilon^{j(\ell)}| = |v^{(\ell)}|. \tag{4.3.12}$$

For all $j \in \{0, \ldots, p-1\}$, define the following $d \times e$ matrix:

$$E^j := \begin{pmatrix} \epsilon^{j(0)} \\ \vdots \\ \epsilon^{j(d-1)} \end{pmatrix}.$$

Recall that e is a multiple of η (and $e = \eta p$). We write the matrix V with η blocks of the same size $V = \begin{pmatrix} V^{[0]} & \cdots & V^{[\eta-1]} \end{pmatrix}$, and we use the same block structure for the matrices $E^j = \begin{pmatrix} E^{j[0]} & \cdots & E^{j[\eta-1]} \end{pmatrix}$. As a consequence of (4.3.11), we have

$$|V^{[i]}| = \frac{1}{p} \sum_{j=0}^{p-1} |E^{j[i]}|, \tag{4.3.13}$$

for all $i \in \{0, \ldots, \eta-1\}$.

Proposition 4.3.14. *We keep the previous notation. In addition to the hypotheses $|v^{(\ell)}| \in \mathbb{N}$ for all $\ell \in \{0, \ldots, d-1\}$, assume that for all $i \in \{0, \ldots, \eta-1\}$ we have $|V^{[i]}| \in \mathbb{N}$. Then we can choose the vectors $\epsilon^{j(\ell)}$ for all $j \in \{0, \ldots, p-1\}$ and $\ell \in \{0, \ldots, d-1\}$ such that the previous properties (4.3.11) and (4.3.12) still hold, together with*

$$|E^{j[i]}| = |V^{[i]}|, \tag{4.3.15}$$

for all $j \in \{0, \ldots, p-1\}$ and $i \in \{0, \ldots, \eta-1\}$.

Example 4.3.16. Take $p = 4$ and $d = 2$. With the following matrix:

$$V := \frac{1}{4} \left(\begin{array}{cccc|cccc} 1 & 2 & 2 & 1 & 2 & 3 & 0 & 1 \\ 0 & 2 & 1 & 3 & 1 & 3 & 2 & 0 \end{array} \right) = \begin{pmatrix} v^{(0)} \\ v^{(1)} \end{pmatrix} = \left(\begin{array}{c|c} V^{[0]} & V^{[1]} \end{array} \right),$$

we have $|v^{(0)}| = |v^{(1)}| = |V^{[0]}| = |V^{[1]}| = 3$. The vectors $\epsilon^{j(\ell)}$ constructed as in the proof of Lemma 4.3.9 are the following:

$$\epsilon^{0(0)} = (1,0,1,0,0,1,0,0), \qquad \epsilon^{0(1)} = (0,1,0,1,0,1,0,0),$$
$$\epsilon^{1(0)} = (0,1,0,1,0,1,0,0), \qquad \epsilon^{1(1)} = (0,1,0,1,0,1,0,0),$$
$$\epsilon^{2(0)} = (0,1,0,0,1,1,0,0), \qquad \epsilon^{2(1)} = (0,0,1,0,1,0,1,0),$$
$$\epsilon^{3(0)} = (0,0,1,0,1,0,0,1), \qquad \epsilon^{3(1)} = (0,0,0,1,0,1,1,0).$$

Thus, we have

$$E^0 = \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right) = \begin{pmatrix} \epsilon^{0(0)} \\ \epsilon^{0(1)} \end{pmatrix} = \left(E^{0[0]} \mid E^{0[1]} \right),$$
$$E^1 = \left(\begin{array}{cccc|cccc} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right) = \begin{pmatrix} \epsilon^{1(0)} \\ \epsilon^{1(1)} \end{pmatrix} = \left(E^{1[0]} \mid E^{1[1]} \right),$$
$$E^2 = \left(\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right) = \begin{pmatrix} \epsilon^{2(0)} \\ \epsilon^{2(1)} \end{pmatrix} = \left(E^{2[0]} \mid E^{2[1]} \right),$$
$$E^3 = \left(\begin{array}{cccc|cccc} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{array}\right) = \begin{pmatrix} \epsilon^{3(0)} \\ \epsilon^{3(1)} \end{pmatrix} = \left(E^{3[0]} \mid E^{3[1]} \right).$$

However, we have $|E^{0[0]}| = 4 \neq |V^{[0]}|$, thus these vectors $\epsilon^{j(\ell)}$ do not satisfy the condition (4.3.15) of Proposition 4.3.14. Let us consider the two compatible submatrices indicated by the coloured entries. Define $A := \begin{pmatrix} E^{0[0]} \\ E^{2[0]} \end{pmatrix}$ and $B := \begin{pmatrix} E^{0[1]} \\ E^{2[1]} \end{pmatrix}$ (respectively $C := \begin{pmatrix} E^{1[0]} \\ E^{3[0]} \end{pmatrix}$ and $D := \begin{pmatrix} E^{1[1]} \\ E^{3[1]} \end{pmatrix}$) and set $(\widetilde{A}, \widetilde{B}) := \gamma_{1,1,1}(A, B)$ (resp. $(\widetilde{C}, \widetilde{D}) := \gamma_{1,2,1}(C, D)$). We have

$$E^0 = \left(A^+ \mid B^+ \right),$$
$$E^1 = \left(C^+ \mid D^+ \right),$$
$$E^2 = \left(A^- \mid B^- \right),$$
$$E^3 = \left(C^- \mid D^- \right),$$

and

$$\widetilde{E}^0 := \left(\widetilde{A}^+ \mid \widetilde{B}^+ \right) = \left(\begin{array}{cccc|cccc} 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right),$$
$$\widetilde{E}^1 := \left(\widetilde{C}^+ \mid \widetilde{D}^+ \right) = \left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right),$$
$$\widetilde{E}^2 := \left(\widetilde{A}^- \mid \widetilde{B}^- \right) = \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right),$$
$$\widetilde{E}^3 := \left(\widetilde{C}^- \mid \widetilde{D}^- \right) = \left(\begin{array}{cccc|cccc} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{array}\right).$$

The vectors $\widetilde{\epsilon}^{j(\ell)}$ defined for all $j \in \{0, \ldots, 3\}$ and $\ell \in \{0, 1\}$ by $\widetilde{E}^j = \begin{pmatrix} \widetilde{\epsilon}^{j(0)} \\ \widetilde{\epsilon}^{j(1)} \end{pmatrix}$ satisfy (4.3.11) and (4.3.12), together with the condition (4.3.15) of Proposition 4.3.14. In general, the existence of such interchanges will be given by Corollary 4.3.8.

The remaining part of this subsection is now devoted to the proof of Proposition 4.3.14. First, note that the interchanges $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ that are compatible with the block decomposition

$$\begin{pmatrix} E^0 \\ \vdots \\ E^{p-1} \end{pmatrix} = \left(\begin{array}{c|c|c} E^{0[0]} & \dots & E^{0[\eta-1]} \\ \hline \vdots & \vdots & \vdots \\ \hline E^{(p-1)[0]} & \dots & E^{(p-1)[\eta-1]} \end{array}\right), \tag{4.3.17}$$

do not affect properties (4.3.11) et (4.3.12). However, these interchanges change the value of some $|E^{j[i]}|$, as described in Proposition 4.3.4. Thus, it suffices to prove that there exists a sequence of compatible interchanges that modifies each $|E^{j[i]}|$ to $|V^{[i]}|$. We endow $\mathbb{N} \times \mathbb{N}^*$ with the usual lexicographic order. We will use an induction on $(\Delta, N) \in \mathbb{N} \times \mathbb{N}^*$, where

$$\Delta := \max \left\{ |E^{j[i]}| - |E^{j'[i]}| : i \in \{0, \dots, \eta - 1\}, j, j' \in \{0, \dots, p-1\} \right\} \in \mathbb{N},$$

and

$$N := \# \left\{ (i, j, j') \in \{0, \dots, \eta - 1\} \times \{0, \dots, p-1\}^2 : |E^{j[i]}| - |E^{j'[i]}| = \Delta \right\} \in \mathbb{N}^*.$$

Define

$$M := \max \left\{ |E^{j[i]}| : i \in \{0, \dots, \eta - 1\}, j \in \{0, \dots, p-1\} \right\},$$
$$m := \min \left\{ |E^{j[i]}| : i \in \{0, \dots, \eta - 1\}, j \in \{0, \dots, p-1\} \right\},$$

and

$$N_{\max} := \# \left\{ (i, j) \in \{0, \dots, \eta - 1\} \times \{0, \dots, p-1\} : |E^{j[i]}| = M \right\},$$
$$N_{\min} := \# \left\{ (i, j) \in \{0, \dots, \eta - 1\} \times \{0, \dots, p-1\} : |E^{j[i]}| = m \right\}.$$

We have $\Delta = M - m$ and $N = N_{\max} N_{\min}$. If $\Delta = 0$ then by (4.3.13) we have $|E^{j[i]}| = |V^{[i]}|$ for all i, j so the proof is over. Assume $\Delta \geq 1$ and let $i_0 \in \{0, \dots, e-1\}$ and $j_0, j_0' \in \{0, \dots, p-1\}$ such that $|E^{j_0[i_0]}| - |E^{j_0'[i_0]}| = \Delta$. We now consider the matrix

$$\begin{pmatrix} E^{j_0} \\ E^{j_0'} \end{pmatrix} = \begin{pmatrix} E^{j_0[0]} & \dots & E^{j_0[i_0]} & \dots & E^{j_0[\eta-1]} \\ E^{j_0'[0]} & \dots & E^{j_0'[i_0]} & \dots & E^{j_0'[\eta-1]} \end{pmatrix},$$

given by the j_0th and j_0'th block-rows of the matrix of (4.3.17). We consider the family $(A_i)_{0 \leq i \leq \eta-1}$ of matrices with the same even number of rows defined by

$$A_i = \begin{pmatrix} A_i^+ \\ A_i^- \end{pmatrix} := \begin{pmatrix} E^{j_0[i]} \\ E^{j_0'[i]} \end{pmatrix},$$

for all $i \in \{0, \dots, \eta - 1\}$. The hypotheses of Corollary 4.3.8 are satisfied, thanks to the definition of i_0 and (4.3.12) (note that $R_\ell(E^{j_0}) = \epsilon^{j_0(\ell)}$ and $R_\ell(E^{j_0'}) = \epsilon^{j_0'(\ell)}$). Hence, we can find a sequence of distinct integers $i_1, \dots, i_s$ distinct from i_0 with $|A_{i_s}^+| < |A_{i_s}^-|$ and

$$A_{i_0} \models \dots \models A_{i_s}.$$

Let $(\widetilde{A}_{i_0}, \dots, \widetilde{A}_{i_s}) \in \Gamma(A_{i_0}, \dots, A_{i_s})$. By Corollary 4.3.5, we know that

$$|\widetilde{A}_{i_t}^+| = |A_{i_t}^+|,$$
$$|\widetilde{A}_{i_t}^-| = |A_{i_t}^-|, \tag{4.3.18}$$

for all $t \in \{1, \dots, s-1\}$. Moreover, we have

$$|\widetilde{A}_{i_0}^+| = |A_{i_0}^+| - 1, \qquad |\widetilde{A}_{i_0}^-| = |A_{i_0}^-| + 1, \tag{4.3.19a}$$
$$|\widetilde{A}_{i_s}^+| = |A_{i_s}^+| + 1, \qquad |\widetilde{A}_{i_s}^-| = |A_{i_s}^-| - 1. \tag{4.3.19b}$$

We now want to evaluate the new values $\widetilde{\Delta}$ and $\widetilde{N}$ of Δ and N that we obtain and prove that $(\widetilde{\Delta}, \widetilde{N})$ is strictly less than (Δ, N). We have

$$\widetilde{\Delta} = \max\left\{|\widetilde{E}^{j[i]}| - |\widetilde{E}^{j'[i]}| : i \in \{0, \dots, \eta-1\}, j, j' \in \{0, \dots, p-1\}\right\} \in \mathbb{N},$$

and

$$\widetilde{N} = \#\left\{(i, j, j') \in \{0, \dots, \eta-1\} \times \{0, \dots, p-1\}^2 : |\widetilde{E}^{j[i]}| - |\widetilde{E}^{j'[i]}| = \widetilde{\Delta}\right\} \in \mathbb{N}^*,$$

where

$$\widetilde{E}^{j[i]} := \begin{cases} \widetilde{A}_{i_t}^+ & \text{if } i = i_t \text{ for some } t \in \{0, \dots, s\} \text{ and } j = j_0, \\ \widetilde{A}_{i_t}^- & \text{if } i = i_t \text{ for some } t \in \{0, \dots, s\} \text{ and } j = j_0', \\ E^{j[i]} & \text{otherwise.} \end{cases}$$

Moreover, with

$$\widetilde{M} := \max\left\{|\widetilde{E}^{j[i]}| : i \in \{0, \dots, \eta-1\}, j \in \{0, \dots, p-1\}\right\},$$
$$\widetilde{m} := \min\left\{|\widetilde{E}^{j[i]}| : i \in \{0, \dots, \eta-1\}, j \in \{0, \dots, p-1\}\right\},$$

and

$$\widetilde{N}_{\max} := \#\left\{(i, j) \in \{0, \dots, \eta-1\} \times \{0, \dots, p-1\} : |\widetilde{E}^{j[i]}| = \widetilde{M}\right\},$$
$$\widetilde{N}_{\min} := \#\left\{(i, j) \in \{0, \dots, \eta-1\} \times \{0, \dots, p-1\} : |\widetilde{E}^{j[i]}| = \widetilde{m}\right\},$$

we have $\widetilde{\Delta} = \widetilde{M} - \widetilde{N}$ and $\widetilde{N} = \widetilde{N}_{\max}\widetilde{N}_{\min}$. Note that by (4.3.18), for all $i \in \{0, \dots, \eta-1\}$ and $j \in \{0, \dots, p-1\}$ we have

$$|\widetilde{E}^{j[i]}| = |E^{j[i]}|, \qquad \text{if } i \notin \{i_0, i_s\} \text{ or } j \notin \{j_0, j_0'\}. \tag{4.3.20}$$

By the assumption $|V^{[i]}| \in \mathbb{N}$ and (4.3.13), thanks to Lemma 4.3.10 we know that $\Delta = |A_{i_0}^+| - |A_{i_0}^-| = M - m \geq 2$. Hence, by (4.3.19a) we have

$$m < |\widetilde{A}_{i_0}^-| \leq |\widetilde{A}_{i_0}^+| < M. \tag{4.3.21}$$

Furthermore, since $m \leq |A_{i_s}^+| < |A_{i_s}^-| \leq M$, by (4.3.19b) we have

$$m < |\widetilde{A}_{i_s}^+| \leq |A_{i_s}^-| \leq M, \tag{4.3.22a}$$
$$m \leq |A_{i_s}^+| \leq |\widetilde{A}_{i_s}^-| < M. \tag{4.3.22b}$$

Equations (4.3.20), (4.3.21) and (4.3.22) prove that $\widetilde{M} \leq M$ and $\widetilde{m} \geq m$, thus $\widetilde{\Delta} \leq \Delta$. If $\widetilde{\Delta} < \Delta$ then $(\widetilde{\Delta}, \widetilde{N}) < (\Delta, N)$, thus we now assume that $\widetilde{\Delta} = \Delta$, that is, $\widetilde{M} = M$ and $\widetilde{m} = m$. By (4.3.20) we have

$$N_{\max} - \widetilde{N}_{\max} = \#\left\{(i, j) \in \{i_0, i_s\} \times \{j_0, j_0'\} : |E^{j[i]}| = M\right\}$$
$$- \#\left\{(i, j) \in \{i_0, i_s\} \times \{j_0, j_0'\} : |\widetilde{E}^{j[i]}| = M\right\}.$$

Thus,

$$N_{\max} - \widetilde{N}_{\max} = 1 + \delta_{|A^-_{i_s}|,M} - \#\left\{(i,j) \in \{i_0, i_s\} \times \{j_0, j_0'\} : |\widetilde{E}^{j[i]}| = M\right\},$$

where δ is the Kronecker symbol. By (4.3.21) and (4.3.22), we obtain

$$N_{\max} - \widetilde{N}_{\max} = 1 + \delta_{|A^-_{i_s}|,M} - \delta_{|\widetilde{A}^+_{i_s}|,M}. \tag{4.3.23}$$

By (4.3.22a), we know that

$$\delta_{|\widetilde{A}^+_{i_s}|,M} \leq \delta_{|A^-_{i_s}|,M},$$

thus (4.3.23) yields $N_{\max} - \widetilde{N}_{\max} \geq 1$. Similarly, we have $N_{\min} - \widetilde{N}_{\min} \geq 1$. Finally, we obtain $\widetilde{N} = \widetilde{N}_{\max}\widetilde{N}_{\min} < N_{\max}N_{\min} = N$ and thus $(\widetilde{\Delta}, \widetilde{N}) = (\Delta, \widetilde{N}) < (\Delta, N)$. By induction, this concludes the proof of Proposition 4.3.14.

4.3.3 A few inequalities

We will prove some inequalities that we will use to prove Theorem 4.2.31. The setting of the first one is reminiscent of Lemma 4.3.9 and the following ones use convexity. Recall that $\|\cdot\|$ is the euclidean norm on $\mathbb{R}^n$ and denote by $\langle \cdot, \cdot \rangle$ the associated scalar product.

Lemma 4.3.24. *Let $n \in \mathbb{N}^*$ and $h : \mathbb{R}^n \to \mathbb{R}$ be a function such that $h - \frac{1}{2}\|\cdot\|^2$ is affine. Let $v \in \mathbb{R}^n$ and suppose that $\epsilon^0, \dots, \epsilon^{p-1} \in \{0,1\}^n$ satisfy $v = \frac{1}{p}\sum_{j=0}^{p-1} \epsilon^j$ and $|\epsilon^j| = \|\epsilon^j\|^2 = |v|$ for all $j \in \{0, \dots, p-1\}$. For any $a \in \mathbb{R}^n$ we have*

$$h(a+v) - \frac{1}{p}\sum_{j=0}^{p-1} h(a+\epsilon^j) = \frac{\|v\|^2 - |v|}{2}.$$

More specifically, there exists $j \in \{0, \dots, p-1\}$ (depending on a) such that

$$h(a+\epsilon^j) \leq h(a+v) + \frac{|v| - \|v\|^2}{2}.$$

Proof. Denote by $\Delta := h(a+v) - \frac{1}{p}\sum_{j=0}^{p-1} h(a+\epsilon^j)$ the left-hand side of the equality. Note that the Hessian matrix of the second partial derivatives of h is the identity matrix. More precisely, since h is a degree 2 polynomial, the Taylor formula reads

$$h(a+w) = h(a) + \langle \nabla h(a), w \rangle + \frac{1}{2}\|w\|^2, \qquad \text{for all } w \in \mathbb{R}^n,$$

where $\nabla h(a)$ denotes the gradient of h at a. Since $v = \frac{1}{p}\sum_{j=0}^{p-1} \epsilon^j$, the quantity that defines Δ vanishes at the affine level, hence

$$\Delta = \frac{1}{2}\left(\|v\|^2 - \frac{1}{p}\sum_{j=0}^{p-1}\|\epsilon^j\|^2\right).$$

We conclude since $\|\epsilon^j\|^2 = |v|$. The second assertion is straightforward. □

The next inequalities involve convexity. The first one is a particular case of a Jensen's inequality for convex functions. The reader may refer to [MeNi, Theorem 4]; we include a proof for completeness.

Lemma 4.3.25. *Let $n \in \mathbb{N}^*$ and $m \in \mathbb{R}$. Let $h : \mathbb{R}^n \to \mathbb{R}$ such that $h - \frac{m}{2}\|\cdot\|^2$ is convex. For any $x_0, \dots, x_{p-1} \in \mathbb{R}^n$ we have*

$$h(\overline{x}) \leq \frac{1}{p}\sum_{j=0}^{p-1} h(x_j) - \frac{m}{2p}\sum_{j=0}^{p-1}\|x_j - \overline{x}\|^2,$$

where $\overline{x} := \frac{1}{p}\sum_{j=0}^{p-1} x_j$.

Proof. Since $h - \frac{m}{2}\|\cdot\|^2$ is convex, we have

$$h(\overline{x}) - \frac{m}{2}\|\overline{x}\|^2 \leq \frac{1}{p}\sum_{j=0}^{p-1} h(x_j) - \frac{m}{2p}\sum_{j=0}^{p-1}\|x_j\|^2.$$

Thus,

$$\begin{aligned}
h(\overline{x}) &\leq \frac{1}{p}\sum_{j=0}^{p-1} h(x_j) - \frac{m}{2p}\left[\sum_{j=0}^{p-1}\|x_j\|^2 - p\|\overline{x}\|^2\right] \\
&\leq \frac{1}{p}\sum_{j=0}^{p-1} h(x_j) - \frac{m}{2p}\left[\sum_{j=0}^{p-1}\|x_j - \overline{x}\|^2 + 2\sum_{j=0}^{p-1}\langle x_j, \overline{x}\rangle - 2p\|\overline{x}\|^2\right] \\
&\leq \frac{1}{p}\sum_{j=0}^{p-1} h(x_j) - \frac{m}{2p}\left[\sum_{j=0}^{p-1}\|x_j - \overline{x}\|^2 + 2p\langle \overline{x}, \overline{x}\rangle - 2p\|\overline{x}\|^2\right] \\
&\leq \frac{1}{p}\sum_{j=0}^{p-1} h(x_j) - \frac{m}{2p}\sum_{j=0}^{p-1}\|x_j - \overline{x}\|^2.
\end{aligned}$$

□

Remark 4.3.26. The real number m of Lemma 4.3.25 is usually taken to be positive. In this case, the map h is convex and we say that it is *m-strongly convex*. We have stated Lemma 4.3.25 for a general m since we will need it to be negative in the proof of Lemma 4.3.27.

For any $x \in \mathbb{R}$, we denote by $\{x\} \in [0, 1[$ its fractional part. We have $\{x\} := x - \lfloor x \rfloor$, where $\lfloor x \rfloor \in \mathbb{Z}$ is the greatest integer less than or equal to x.

Lemma 4.3.27. *Let $x_0, \dots, x_{p-1} \in \mathbb{Z}$ be integers and let $\overline{x} := \frac{1}{p}\sum_{j=0}^{p-1} x_j$. With $v := \{\overline{x}\}$ we have*

$$v - v^2 \leq \frac{1}{p}\sum_{j=0}^{p-1}(x_j - \overline{x})^2.$$

Proof. Let us consider the function $\phi : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto \{x\} - \{x\}^2 + x^2$. It is continuous on $\mathbb{R} \setminus \mathbb{Z}$, and in fact continuous on $\mathbb{R}$ since $\lim_{x \to n^-} \phi(x) = \lim_{x \to n^+} \phi(x) = n^2$ for any $n \in \mathbb{Z}$. Moreover,

$$\phi(x) = x - \lfloor x \rfloor - (x^2 - 2\lfloor x \rfloor x + \lfloor x \rfloor^2) + x^2 = (1 + 2\lfloor x \rfloor)x - \lfloor x \rfloor(1 + \lfloor x \rfloor).$$

Thus, the function ϕ is affine on each interval $[n, n+1[$ for $n \in \mathbb{Z}$, with slope $2n + 1$. Hence, the function ϕ is continuous with non-decreasing left derivative thus ϕ is convex. Applying Lemma 4.3.25 with $n := 1$, $m := -2$ and $h := \{\cdot\} - \{\cdot\}^2$ we obtain

$$v - v^2 \leq \frac{1}{p}\sum_{j=0}^{p-1}\left(\{x_j\} - \{x_j\}^2\right) + \frac{1}{p}\sum_{j=0}^{p-1}(x_j - \overline{x})^2.$$

For any $j \in \{0, \dots, p-1\}$ we have $x_j \in \mathbb{Z}$ thus $\{x_j\} = 0$ and we conclude. □

4.4 Proof of the main theorem

We are now ready to prove Theorem 4.2.31, which we repeat here for the convenience of the reader.

Theorem 4.2.31. *Let* $\boldsymbol{\lambda}$ *be an* r*-partition and let* $\alpha := \alpha_\kappa(\boldsymbol{\lambda}) \in Q^+$. *Assume that* κ *is compatible with* (d, η, p). *If* $\sigma \cdot \alpha = \alpha$ *then there is an* r*-partition* $\boldsymbol{\mu} \in \mathcal{P}_\alpha^\kappa$ *with* ${}^\sigma\boldsymbol{\mu} = \boldsymbol{\mu}$.

Let $\boldsymbol{\lambda}$ be an r-partition and assume that the multicharge $\kappa \in (\mathbb{Z}/e\mathbb{Z})^r$ is compatible with (d, η, p). Recalling the reduction step Proposition 4.2.38, we assume that $\boldsymbol{\lambda}$ is an e-multicore. We define

$$\begin{aligned} \alpha &:= \alpha_\kappa(\boldsymbol{\lambda}), \\ x^{(k)} &:= x^{(k)}(\boldsymbol{\lambda}), && \text{for all } k \in \{0, \dots, r-1\}, \\ n^i &:= n_\kappa^i(\boldsymbol{\lambda}), && \text{for all } i \in \{0, \dots, e-1\}. \end{aligned}$$

In the whole section, we assume that $\sigma \cdot \alpha = \alpha$. There will be four steps in the proof, each step corresponding to one subsection. First, we will give an expression of n^0 in terms of the abacus variables $x^{(0)}, \dots, x^{(r-1)}$, which takes into account the σ-stability of α. We will then give a key lemma, followed by a naive (but useful) attempt to prove the theorem. Finally, we will use the results of Section 4.3 to conclude the proof.

4.4.1 Using shift invariance

In this subsection, we will write n^0 in terms of $x_i^{(k)}$ for $k \in \{0, \dots, r-1\}$ and $i \in \{0, \dots, e-1\}$ (Lemma 4.2.25). The difference with the equality of Lemma 4.2.25 is that α is now assumed to be σ-stable, which will allow us to make the expression symmetric. The map $(\mathbb{R}^e)^r \to \mathbb{R}$ that we obtain will be later used to apply the convexity results of §4.3.3.

Recall from §4.2.4 that we have some linear forms $L_0, \dots, L_{e-1}$ that satisfy (4.2.26):

$$n^0 = \sum_{k=0}^{r-1} \left[\frac{1}{2}\|x^{(k)}\|^2 - L_{-\kappa_k}(x^{(k)})\right].$$

Since $\sigma \cdot \alpha = \alpha$, for all $j_0 \in \{0, \dots, p-1\}$ we have $n_\kappa^0(\boldsymbol{\lambda}) = n_\kappa^0({}^{\sigma^{-j_0}}\boldsymbol{\lambda})$ by Lemma 4.2.29. We deduce that

$$\begin{aligned} n^0 &= \sum_{k=0}^{r-1} \left[\frac{1}{2}\|x^{(k+j_0 d)}\|^2 - L_{-\kappa_k}(x^{(k+j_0 d)})\right] \\ &= \sum_{k=0}^{r-1} \left[\frac{1}{2}\|x^{(k)}\|^2 - L_{-\kappa_{k-j_0 d}}(x^{(k)})\right]. \end{aligned}$$

Averaging on $j_0 \in \{0, \dots, p-1\}$, we obtain

$$n^0 = \sum_{k=0}^{r-1} \left[\frac{1}{2}\|x^{(k)}\|^2 - \widetilde{L}_{\overline{k}}(x^{(k)})\right],$$

where $\widetilde{L}_{\overline{k}}$ is a linear form that depends only on the residue $\overline{k} \in \{0, \dots, d-1\}$ of k modulo d. Now, if for $\ell \in \{0, \dots, d-1\}$ we consider the map defined on $\mathbb{R}^e$ by

$$g_\ell : x \mapsto \frac{1}{2}\|x\|^2 - \widetilde{L}_\ell(x), \tag{4.4.1}$$

we have

$$n^0 = \sum_{\ell=0}^{d-1}\sum_{j=0}^{p-1} g_\ell(x^{(\ell+jd)}) =: f(x^{(0)}, \dots, x^{(r-1)}). \tag{4.4.2}$$

The map $f : (\mathbb{R}^e)^r \to \mathbb{R}$ is of the form $f = \frac{1}{2}\|\cdot\|^2 - L$ where L is a linear form. Moreover, define

$$\begin{aligned} f^{\langle p\rangle}(x^{(0)}, \dots, x^{(d-1)}) &:= \sum_{\ell=0}^{d-1} g_\ell(x^{(\ell)}) \\ &= \frac{1}{p} f(x^{(0)}, \dots, x^{(d-1)}, \dots, x^{(0)}, \dots, x^{(d-1)}), \end{aligned} \tag{4.4.3}$$

where, in the expression $f(x^{(0)}, \dots, x^{(d-1)}, \dots, x^{(0)}, \dots, x^{(d-1)})$ the sequence $x^{(0)}, \dots, x^{(d-1)}$ is repeated p times. Like f, the map $f^{\langle p\rangle} : (\mathbb{R}^e)^d \to \mathbb{R}$ is of the form $\frac{1}{2}\|\cdot\|^2 - L^{\langle p\rangle}$, where $L^{\langle p\rangle}$ is a linear form. Note that for all $j \in \{0, \dots, p-1\}$ we have

$$f^{\langle p\rangle}(x^{(jd)}, \dots, x^{(d-1+jd)}) = \sum_{\ell=0}^{d-1} g_\ell(x^{(\ell+jd)}),$$

hence, by (4.4.2) we deduce that

$$f(x^{(0)}, \dots, x^{(r-1)}) = \sum_{j=0}^{p-1} f^{\langle p\rangle}(x^{(jd)}, \dots, x^{(d-1+jd)}).$$

4.4.2 Key lemma

Lemma 4.4.5 that we will give in this subsection is the key to our proof of Theorem 4.2.31. Recall that $\alpha = \alpha_\kappa(\boldsymbol{\lambda})$ satisfies $\sigma \cdot \alpha = \alpha$. For any $i \in \{0, \dots, \eta - 1\}$, define

$$\delta_i := n^i - n^{i+1}.$$

The σ-stability of α implies that $\delta_i = n^{i+j_0\eta} - n^{i+j_0\eta+1}$ for all $j_0 \in \{0, \dots, p-1\}$. We deduce from Lemma 4.2.25 and the compatibility of κ with (d, η, p) (cf. (4.2.20)) that

$$\delta_i = \sum_{k=0}^{r-1} x^{(k)}_{i+j_0\eta-\kappa_k} = \sum_{\ell=0}^{d-1}\sum_{j=0}^{p-1} x^{(\ell+jd)}_{i+(j_0-j)\eta-\kappa_\ell}, \tag{4.4.4}$$

for all $j_0 \in \{0, \dots, p-1\}$.

As noted in Remark 4.2.39, the stuttering r-partition $\boldsymbol{\mu}$ of Theorem 4.2.31, which satisfies $\alpha_\kappa(\boldsymbol{\mu}) = \alpha$, is not necessary an e-multicore. The following lemma shows that, to prove Theorem 4.2.31, it suffices to find a stuttering e-multicore $\boldsymbol{\nu}$ such that $\alpha_\kappa(\boldsymbol{\nu}) = \alpha - h(\alpha_0 + \dots + \alpha_{e-1})$ for some $h \in \mathbb{N}$.

Lemma 4.4.5. *Suppose that $z^{(0)}, \dots, z^{(d-1)} \in \mathbb{Z}^e_0$ are such that*

$$p f^{\langle p\rangle}(z^{(0)}, \dots, z^{(d-1)}) \leq f(x^{(0)}, \dots, x^{(r-1)}), \tag{4.4.6}$$

and

$$\sum_{\ell=0}^{d-1}\sum_{j=0}^{p-1} z^{(\ell)}_{i-j\eta-\kappa_\ell} = \delta_i, \tag{4.4.7}$$

for all $i \in \{0, \dots, \eta - 1\}$. Then Theorem 4.2.31 is true for the e-multicore $\boldsymbol{\lambda}$: we can find an r-partition $\boldsymbol{\mu}$ such that $\alpha_\kappa(\boldsymbol{\mu}) = \alpha$ and ${}^\sigma\boldsymbol{\mu} = \boldsymbol{\mu}$.

Proof. For any $\ell \in \{0,\dots,d-1\}$ and $j \in \{1,\dots,p-1\}$, define $z^{(\ell+jd)} := z^{(\ell)} \in \mathbb{Z}_0^e$. For each $k \in \{0,\dots,r-1\}$, let $\overline{\mu}^{(k)}$ be the e-core of parameter $z^{(k)}$. We obtain an e-multicore $\overline{\boldsymbol{\mu}} = (\overline{\mu}^{(0)},\dots,\overline{\mu}^{(r-1)})$ that satisfies ${}^{\sigma}\overline{\boldsymbol{\mu}} = \overline{\boldsymbol{\mu}}$. For any $i \in \{0,\dots,e-1\}$, we define $m^i := n^i_\kappa(\overline{\boldsymbol{\mu}})$. Since κ is compatible with (d,η,p), we have $\sum_{\ell=0}^{d-1}\sum_{j=0}^{p-1} z^{(\ell)}_{i-j\eta-\kappa_\ell} = \sum_{k=0}^{r-1} z^{(k)}_{i-\kappa_k}$. By Lemma 4.2.25 and the assumption (4.4.7), we deduce that

$$m^i - m^{i+1} = \delta_i,$$

for all $i \in \{0,\dots,\eta-1\}$. Hence, for all $i \in \{0,\dots,\eta-1\}$ we have $m^i - m^{i+1} = n^i - n^{i+1}$ thus

$$m^0 - m^i = n^0 - n^i. \tag{4.4.8}$$

The above equality is also true for any $i \in \{0,\dots,e-1\}$ since $n^i = n^{i+\eta}$ and $m^i = m^{i+\eta}$ (by Lemma 4.2.29). Recalling the definition of f (respectively $f^{\langle p\rangle}$) given at (4.4.2) (resp. (4.4.3)), the assumption (4.4.6) implies

$$m^0 \leq n^0.$$

Hence, as in the proof of Proposition 4.2.38 we can construct an r-partition $\boldsymbol{\mu} = (\mu^{(0)},\dots,\mu^{(r-1)})$ such that ${}^{\sigma}\boldsymbol{\mu} = \boldsymbol{\mu}$ and:

- the partition $\mu^{(0)}$ is obtained by adding $n^0 - m^0$ times an η-rim hook to $\overline{\mu}^{(0)}$;
- we have $\mu^{(j)} = \overline{\mu}^{(j)}$ for all $j \in \{1,\dots,d-1\}$.

Finally, by Lemma 4.2.37 and (4.4.8) we obtain

$$\begin{aligned}
\alpha_\kappa(\boldsymbol{\mu}) &= \alpha_\kappa(\overline{\boldsymbol{\mu}}) + (n^0 - m^0)(\alpha_0 + \cdots + \alpha_{e-1}) \\
&= \sum_{i=0}^{e-1} m^i \alpha_i + \sum_{i=0}^{e-1} (n^0 - m^0)\alpha_i \\
&= \sum_{i=0}^{e-1} (n^0 + m^i - m^0)\alpha_i \\
&= \sum_{i=0}^{e-1} n^i \alpha_i \\
&= \alpha,
\end{aligned}$$

thus we conclude. □

4.4.3 Naive attempt

We will use the convexity of the map $f : (\mathbb{R}^e)^r \to \mathbb{R}$ to obtain some parameters $\tilde{z}^{(0)},\dots,\tilde{z}^{(d-1)}$ that almost satisfy the conditions of Lemma 4.4.5. These parameters will not necessary be integers: we will fix this in the next section.

Proposition 4.4.9. *For any $\ell \in \{0,\dots,d-1\}$, we define*

$$\tilde{z}^{(\ell)} := \frac{1}{p}\sum_{j=0}^{p-1} x^{(\ell+jd)} \in \frac{1}{p}\mathbb{Z}^e.$$

We have

$$pf^{\langle p\rangle}\big(\tilde{z}^{(0)},\dots,\tilde{z}^{(d-1)}\big) \leq f\big(x^{(0)},\dots,x^{(r-1)}\big) - \frac{1}{2}\sum_{\ell=0}^{d-1}\sum_{j=0}^{p-1} \big\|x^{(\ell+jd)} - \tilde{z}^{(\ell)}\big\|^2.$$

Proof. Let $\ell \in \{0, \dots, d-1\}$ and let $k \in \{0, \dots, r-1\}$ be of residue ℓ modulo d. Recall the definition of the map $g_\ell : \mathbb{R}^e \to \mathbb{R}$ given in (4.4.1). The map $g_\ell - \frac{1}{2}\|\cdot\|^2$ is convex, thus by Lemma 4.3.25 we deduce that

$$g_\ell\big(\tilde{z}^{(\ell)}\big) \leq \frac{1}{p}\sum_{j=0}^{p-1} g_\ell\big(x^{(\ell+jd)}\big) - \frac{1}{2p}\sum_{j=0}^{p-1}\big\|x^{(\ell+jd)} - \tilde{z}^{(\ell)}\big\|^2.$$

Summing over all $\ell \in \{0, \dots, d-1\}$, we obtain

$$f^{\langle p\rangle}\big(\tilde{z}^{(0)}, \dots, \tilde{z}^{(d-1)}\big) \leq \frac{1}{p} f\big(x^{(0)}, \dots, x^{(r-1)}\big) - \frac{1}{2p}\sum_{\ell=0}^{d-1}\sum_{j=0}^{p-1}\big\|x^{(\ell+jd)} - \tilde{z}^{(\ell)}\big\|^2.$$

Multiplying by p gives the desired result. □

Remark 4.4.10. The inequality of Proposition 4.4.9 is in fact an equality since $g_\ell - \frac{1}{2}\|\cdot\|^2$ is linear.

Let us now try to verify the hypotheses of Lemma 4.4.5 with the parameters $\tilde{z}^{(0)}, \dots, \tilde{z}^{(d-1)} \in \frac{1}{p}\mathbb{Z}^e$ of Proposition 4.4.9. First, for each $\ell \in \{0, \dots, d-1\}$ we have

$$\big|\tilde{z}^{(\ell)}\big| = \frac{1}{p}\sum_{j=0}^{p-1}\big|x^{(\ell+jd)}\big| = \frac{1}{p}\sum_{j=0}^{p-1} 0 = 0. \tag{4.4.11}$$

Moreover, since $\big\|x^{(\ell+jd)} - x^{(\ell+j'd)}\big\| \geq 0$ we deduce from the inequality of Proposition 4.4.9 that

$$p f^{\langle p\rangle}\big(\tilde{z}^{(0)}, \dots, \tilde{z}^{(d-1)}\big) \leq f\big(x^{(0)}, \dots, x^{(r-1)}\big). \tag{4.4.12}$$

Finally, for each $i \in \{0, \dots, \eta-1\}$ we have, using (4.4.4),

$$\begin{aligned}
\sum_{\ell=0}^{d-1}\sum_{j=0}^{p-1}\tilde{z}^{(\ell)}_{i-j\eta-\kappa_\ell} &= \frac{1}{p}\sum_{\ell=0}^{d-1}\sum_{j=0}^{p-1}\sum_{j'=0}^{p-1} x^{(\ell+j'd)}_{i-j\eta-\kappa_\ell} \\
&= \frac{1}{p}\sum_{j=0}^{p-1}\sum_{\ell=0}^{d-1}\sum_{j'=0}^{p-1} x^{(\ell+j'd)}_{i+(\underbrace{-j+j'}_{=:j_0}-j')\eta-\kappa_\ell} \\
&= \frac{1}{p}\sum_{j_0=0}^{p-1}\left(\sum_{\ell=0}^{d-1}\sum_{j'=0}^{p-1} x^{(\ell+j'd)}_{i+(j_0-j')\eta-\kappa_\ell}\right) \\
&= \frac{1}{p}\sum_{j_0=0}^{p-1}\delta_i \\
&= \delta_i.
\end{aligned} \tag{4.4.13}$$

Hence, all hypotheses are satisfied but one: the parameters $\tilde{z}^{(0)}, \dots, \tilde{z}^{(d-1)} \in \frac{1}{p}\mathbb{Z}^e_0$ are not necessary in $\mathbb{Z}^e_0$.

4.4.4 Rectification of the parameters

We will construct from $\tilde{z}^{(0)}, \dots, \tilde{z}^{(d-1)} \in \frac{1}{p}\mathbb{Z}^e_0$ (defined in Proposition 4.4.9) some elements $z^{(0)}, \dots, z^{(d-1)} \in \mathbb{Z}^e_0$ that satisfy all the assumptions of Lemma 4.4.5. To that end, we will approximate $\tilde{z}^{(0)}, \dots, \tilde{z}^{(d-1)}$ using Proposition 4.3.14, and we will control the value of $f\big(z^{(0)}, \dots, z^{(d-1)}\big)$ using Lemma 4.3.24. The remaining of this subsection in now devoted to the proof of the following proposition.

Proposition 4.4.14. *There exist elements* $z^{(0)}, \ldots, z^{(d-1)} \in \mathbb{Z}_0^e$ *such that*

$$\sum_{\ell=0}^{d-1} \sum_{j=0}^{p-1} z_{i-j\eta-\kappa_\ell}^{(\ell)} = \delta_i,$$

for all $i \in \{0, \ldots, \eta - 1\}$ *and*

$$f^{\langle p \rangle}(z^{(0)}, \ldots, z^{(d-1)}) \leq f^{\langle p \rangle}(\widetilde{z}^{(0)}, \ldots, \widetilde{z}^{(d-1)}) + \frac{1}{2p} \sum_{\ell=0}^{d-1} \sum_{j=0}^{p-1} \|x^{(\ell+jd)} - \widetilde{z}^{(\ell)}\|^2.$$

Let $\ell \in \{0, \ldots, d-1\}$. Since $\widetilde{z}^{(\ell)} \in \frac{1}{p}\mathbb{Z}^e$, we know that for any $i \in \{0, \ldots, \eta - 1\}$ and $j \in \{0, \ldots, p-1\}$ there exist unique elements $m_{j+ip}^{(\ell)} \in \mathbb{Z}$ and $w_{j+ip}^{(\ell)} \in \{0, \ldots, p-1\}$ such that

$$\widetilde{z}_{i-j\eta-\kappa_\ell}^{(\ell)} = m_{j+ip}^{(\ell)} + \frac{w_{j+ip}^{(\ell)}}{p}. \tag{4.4.15}$$

The fractional part of $\widetilde{z}_{i-j\eta-\kappa_\ell}^{(\ell)}$ is

$$\{\widetilde{z}_{i-j\eta-\kappa_\ell}^{(\ell)}\} = \frac{w_{j+ip}^{(\ell)}}{p} =: v_{j+ip}^{(\ell)}. \tag{4.4.16}$$

For each $\ell \in \{0, \ldots, d-1\}$, we have two e-tuples $m^{(\ell)} := (m_0^{(\ell)}, \ldots, m_{e-1}^{(\ell)})$ and $v^{(\ell)} := (v_0^{(\ell)}, \ldots, v_{e-1}^{(\ell)})$. Let π_ℓ be the permutation of $\{0, \ldots, e-1\}$ defined by

$$\pi_\ell(j + ip) := i - j\eta - \kappa_\ell,$$

for all $i \in \{0, \ldots, \eta - 1\}$ and $j \in \{0, \ldots, p-1\}$. Permuting the components of e-tuples according to $\pi_0, \ldots, \pi_{d-1}$, we obtain a map $\widetilde{f}^{\langle p \rangle} : (\mathbb{R}^e)^d \to \mathbb{R}$ that satisfies

$$\widetilde{f}^{\langle p \rangle}(m^{(0)} + v^{(0)}, \ldots, m^{(d-1)} + v^{(d-1)}) = f^{\langle p \rangle}(\widetilde{z}^{(0)}, \ldots, \widetilde{z}^{(d-1)}).$$

To match with the setting of §4.3.2, we define the two following $d \times e$ matrices:

$$M = \begin{pmatrix} m^{(0)} \\ \vdots \\ m^{(d-1)} \end{pmatrix}, \qquad V = \begin{pmatrix} v^{(0)} \\ \vdots \\ v^{(d-1)} \end{pmatrix},$$

so that

$$\widetilde{f}^{\langle p \rangle}(M + V) = f^{\langle p \rangle}(\widetilde{z}^{(0)}, \ldots, \widetilde{z}^{(d-1)}). \tag{4.4.17}$$

Like $f^{\langle p \rangle}$, the map $\widetilde{f}^{\langle p \rangle}$ defined on the $d \times e$ matrices is of the form $\frac{1}{2}\|\cdot\|^2 - \widetilde{L}$ where $\|\cdot\|^2$ is the sum of the squares of the entries and $\widetilde{L}$ is a linear form. We now write the matrix V blockwise in the same fashion as for Proposition 4.3.14. That is,

$$V = \begin{pmatrix} v^{(0)} \\ \vdots \\ v^{(d-1)} \end{pmatrix} = \begin{pmatrix} V^{[0]} & \cdots & V^{[\eta-1]} \end{pmatrix},$$

where

$$V^{[i]} = \begin{pmatrix} v_{ip}^{(0)} & \cdots & v_{p-1+ip}^{(0)} \\ \vdots & \vdots & \vdots \\ v_{ip}^{(d-1)} & \cdots & v_{p-1+ip}^{(d-1)} \end{pmatrix},$$

for any $i \in \{0, \dots, \eta - 1\}$. We now check that V satisfies the assumptions of Proposition 4.3.14. First, for any $\ell \in \{0, \dots, d-1\}$ the element $v^{(\ell)}$ satisfies $|v^{(\ell)}| \geq 0$ since its entries are non-negative. Furthermore,

$$\begin{aligned} |v^{(\ell)}| &= \sum_{i=0}^{\eta-1} \sum_{j=0}^{p-1} v_{j+ip}^{(\ell)} \\ &= \sum_{i=0}^{\eta-1} \sum_{j=0}^{p-1} (\tilde{z}_{i-j\eta-\kappa_\ell}^{(\ell)} - m_{j+ip}^{(\ell)}) \qquad \text{(by (4.4.15), (4.4.16))} \\ &= \sum_{i=0}^{e-1} \tilde{z}_i^{(\ell)} - \sum_{i=0}^{\eta-1} \sum_{j=0}^{p-1} m_{j+ip}^{(\ell)} \\ &= |\tilde{z}^{(\ell)}| - \sum_{i=0}^{\eta-1} \sum_{j=0}^{p-1} m_{j+ip}^{(\ell)}. \end{aligned}$$

Hence, we have $|v^{(\ell)}| \in \mathbb{Z}$ since $|\tilde{z}^{(\ell)}| = 0$ and $m_{j+ip}^{(\ell)} \in \mathbb{Z}$, thus $|v^{(\ell)}| \in \mathbb{N}$. The same argument shows that $|V^{[i]}| \in \mathbb{N}$ for any $i \in \{0, \dots, \eta - 1\}$ since

$$\begin{aligned} |V^{[i]}| &= \sum_{\ell=0}^{d-1} \sum_{j=0}^{p-1} v_{j+ip}^{(\ell)} \\ &= \sum_{\ell=0}^{d-1} \sum_{j=0}^{p-1} \tilde{z}_{i-j\eta-\kappa_\ell}^{(\ell)} - \sum_{\ell=0}^{d-1} \sum_{j=0}^{p-1} m_{j+ip}^{(\ell)} \\ &= \delta_i - \sum_{\ell=0}^{d-1} \sum_{j=0}^{p-1} m_{j+ip}^{(\ell)}. \end{aligned}$$

Thus, we can apply Proposition 4.3.14. There exist vectors $\epsilon^{j(\ell)} \in \{0,1\}^e$ for all $j \in \{0, \dots, p-1\}$ and $\ell \in \{0, \dots, d-1\}$ such that

$$\frac{1}{p} \sum_{j=0}^{p-1} \epsilon^{j(\ell)} = v^{(\ell)},$$

$$|\epsilon^{j(\ell)}| = |v^{(\ell)}|, \tag{4.4.18}$$

$$|E^{j[i]}| = |V^{[i]}|, \qquad \text{for all } i \in \{0, \dots, \eta - 1\}. \tag{4.4.19}$$

In the above equality, the matrices $E^{j[i]}$ for any $i \in \{0, \dots, \eta - 1\}$ are defined by the same block decomposition as V:

$$E^j := \begin{pmatrix} \epsilon^{j(0)} \\ \vdots \\ \epsilon^{j(d-1)} \end{pmatrix} = \begin{pmatrix} E^{j[0]} & \cdots & E^{j[\eta-1]} \end{pmatrix},$$

in particular each $E^{j[i]}$ has size $d \times p$. The map $\widetilde{f}^{\langle p \rangle}$ and the matrices V and E^j for all $j \in \{0, \dots, p-1\}$ satisfy the assumptions of Lemma 4.3.24. Hence, there exists $j_0 \in \{0, \dots, p-1\}$

such that

$$\widetilde{f}^{\langle p\rangle}(M+E^{j_0}) \leq \widetilde{f}^{\langle p\rangle}(M+V) + \frac{|V|-\|V\|^2}{2}.$$

For each $\ell \in \{0,\ldots,d-1\}$, define the vector $z^{(\ell)}$ by permuting the coordinates of $m^{(\ell)} + \epsilon^{j_0(\ell)}$ via π_ℓ. We have

$$f^{\langle p\rangle}\big(z^{(0)},\ldots,z^{(d-1)}\big) = \widetilde{f}^{\langle p\rangle}(M+E^{j_0}),$$

thus, recalling (4.4.17),

$$f^{\langle p\rangle}\big(z^{(0)},\ldots,z^{(d-1)}\big) \leq f^{\langle p\rangle}\big(\widetilde{z}^{(0)},\ldots,\widetilde{z}^{(d-1)}\big) + \frac{|V|-\|V\|^2}{2}. \tag{4.4.20}$$

We now check that $z^{(0)},\ldots,z^{(d-1)}$ have the properties described in Proposition 4.4.14. First, for any $\ell \in \{0,\ldots,d-1\}$ the vector $z^{(\ell)}$ is a permutation of $m^{(\ell)} + \epsilon^{j_0(\ell)}$. Since $m^{(\ell)} \in \mathbb{Z}^e$ and $\epsilon^{j_0(\ell)} \in \{0,1\}^e$, we have $z^{(\ell)} \in \mathbb{Z}^e$. Moreover,

$$\begin{aligned}
|z^{(\ell)}| &= |m^{(\ell)}| + |\epsilon^{j_0(\ell)}| \\
&= |m^{(\ell)}| + |v^{(\ell)}| && \text{(by (4.4.18))} \\
&= |\widetilde{z}^{(\ell)}| \\
&= 0 && \text{(by (4.4.11))},
\end{aligned}$$

thus $z^{(\ell)} \in \mathbb{Z}_0^e$. The equality condition of Proposition 4.4.14 is satisfied, since for any $i \in \{0,\ldots,\eta-1\}$ we have

$$\begin{aligned}
\sum_{\ell=0}^{d-1}\sum_{j=0}^{p-1} z^{(\ell)}_{i-j\eta-\kappa_\ell} &= \sum_{\ell=0}^{d-1}\sum_{j=0}^{p-1}\Big[m^{(\ell)}_{j+ip} + \epsilon^{j_0(\ell)}_{j+ip}\Big] \\
&= \sum_{\ell=0}^{d-1}\sum_{j=0}^{p-1} m^{(\ell)}_{j+ip} + |E^{j_0[i]}| \\
&= \sum_{\ell=0}^{d-1}\sum_{j=0}^{p-1} m^{(\ell)}_{j+ip} + |V^{[i]}| && \text{(by (4.4.19))} \\
&= \sum_{\ell=0}^{d-1}\sum_{j=0}^{p-1}\Big[m^{(\ell)}_{j+ip} + v^{(\ell)}_{j+ip}\Big] \\
&= \sum_{\ell=0}^{d-1}\sum_{j=0}^{p-1} \widetilde{z}^{(\ell)}_{i-j\eta-\kappa_\ell} \\
&= \delta_i.
\end{aligned}$$

It remains to prove that the value of $f^{\langle p\rangle}\big(z^{(0)},\ldots,z^{(d-1)}\big)$ does not grow too much. We have

$$\frac{|V|-\|V\|^2}{2} = \frac{1}{2}\sum_{\ell=0}^{d-1}\Big[|v^{(\ell)}| - \|v^{(\ell)}\|^2\Big] = \frac{1}{2}\sum_{\ell=0}^{d-1}\sum_{i=0}^{\eta-1}\sum_{j=0}^{p-1}\Big[v^{(\ell)}_{j+ip} - (v^{(\ell)}_{j+ip})^2\Big].$$

Recall the definition of the vectors $\widetilde{z}^{(\ell)}$ for any $\ell \in \{0,\ldots,d-1\}$ given in Proposition 4.4.9. Since for all $i \in \{0,\ldots,\eta-1\}$ and $j \in \{0,\ldots,p-1\}$, each $v^{(\ell)}_{j+ip}$ is the fractional part of

$$\widetilde{z}^{(\ell)}_{i-j\eta-\kappa_\ell} = \frac{1}{p}\sum_{j'=0}^{p-1} x^{(\ell+j'd)}_{i-j\eta-\kappa_\ell},$$

and since each $x^{(\ell+j'd)}_{i-j\eta-\kappa_\ell}$ is an integer, we can apply Lemma 4.3.27. We obtain

$$v^{(\ell)}_{j+ip} - \left(v^{(\ell)}_{j+ip}\right)^2 \le \frac{1}{p}\sum_{j'=0}^{p-1}\left(x^{(\ell+j'd)}_{i-j\eta-\kappa_\ell} - \tilde{z}^{(\ell)}_{i-j\eta-\kappa_\ell}\right)^2,$$

for all $i \in \{0,\dots,\eta-1\}$ and $j \in \{0,\dots,p-1\}$, thus

$$|v^{(\ell)}| - \|v^{(\ell)}\|^2 \le \frac{1}{p}\sum_{j'=0}^{p-1}\|x^{(\ell+j'd)} - \tilde{z}^{(\ell)}\|^2$$

It follows from (4.4.20) that

$$\begin{aligned} f^{\langle p\rangle}(z^{(0)},\dots,z^{(d-1)}) &\le f^{\langle p\rangle}(\tilde{z}^{(0)},\dots,\tilde{z}^{(d-1)}) + \frac{1}{2}\sum_{\ell=0}^{d-1}\left[|v^{(\ell)}| - \|v^{(\ell)}\|^2\right] \\ &\le f^{\langle p\rangle}(\tilde{z}^{(0)},\dots,\tilde{z}^{(d-1)}) + \frac{1}{2p}\sum_{\ell=0}^{d-1}\sum_{j=0}^{p-1}\|x^{(\ell+jd)} - \tilde{z}^{(\ell)}\|^2, \end{aligned}$$

as desired.

4.4.5 Proof of the main theorem

We now conclude the proof of Theorem 4.2.31. Let $z^{(0)},\dots,z^{(d-1)} \in \mathbb{Z}^e_0$ be as in Proposition 4.4.14. They satisfy

$$\sum_{\ell=0}^{d-1}\sum_{j=0}^{p-1} z^{(\ell)}_{i-j\eta-\kappa_\ell} = \delta_i, \tag{4.4.21}$$

for all $i \in \{0,\dots,\eta-1\}$ and

$$f^{\langle p\rangle}(z^{(0)},\dots,z^{(d-1)}) \le f^{\langle p\rangle}(\tilde{z}^{(0)},\dots,\tilde{z}^{(d-1)}) + \frac{1}{2p}\sum_{\ell=0}^{d-1}\sum_{j=0}^{p-1}\|x^{(\ell+jd)} - \tilde{z}^{(\ell)}\|^2.$$

Since, by Proposition 4.4.9, we have

$$pf^{\langle p\rangle}(\tilde{z}^{(0)},\dots,\tilde{z}^{(d-1)}) \le f(x^{(0)},\dots,x^{(r-1)}) - \frac{1}{2}\sum_{\ell=0}^{d-1}\sum_{j=0}^{p-1}\|x^{(\ell+jd)} - \tilde{z}^{(\ell)}\|^2,$$

we obtain

$$\begin{aligned} pf^{\langle p\rangle}(z^{(0)},\dots,z^{(d-1)}) \le f(x^{(0)},\dots,x^{(r-1)}) &- \frac{1}{2}\sum_{\ell=0}^{d-1}\sum_{j=0}^{p-1}\|x^{(\ell+jd)} - \tilde{z}^{(\ell)}\|^2 \\ &+ \frac{1}{2}\sum_{\ell=0}^{d-1}\sum_{j=0}^{p-1}\|x^{(\ell+jd)} - \tilde{z}^{(\ell)}\|^2, \end{aligned}$$

thus

$$pf^{\langle p\rangle}(z^{(0)},\dots,z^{(d-1)}) \le f(x^{(0)},\dots,x^{(r-1)}).$$

Remark 4.4.22. The error term $\frac{1}{2}\sum_{\ell=0}^{d-1}\sum_{j=0}^{p-1}\|x^{(\ell+jd)} - \tilde{z}^{(\ell)}\|^2$ vanished thanks the strong convexity inequality of Proposition 4.4.9, the "basic" convexity inequality (4.4.12) being not accurate enough.

The above inequality, together with (4.4.21), prove that the elements $z^{(0)},\dots,z^{(d-1)} \in \mathbb{Z}^e_0$ satisfy the hypotheses of Lemma 4.4.5. Hence, Theorem 4.2.31 is proved for the e-multicore $\boldsymbol{\lambda}$. Recalling the reduction step from r-partitions to e-multicores, Proposition 4.2.38, we conclude that Theorem 4.2.31 is true for any r-partition.

4.5 Applications

We assume that the multicharge κ is compatible with (d, η, p) (cf. (4.2.20) and (4.2.21)). We consider the weight $\mathbf{\Lambda} \in \mathbb{N}^I$ given by

$$\Lambda_i := \#\{k \in \{0, \dots, r-1\}, \kappa_k = i\}, \tag{4.5.1}$$

for all $i \in I$. The compatibility condition (4.2.20) for κ gives

$$\Lambda_{i+\eta} = \Lambda_i,$$

for all $i \in I$, thus the Ariki–Koike algebra $\mathrm{H}_n^{\mathbf{\Lambda}}(q) = \mathrm{H}_n^{\mathbf{\Lambda}}(q, \zeta)$ and its subalgebra $\mathrm{H}_{p,n}^{\mathbf{\Lambda}}(q)$ are well-defined (see Definition 2.2.41), where $\zeta := q^\eta$ is a primitive pth root of unity. Note that $p' = 1$ and the cyclotomic relation (2.2.25) in $\mathrm{H}_n^{\mathbf{\Lambda}}(q)$ is

$$\prod_{i \in I} (S - q^i)^{\Lambda_i} = \prod_{k=0}^{r-1} (S - q^{\kappa_k}) = 0.$$

We present two applications of Theorem 4.2.31 and Corollary 4.2.34. First, we will recall the definition of cellular algebras, as introduced by Graham and Lehrer [GrLe]. The algebra $\mathrm{H}_n^{\mathbf{\Lambda}}(q)$ and its blocks $\mathrm{H}_\alpha^{\mathbf{\Lambda}}(q)$ for $\alpha \in Q^+$ are examples of cellular algebras. We are interested in the fixed point subalgebras $\mathrm{H}_{p,[\alpha]}^{\mathbf{\Lambda}}(q)$ (respectively $\mathrm{H}_{p,n}^{\mathbf{\Lambda}}(q)$) of $\mathrm{H}_{[\alpha]}^{\mathbf{\Lambda}}(q)$ (resp. $\mathrm{H}_n^{\mathbf{\Lambda}}(q)$) for the algebra homomorphism σ. Recall that we gave in Proposition 4.5.18 bases for these algebras. In §4.5.2.4, we prove that if $\#[\alpha] = p$ (resp. if p and n are coprime) then $\mathrm{H}_{p,[\alpha]}^{\mathbf{\Lambda}}(q)$ (resp. $\mathrm{H}_{p,n}^{\mathbf{\Lambda}}(q)$) is cellular. Otherwise, using Corollary 4.2.34 we show that if in addition p is odd then none of these bases of $\mathrm{H}_{p,[\alpha]}^{\mathbf{\Lambda}}(q)$ are *adapted* cellular (see §4.5.2.5). Finally, in §4.5.3 we will study the restriction of *Specht modules* of $\mathrm{H}_{[\alpha]}^{\mathbf{\Lambda}}(q)$.

4.5.1 Cellular algebras

Let A be an associative unitary finite-dimensional F-algebra. A *cellular datum* for the algebra A is a triple $(\Lambda, \mathcal{T}, c)$ such that:

- the element $\Lambda = (\Lambda, \geq)$ is a finite partially ordered set;
- for any $\lambda \in \Lambda$ we have an indexing set $\mathcal{T}(\lambda)$ and distinct elements $c_{\mathfrak{st}}^\lambda$ for all $\mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)$ such that
$$\left\{c_{\mathfrak{st}}^\lambda : \lambda \in \Lambda, \mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)\right\},$$
is a basis of A as an F-module;
- for any $\lambda \in \Lambda, \mathfrak{t} \in \mathcal{T}(\lambda)$ and $a \in A$, there exist scalars $r_{\mathfrak{tv}}(a) \in F$ such that for all $\mathfrak{s} \in \mathcal{T}(\lambda)$,
$$c_{\mathfrak{st}}^\lambda a = \sum_{\mathfrak{v} \in \mathcal{T}(\lambda)} r_{\mathfrak{tv}}(a) c_{\mathfrak{sv}}^\lambda \pmod{A^{>\lambda}},$$
where $A^{>\lambda}$ is the F-module spanned by $\{c_{\mathfrak{ab}}^\mu : \mu > \lambda \text{ and } \mathfrak{a}, \mathfrak{b} \in \mathcal{T}(\mu)\}$;
- the F-linear map $* : A \to A$ determined by $(c_{\mathfrak{st}}^\lambda)^* := c_{\mathfrak{ts}}^\lambda$ for all $\lambda \in \Lambda$ and $\mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)$ is an anti-automorphism of the algebra A.

We say that A is a *cellular algebra* if it has a cellular datum. We say that a basis $\mathcal{B}$ of A is *cellular* if it coincides with $\{c_{\mathfrak{st}}^\lambda : \lambda \in \Lambda, \mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)\}$ where $(\Lambda, \mathcal{T}, c)$ is a cellular datum for A.

Remark 4.5.2. If $(\Lambda, \mathcal{T}, c)$ is a cellular datum for A then

$$\dim A = \sum_{\lambda \in \Lambda} \#\mathcal{T}(\lambda)^2.$$

Lemma 4.5.3. *Let $(\Lambda, \mathcal{T}, c)$ be a cellular datum of A and let $*$ be the corresponding anti-automorphism. The cardinality of*

$$\left\{c_{\mathfrak{s}\mathfrak{t}}^{\lambda} : \lambda \in \Lambda, \mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda), \left(c_{\mathfrak{s}\mathfrak{t}}^{\lambda}\right)^* = c_{\mathfrak{s}\mathfrak{t}}^{\lambda}\right\},$$

is $\sum_{\lambda \in \Lambda} \#\mathcal{T}(\lambda)$.

Proof. Since $\left(c_{\mathfrak{s}\mathfrak{t}}^{\lambda}\right)^* = c_{\mathfrak{t}\mathfrak{s}}^{\lambda}$, we have $\left(c_{\mathfrak{s}\mathfrak{t}}^{\lambda}\right)^* = c_{\mathfrak{s}\mathfrak{t}}^{\lambda}$ if and only if $\mathfrak{s} = \mathfrak{t}$. ☐

Assume that $(\Lambda, \mathcal{T}, c)$ is a cellular datum for A. By [GrLe], for each $\lambda \in \Lambda$ we have an A-module $\mathcal{S}^\lambda$, called *cell module*, endowed with a certain bilinear form b_λ whose radical is an A-module. Moreover, if $\mathcal{D}^\lambda$ denotes the quotient of $\mathcal{S}^\lambda$ by the radical of b_λ, the set $\{\mathcal{D}^\lambda : \lambda \in \Lambda, \mathcal{D}^\lambda \neq \{0\}\}$ is a complete family of non-isomorphic irreducible A-modules.

4.5.2 Cellularity of the fixed point subalgebra

We will first give more definitions from combinatorics, and recall the existence of a particular cellular datum for $\mathrm{H}_n^{\mathbf{\Lambda}}(q)$ and its blocks $\mathrm{H}_\alpha^{\mathbf{\Lambda}}(q)$. Then, we will construct bases for the algebra $\mathrm{H}_{p,[\alpha]}^{\mathbf{\Lambda}}(q)$ and study its cellularity. We will use the following notation:

$$Q_n^\kappa := \left\{\alpha \in Q^+ : \text{there exists } \boldsymbol{\lambda} \in \mathcal{P}_n^\kappa \text{ such that } \alpha_\kappa(\boldsymbol{\lambda}) = \alpha\right\},$$

so that the blocks of $\mathrm{H}_n^{\mathbf{\Lambda}}(q)$ are the subalgebras $\mathrm{H}_\alpha^{\mathbf{\Lambda}}(q)$ for any $\alpha \in Q_n^\kappa$.

4.5.2.1 Tableaux

Let $\boldsymbol{\lambda} = \left(\lambda^{(0)}, \dots, \lambda^{(r-1)}\right)$ be an r-partition of n. Recall that we defined in §4.2.1 and §4.2.4 the Young diagram $\mathcal{Y}(\boldsymbol{\lambda})$ of $\boldsymbol{\lambda}$. A $\boldsymbol{\lambda}$*-tableau* is a bijection $\mathfrak{t} = \left(\mathfrak{t}^{(0)}, \dots, \mathfrak{t}^{(r-1)}\right) : \mathcal{Y}(\boldsymbol{\lambda}) \to \{1, \dots, n\}$. The κ*-residue sequence* of a $\boldsymbol{\lambda}$-tableau $\mathfrak{t}$ is the sequence

$$\operatorname{res}_\kappa(\mathfrak{t}) := \left(\operatorname{res}_\kappa(\mathfrak{t}^{-1}(a))\right)_{a \in \{1, \dots, n\}}.$$

A $\boldsymbol{\lambda}$-tableau $\mathfrak{t} : \mathcal{Y}(\boldsymbol{\lambda}) \to \{1, \dots, n\}$ is *standard* if the value of $\mathfrak{t}$ increases along the rows and down the columns of $\mathcal{Y}(\boldsymbol{\lambda})$. We denote by $\mathcal{T}(\boldsymbol{\lambda})$ the set of standard $\boldsymbol{\lambda}$-tableaux.

Example 4.5.4. We take $r = p = 2$ and we consider the bipartition $\boldsymbol{\lambda} := \left((4,1),(1)\right)$. The map $\mathfrak{t} : \mathcal{Y}(\boldsymbol{\lambda}) \to \{1, \dots, 6\}$ described by

$$\begin{array}{|c|c|c|c|}\hline 1 & 5 & 4 & 6 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|}\hline 3 \\ \hline\end{array},$$

is a $\boldsymbol{\lambda}$-tableau (we warn the reader that we represented in the same way the multiset of residues associated with a multipartition), but it is not standard. The tableau $\mathfrak{s} : \mathcal{Y}(\boldsymbol{\lambda}) \to \{1, \dots, 6\}$ described by

$$\begin{array}{|c|c|c|c|}\hline 1 & 4 & 5 & 6 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|}\hline 3 \\ \hline\end{array},$$

is standard. With $\kappa = (0,2)$ and $e = 4 = 2\eta$, the residue sequence of $\mathfrak{s}$ is $\operatorname{res}_\kappa(\mathfrak{s}) = (0,3,2,1,2,3)$.

Mimicking Definition 4.2.28, we define the *shift* of a $\boldsymbol{\lambda}$-tableau $\mathfrak{t} = (\mathfrak{t}^{(0)}, \dots, \mathfrak{t}^{(r-1)})$ by

$$^{\sigma}\mathfrak{t} := \left(\mathfrak{t}^{(r-d)}, \dots, \mathfrak{t}^{(r-1)}, \mathfrak{t}^{(0)}, \dots, \mathfrak{t}^{(r-d+1)}\right),$$

and we denote by $[\mathfrak{t}]$ the orbit of $\mathfrak{t}$ under the action of σ. Note that $^{\sigma}\mathfrak{t}$ is a $^{\sigma}\boldsymbol{\lambda}$-tableau, which is standard if $\mathfrak{t}$ is standard. In particular the set $\mathcal{T}[\boldsymbol{\lambda}] := \cup_{\boldsymbol{\mu}\in[\boldsymbol{\lambda}]}\mathcal{T}(\boldsymbol{\mu})$ is stable under σ and there is a well-defined equivalence relation $\sim$ on $\mathcal{T}[\boldsymbol{\lambda}]$ generated by $\mathfrak{t} \sim {}^{\sigma}\mathfrak{t}$. We write $\mathfrak{T}[\boldsymbol{\lambda}] := \mathcal{T}[\boldsymbol{\lambda}]/\!\sim$ for the set of equivalence classes. Choose a lift $\phi : \mathfrak{T}[\boldsymbol{\lambda}] \to \mathcal{T}[\boldsymbol{\lambda}]$ of the canonical projection $\mathcal{T}[\boldsymbol{\lambda}] \to \mathfrak{T}[\boldsymbol{\lambda}]$. In other words, if $\mathfrak{t}$ is any standard $\boldsymbol{\lambda}$-tableau then $\phi([\mathfrak{t}]) \in [\mathfrak{t}]$. For any $j \in \{0, \dots, p-1\}$, we define

$$\mathcal{T}_j^{\phi}(\boldsymbol{\lambda}) := \left\{\mathfrak{t} \in \mathcal{T}(\boldsymbol{\lambda}) : \phi([\mathfrak{t}]) = {}^{\sigma^j}\mathfrak{t}\right\}.$$

Note that the set $\mathcal{T}_j^{\phi}(\boldsymbol{\lambda})$ may be empty for some $j \in \{0, \dots, p-1\}$, but we have a partition $\mathcal{T}(\boldsymbol{\lambda}) = \sqcup_{j=0}^{p-1}\mathcal{T}_j^{\phi}(\boldsymbol{\lambda})$. Moreover:

$$\text{if } \mathfrak{t} \in \mathcal{T}_j^{\phi}(\boldsymbol{\lambda}) \text{ then } {}^{\sigma}\mathfrak{t} \in \mathcal{T}_{j-1}^{\phi}({}^{\sigma}\boldsymbol{\lambda}). \tag{4.5.5}$$

We have

$$\#\mathcal{T}_0^{\phi}[\boldsymbol{\lambda}] = \frac{1}{p}\#\mathcal{T}[\boldsymbol{\lambda}], \tag{4.5.6}$$

where $\mathcal{T}_0^{\phi}[\boldsymbol{\lambda}] := \cup_{\boldsymbol{\mu}\in[\boldsymbol{\lambda}]}\mathcal{T}_0^{\phi}(\boldsymbol{\mu}) = \{\mathfrak{t} \in \mathcal{T}[\boldsymbol{\lambda}] : \phi([\mathfrak{t}]) = \mathfrak{t}\}$. In particular, the cardinality of $\mathcal{T}_0^{\phi}[\boldsymbol{\lambda}]$ does not depend on ϕ and we may abuse notation by writing $\#\mathcal{T}_0[\boldsymbol{\lambda}]$ instead of $\mathcal{T}_0^{\phi}[\boldsymbol{\lambda}]$. Since $\#\mathcal{T}(\boldsymbol{\lambda}) = \frac{1}{\#[\boldsymbol{\lambda}]}\#\mathcal{T}[\boldsymbol{\lambda}]$, we also deduce that

$$\#\mathcal{T}(\boldsymbol{\lambda}) = \frac{p}{\#[\boldsymbol{\lambda}]}\mathcal{T}_0^{\phi}[\boldsymbol{\lambda}]. \tag{4.5.7}$$

Example 4.5.8. Recall that the multicharge κ is compatible with (d, η, p). For any $\mathfrak{t} \in \mathcal{T}[\boldsymbol{\lambda}]$, the compatibility condition (4.2.21) ensures that there exists a unique standard tableau $\widetilde{\phi}(\mathfrak{t}) \in [\mathfrak{t}]$ such that 1 is in the image of the first d components of $\widetilde{\phi}(\mathfrak{t})$, that is, such that there exists $c \in \{0, \dots, d-1\}$ with $\widetilde{\phi}(\mathfrak{t})\big((0,0,c)\big) = 1$. Note that when $d = 1$ (*i.e.* when $r = p$), this condition is the same as $\operatorname{res}_{\kappa}(\widetilde{\phi}^{-1}(1)) = \kappa_0$. The map $\widetilde{\phi} : \mathcal{T}[\boldsymbol{\lambda}] \to \mathcal{T}[\boldsymbol{\lambda}]$ is constant on the equivalent classes of $\sim$. Thus, it factorises to a map $\phi : \mathfrak{T}[\boldsymbol{\lambda}] \to \mathcal{T}[\boldsymbol{\lambda}]$ that lifts the natural projection. In this case, for any $j \in \{0, \dots, p-1\}$ we have

$$\mathcal{T}_j^{\phi}(\boldsymbol{\lambda}) = \left\{\mathfrak{t} \in \mathcal{T}(\boldsymbol{\lambda}) : \text{there exists } c \in \{(p-j)d, \dots, (p-j+1)d-1\} \text{ such that } \mathfrak{t}\big((0,0,c)\big) = 1\right\}.$$

We will see in §4.5.2.4 another example of a lift ϕ of the natural projection.

Remark 4.5.9. Here, we chose ϕ to be a map $\mathfrak{T}[\boldsymbol{\lambda}] \to \mathcal{T}[\boldsymbol{\lambda}]$. If $\mathcal{P}$ is any subset of $\mathcal{P}_n^{\kappa}/\!\sim$, the equivalence relation $\sim$ is also defined on $\cup_{[\boldsymbol{\lambda}]\in\mathcal{P}}\mathcal{T}[\boldsymbol{\lambda}]$ and the equivalence classes are in natural bijection with $\cup_{[\boldsymbol{\lambda}]\in\mathcal{P}}\mathfrak{T}[\boldsymbol{\lambda}]$. Thus, we can allow ϕ to be a lift $\cup_{[\boldsymbol{\lambda}]\in\mathcal{P}}\mathfrak{T}[\boldsymbol{\lambda}] \to \cup_{[\boldsymbol{\lambda}]\in\mathcal{P}}\mathcal{T}[\boldsymbol{\lambda}]$.

4.5.2.2 Cellular datum for the Ariki–Koike algebra

It is known that we can find a family

$$\left\{c_{\mathfrak{s}\mathfrak{t}}^{\boldsymbol{\lambda}} : \boldsymbol{\lambda} \in \mathcal{P}_n^{\kappa} \text{ and } \mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\boldsymbol{\lambda})\right\}, \tag{4.5.10}$$

that form a cellular basis of $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q)$ (cf. [DJM]). Recall from Section 2.2 the algebra automorphism $\sigma : \mathrm{H}_n^{\boldsymbol{\Lambda}}(q) \to \mathrm{H}_n^{\boldsymbol{\Lambda}}(q)$ of order p. Let $\boldsymbol{\eta}$ be the n-tuple $(\eta, \dots, \eta)$ considered as an element of

$(\mathbb{Z}/e\mathbb{Z})^n$. By [BrKl-a] (see also Chapter 2), we know that the algebra $\mathrm{H}_n^{\mathbf{\Lambda}}(q)$ is generated by some elements

$$
\begin{aligned}
&e(\boldsymbol{i}), && \text{for any } \boldsymbol{i} \in (\mathbb{Z}/e\mathbb{Z})^n,\\
&\psi_a, && \text{for any } a \in \{1, \dots, n-1\},\\
&y_a, && \text{for any } a \in \{1, \dots, n\},
\end{aligned}
$$

the "Khovanov–Lauda generators", for which

$$\sigma(e(\boldsymbol{i})) = e(\boldsymbol{i} - \boldsymbol{\eta}), \qquad \text{for any } \boldsymbol{i} \in (\mathbb{Z}/e\mathbb{Z})^n, \tag{4.5.11a}$$

$$\sigma(\psi_a) = \psi_a, \qquad \text{for any } a \in \{1, \dots, n-1\}, \tag{4.5.11b}$$

$$\sigma(y_a) = y_a, \qquad \text{for any } a \in \{1, \dots, n\}. \tag{4.5.11c}$$

The elements $\{e(\boldsymbol{i}) : \boldsymbol{i} \in (\mathbb{Z}/e\mathbb{Z})^n\}$ form a complete system of orthogonal idempotents, that is,

$$e(\boldsymbol{i})^2 = e(\boldsymbol{i}), \qquad \text{for any } \boldsymbol{i} \in (\mathbb{Z}/e\mathbb{Z})^n, \tag{4.5.12a}$$

$$e(\boldsymbol{i})e(\boldsymbol{j}) = 0, \qquad \text{for any } \boldsymbol{i} \neq \boldsymbol{j} \in (\mathbb{Z}/e\mathbb{Z})^n, \tag{4.5.12b}$$

$$\sum_{\boldsymbol{i} \in (\mathbb{Z}/e\mathbb{Z})^n} e(\boldsymbol{i}) = 1. \tag{4.5.12c}$$

Among the generators $e(\boldsymbol{i})$ for any $\boldsymbol{i} \in (\mathbb{Z}/e\mathbb{Z})^n$, we know exactly the ones that are non-zero (see [HuMa10, 4.1.Lemma]).

Lemma 4.5.13. *For any $\boldsymbol{i} \in (\mathbb{Z}/e\mathbb{Z})^n$, the idempotent $e(\boldsymbol{i}) \in \mathrm{H}_n^{\mathbf{\Lambda}}(q)$ is non-zero if and only if there exist $\boldsymbol{\lambda} \in \mathcal{P}_n^\kappa$ and $\mathfrak{t} \in \mathcal{T}(\boldsymbol{\lambda})$ such that $\boldsymbol{i} = \mathrm{res}_\kappa(\mathfrak{t})$.*

There is a well-defined algebra anti-automorphism $* : \mathrm{H}_n^{\mathbf{\Lambda}}(q) \to \mathrm{H}_n^{\mathbf{\Lambda}}(q)$, which we now fix, that is the identity on each Khovanov–Lauda generator (see [HuMa10, §5.1]). We can find a cellular basis of $\mathrm{H}_n^{\mathbf{\Lambda}}(q)$ of the form (4.5.10) such that the associated anti-automorphism is the map $*$, with the additional property

$$c_{\mathfrak{s}\mathfrak{t}}^{\boldsymbol{\lambda}} \in e(\mathrm{res}_\kappa(\mathfrak{s}))\mathrm{H}_n^{\mathbf{\Lambda}}(q)e(\mathrm{res}_\kappa(\mathfrak{t})), \tag{4.5.14}$$

for all $\boldsymbol{\lambda} \in \mathcal{P}_n^\kappa$ and $\mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\boldsymbol{\lambda})$ (see [HuMa10] and also [Bow]). Note that we recover the result of Lemma 4.5.13. We now fix such a cellular basis.

Remark 4.5.15. The cellular bases that are constructed in [HuMa10, Bow] are *graded* cellular bases: the algebra $\mathrm{H}_n^{\mathbf{\Lambda}}(q)$ is $\mathbb{Z}$-graded ([Rou, BrKl-a]) and there exists a map $\deg : \coprod_{\boldsymbol{\lambda} \in \mathcal{P}_n^\kappa} \mathcal{T}(\boldsymbol{\lambda}) \to \mathbb{Z}$ such that $c_{\mathfrak{s}\mathfrak{t}}^{\boldsymbol{\lambda}}$ is homogeneous of degree $\deg \mathfrak{s} + \deg \mathfrak{t}$ (see also Section 5.1). These graded cellular bases seem to be more adapted to σ than the ungraded one of [DJM]: if $\mathrm{H}_n^{\mathbf{\Lambda}}(q)$ is semi-simple, we can prove that σ permutes the elements of the graded basis but its action on the ungraded basis is more complicated.

The condition (4.5.14) allows us to give a more precise description of this cellular structure for $\mathrm{H}_n^{\mathbf{\Lambda}}(q)$. For any $\alpha \in Q^+$ with $|\alpha| = n$, the subalgebra

$$\mathrm{H}_\alpha^{\mathbf{\Lambda}}(q) = \sum_{\boldsymbol{i},\boldsymbol{j} \in I^\alpha} e(\boldsymbol{i})\mathrm{H}_n^{\mathbf{\Lambda}}(q)e(\boldsymbol{j}) \subseteq \mathrm{H}_n^{\mathbf{\Lambda}}(q),$$

is a block of $\mathrm{H}_n^{\mathbf{\Lambda}}(q)$ if $\alpha \in Q_n^\kappa$ and $\{0\}$ otherwise (as we noticed in Remark 2.3.10 by [LyMa]). By (4.5.14), when $\alpha \in Q_n^\kappa$ the algebra $\mathrm{H}_\alpha^{\mathbf{\Lambda}}(q)$ is cellular, with cellular basis

$$\left\{c_{\mathfrak{s}\mathfrak{t}}^{\boldsymbol{\lambda}} : \boldsymbol{\lambda} \in \mathcal{P}_\alpha^\kappa \text{ and } \mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\boldsymbol{\lambda})\right\}$$

(cf. [HuMa10, Corollary 5.12]).

4.5.2.3 Subalgebras of fixed points

Recall that $\mathrm{H}^{\mathbf{\Lambda}}_{p,n}(q)$ is the subalgebra of the fixed points of $\sigma : \mathrm{H}^{\mathbf{\Lambda}}_n(q) \to \mathrm{H}^{\mathbf{\Lambda}}_n(q)$. If $\mu : \mathrm{H}^{\mathbf{\Lambda}}_n(q) \to \mathrm{H}^{\mathbf{\Lambda}}_n(q)$ is the linear map defined by $\mu \coloneqq \sum_{j=0}^{p-1} \sigma^j$, we have $\mu(\mathrm{H}^{\mathbf{\Lambda}}_n(q)) = \mathrm{H}^{\mathbf{\Lambda}}_{p,n}(q)$.

Remark 4.5.16. We warn the reader that the map that we denoted by μ in [Ro16] is the map $\frac{1}{p}\mu$. Note that p is invertible in $F^\times$ since F has a primitive pth root of unity (namely q^η).

We now look at the blocks of $\mathrm{H}^{\mathbf{\Lambda}}_n(q)$. Let $\alpha \in Q^\kappa_n$ and denote by $[\alpha]$ the orbit of α under the action of σ (cf. Definition 4.2.27). The subalgebra $\mathrm{H}^{\mathbf{\Lambda}}_\alpha(q) \subseteq \mathrm{H}^{\mathbf{\Lambda}}_n(q)$ is not necessarily stable under σ. However, by (4.5.11a) and as in Section 1.4, we have

$$\sigma(\mathrm{H}^{\mathbf{\Lambda}}_\alpha(q)) \subseteq \mathrm{H}^{\mathbf{\Lambda}}_{\sigma\cdot\alpha}(q), \tag{4.5.17}$$

thus the subalgebra

$$\mathrm{H}^{\mathbf{\Lambda}}_{[\alpha]}(q) \coloneqq \bigoplus_{\beta\in[\alpha]} \mathrm{H}^{\mathbf{\Lambda}}_\beta(q)$$

of $\mathrm{H}^{\mathbf{\Lambda}}_n(q)$ is stable under σ and contains $\mathrm{H}^{\mathbf{\Lambda}}_\alpha(q)$. Similarly, we define $\mathcal{P}^\kappa_{[\alpha]} \coloneqq \cup_{\beta\in[\alpha]}\mathcal{P}^\kappa_\beta$. Note that by Lemma 4.2.29 we have $[\boldsymbol{\lambda}] \subseteq \mathcal{P}^\kappa_{[\alpha]}$. Hence, as for the tableaux, there is a well-defined equivalence relation $\sim$ on $\mathcal{P}^\kappa_{[\alpha]}$ generated by $\boldsymbol{\lambda} \sim {}^\sigma\boldsymbol{\lambda}$. We write $\mathfrak{P}^\kappa_{[\alpha]} \coloneqq \mathcal{P}^\kappa_{[\alpha]}/{\sim}$ for the set of equivalence classes. As in §4.5.2.2, the algebra $\mathrm{H}^{\mathbf{\Lambda}}_{[\alpha]}(q)$ is cellular, with cellular basis $\{c^{\boldsymbol{\lambda}}_{\mathfrak{st}} : \boldsymbol{\lambda} \in \mathcal{P}^\kappa_{[\alpha]}$ and $\mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\boldsymbol{\lambda})\}$. Moreover, if $\mathrm{H}^{\mathbf{\Lambda}}_{p,[\alpha]}(q) \subseteq \mathrm{H}^{\mathbf{\Lambda}}_{[\alpha]}(q)$ denotes the subalgebra of fixed points of $\sigma : \mathrm{H}^{\mathbf{\Lambda}}_{[\alpha]}(q) \to \mathrm{H}^{\mathbf{\Lambda}}_{[\alpha]}(q)$ then $\mathrm{H}^{\mathbf{\Lambda}}_{p,[\alpha]}(q) = \mu(\mathrm{H}^{\mathbf{\Lambda}}_{[\alpha]}(q))$.

Proposition 4.5.18. *Let*

$$\phi : \bigcup_{[\boldsymbol{\lambda}]\in\mathfrak{P}^\kappa_{[\alpha]}} \mathfrak{T}[\boldsymbol{\lambda}] \longrightarrow \bigcup_{[\boldsymbol{\lambda}]\in\mathfrak{P}^\kappa_{[\alpha]}} \mathcal{T}[\boldsymbol{\lambda}],$$

be a lift of the canonical projection. The family

$$\left\{\mu(c^{\boldsymbol{\lambda}}_{\mathfrak{st}}) : \boldsymbol{\lambda} \in \mathcal{P}^\kappa_{[\alpha]}, \mathfrak{s} \in \mathcal{T}(\boldsymbol{\lambda}), \mathfrak{t} \in \mathcal{T}^\phi_0(\boldsymbol{\lambda})\right\}, \tag{4.5.19}$$

is an F-basis of $\mathrm{H}^{\mathbf{\Lambda}}_{p,[\alpha]}(q)$.

Proof. Recall that p is invertible in F (see Remark 4.5.16). It suffices to prove that the family

$$\left\{\sigma^j(c^{\boldsymbol{\lambda}}_{\mathfrak{st}}) : j \in \{0,\dots,p-1\}, \boldsymbol{\lambda} \in \mathcal{P}^\kappa_{[\alpha]}, \mathfrak{s} \in \mathcal{T}(\boldsymbol{\lambda}), \mathfrak{t} \in \mathcal{T}^\phi_0(\boldsymbol{\lambda})\right\},$$

is an F-basis of $\mathrm{H}^{\mathbf{\Lambda}}_{[\alpha]}(q)$. For any $j \in \{0,\dots,p-1\}$, define the idempotent

$$e^\phi_j \coloneqq \sum_{\boldsymbol{\lambda}\in\mathcal{P}^\kappa_{[\alpha]}} \sum_{\mathfrak{t}\in\mathcal{T}^\phi_j(\boldsymbol{\lambda})} e(\mathrm{res}_\kappa(\mathfrak{t})).$$

The family $\{c^{\boldsymbol{\lambda}}_{\mathfrak{st}} : \boldsymbol{\lambda} \in \mathcal{P}^\kappa_{[\alpha]}, \mathfrak{s} \in \mathcal{T}(\boldsymbol{\lambda}), \mathfrak{t} \in \mathcal{T}^\phi_0(\boldsymbol{\lambda})\}$ is an F-basis of $\mathrm{H}^{\mathbf{\Lambda}}_{[\alpha]}(q)e^\phi_0$. Since κ is compatible with (d,η,p), for any $\boldsymbol{\lambda} \in \mathcal{P}^\kappa_{[\alpha]}$ and any $\boldsymbol{\lambda}$-tableau $\mathfrak{t}$ we have

$$\mathrm{res}_\kappa({}^\sigma\mathfrak{t}) = \mathrm{res}_\kappa(\mathfrak{t}) + \boldsymbol{\eta}.$$

Using (4.5.5), we deduce that $\sigma(e^\phi_j) = e^\phi_{j+1}$ for all $j \in \{0,\dots,p-1\}$. Hence the family $\{\sigma^j(c^{\boldsymbol{\lambda}}_{\mathfrak{st}}) : \boldsymbol{\lambda} \in \mathcal{P}^\kappa_{[\alpha]}, \mathfrak{s} \in \mathcal{T}(\boldsymbol{\lambda}), \mathfrak{t} \in \mathcal{T}^\phi_0(\boldsymbol{\lambda})\}$ is an F-basis of $\mathrm{H}^{\mathbf{\Lambda}}_{[\alpha]}(q)e_j$. By (4.5.12c) and Lemma 4.5.13 we have $\sum_{j=0}^{p-1} e^\phi_j = 1$ thus $\mathrm{H}^{\mathbf{\Lambda}}_{[\alpha]}(q) = \oplus_{j=0}^{p-1}\mathrm{H}^{\mathbf{\Lambda}}_{[\alpha]}(q)e^\phi_j$ and we conclude. ☐

Remark 4.5.20. Recall from Remark 4.5.15 that the algebra $\mathrm{H}^{\mathbf{\Lambda}}_{[\alpha]}(q)$ is $\mathbb{Z}$-graded. By Section 2.4, the algebra $\mathrm{H}^{\mathbf{\Lambda}}_{p,[\alpha]}(q)$ is also $\mathbb{Z}$-graded and the basis (4.5.19) is homogeneous.

We will prove the following partial alternative:

- if $\#[\alpha] = p$, the family (4.5.19) is a (graded) cellular basis of $\mathrm{H}^{\mathbf{\Lambda}}_{p,[\alpha]}(q)$, for a particular choice of lift ϕ (§4.5.2.4);
- if $\#[\alpha] < p$ and p is odd, for any lift ϕ the family (4.5.19) is not an *adapted* cellular basis of $\mathrm{H}^{\mathbf{\Lambda}}_{p,[\alpha]}(q)$, in the sense of Definition 4.5.28 (§4.5.2.5).

4.5.2.4 Cellular basis in the full orbit case

Let $\alpha \in Q^\kappa_n$ and assume that $\#[\alpha] = p$. By Lemma 4.2.29, given $\boldsymbol{\lambda} \in \mathcal{P}^\kappa_{[\alpha]}$ we know that for any $\mathfrak{t} \in \mathcal{T}(\boldsymbol{\lambda})$ there is a unique standard tableau $\mathfrak{t}_\alpha \in [\mathfrak{t}]$ whose underlying r-partition is in $\mathcal{P}^\kappa_\alpha$. We have in fact $\mathfrak{t}_\alpha \in \mathcal{T}(\boldsymbol{\lambda}_\alpha)$, where $\boldsymbol{\lambda}_\alpha$ is the unique element of $[\boldsymbol{\lambda}]$ that is in $\mathcal{P}^\kappa_\alpha$. We obtain a map

$$\phi : \left| \begin{array}{ccc} \bigcup_{[\boldsymbol{\lambda}]\in\mathfrak{P}^\kappa_{[\alpha]}} \mathfrak{T}[\boldsymbol{\lambda}] & \longrightarrow & \bigcup_{[\boldsymbol{\lambda}]\in\mathfrak{P}^\kappa_{[\alpha]}} \mathcal{T}[\boldsymbol{\lambda}] \\ \mathfrak{t} & \longmapsto & \mathfrak{t}_\alpha \end{array} \right. ,$$

that lifts the natural projection. For any $\boldsymbol{\lambda} \in \mathcal{P}^\kappa_{[\alpha]}$, we have

$$\mathcal{T}^\phi_0(\boldsymbol{\lambda}) = \begin{cases} \mathcal{T}(\boldsymbol{\lambda}), & \text{if } \boldsymbol{\lambda} \in \mathcal{P}^\kappa_\alpha, \\ \emptyset, & \text{otherwise.} \end{cases}$$

The basis (4.5.19) of $\mathrm{H}^{\mathbf{\Lambda}}_{p,[\alpha]}(q)$ that we obtain is thus

$$\left\{\mu(c^{\boldsymbol{\lambda}}_{\mathfrak{st}}) : \boldsymbol{\lambda} \in \mathcal{P}^\kappa_\alpha \text{ and } \mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\boldsymbol{\lambda})\right\}. \tag{4.5.21}$$

For any $\boldsymbol{\lambda} \in \mathcal{P}^\kappa_\alpha$ and $\mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\boldsymbol{\lambda})$, we set $d^{\boldsymbol{\lambda}}_{\mathfrak{st}} \coloneqq \mu(c^{\boldsymbol{\lambda}}_{\mathfrak{st}})$. Recall that $(\mathcal{P}^\kappa_\alpha, \mathcal{T}, c)$ is a cellular datum for $\mathrm{H}^{\mathbf{\Lambda}}_\alpha(q)$.

Proposition 4.5.22. *Recall that $\#[\alpha] = p$. The triple $(\mathcal{P}^\kappa_\alpha, \mathcal{T}, d)$ is a cellular datum for $\mathrm{H}^{\mathbf{\Lambda}}_{p,[\alpha]}(q)$.*

Proof. It suffices to prove that μ commutes with $*$ and induces an algebra isomorphism between $\mathrm{H}^{\mathbf{\Lambda}}_\alpha(q)$ and $\mathrm{H}^{\mathbf{\Lambda}}_{p,[\alpha]}(q)$. The first point is clear: indeed, since $*$ fixes each Khovanov–Lauda generator and by the action of σ on these generators (cf. (4.5.11)) we know that $*$ and σ commute. Now, the restriction of μ to $\mathrm{H}^{\mathbf{\Lambda}}_\alpha(q)$ is an algebra homomorphism. Indeed, for any $j \in \{1, \dots, p-1\}$ we have $\alpha \neq \sigma^j \cdot \alpha$ since $\#[\alpha] = p$, hence for any $h, h' \in \mathrm{H}^{\mathbf{\Lambda}}_\alpha(q)$ we have $h\sigma^j(h') = 0$ (recall (4.5.12b) and (4.5.17)). We conclude since by (4.5.21), μ sends a basis of $\mathrm{H}^{\mathbf{\Lambda}}_\alpha(q)$ onto a basis of $\mathrm{H}^{\mathbf{\Lambda}}_{p,[\alpha]}(q)$. ☐

Corollary 4.5.23. *If p and n are coprime then the algebra $\mathrm{H}^{\mathbf{\Lambda}}_{p,n}(q)$ is cellular.*

Proof. Let us first prove that $\#[\beta] = p$ for all $\beta \in Q^+$ with $|\beta| = n$. If $\#[\beta] = p'$ then p' divides p and we can write

$$\beta = \sum_{i=0}^{p'\eta-1} \beta_i(\alpha_i + \alpha_{p'\eta+i} + \cdots + \alpha_{(d-1)p'\eta+i}),$$

where $d := \frac{p}{p'}$ and $\beta_0, \dots, \beta_{p'\eta-1} \in \mathbb{N}$. We deduce that

$$n = |\beta| = d \sum_{i=0}^{p'\eta-1} \beta_i,$$

hence d divides n. But d also divides p thus $d = 1$ and $p' = p$ as desired. Hence, each subalgebra appearing in the following decomposition

$$\mathrm{H}^{\mathbf{\Lambda}}_{p,n}(q) = \bigoplus_{[\beta] \in \mathfrak{Q}^\kappa_n} \mathrm{H}^{\mathbf{\Lambda}}_{p,[\beta]}(q), \tag{4.5.24}$$

is cellular by Proposition 4.5.22, where $\mathfrak{Q}^\kappa_n$ is the quotient of Q^κ_n by the equivalence relation $\sim$ generated by $\beta \sim \sigma \cdot \beta$ for all $\beta \in Q^\kappa_n$. We now easily check that $\mathrm{H}^{\mathbf{\Lambda}}_{p,n}(q)$ is cellular, using the following fact: for any $[\beta] \neq [\beta'] \in \mathfrak{Q}^\kappa_n$ we have $hh' = 0$ for all $h \in \mathrm{H}^{\mathbf{\Lambda}}_{[\beta]}(q)$ and $h' \in \mathrm{H}^{\mathbf{\Lambda}}_{[\beta']}(q)$ (cf. (4.5.12b)). □

4.5.2.5 Adapted cellularity

Let $\alpha \in Q^\kappa_n$ and let ϕ be as in Proposition 4.5.18. By (4.5.7), we have

$$\begin{aligned}
\dim \mathrm{H}^{\mathbf{\Lambda}}_{p,[\alpha]}(q) &= \sum_{\boldsymbol{\lambda} \in \mathcal{P}^\kappa_{[\alpha]}} (\#\mathcal{T}(\boldsymbol{\lambda}))(\#\mathcal{T}^\phi_0(\boldsymbol{\lambda})) \\
&= \sum_{\boldsymbol{\lambda} \in \mathcal{P}^\kappa_{[\alpha]}} \frac{p}{\#[\boldsymbol{\lambda}]} (\#\mathcal{T}^\phi_0[\boldsymbol{\lambda}])(\#\mathcal{T}^\phi_0(\boldsymbol{\lambda})) \\
&= \sum_{[\boldsymbol{\lambda}] \in \mathfrak{P}^\kappa_{[\alpha]}} \frac{p}{\#[\boldsymbol{\lambda}]} (\#\mathcal{T}^\phi_0[\boldsymbol{\lambda}]) \sum_{\boldsymbol{\mu} \in [\boldsymbol{\lambda}]} \#\mathcal{T}^\phi_0(\boldsymbol{\mu}) \\
&= \sum_{[\boldsymbol{\lambda}] \in \mathfrak{P}^\kappa_{[\alpha]}} \frac{p}{\#[\boldsymbol{\lambda}]} (\#\mathcal{T}^\phi_0[\boldsymbol{\lambda}])^2.
\end{aligned}$$

Recalling that $\#\mathcal{T}^\phi_0[\boldsymbol{\lambda}]$ does not depend on ϕ, we obtain

$$\dim \mathrm{H}^{\mathbf{\Lambda}}_{p,[\alpha]}(q) = \sum_{[\boldsymbol{\lambda}] \in \mathfrak{P}^\kappa_{[\alpha]}} \frac{p}{\#[\boldsymbol{\lambda}]} (\#\mathcal{T}_0[\boldsymbol{\lambda}])^2. \tag{4.5.25}$$

Remark 4.5.26. With (4.5.7) and Remark 4.5.2 we obtain the equality $\dim \mathrm{H}^{\mathbf{\Lambda}}_{p,[\alpha]}(q) = \frac{1}{p} \dim \mathrm{H}^{\mathbf{\Lambda}}_{[\alpha]}(q)$.

Suppose that there exists a cellular datum $(\Lambda, \mathcal{T}, c)$ for $\mathrm{H}^{\mathbf{\Lambda}}_{p,[\alpha]}(q)$. Remark 4.5.2 and (4.5.25) give two ways to write $\dim \mathrm{H}^{\mathbf{\Lambda}}_{p,[\alpha]}(q)$ as a sum of squares:

$$\dim \mathrm{H}^{\mathbf{\Lambda}}_{p,[\alpha]}(q) = \sum_{\lambda \in \Lambda} \#\mathcal{T}(\lambda)^2 = \sum_{[\boldsymbol{\lambda}] \in \mathfrak{P}^\kappa_{[\alpha]}} \frac{p}{\#[\boldsymbol{\lambda}]} (\#\mathcal{T}_0[\boldsymbol{\lambda}])^2.$$

These two sums have the same terms up to reordering if and only if for all $[\boldsymbol{\lambda}] \in \mathfrak{P}^\kappa_{[\alpha]}$, there exist $\lambda_{[\boldsymbol{\lambda}],1}, \dots, \lambda_{[\boldsymbol{\lambda}],\frac{p}{\#[\boldsymbol{\lambda}]}} \in \Lambda$ such that

$$\#\mathcal{T}(\lambda_{[\boldsymbol{\lambda}],j}) = \#\mathcal{T}_0[\boldsymbol{\lambda}], \qquad \text{for all } j \in \left\{1, \dots, \frac{p}{\#[\boldsymbol{\lambda}]}\right\}, \tag{4.5.27a}$$

and

$$\left\{\lambda_{[\boldsymbol{\lambda}],j} : [\boldsymbol{\lambda}] \in \mathfrak{P}^\kappa_{[\alpha]} \text{ and } j \in \left\{1, \dots, \frac{p}{\#[\boldsymbol{\lambda}]}\right\}\right\} = \Lambda. \tag{4.5.27b}$$

Recall that the anti-automorphism $* : \mathrm{H}^{\mathbf{\Lambda}}_n(q) \to \mathrm{H}^{\mathbf{\Lambda}}_n(q)$ was fixed in §4.5.2.2.

Definition 4.5.28. Suppose that $(\Lambda, \mathcal{T}, c)$ is a cellular datum for $\mathrm{H}^{\boldsymbol{\Lambda}}_{p,[\alpha]}(q)$. We say that $(\Lambda, \mathcal{T}, c)$ is an *adapted* cellular datum if for all $[\boldsymbol{\lambda}] \in \mathfrak{P}^{\kappa}_n$, there exist $\lambda_{[\boldsymbol{\lambda}],1}, \dots, \lambda_{[\boldsymbol{\lambda}],\frac{p}{\#[\boldsymbol{\lambda}]}} \in \Lambda$ such that the conditions (4.5.27) are satisfied, together with $\left(c^{\lambda}_{\mathfrak{st}}\right)^* = c^{\lambda}_{\mathfrak{ts}}$ for all $\lambda \in \Lambda$ and $\mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)$.

We say that a basis $\mathcal{B}$ of $\mathrm{H}^{\boldsymbol{\Lambda}}_{p,[\alpha]}(q)$ is *adapted cellular* if there exists an adapted cellular datum $(\Lambda, \mathcal{T}, c)$ for $\mathrm{H}^{\boldsymbol{\Lambda}}_{p,[\alpha]}(q)$ such that $\mathcal{B}$ coincides with $\{c^{\lambda}_{\mathfrak{st}} : \lambda \in \Lambda \text{ and } \mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)\}$.

Lemma 4.5.29. *Let $\boldsymbol{\lambda} \in \mathcal{P}^{\kappa}_n$ and $\mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\boldsymbol{\lambda})$. Then $\mu(c^{\boldsymbol{\lambda}}_{\mathfrak{st}})^* = \mu(c^{\boldsymbol{\lambda}}_{\mathfrak{st}})$ if and only if*

$$\begin{aligned} &\mathfrak{s} = \mathfrak{t}, && \text{if } p \text{ is odd},\\ &\mathfrak{s} = \mathfrak{t} \text{ or } \sigma^{p/2}(c^{\boldsymbol{\lambda}}_{\mathfrak{st}}) = c^{\boldsymbol{\lambda}}_{\mathfrak{ts}}, && \text{if } p \text{ is even}. \end{aligned}$$

Proof. Since μ and $*$ commute, we have

$$\mu(c^{\boldsymbol{\lambda}}_{\mathfrak{st}})^* = \mu(c^{\boldsymbol{\lambda}}_{\mathfrak{ts}}) = \sum_{j=0}^{p-1} \sigma^j(c^{\boldsymbol{\lambda}}_{\mathfrak{ts}}).$$

Thus, if $\mu(c^{\boldsymbol{\lambda}}_{\mathfrak{st}})^* = \mu(c^{\boldsymbol{\lambda}}_{\mathfrak{st}})$ then

$$\sum_{j=0}^{p-1} \sigma^j(c^{\boldsymbol{\lambda}}_{\mathfrak{ts}}) = \sum_{j=0}^{p-1} \sigma^j(c^{\boldsymbol{\lambda}}_{\mathfrak{st}}).$$

By (4.5.11a), (4.5.12a), (4.5.12b) and (4.5.14), we deduce that there exists $j \in \{0, \dots, p-1\}$ such that

$$c^{\boldsymbol{\lambda}}_{\mathfrak{ts}} = \sigma^j(c^{\boldsymbol{\lambda}}_{\mathfrak{st}}). \tag{4.5.30}$$

Since σ and $*$ commute, we obtain

$$c^{\boldsymbol{\lambda}}_{\mathfrak{st}} = \sigma^j(c^{\boldsymbol{\lambda}}_{\mathfrak{ts}}),$$

thus,

$$\sigma^j(c^{\boldsymbol{\lambda}}_{\mathfrak{st}}) = \sigma^{2j}(c^{\boldsymbol{\lambda}}_{\mathfrak{ts}}).$$

Combining with (4.5.30), we obtain

$$c^{\boldsymbol{\lambda}}_{\mathfrak{ts}} = \sigma^{2j}(c^{\boldsymbol{\lambda}}_{\mathfrak{ts}}).$$

By (4.5.11a), (4.5.12a) and (4.5.12b) and since $\boldsymbol{\eta} \in (\mathbb{Z}/e\mathbb{Z})^n$ has order p, this equality implies that $2j \in \{0, p\}$. If p is odd then $j = 0$ and (4.5.30) yield $c^{\boldsymbol{\lambda}}_{\mathfrak{ts}} = c^{\boldsymbol{\lambda}}_{\mathfrak{st}}$ thus $\mathfrak{s} = \mathfrak{t}$. If p is even then $j \in \{0, \frac{p}{2}\}$ and similarly we conclude using (4.5.30). The converse is straightforward. □

Given the result of §4.5.2.4, it seems natural to look for a cellular basis for $\mathrm{H}^{\boldsymbol{\Lambda}}_{p,[\alpha]}(q)$ of the form (4.5.19). The following proposition uses Corollary 4.2.34 to give a partial answer to this problem.

Proposition 4.5.31. *If $\#[\alpha] < p$ and p is odd then the basis (4.5.19)*

$$\left\{\mu(c^{\boldsymbol{\lambda}}_{\mathfrak{st}}) : \boldsymbol{\lambda} \in \mathcal{P}^{\kappa}_{[\alpha]}, \mathfrak{s} \in \mathcal{T}(\boldsymbol{\lambda}), \mathfrak{t} \in \mathcal{T}^{\phi}_0(\boldsymbol{\lambda})\right\},$$

of $\mathrm{H}^{\boldsymbol{\Lambda}}_{p,[\alpha]}(q)$ is not adapted cellular.

Proof. Let N be the cardinality of

$$\left\{\mu(c^{\boldsymbol{\lambda}}_{\mathfrak{st}}) : \boldsymbol{\lambda} \in \mathcal{P}^{\kappa}_{[\alpha]}, \mathfrak{s} \in \mathcal{T}(\boldsymbol{\lambda}), \mathfrak{t} \in \mathcal{T}^{\phi}_0(\boldsymbol{\lambda}), \mu(c^{\boldsymbol{\lambda}}_{\mathfrak{st}})^* = \mu(c^{\boldsymbol{\lambda}}_{\mathfrak{st}})\right\}.$$

Assume that the basis (4.5.19) is adapted cellular with associated cellular datum $(\Lambda, \mathcal{T}, c)$. Lemma 4.5.3 yields, with the notation of Definition 4.5.28,

$$\begin{aligned} N &= \sum_{\lambda \in \Lambda} \#\mathcal{T}(\lambda) \\ &= \sum_{[\boldsymbol{\lambda}] \in \mathfrak{P}^{\kappa}_{[\alpha]}} \sum_{j=1}^{\frac{p}{\#[\boldsymbol{\lambda}]}} \#\mathcal{T}(\lambda_{[\boldsymbol{\lambda}],j}) \\ &= \sum_{[\boldsymbol{\lambda}] \in \mathfrak{P}^{\kappa}_{[\alpha]}} \sum_{j=1}^{\frac{p}{\#[\boldsymbol{\lambda}]}} \#\mathcal{T}_0[\boldsymbol{\lambda}] \\ &= \sum_{[\boldsymbol{\lambda}] \in \mathfrak{P}^{\kappa}_{[\alpha]}} \frac{p}{\#[\boldsymbol{\lambda}]} \#\mathcal{T}_0[\boldsymbol{\lambda}]. \end{aligned}$$

We have $\frac{p}{\#[\boldsymbol{\lambda}]} \geq 1$ for all $[\boldsymbol{\lambda}] \in \mathfrak{P}^{\kappa}_{[\alpha]}$. Moreover, since $\#[\alpha] < p$ we know by Corollary 4.2.34 that there exists $[\boldsymbol{\lambda}] \in \mathfrak{P}^{\kappa}_{[\alpha]}$ such that $\frac{p}{\#[\boldsymbol{\lambda}]} > 1$. Thus, we obtain

$$N > \sum_{[\boldsymbol{\lambda}] \in \mathfrak{P}^{\kappa}_{[\alpha]}} \#\mathcal{T}_0[\boldsymbol{\lambda}]. \tag{4.5.32}$$

But now p is odd, thus by Lemma 4.5.29 we know that

$$\mu(c^{\boldsymbol{\lambda}}_{\mathfrak{st}})^* = c^{\boldsymbol{\lambda}}_{\mathfrak{st}} \iff \mathfrak{s} = \mathfrak{t},$$

for all $\boldsymbol{\lambda} \in \mathcal{P}^{\kappa}_{[\alpha]}$, $\mathfrak{s} \in \mathcal{T}(\boldsymbol{\lambda})$ and $\mathfrak{t} \in \mathcal{T}^{\phi}_0(\boldsymbol{\lambda})$. Hence, the only elements of the basis (4.5.19) that are fixed by the $*$ anti-automorphism are the $\mu(c^{\boldsymbol{\lambda}}_{\mathfrak{ss}})$ for all $\boldsymbol{\lambda} \in \mathcal{P}^{\kappa}_{[\alpha]}$ and $\mathfrak{s} \in \mathcal{T}^{\phi}_0(\boldsymbol{\lambda})$. We obtain

$$N = \sum_{\boldsymbol{\lambda} \in \mathcal{P}^{\kappa}_{[\alpha]}} \#\mathcal{T}^{\phi}_0(\boldsymbol{\lambda}) = \sum_{\boldsymbol{\lambda} \in \mathcal{P}^{\kappa}_{[\alpha]}} \#\mathcal{T}_0(\boldsymbol{\lambda}) = \sum_{[\boldsymbol{\lambda}] \in \mathfrak{P}^{\kappa}_{[\alpha]}} \#\mathcal{T}_0[\boldsymbol{\lambda}],$$

which contradicts (4.5.32). □

Remark 4.5.33. We can also define an adapted cellularity for $\mathrm{H}^{\boldsymbol{\Lambda}}_{p,n}(q)$, similarly to Definition 4.5.28. Using Proposition 4.1.1, we can show that if p and n are not coprime and p is odd, then the basis of $\mathrm{H}^{\boldsymbol{\Lambda}}_{p,n}(q)$ that we obtain from (4.5.19) and (4.5.24) is not adapted cellular. Note that, under these conditions, there can exist an $\alpha \in Q^{\kappa}_n$ with $\#[\alpha] = p$, so that the subalgebra $\mathrm{H}^{\boldsymbol{\Lambda}}_{p,[\alpha]}(q)$ is cellular (cf. §4.5.2.4). This explains why we are dealing with $\mathrm{H}^{\boldsymbol{\Lambda}}_{p,[\alpha]}(q)$ and not only with $\mathrm{H}^{\boldsymbol{\Lambda}}_{p,n}(q)$.

4.5.3 Restriction of Specht modules

Since we have a cellular datum $(\mathcal{P}^{\kappa}_n, \mathcal{T}, c)$ for the algebra $\mathrm{H}^{\boldsymbol{\Lambda}}_n(q)$, we have a collection of cell modules $\{\mathcal{S}^{\boldsymbol{\lambda}} : \boldsymbol{\lambda} \in \mathcal{P}^{\kappa}_n\}$. In this case, the cell modules are called *Specht modules*. The algebra $\mathrm{H}^{\boldsymbol{\Lambda}}_{p,n}(q)$ is not known to be cellular in general, but Hu and Mathas [HuMa12] defined what they also called *Specht modules* for $\mathrm{H}^{\boldsymbol{\Lambda}}_{p,n}(q)$. It is a family

$$\left\{\mathcal{S}^{\boldsymbol{\lambda}}_j : j \in \{0, \dots, \frac{p}{\#[\boldsymbol{\lambda}]} - 1\}\right\},$$

of $\mathrm{H}^{\boldsymbol{\Lambda}}_{p,n}(q)$-modules, the restriction of $\mathcal{S}^{\boldsymbol{\lambda}}$ to a $\mathrm{H}^{\boldsymbol{\Lambda}}_{p,n}(q)$-module being

$$\mathcal{S}^{\boldsymbol{\lambda}}_0 \oplus \cdots \oplus \mathcal{S}^{\boldsymbol{\lambda}}_{\frac{p}{\#[\boldsymbol{\lambda}]}-1}, \tag{4.5.34}$$

for any $\boldsymbol{\lambda} \in \mathcal{P}_n^\kappa$. Moreover, for any $j, j' \in \{0, \dots, \frac{p}{\#[\boldsymbol{\lambda}]} - 1\}$, the $\mathrm{H}_{p,n}^{\boldsymbol{\Lambda}}(q)$-modules $\mathcal{S}_j^{\boldsymbol{\lambda}}$ and $\mathcal{S}_{j'}^{\boldsymbol{\lambda}}$ are isomorphic up to a twist of the action of $\mathrm{H}_{p,n}^{\boldsymbol{\Lambda}}(q)$. The purpose of the name "Specht module" is that each irreducible $\mathrm{H}_{p,n}^{\boldsymbol{\Lambda}}(q)$-module is isomorphic to the head of a $\mathcal{S}_j^{\boldsymbol{\lambda}}$.

By Proposition 4.1.1, we know that the maximal number of summands in (4.5.34) is $\gcd(p, n)$ when we restrict a Specht module of $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q)$. Our result Corollary 4.2.34 refines this result.

Proposition 4.5.35. *For any $\alpha \in Q_n^\kappa$, the maximal number of summands in (4.5.34) is $\frac{p}{\#[\alpha]}$ when we restrict a Specht module $\mathcal{S}^{\boldsymbol{\lambda}}$ with $\boldsymbol{\lambda} \in \mathcal{P}_{[\alpha]}^\kappa$, that is, when we restrict a Specht module of $\mathrm{H}_{[\alpha]}^{\boldsymbol{\Lambda}}(q)$.*

Chapter 5

Works in progress

In this short chapter, we quickly describe our works in progress.

5.1 Cellularity of the Hecke algebra of type $G(r,p,n)$

This is joint work with Jun Hu and Andrew Mathas. We began in §4.5.2 a quick study of the cellularity of the algebra $\mathrm{H}^{\boldsymbol{\Lambda}}_{p,[\alpha]}(q) = \mathrm{H}^{\boldsymbol{\Lambda}}_{[\alpha]}(q)^{\sigma}$. However, except for an easy case (see §4.5.2), we could not find a basis with a "cellular" shape (this is precisely our "adapted cellularity" from §4.5.2.5).

The main reason is that σ behaves very badly in general towards the graded cellular basis $\{c^{\boldsymbol{\lambda}}_{\mathfrak{st}}\}$ of Hu and Mathas [HuMa10]. Note that, in the semisimple case, we can prove that the map σ just permutes the elements of the latter cellular basis. We would like this situation to always happen.

To that extent, it seems that one of the graded cellular basis introduced by Webster [We] and Bowman [Bow], constructed from the *diagrammatic Cherednik algebra*, behaves well with respect to σ. Note that the underlying poset order is different from the one used in [HuMa10]. As in the semisimple case, the map σ just permutes the basis elements. We obtain a basis of $\mathrm{H}^{\boldsymbol{\Lambda}}_{p,[\alpha]}(q)$ of the form $\{c^{\lambda}_{\mathfrak{st}} : \lambda \in \Lambda, \mathfrak{s}, t \in \mathcal{T}(\lambda)\}$ that almost satisfies the cellularity axioms: the condition $(c^{\lambda}_{\mathfrak{st}})^* = c^{\lambda}_{\mathfrak{ts}}$ has to be changed. We thus define a slightly more general notion than the cellularity, with similar applications to representation theory.

5.2 A disjoint quiver isomorphism for cyclotomic quiver Hecke algebras of type B

This is joint work with Loïc Poulain d'Andecy and Ruari Walker. In [PAWal], Poulain d'Andecy and Walker proved an analogue of Theorem 2.3.16 for cyclotomic quotients of affine Hecke algebras of type B and cyclotomic quiver Hecke algebras of type B, the latter being a generalisation of a family of algebras introduced by Varagnolo and Vasserot [VaVa]. The aim is now to give the analogue of Theorem 1.3.57 concering "disjoint quiver" Hecke algebras for these cyclotomic quiver Hecke algebras of type B. As with Theorem 2.3.19, this would yield a Morita equivalence result, a new one this time, for cyclotomic quotients of affine Hecke algebras of type B.

Annexe A

Version abrégée

Introduction

Généralisant les groupes de réflexions réelles, aussi appelés groupes de Coxeter finis, les groupes de réflexions complexes sont des groupes finis engendrés par des réflexions complexes, c'est-à-dire, des endomorphismes de $\mathbb{C}^n$ d'ordre fini, différents de l'identité et possédant un hyperplan de points fixes. Comme pour les groupes de réflexions réelles, les groupes de réflexions complexes sont entièrement classifiés ([ShTo]). Cette classification consiste en une série infinie $\{G(r,p,n)\}$ où r,p,n sont des entiers strictement positifs avec $r = dp$ pour un $d \in \mathbb{N}^*$, série à laquelle s'ajoutent 34 exceptions.

Avec pour but de généraliser la construction des algèbres d'Iwahori–Hecke, Broué–Malle [BrMa] et Broué, Malle et Rouquier [BMR] ont défini ces déformations pour tous les groupes de réflexions complexes, connues sous le nom d'*algèbres de Hecke*. De telles déformations $\mathrm{H}_n(q,\boldsymbol{u})$ ont également été construites par Ariki et Koike [ArKo] pour le cas particulier $G(r,1,n)$, plus connues sous le nom d'*algèbres d'Ariki–Koike*, où q et $\boldsymbol{u} = (u_1,\dots,u_r)$ sont des paramètres, suivi d'Ariki [Ar95] qui a fait la même chose pour $G(r,p,n)$. En particulier, pour un choix adéquat de paramètre $\boldsymbol{u}$ et de poids $\boldsymbol{\Lambda}$ de niveau r (c'est-à-dire, une suite finie d'entiers positifs de somme r), cette algèbre de Hecke $\mathrm{H}^{\boldsymbol{\Lambda}}_{p,n}(q)$ de $G(r,p,n)$ peut être vue comme une sous-algèbre de $\mathrm{H}^{\boldsymbol{\Lambda}}_n(q) \coloneqq \mathrm{H}_n(q,\boldsymbol{u})$.

Également dans le but d'étudier les groupes de Chevalley finis, Yokonuma [Yo] a introduit les algèbres de Yokonuma–Hecke. Elles sont définies comme algèbre du centralisateur d'une représentation induite de la représentation triviale sur un sous-groupe *unipotent* maximal, contrairement aux algèbres d'Iwahori–Hecke. De façon similaire aux algèbres d'Ariki–Koike, les algèbres de Yokonuma–Hecke de type A peuvent être vues comme des déformations de l'algèbre de groupe de $G(r,1,n)$. Cependant, cette fois la structure de produit en couronne $G(r,1,n) \simeq (\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n$ apparait dans la définition par générateurs et relations.

Intéressons-nous maintenant à la théorie des représentations des algèbres introduites ci-avant. Tout d'abord, rappelons quelques faits de théorie des représentations du groupe symétrique $\mathfrak{S}_n$ sur n lettres. Nous savons depuis Frobenius que les représentations irréductibles $\{\mathcal{D}^\lambda\}_\lambda$ de $\mathfrak{S}_n$ sur un corps de caractéristique 0 sont paramétrées par les *partitions* de n, c'est-à-dire, par les suites finies $\lambda = (\lambda_0 \geq \cdots \geq \lambda_{h-1} > 0)$ d'entiers strictement positifs de somme $|\lambda| \coloneqq \lambda_0 + \cdots + \lambda_{h-1} = n$. Lorsque le corps de base est de caractéristique un nombre premier p, les représentations irréductibles $\{\mathcal{D}^\lambda\}_\lambda$ sont maintenant indexées par les partitions *p-régulières* de n, c'est-à-dire, les partitions de n avec aucune composante répétée p fois ou plus. Cependant, dans ce cas certaines représentations peuvent ne pas s'écrire comme somme directe de représentations irréductibles. Ainsi, il est également intéressant d'étudier les *blocs* de l'algèbre du groupe, c'est-à-dire, les idéaux bilatères indécomposables. Les blocs partitionnent à la fois l'ensemble des représentations irréductibles et celui des représentations indécomposables. Brauer et Robinson ont montré que ces blocs sont paramétrés par les *p-cœurs* des partitions de n, prouvant ainsi la « conjecture de Nakayama ». Nous renvoyons à [JamKe] pour de plus amples détails sur la théorie des représentations du groupe symétrique.

Si $\mathbf{\Lambda}$ est un poids de niveau $r = 1$, la théorie des représentations de $\mathrm{H}_n^{\mathbf{\Lambda}}(q)$ est similaire à celle du groupe symétrique : si $\mathrm{H}_n^{\mathbf{\Lambda}}(q)$ est semi-simple alors ses modules irréductibles $\{\mathcal{D}^\lambda\}_\lambda$ sont paramétrés par les partitions de n. Dans le cas modulaire, les modules irréductibles (respectivement les blocs) sont paramétrés par les partitions de n qui sont e-régulières (resp. par les e-cœurs des partitions de n), où $e \in \mathbb{N}$ est le plus petit entier positif tel que $1+q+\cdots+q^{e-1} = 0$. Considérons maintenant un poids $\mathbf{\Lambda}$ de niveau arbitraire. Dans le cas semi-simple, Ariki et Koike ont déterminé tous les modules irréductibles de $\mathrm{H}_n^{\mathbf{\Lambda}}(q)$. Ils sont paramétrés par les r-*partitions* de n, c'est-à-dire, les r-uplets $\boldsymbol{\lambda} = (\lambda^{(0)}, \ldots, \lambda^{(r-1)})$ de partitions avec $|\boldsymbol{\lambda}| \coloneqq |\lambda^{(0)}| + \cdots + |\lambda^{(r-1)}| = n$. Le cas modulaire a été traité par Ariki et Mathas [ArMa, Ar01], et également par Graham et Lehrer [GrLe] ainsi que Dipper, James et Mathas [DJM], grâce à la théorie des algèbres cellulaires. Cette théorie produit une collection de *modules cellulaires*, également appelés dans ce cas *modules de Specht*. Ces modules permettent de construire une famille complète de $\mathrm{H}_n^{\mathbf{\Lambda}}(q)$-modules irréductibles $\{\mathcal{D}^{\boldsymbol{\lambda}}\}_{\boldsymbol{\lambda}}$. Cette famille peut être indexée par une généralisation non triviale des partitions e-régulières : on parle de r-partitions de *Kleshchev* (voir [ArMa, Ar01]). De même, la généralisation naturelle des e-cœurs aux r-partitions, les e-multicœurs, ne paramètre pas en général les blocs de $\mathrm{H}_n^{\mathbf{\Lambda}}(q)$. Lyle et Mathas [LyMa] ont en fait prouvé que les blocs de $\mathrm{H}_n^{\mathbf{\Lambda}}(q)$ sont paramétrés par les multi-ensembles de κ-résidus modulo e des r-partitions de n, où $\kappa \in (\mathbb{Z}/e\mathbb{Z})^r$ est une multi-charge associée à $\mathbf{\Lambda}$. Finalement, une avancée majeure dans la théorie des représentations de $\mathrm{H}_n^{\mathbf{\Lambda}}(q)$ fut un théorème d'Ariki [Ar96], prouvant ainsi une conjecture de Lascoux, Leclerc et Thibon [LLT]. En caractéristique 0, ce théorème a la conséquence suivante : il est équivalent de déterminer la matrice de décomposition de $\mathrm{H}_n^{\mathbf{\Lambda}}(q)$ ou de déterminer la base canonique d'un certain $\widehat{\mathfrak{sl}}_e$-module $L(\mathbf{\Lambda})$ de plus haut poids, où $\widehat{\mathfrak{sl}}_e$ désigne l'algèbre de Kac–Moody de type $A_{e-1}^{(1)}$. Avec les travaux de Lascoux, Leclerc et Thibon [LLT] et Jacon [Jac05] permettant de calculer cette base canonique, nous pouvons donc déterminer explicitement la matrice de décomposition de $\mathrm{H}_n^{\mathbf{\Lambda}}(q)$ (voir aussi Uglov [Ug]).

Dans le cas semi-simple, Ariki [Ar95] a utilisé la théorie de Clifford pour déterminer tous les modules irréductibles pour $\mathrm{H}_{p,n}^{\mathbf{\Lambda}}(q)$. Dans le cas modulaire, Genet et Jacon [GeJac] et Chlouveraki et Jacon [ChJac] ont donné une paramétrisation des modules simples de $\mathrm{H}_{p,n}^{\mathbf{\Lambda}}(q)$ sur $\mathbb{C}$, et Hu [Hu04, Hu07] les a classifiés sur un corps contenant une racine primitive p-ième de l'unité. De surcroît, Hu et Mathas [HuMa09, HuMa12] ont donné une procédure pour calculer la matrice de décomposition de $\mathrm{H}_{p,n}^{\mathbf{\Lambda}}(q)$ en caractéristique 0, sous une hypothèse de séparation où l'algèbre de Hecke n'est pas semi-simple en général. Mentionnons également les travaux de Geck [Ge00] en type D, qui correspond au cas $r = p = 2$. Toutes ces études de la théorie des représentations de $\mathrm{H}_{p,n}^{\mathbf{\Lambda}}(q)$ font intervenir l'application de *décalage* sur les r-partitions, définie par

$$^{\sigma}\boldsymbol{\lambda} \coloneqq (\lambda^{(r-d)}, \ldots, \lambda^{(r-1)}, \lambda^{(0)}, \ldots, \lambda^{(r-d-1)}),$$

pour toute r-partition $\boldsymbol{\lambda} = (\lambda^{(0)}, \ldots, \lambda^{(r-1)})$, où $r = dp$. Si $\boldsymbol{\lambda}$ est une r-partition de Kleshchev de n, la restriction du $\mathrm{H}_n^{\mathbf{\Lambda}}(q)$-module irréductible $\mathcal{D}^{\boldsymbol{\lambda}}$ en un $\mathrm{H}_{p,n}^{\mathbf{\Lambda}}(q)$-module est isomorphe à une somme de modules irréductibles, dont le nombre dépend de la cardinalité de l'orbite $[\boldsymbol{\lambda}]$ de $\boldsymbol{\lambda}$ sous l'action de σ.

En ce qui concerne les algèbres de Yokonuma–Hecke, leur présentation naturelle en type A a été transformée depuis les travaux originaux de Yokonuma (voir [Ju98, Ju04, JuKa, ChPA14, ChPou]). La théorie des représentations des algèbres de Yokonuma–Hecke a d'abord été étudiée par Thiem [Th04, Th05, Th07], et [ChPA14, ChPA15] ont adopté une approche combinatoire pour le type A. Dans le papier pré-cité [ChPA15], Chlouveraki et Poulain d'Andecy ont introduit des généralisations de ces algèbres : les algèbres de Yokonuma–Hecke affine et leurs quotients cyclotomiques, qui généralisent les algèbres de Hecke affine de type A et les algèbres d'Ariki–Koike respectivement. L'intérêt porté sur les algèbres des Yokonuma–Hecke a récemment grandi : dans [CJKL] (voir également [PAWag]), les auteurs ont défini un invariant d'entrelacs à partir des algèbres de Yokonuma–Hecke, invariant qui est plus fort que ceux connus grâce aux algèbres de Iwahori–Hecke classiques en type A, comme le polynôme HOMFLYPT, et aux algèbres d'Ariki–Koike. Finalement, mentionnons les travaux Juyamaya [Ju99] sur l'algèbre des *tresses et liens*, une sous-algèbre particulière de l'algèbre de Yokonuma–Hecke. Cette construction a été généralisée par Marin [Mar] dans le cas des groupes de réflexions complexes.

Un nouvel aspect de la théorie des représentations des algèbres d'Ariki–Koike est apparu à la fin des années 2000. En partie motivés par le théorème d'Ariki, Khovanov et Lauda [KhLau09, KhLau11] et Rouquier [Rou] ont indépendamment introduit l'algèbre $\mathrm{R}_n(\Gamma)$, connue sous le nom d'*algèbre de Hecke carquois* ou *algèbre KLR*. Ils ont établi un résultat de catégorification,

$$\mathcal{U}_v^-(\mathfrak{g}_\Gamma) \simeq \bigoplus_{n\geq 0} [\mathrm{Proj}(\mathrm{R}_n(\Gamma))],$$

où $\mathcal{U}_v^-(\mathfrak{g}_\Gamma)$ est la partie négative du groupe quantique de $\mathfrak{g}_\Gamma$, l'algèbre de Kac–Moody associée au carquois Γ, et $[\mathrm{Proj}(\mathrm{R}_n(\Gamma))]$ désigne le groupe de Grothendieck de la catégorie additive des $\mathrm{R}_n(\Gamma)$-modules projectifs gradués finiment engendrés. De plus, en considérant des quotients cyclotomiques $\mathrm{R}_n^{\mathbf{\Lambda}}(\Gamma)$ des algèbres de Hecke carquois, Kang et Kashiwara [KanKa] ont montré un résultat de catégorification pour les $\mathcal{U}(\mathfrak{g})$-modules de plus haut poids, comme conjecturé dans [KhLau09]. Plus précisément, pour chaque poids dominant $\mathbf{\Lambda}$ l'algèbre $\mathrm{R}_n(\Gamma)$ a un quotient cyclotomique $\mathrm{R}_n^{\mathbf{\Lambda}}(\Gamma)$ qui catégorifie le module de plus haut poids correspondant $L(\mathbf{\Lambda})$.

Si Γ est un carquois de type $A_{e-1}^{(1)}$, les résultats précédents mettent donc en évidence une connexion entre l'algèbre d'Ariki–Koike $\mathrm{H}_n^{\mathbf{\Lambda}}(q)$ et $\mathrm{R}_n^{\mathbf{\Lambda}}(\Gamma)$. Un pas important dans la compréhension de cette connexion, et donc des algèbres de Hecke carquois cyclotomiques, a été fait par Brundan et Kleshchev [BrKl-a] et indépendamment par Rouquier [Rou]. Les deux premiers auteurs ont montré que les algèbres d'Ariki–Koike sont des cas particuliers d'algèbres de Hecke carquois cyclotomiques, donnant toute une famille d'isomorphismes explicites. Rouquier a lui également donné une version affine de cet isomorphisme. Brundan et Kleshchev ont remarqué que l'algèbre d'Ariki–Koike hérite de la $\mathbb{Z}$-graduation naturelle sur l'algèbre de Hecke carquois cyclotomique. Cette graduation permet alors d'étudier la théorie des représentations *graduées* des algèbres d'Ariki–Koike. Ils ont également montré une version graduée du théorème de catégorification d'Ariki. Par ailleurs, inspirés par les travaux de Brundan et Kleshchev, Hu et Mathas [HuMa10] ont construit une base cellulaire graduée de $\mathrm{H}_n^{\mathbf{\Lambda}}(q)$. Ce fut le premier exemple de base homogène de $\mathrm{H}_n^{\mathbf{\Lambda}}(q)$.

Dans cette thèse, notre but est de généraliser certains des résultats précédents. Les travaux présentés sont une compilation des articles [Ro16, Ro17-a, Ro17-b]. Tout d'abord, dans la Section A.1 nous montrons quelques résultats sur les algèbres de Hecke carquois cyclotomiques. Plus précisément, nous étudions les algèbres de Hecke carquois cyclotomiques où le carquois n'est pas connexe (Théorème A.1.2.6), ainsi que les sous-algèbres des points fixes pour des automorphismes construits à partir d'automorphismes de carquois d'ordre fini (Théorèmes A.1.3.5 et A.1.3.15). Dans la Section A.2, nous donnons une présentation de « type » Hecke carquois cyclotomique pour $\mathrm{H}_n^{\mathbf{\Lambda}}(q)$ (Corollaire A.2.3.2). En particulier, cette algèbre est une sous-algèbre graduée de $\mathrm{H}_n^{\mathbf{\Lambda}}(q)$. Nous retrouvons également un résultat important d'équivalence de Morita entre algèbres d'Ariki–Koike, dont la démonstration diffère de l'originale [DiMa] (Théorème A.2.2.4). Dans la Section A.3, nous montrons que les algèbres de Yokonuma–Hecke cyclotomiques sont un cas particulier d'algèbres de Hecke carquois cyclotomiques. Le carquois est le même que dans le cas Ariki–Koike, c'est-à-dire, donné par une union disjointe de carquois cycliques (Théorème A.3.2.3). Finalement, la Section A.4 est en très grande partie indépendante du reste et traite d'un problème purement combinatoire. Nous y étudions le lien entre le nombre d'éléments de l'orbite des multi-partitions ainsi que celui de leurs multi-ensembles de résidus par rapport à l'action de décalage. Le résultat principal est donné avec le Théorème A.4.1.8. Nous appliquons ensuite ces résultats à la théorie des représentations de $\mathrm{H}^{\mathbf{\Lambda}}_{p,n}(q)$. Finalement, la très courte Section A.5 décrit nos travaux en cours.

Notations

Soit $\mathcal{K}$ un ensemble et $n \in \mathbb{N}^*$. Une *$\mathcal{K}$-composition de n* est une famille à support fini d'entiers naturels indexée par $\mathcal{K}$, de somme n. Nous écrivons $\alpha \models_{\mathcal{K}} n$ si $\alpha = (\alpha_k) \in \mathbb{N}^{(\mathcal{K})}$ est une $\mathcal{K}$-composition de n. Pour tout $d \in \mathbb{N}^*$, nous écrivons $\models_d$ au lieu de $\models_{\{1,\dots,d\}}$. Un *poids* est une famille à support fini $\mathbf{\Lambda} = (\Lambda_k) \in \mathbb{N}^{(\mathcal{K})}$ d'entiers naturels indexées par $\mathcal{K}$. La *longueur* d'un poids $\mathbf{\Lambda} = (\Lambda_k)_{k\in\mathcal{K}} \in \mathbb{N}^{(\mathcal{K})}$ est $\ell(\mathbf{\Lambda}) \coloneqq \sum_{k\in\mathcal{K}} \Lambda_k$.

Soit $\alpha \models_{\mathcal{K}} n$. Nous désignons par $\mathcal{K}^\alpha$ le sous-ensemble de $\mathcal{K}^n$ constitué des éléments $\boldsymbol{k} = (k_1, \dots, k_n) \in \mathcal{K}^n$ tels que pour tout $k \in \mathcal{K}$, il y a exactement α_k entiers $a \in \{1, \dots, n\}$ tels que $k_a = k$. Remarquons que chaque ensemble $\mathcal{K}^\alpha$ est fini.

Nous désignons par F un corps et nous considérons $q \in F^\times$. Mise à part la Section 3.6, nous avons toujours $q \neq 1$. Nous considérons l'élément $e \in \mathbb{N}^* \cup \{\infty\}$ minimal tel que $1+q+\cdots+q^{e-1} = 0$. Si $q \neq 1$ et $e \neq \infty$ alors q est une racine primitive e-ième de l'unité. Nous utiliserons intensivement l'ensemble suivant :

$$I \coloneqq \begin{cases} \mathbb{Z}/e\mathbb{Z}, & \text{si } e \neq \infty, \\ \mathbb{Z}, & \text{sinon.} \end{cases}$$

A.1 Algèbres de Hecke carquois

Cette section est une adaptation de [Ro16, Ro17-a]. Étant donné un carquois Γ sans boucle, nous définissons en §A.1.1 l'algèbre de Hecke carquois $\mathrm{R}_n(\Gamma)$ et ses quotients cyclotomiques $\mathrm{R}^{\mathbf{\Lambda}}_n(\Gamma)$. En §A.1.2, nous donnons un théorème de décomposition suivant les composantes connexes de Γ (Théorème A.1.2.6). En §A.1.3 nous étudions la sous-algèbre des points fixes pour un automorphisme construit à partir d'un automorphisme de carquois (voir les Théorèmes A.1.3.5 et A.1.3.15).

A.1.1 Définitions

Soient A un anneau commutatif et u, v deux indéterminées sur A. Soit Γ un carquois sans boucle, d'ensemble de sommets K (non nécessairement fini). Pour chaque $k \neq k' \in K$, notons $d_{k,k'}$ le nombre de flèches de k vers k'. La *matrice de Cartan* de Γ est la matrice $C = (c_{k,k'})_{k,k'\in K}$ définie par

$$c_{k,k'} \coloneqq \begin{cases} 2, & \text{si } k = k', \\ -d_{k,k'} - d_{k',k}, & \text{sinon,} \end{cases}$$

pour tout $k, k' \in K$. Suivant [Rou, §3.2.4], au carquois Γ nous associons la famille de polynômes bivariés $(Q_{k,k'})_{k,k'\in K}$ définie par $Q_{k,k} \coloneqq 0$ et

$$Q_{k,k'}(u, v) \coloneqq (-1)^{d_{k,k'}} (u - v)^{-c_{k,k'}},$$

pour tout $k \neq k' \in K$. Notons que $Q_{k,k'}(u,v) = Q_{k',k}(v,u)$ pour tout $k, k' \in K$.

Soit maintenant $\alpha \models_K n$. L'*algèbre de Hecke carquois* $\mathrm{R}_\alpha(\Gamma)$ est la A-algèbre associative unitaire de partie génératrice

$$\{e(\boldsymbol{k})\}_{\boldsymbol{k} \in K^\alpha} \cup \{y_1, \dots, y_n\} \cup \{\psi_1, \dots, \psi_{n-1}\}, \tag{A.1.1.1}$$

soumise aux relations

$$\sum_{\boldsymbol{k} \in K^\alpha} e(\boldsymbol{k}) = 1, \tag{A.1.1.2a}$$

$$e(\boldsymbol{k})e(\boldsymbol{k'}) = \delta_{\boldsymbol{k},\boldsymbol{k'}} e(\boldsymbol{k}), \tag{A.1.1.2b}$$

$$y_a e(\boldsymbol{k}) = e(\boldsymbol{k}) y_a, \tag{A.1.1.2c}$$

$$\psi_b e(\boldsymbol{k}) = e(s_b \cdot \boldsymbol{k}) \psi_b, \tag{A.1.1.2d}$$

$$y_a y_{a'} = y_{a'} y_a, \tag{A.1.1.2e}$$

$$\psi_b y_a = y_a \psi_b, \qquad \text{si } a \neq b, b+1, \tag{A.1.1.2f}$$

$$\psi_b \psi_{b'} = \psi_{b'} \psi_b, \qquad \text{si } |b - b'| > 1, \tag{A.1.1.2g}$$

$$\psi_b y_{b+1} e(\boldsymbol{k}) = \begin{cases} (y_b \psi_b + 1) e(\boldsymbol{k}), & \text{si } k_b = k_{b+1}, \\ y_b \psi_b e(\boldsymbol{k}), & \text{si } k_b \neq k_{b+1}, \end{cases} \tag{A.1.1.2h}$$

$$y_{b+1} \psi_b e(\boldsymbol{k}) = \begin{cases} (\psi_b y_b + 1) e(\boldsymbol{k}), & \text{si } k_b = k_{b+1}, \\ \psi_b y_b e(\boldsymbol{k}), & \text{si } k_b \neq k_{b+1}, \end{cases} \tag{A.1.1.2i}$$

ainsi que

$$\psi_b^2 e(\boldsymbol{k}) = Q_{k_b, k_{b+1}}(y_b, y_{b+1}) e(\boldsymbol{k}), \tag{A.1.1.3a}$$

$$\psi_{c+1}\psi_c\psi_{c+1} e(\boldsymbol{k}) = \begin{cases} \psi_c\psi_{c+1}\psi_c e(\boldsymbol{k}) + \frac{Q_{k_c,k_{c+1}}(y_c,y_{c+1}) - Q_{k_{c+2},k_{c+1}}(y_{c+2},y_{c+1})}{y_c - y_{c+2}} e(\boldsymbol{k}), & \text{si } k_c = k_{c+2}, \\ \psi_c\psi_{c+1}\psi_c e(\boldsymbol{k}), & \text{sinon,} \end{cases} \tag{A.1.1.3b}$$

pour tout $\boldsymbol{k} \in K^\alpha$, $a, a' \in \{1, \dots, n\}$, $b, b' \in \{1, \dots, n-1\}$ et $c \in \{1, \dots, n-2\}$, où s_b est la transposition $(b, b+1) \in \mathfrak{S}_n$. Nous définissons également

$$\mathrm{R}_n(\Gamma) \coloneqq \bigoplus_{\alpha \models_K n} \mathrm{R}_\alpha(\Gamma).$$

Remarquons que pour chaque $\alpha \models_K n$, l'idempotent central

$$e(\alpha) \coloneqq \sum_{\boldsymbol{k} \in K^\alpha} e(\boldsymbol{k}) \in \mathrm{R}_n(\Gamma), \tag{A.1.1.4}$$

vérifie

$$e(\alpha)\mathrm{R}_n(\Gamma) \simeq \mathrm{R}_\alpha(\Gamma).$$

Nous serons particulièrement intéressés par les carquois Γ ne possédant que des arêtes orientées simples. Dans ce cas, dans $\mathrm{R}_\alpha(\Gamma)$ les relations (A.1.1.3) deviennent

$$\psi_b^2 e(\boldsymbol{k}) = \begin{cases} 0, & \text{si } k_b = k_{b+1}, \\ e(\boldsymbol{k}), & \text{si } k_b \not\to k_{b+1}, \\ (y_{b+1} - y_b)e(\boldsymbol{k}), & \text{si } k_b \to k_{b+1}, \\ (y_b - y_{b+1})e(\boldsymbol{k}), & \text{si } k_b \leftarrow k_{b+1}, \\ (y_{b+1} - y_b)(y_b - y_{b+1})e(\boldsymbol{k}), & \text{si } k_b \leftrightarrows k_{b+1}, \end{cases} \tag{A.1.1.5a}$$

$$\psi_{c+1}\psi_c\psi_{c+1} e(\boldsymbol{k}) = \begin{cases} (\psi_c\psi_{c+1}\psi_c - 1)e(\boldsymbol{k}), & \text{si } k_{c+2} = k_c \to k_{c+1}, \\ (\psi_c\psi_{c+1}\psi_c + 1)e(\boldsymbol{k}), & \text{si } k_{c+2} = k_c \leftarrow k_{c+1}, \\ (\psi_c\psi_{c+1}\psi_c + 2y_{c+1} - y_c - y_{c+2})e(\boldsymbol{k}), & \text{si } k_{c+2} = k_c \leftrightarrows k_{c+1}, \\ \psi_c\psi_{c+1}\psi_c e(\boldsymbol{k}), & \text{sinon}, \end{cases} \tag{A.1.1.5b}$$

pour tout $\boldsymbol{k} \in K^\alpha$, $b \in \{1, \dots, n-1\}$ et $c \in \{1, \dots, n-2\}$, où :
- nous écrivons $k \not\to k'$ quand $k \neq k'$ et ni (k, k') ni (k', k) ne sont des arêtes de Γ ;
- nous écrivons $k \to k'$ quand (k, k') est une arête de Γ et pas (k', k) ;
- nous écrivons $k \leftarrow k'$ quand (k', k) est une arête de Γ et pas (k, k') ;
- nous écrivons $k \leftrightarrows k'$ quand (k, k') et (k', k) sont des arêtes de Γ.

Donnons maintenant une propriété remarquable des algèbres de Hecke carquois. La preuve consiste seulement en une vérification de chaque relation.

Proposition A.1.1.6. *L'algèbre de Hecke carquois $\mathrm{R}_\alpha(\Gamma)$ est $\mathbb{Z}$-graduée via*

$$\begin{aligned} \deg e(\boldsymbol{k}) &= 0, & & \\ \deg y_a e(\boldsymbol{k}) &= 2, & & \textit{pour tout } a \in \{1, \dots, n\}, \\ \deg \psi_a e(\boldsymbol{k}) &= -c_{k_a, k_{a+1}}, & & \textit{pour tout } a \in \{1, \dots, n-1\}, \end{aligned}$$

pour tout $\boldsymbol{k} \in K^\alpha$.

Soit maintenant $\boldsymbol{\Lambda} = (\Lambda_k)_{k \in K} \in \mathbb{N}^{(K)}$ un poids et définissons un cas particulier de quotient cyclotomique de $\mathrm{R}_\alpha(\Gamma)$.

Définition A.1.1.7. L'algèbre de Hecke carquois cyclotomique $\mathrm{R}_\alpha^{\boldsymbol{\Lambda}}(\Gamma)$ est le quotient de l'algèbre de Hecke $\mathrm{R}_\alpha(\Gamma)$ par l'idéal bilatère $\mathcal{I}_\alpha^{\boldsymbol{\Lambda}}$ engendré par les relations

$$y_1^{\Lambda_{k_1}} e(\boldsymbol{k}) = 0, \tag{A.1.1.8}$$

pour tout $\boldsymbol{k} \in K^\alpha$.

La graduation sur $\mathrm{R}_\alpha(\Gamma)$ donne une graduation sur $\mathrm{R}_\alpha^{\boldsymbol{\Lambda}}(\Gamma)$. Comme dans le cas non cyclotomique, nous pouvons définir

$$\mathrm{R}_n^{\boldsymbol{\Lambda}}(\Gamma) \coloneqq \bigoplus_{\alpha \models_K n} \mathrm{R}_\alpha^{\boldsymbol{\Lambda}}(\Gamma),$$

et on a $e(\alpha)\mathrm{R}_n^{\boldsymbol{\Lambda}}(\Gamma) \simeq \mathrm{R}_\alpha^{\boldsymbol{\Lambda}}(\Gamma)$.

Lemme A.1.1.9 ([BrKl-a]). *Pour tout $a \in \{1, \dots, n\}$, les éléments $y_a \in \mathrm{R}_\alpha^{\boldsymbol{\Lambda}}(\Gamma)$ sont nilpotents.*

A.1.2 Décomposition dans le cas de carquois disjoints

Soit Γ un carquois fini. Écrivons Γ comme l'union disjointe de d sous-carquois propres $\Gamma^1, \dots, \Gamma^d$. Notre but est de donner un isomorphisme entre $\mathrm{R}_n^{\boldsymbol{\Lambda}}(\Gamma)$ et des quotients cyclotomiques des algèbres $\mathrm{R}_{n'}(\Gamma^j)$.

A.1.2.1 Cadre

Définissions $J = \mathbb{Z}/d\mathbb{Z} \simeq \{1, \dots, d\}$. Considérons une partition de K en d parts $K = \sqcup_{j\in J} K_j$. Soit $\lambda \vDash_d n$.

Définition A.1.2.1. Soit $\boldsymbol{k} \in K^n$ et $\mathsf{t} \in J^n$.

— Nous disons que $\boldsymbol{k}$ est un *étiquetage* de t quand

$$k_a \in K_{\mathsf{t}_a},$$

pour tout $a \in \{1, \dots, n\}$. Nous désignons par K^{t} l'ensemble des éléments de K^n qui sont des étiquetages de t.

— Nous disons que t est de *forme* $\lambda \vDash_d n$ et nous écrivons $[\mathsf{t}] = \lambda$ si pour tout $j \in J$ il y a exactement λ_j composantes de t égales à j, c'est-à-dire,

$$\#\{a \in \{1, \dots, n\} : \mathsf{t}_a = j\} = \lambda_j,$$

pour tout $j \in J$. Nous désignons par J^λ l'ensemble des éléments de J^n de forme λ.

L'ensemble J^λ est de cardinalité

$$m_\lambda \coloneqq \frac{n!}{\lambda_1! \dots \lambda_d!}. \tag{A.1.2.2}$$

Désignons par $\mathsf{t}^\lambda \in J^\lambda$ l'élément canonique de forme λ, donné par

$$\mathsf{t}^\lambda \coloneqq (1, \dots, 1, \dots, d, \dots, d),$$

où chaque $j \in J$ apparait exactement λ_j fois. Finalement, pour chaque $\mathsf{t} \in J^\lambda$ nous définissions l'idempotent suivant :

$$e(\mathsf{t}) \coloneqq \sum_{\boldsymbol{k} \in K^{\mathsf{t}}} e(\boldsymbol{k}) \in \mathrm{R}_n(\Gamma),$$

et nous écrivons $e_\lambda \coloneqq e(\mathsf{t}^\lambda)$.

Proposition A.1.2.3. *Soit $\mathsf{t} \in J^\lambda$. Il y a un unique élément $\pi_{\mathsf{t}} \in \mathfrak{S}_n$ de longueur minimale vérifiant*

$$\pi_{\mathsf{t}} \cdot \mathsf{t} = \mathsf{t}^\lambda.$$

Proposition A.1.2.4. *Soit $\mathsf{t} \in J^\lambda$. Les égalités suivantes sont vérifiées :*

$$\psi_{\pi_{\mathsf{t}}^{-1}} \psi_{\pi_{\mathsf{t}}} e(\mathsf{t}) = e(\mathsf{t}),$$
$$\psi_{\pi_{\mathsf{t}}} \psi_{\pi_{\mathsf{t}}^{-1}} e_\lambda = e_\lambda.$$

A.1.2.2 Isomorphisme de décomposition

Soit $\lambda \vDash_d n$ une d-composition de n. Définissons l'algèbre suivante :

$$\mathrm{R}_\lambda(\Gamma) \coloneqq \mathrm{R}_{\lambda_1}(\Gamma^1) \otimes \cdots \otimes \mathrm{R}_{\lambda_d}(\Gamma^d).$$

Théorème A.1.2.5. *On peut identifier $\mathrm{R}_\lambda(\Gamma)$ à la sous-algèbre $e_\lambda \mathrm{R}_n(\Gamma) e_\lambda$ (d'unité e_λ) de $\mathrm{R}_n(\Gamma)$.*

Théorème A.1.2.6. *Nous avons l'isomorphisme de A-algèbres suivant :*

$$\mathrm{R}_n(\Gamma) \simeq \bigoplus_{\lambda \vDash_d n} \mathrm{Mat}_{m_\lambda} \mathrm{R}_\lambda(\Gamma).$$

Indexons les lignes et les colonnes des éléments de $\mathrm{Mat}_{m_\lambda}\mathrm{R}_\lambda(\Gamma)$ par $(\mathfrak{t}', \mathfrak{t}) \in (J^\lambda)^2$ et désignons par $E_{\mathfrak{t}',\mathfrak{t}}$ la matrice avec un 1 en position $(\mathfrak{t}', \mathfrak{t})$ et des 0 partout ailleurs. La clé de la démonstration du Théorème A.1.2.6 est l'isomorphisme de A-modules suivant :

$$e(\mathfrak{t}')\mathrm{R}_n(\Gamma)e(\mathfrak{t}) \simeq \mathrm{R}_\lambda(\Gamma)E_{\mathfrak{t}',\mathfrak{t}},$$

où $\mathfrak{t}, \mathfrak{t}' \in J^\lambda$. Pour cela, nous utilisons

$$\Phi_{\mathfrak{t}',\mathfrak{t}} : \mathrm{R}_\lambda(\Gamma)E_{\mathfrak{t}',\mathfrak{t}} \to e(\mathfrak{t}')\mathrm{R}_n(\Gamma)e(\mathfrak{t}),$$
$$\Psi_{\mathfrak{t}',\mathfrak{t}} : e(\mathfrak{t}')\mathrm{R}_n(\Gamma)e(\mathfrak{t}) \to \mathrm{R}_\lambda(\Gamma)E_{\mathfrak{t}',\mathfrak{t}},$$

définis par :

$$\Phi_{\mathfrak{t}',\mathfrak{t}}(vE_{\mathfrak{t}',\mathfrak{t}}) := \psi_{\pi_{\mathfrak{t}'}^{-1}} v \psi_{\pi_{\mathfrak{t}}}, \qquad \text{pour tout } v \in \mathrm{R}_\lambda(\Gamma),$$
$$\Psi_{\mathfrak{t}',\mathfrak{t}}(w) := (\psi_{\pi_{\mathfrak{t}'}} w \psi_{\pi_{\mathfrak{t}}^{-1}})E_{\mathfrak{t}',\mathfrak{t}}, \qquad \text{pour tout } w \in e(\mathfrak{t}')\mathrm{R}_n(\Gamma)e(\mathfrak{t}).$$

Remarque A.1.2.7. Notre application $\Phi_{\mathfrak{t}',\mathfrak{t}}$ est similaire à celle définie en [SVV, (17)].

Nous pouvons aisément donner une version cyclotomique du Théorème A.1.2.6. Le résultat obtenu peut alors être retrouvé en utilisant le résultat plus général [SVV, Theorem 3.15].

A.1.3 Sous-algèbre des points fixes

Soit Γ un carquois d'ensemble de sommets K. Soit σ une permutation de K d'ordre fini $p \in \mathbb{N}^*$. Supposons que σ est un automorphisme du carquois Γ, c'est-à-dire, pour tout $k, k' \in K$ avec $k \neq k'$ il y a autant de flèches de k vers k' que de $\sigma(k)$ vers $\sigma(k')$. Si $Q = (Q_{k,k'})_{k,k' \in K}$ est la famille de polynômes bivariés à coefficients dans A associés à Γ comme en Section A.1.1, cette condition s'écrit

$$Q_{\sigma(k),\sigma(k')} = Q_{k,k'}, \tag{A.1.3.1}$$

pour tout $k, k' \in K$.

Définition A.1.3.2. Pour tout $\alpha \models_K n$, la K-composition $\sigma \cdot \alpha$ de n est définie par

$$(\sigma \cdot \alpha)_k := \alpha_{\sigma^{-1}(k)},$$

pour tout $k \in K$.

Nous allons maintenant expliquer comment σ induit un automorphisme d'algèbres de Hecke carquois (cyclotomiques). Nous donnerons également une présentation de l'algèbre des points fixes.

A.1.3.1 Cas affine

Théorème A.1.3.3. *Soit $\alpha \models_K n$. Il y a un morphisme d'algèbres bien défini $\sigma : \mathrm{R}_\alpha(\Gamma) \to \mathrm{R}_{\sigma\cdot\alpha}(\Gamma)$ donné par*

$$\sigma(e(\boldsymbol{k})) := e(\sigma(\boldsymbol{k})), \qquad \textit{pour tout } \boldsymbol{k} \in K^\alpha,$$
$$\sigma(y_a) := y_a, \qquad \textit{pour tout } a \in \{1, \ldots, n\},$$
$$\sigma(\psi_a) := \psi_a, \qquad \textit{pour tout } a \in \{1, \ldots, n-1\}.$$

Soit $[\alpha]$ l'orbite de α sous l'action de $\langle\sigma\rangle$. Pour chaque $\alpha \models_K n$, définissions le sous-ensemble fini suivant de K^n :

$$K^{[\alpha]} := \bigsqcup_{\beta\in[\alpha]} K^\beta,$$

et de façon similaire, l'algèbre unitaire suivante :

$$\mathrm{R}_{[\alpha]}(\Gamma) := \bigoplus_{\beta\in[\alpha]} \mathrm{R}_\beta(\Gamma).$$

Nous obtenons un *automorphisme* $\sigma : \mathrm{R}_{[\alpha]}(\Gamma) \to \mathrm{R}_{[\alpha]}(\Gamma)$ d'ordre p. Considérons maintenant la relation d'équivalence $\sim$ sur K engendrée par $k \sim \sigma(k)$ pour tout $k \in K$. Cette relation s'étend à $K^{[\alpha]}$ via $\boldsymbol{k} \sim \sigma(\boldsymbol{k})$ pour tout $\boldsymbol{k} \in K^{[\alpha]}$. Écrivons $K_\sigma^{[\alpha]}$ pour l'ensemble quotient.

Définition A.1.3.4. Pour chaque $\gamma \in K_\sigma^{[\alpha]}$, définissons

$$e(\gamma) := \sum_{\boldsymbol{k}\in\gamma} e(\boldsymbol{k}).$$

Ces éléments $e(\gamma)$ sont des points fixes de σ. Donnons maintenant une présentation de l'algèbre $\mathrm{R}_{[\alpha]}(\Gamma)^\sigma$.

Théorème A.1.3.5. *L'algèbre $\mathrm{R}_{[\alpha]}(\Gamma)^\sigma$ a la présentation suivante. L'ensemble générateur est*

$$\{e(\gamma)\}_{\gamma\in K_\sigma^{[\alpha]}} \cup \{y_1, \dots, y_n\} \cup \{\psi_1, \dots, \psi_{n-1}\}, \tag{A.1.3.6}$$

et les relations sont

$$\sum_{\gamma\in K_\sigma^{[\alpha]}} e(\gamma) = 1, \tag{A.1.3.7a}$$

$$e(\gamma)e(\gamma') = \delta_{\gamma,\gamma'} e(\gamma), \tag{A.1.3.7b}$$

$$y_a e(\gamma) = e(\gamma) y_a, \tag{A.1.3.7c}$$

$$\psi_b e(\gamma) = e(s_b \cdot \gamma)\psi_b, \tag{A.1.3.7d}$$

$$y_a y_{a'} = y_{a'} y_a, \tag{A.1.3.7e}$$

$$\psi_b y_a = y_a \psi_b, \qquad \textit{si } a \neq b, b+1, \tag{A.1.3.7f}$$

$$\psi_b \psi_{b'} = \psi_{b'} \psi_b, \qquad \textit{si } |b - b'| > 1, \tag{A.1.3.7g}$$

$$\psi_b y_{b+1} e(\gamma) = \begin{cases} (y_b\psi_b + 1)e(\gamma), & \textit{si } \gamma_b = \gamma_{b+1}, \\ y_b\psi_b e(\gamma), & \textit{si } \gamma_b \neq \gamma_{b+1}, \end{cases} \tag{A.1.3.7h}$$

$$y_{b+1}\psi_b e(\gamma) = \begin{cases} (\psi_b y_b + 1)e(\gamma), & \textit{si } \gamma_b = \gamma_{b+1}, \\ \psi_b y_b e(\gamma), & \textit{si } \gamma_b \neq \gamma_{b+1}, \end{cases} \tag{A.1.3.7i}$$

et

$$\psi_b^2 e(\gamma) = Q_{\gamma_b,\gamma_{b+1}}(y_b, y_{b+1})e(\gamma), \tag{A.1.3.8a}$$

$$\psi_{c+1}\psi_c\psi_{c+1} e(\gamma) = \begin{cases} \psi_c\psi_{c+1}\psi_c e(\gamma) + \frac{Q_{\gamma_c,\gamma_{c+1}}(y_c,y_{c+1}) - Q_{\gamma_{c+2},\gamma_{c+1}}(y_{c+2},y_{c+1})}{y_c - y_{c+2}} e(\gamma), & \textit{si } \gamma_c = \gamma_{c+2}, \\ \psi_c\psi_{c+1}\psi_c e(\gamma), & \textit{sinon}, \end{cases} \tag{A.1.3.8b}$$

pour tout $\gamma \in K_\sigma^{[\alpha]}$, $a, a' \in \{1, \dots, n\}$, $b, b' \in \{1, \dots, n-1\}$ et $c \in \{1, \dots, n-2\}$.

Remarquons que les relations (A.1.3.7) et (A.1.3.8) de $\mathrm{R}_{[\alpha]}(\Gamma)^\sigma$ ressemblent aux relations (A.1.1.2) et (A.1.1.3) de $\mathrm{R}_\alpha(\Gamma)$. Cependant, l'ensemble d'indices pour les idempotents n'est en général plus un sous-ensemble $\mathfrak{S}_n$-stable de $\mathcal{I}^n$ avec $\mathcal{I}$ est un certain ensemble d'indices. La stratégie de preuve du Théorème A.1.3.5 est de comparer une base de $\mathrm{R}_{[\alpha]}(\Gamma)^\sigma$ avec une famille génératrice de l'algèbre définie par les générateurs (A.1.3.6) et les relations (A.1.3.7)–(A.1.3.8).

Dans le cas où Γ est un carquois (sans boucle et) sans flèche multiple, les relations (A.1.3.8) deviennent, avec les notations de §A.1.1,

$$\psi_b^2 e(\gamma) = \begin{cases} 0, & \text{si } \gamma_b = \gamma_{b+1}, \\ e(\gamma), & \text{si } \gamma_b \not\!\!- \gamma_{b+1}, \\ (y_{b+1} - y_b)e(\gamma), & \text{si } \gamma_b \to \gamma_{b+1}, \\ (y_b - y_{b+1})e(\gamma), & \text{si } \gamma_b \leftarrow \gamma_{b+1}, \\ (y_{b+1} - y_b)(y_b - y_{b+1})e(\gamma), & \text{si } \gamma_b \leftrightarrows \gamma_{b+1}, \end{cases} \tag{A.1.3.9a}$$

$$\psi_{c+1}\psi_c\psi_{c+1}e(\gamma) = \begin{cases} (\psi_c\psi_{c+1}\psi_c - 1)e(\gamma), & \text{si } \gamma_{c+2} = \gamma_c \to \gamma_{c+1}, \\ (\psi_c\psi_{c+1}\psi_c + 1)e(\gamma), & \text{si } \gamma_{c+2} = \gamma_c \leftarrow \gamma_{c+1}, \\ (\psi_c\psi_{c+1}\psi_c + 2y_{c+1} - y_c - y_{c+2})e(\gamma), & \text{si } \gamma_{c+2} = \gamma_c \leftrightarrows \gamma_{c+1}, \\ \psi_c\psi_{c+1}\psi_c e(\gamma), & \text{sinon}, \end{cases} \tag{A.1.3.9b}$$

pour tout $\gamma \in K_\sigma^{[\alpha]}$, $b \in \{1, \ldots, n-1\}$ et $c \in \{1, \ldots, n-2\}$.

Remarquons que la sous-algèbre $\mathrm{R}_{[\alpha]}(\Gamma)^\sigma$ est une sous-alègbre graduée de $\mathrm{R}_{[\alpha]}(\Gamma)$, puisque σ est homogène. Plus précisément, nous pouvons donner un analogue de la Proposition A.1.1.6 : il y a une unique $\mathbb{Z}$-graduation sur $\mathrm{R}_{[\alpha]}(\Gamma)^\sigma$ telle que $e(\gamma)$ est de degré 0, l'élément y_a est de degré 2 et $\psi_a e(\gamma)$ est de degré $-c_{\gamma_a,\gamma_{a+1}}$.

A.1.3.2 Cas cyclotomique

Soit $\mathbf{\Lambda} \in \mathbb{N}^{(K)}$ un poids. Jusqu'à la fin de cette section, nous faisons l'hypothèse de σ-stabilité suivante sur $\mathbf{\Lambda}$:

$$\Lambda_k = \Lambda_{\sigma(k)}, \tag{A.1.3.10}$$

pour tout $k \in K$. De façon similaire à (A.1.3.1), définissons

$$\mathrm{R}_{[\alpha]}^{\mathbf{\Lambda}}(\Gamma) \coloneqq \bigoplus_{\beta \in [\alpha]} \mathrm{R}_\beta^{\mathbf{\Lambda}}(\Gamma).$$

Cette algèbre est le quotient de $\mathrm{R}_{[\alpha]}(\Gamma)$ par l'idéal bilatère

$$\mathcal{I}_{[\alpha]}^{\mathbf{\Lambda}} \coloneqq \bigoplus_{\beta \in [\alpha]} \mathcal{I}_\beta^{\mathbf{\Lambda}}$$

engendré par les éléments $y_1^{\Lambda_{k_1}} e(\boldsymbol{k})$ pour tout $\boldsymbol{k} \in K^{[\alpha]}$.

Lemme A.1.3.11. *Nous avons un morphisme d'algèbres* $\sigma^{\mathbf{\Lambda}} : \mathrm{R}_{[\alpha]}^{\mathbf{\Lambda}}(\Gamma) \to \mathrm{R}_{[\alpha]}^{\mathbf{\Lambda}}(\Gamma)$*, induit par* $\sigma : \mathrm{R}_{[\alpha]}(\Gamma) \to \mathrm{R}_{[\alpha]}(\Gamma)$.

Nous écrirons souvent également σ pour désigner l'automorphisme $\sigma^{\mathbf{\Lambda}}$.

Définition A.1.3.12. Nous définissons $\mathrm{R}_{[\alpha]}^{\mathbf{\Lambda}}(\Gamma)^\sigma$ comme la A-algèbre des points fixes de $\mathrm{R}_{[\alpha]}^{\mathbf{\Lambda}}(\Gamma)$ sous l'action de l'automorphisme $\sigma^{\mathbf{\Lambda}}$.

Nous pouvons également considérer l'algèbre $(\mathrm{R}_{[\alpha]}(\Gamma)^\sigma)^{\mathbf{\Lambda}}$, le quotient de $\mathrm{R}_{[\alpha]}(\Gamma)^\sigma$ par l'idéal bilatère $\mathcal{I}^{\mathbf{\Lambda}}_{[\alpha],\sigma}$ engendré par les relations suivantes :

$$y_1^{\Lambda_{\gamma_1}} e(\gamma) = 0, \tag{A.1.3.13}$$

pour tout $\gamma \in K_\sigma^{[\alpha]}$.

Lemme A.1.3.14. *Nous avons*

$$\mathcal{I}^{\mathbf{\Lambda}}_{[\alpha]} \cap \mathrm{R}_{[\alpha]}(\Gamma)^\sigma = \mathcal{I}^{\mathbf{\Lambda}}_{[\alpha],\sigma}.$$

Théorème A.1.3.15. *Les algèbres graduées* $\mathrm{R}^{\mathbf{\Lambda}}_{[\alpha]}(\Gamma)^\sigma$ *et* $(\mathrm{R}_{[\alpha]}(\Gamma)^\sigma)^{\mathbf{\Lambda}}$ *sont isomorphes. En particulier, les générateurs* (A.1.3.6) *ainsi que les relations* (A.1.3.7), (A.1.3.8) *et* (A.1.3.13) *donnent une présentation de* $\mathrm{R}^{\mathbf{\Lambda}}_{[\alpha]}(\Gamma)^\sigma$.

Remarquons que ce théorème est énoncé et montré dans [Ro16] sous l'hypothèse que p est inversible dans A. L'énoncé et la preuve sont ici valables en toute caractéristique, la différence principale étant la preuve du Lemme A.1.3.14.

A.2 Algèbres de Hecke de groupes de réflexions complexes

Cette section est une adaptation de [Ro16]. Nous commençons par généraliser un isomorphisme de Brundan et Kleshchev entre l'algèbre de Hecke de type $G(r, 1, n)$ et l'algèbre de Hecke carquois cyclotomique de type A (Théorème A.2.2.2). Nous utilisons ensuite les résultats de la Section A.1 pour donner une présentation de type Hecke carquois cyclotomique pour l'algèbre de Hecke de type $G(r, p, n)$, c'est-à-dire, pour les groupes de réflexions complexes de la série infinie, voir le Théorème A.2.3.1 et le Corollaire A.2.3.2. Nous donnons également un isomorphisme explicite qui réalise une équivalence de Morita bien connue entre algèbres d'Ariki–Koike (cf. [DiMa]), voir Théorème A.2.2.4.

A.2.1 Cas non gradués

Soient $n, r \in \mathbb{N}^*$ et soit $\boldsymbol{u} = (u_1, \ldots, u_r)$ un r-uplet d'éléments de $F^\times$. Nous rappelons ici la définition de l'algèbre d'Ariki–Koike $\mathrm{H}_n(q, \boldsymbol{u})$.

Définition A.2.1.1 ([BrMa, ArKo])**.** L'algèbre $\mathrm{H}_n(q, \boldsymbol{u})$ est la F-algèbre unitaire associative engendrée par les éléments $S, T_1, \ldots, T_{n-1}$, soumis aux relations suivantes :

$$\prod_{k=1}^{r} (S - u_k) = 0, \tag{A.2.1.2a}$$

$$(T_a + 1)(T_a - q) = 0, \tag{A.2.1.2b}$$

$$ST_1ST_1 = T_1ST_1S, \tag{A.2.1.2c}$$

$$ST_a = T_aS, \qquad \text{si } a > 1, \tag{A.2.1.2d}$$

$$T_aT_{a'} = T_{a'}T_a, \qquad \text{si } |a - a'| > 1, \tag{A.2.1.2e}$$

$$T_bT_{b+1}T_b = T_{b+1}T_bT_{b+1}, \tag{A.2.1.2f}$$

pour tout $a, a' \in \{1, \ldots, n\}$ et $b \in \{1, \ldots, n-1\}$.

Suivant la terminologie de [BrMa], nous disons que $\mathrm{H}_n(q, \boldsymbol{u})$ est une *algèbre de Hecke de type* $G(r,1,n)$. Soit $X_1 \coloneqq S$ et définissons pour chaque $a \in \{1, \dots, n-1\}$ l'élément $X_{a+1} \in \mathrm{H}_n(q, \boldsymbol{u})$ par

$$qX_{a+1} \coloneqq T_a X_a T_a.$$

Définissons

$$p' \coloneqq \min\{m \in \mathbb{N}^* : \zeta^m \in \langle q \rangle\} \in \{1, \dots, p\},$$

et

$$J' \coloneqq \{0, \dots, p'-1\}.$$

L'entier p' ne dépend que de p et de e. Finalement, désignons par η l'unique élément de I tel que

$$\zeta^{p'} = q^\eta.$$

Soit $\boldsymbol{\Lambda} = (\Lambda_{i,j}) \in \mathbb{N}^{(I \times J')}$ un poids avec $\ell(\boldsymbol{\Lambda}) = r$ et supposons que les paramètres $u_1, \dots, u_r$ sont choisis tels que la relation (A.2.1.2a) de $\mathrm{H}_n(q, \boldsymbol{u})$ s'écrive

$$\prod_{i \in I} \prod_{j \in J'} \left(S - \zeta^j q^i\right)^{\Lambda_{i,j}} = 0. \tag{A.2.1.3}$$

Définition A.2.1.4. Avec les notations précédentes, nous définissons $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q, \zeta) \coloneqq \mathrm{H}_n(q, \boldsymbol{u})$.

Si $\boldsymbol{\Lambda} \in \mathbb{N}^{(I)}$, l'algèbre $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q, \zeta)$ est définie en posant $\Lambda_{i,j} \coloneqq \Lambda_i$ pour tout $(i,j) \in I \times J'$.

Proposition A.2.1.5. *Supposons que* $\boldsymbol{\Lambda} \in \mathbb{N}^{(I)}$ *vérifie*

$$\Lambda_i = \Lambda_{i+\eta}, \tag{A.2.1.6}$$

pour tout $i \in I$. *Il y a un morphisme d'algèbres bien défini* $\sigma : \mathrm{H}_n^{\boldsymbol{\Lambda}}(q, \zeta) \to \mathrm{H}_n^{\boldsymbol{\Lambda}}(q, \zeta)$ *donné par*

$$\begin{aligned} \sigma(S) &\coloneqq \zeta S, \\ \sigma(T_a) &\coloneqq T_a, & &\text{pour tout } a \in \{1, \dots, n-1\}. \end{aligned}$$

Nous dirons que σ est l'*automorphisme de décalage* de $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q, \zeta)$.

Définition A.2.1.7. Soit $\boldsymbol{\Lambda} \in \mathbb{N}^{(I)}$ vérifiant la condition de stabilité (A.2.1.6). Nous définissons l'algèbre

$$\mathrm{H}_{p,n}^{\boldsymbol{\Lambda}}(q) \coloneqq \mathrm{H}_n^{\boldsymbol{\Lambda}}(q, \zeta)^\sigma,$$

la sous-algèbre de $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q, \zeta)$ constituée des points fixes de σ. C'est une algèbre de Hecke de type $G(r,p,n)$.

Nous pouvons trouver dans [BrMa, Ar95] deux présentations de $\mathrm{H}_{p,n}^{\boldsymbol{\Lambda}}(q)$ par générateurs et relations. Nous montrons en §2.2.3 que ces deux présentations sont isomorphes.

A.2.2 L'isomorphisme de Brundan et Kleshchev

Dans cette section, nous montrons comment l'isomorphisme de Brundan et Kleshchev [BrKl-a] concernant l'algèbre $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q, 1)$ se généralise à $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q, \zeta)$.

A.2.2.1 Énoncé

Considérons le carquois Γ_e défini comme suit :
— l'ensemble des sommets est $\{q^i\}_{i\in I}$;
— il y a une flèche de v vers qv pour chaque sommet v de Γ_e.
Le carquois Γ_e est le carquois cyclique à e sommets si $e < \infty$ et une copie orientée de $\mathbb{Z}$ si $e = \infty$. Considérons maintenant p' éléments non nuls $v_0, \dots, v_{p'-1}$ de F tels que

$$\frac{v_k}{v_l} \notin \langle q \rangle,$$

pour tout $k \neq l$ et considérons le carquois Γ défini comme suit :
— l'ensemble des sommets est $V := \{v_j q^i\}_{i\in I, j\in J'}$;
— il y a une flèche de v vers qv pour chaque sommet v de Γ.
Puisque les éléments v_k sont dans des q-orbits distinctes, l'ensemble des sommets V de Γ s'identifie à $K := I \times J'$. Plus précisément, le carquois Γ est donné par exactement p' copies disjointes de Γ_e. En particulier, il ne dépend que de e et de p'. Nous écrivons donc $\Gamma_{e,p'} := \Gamma$.

Soit $\mathbf{\Lambda} = (\Lambda_k)_{k\in K} \in \mathbb{N}^{(K)}$ un poids de longueur r. Comme dans la définition de $\mathrm{H}_n^{\mathbf{\Lambda}}(q,\zeta)$, choisissons un r-uplet $\boldsymbol{u} \in (F^\times)^r$ donné par exactement $\Lambda_{i,j}$ copies de $v_j q^i$ pour chaque $(i,j) \in I \times J'$ et posons $\mathrm{H}_n^{\mathbf{\Lambda}}(q,\boldsymbol{v}) := \mathrm{H}_n(q,\boldsymbol{u})$. Ainsi, la relation (A.2.1.2a) dans $\mathrm{H}_n(q,\boldsymbol{u})$ est

$$\prod_{i\in I}\prod_{j\in J'} (S - v_j q^i)^{\Lambda_{i,j}} = 0. \tag{A.2.2.1}$$

Théorème A.2.2.2. *Il y a une famille d'isomorphismes (explicites) de F-algèbres*

$$\mathrm{H}_n^{\mathbf{\Lambda}}(q,\boldsymbol{v}) \simeq \mathrm{R}_n^{\mathbf{\Lambda}}(\Gamma_{e,p'}).$$

Brundan et Kleshchev [BrKl-a] ont montré le Théorème A.2.2.2 pour $p = 1$, et leur preuve se généralise directement à ce cadre plus général. Remarquons qu'un isomorphisme comme dans le Théorème A.2.2.2 pour $e < \infty$ a déjà été obtenu par Rouquier [Rou, Corollary 3.20].

Définition A.2.2.3. Soit $\alpha \models_K n$. Notons $e(\boldsymbol{k}) \in \mathrm{H}_n^{\mathbf{\Lambda}}(q,\boldsymbol{v})$ l'image de $e(\boldsymbol{k}) \in \mathrm{R}_n^{\mathbf{\Lambda}}(\Gamma_{e,p'})$ par un isomorphisme du Théorème A.2.2.2. Nous définissons $\mathrm{H}_\alpha^{\mathbf{\Lambda}}(q,\boldsymbol{v}) := e(\alpha)\mathrm{H}_n^{\mathbf{\Lambda}}(q,\boldsymbol{v})$.

Remarquons que $\mathrm{H}_\alpha^{\mathbf{\Lambda}}(q,\boldsymbol{v}) \simeq \mathrm{R}_\alpha^{\mathbf{\Lambda}}(\Gamma_{e,p'})$.

A.2.2.2 Un corollaire inattendu

Pour chaque $j \in J'$, soit $\mathbf{\Lambda}^j$ la restriction de $\mathbf{\Lambda}$ à $I \times \{j\} \simeq I$. Définissons également

$$\mathrm{H}_n^{\mathbf{\Lambda}^j}(q) := \mathrm{H}_n^{\mathbf{\Lambda}^j}(q, \boldsymbol{v}_{\mathrm{triv}}),$$

où $\boldsymbol{v}_{\mathrm{triv}}$ a seulement une coordonnée non nulle, égale à 1. Combinant la version cyclotomique du Théorème A.1.2.6 et le Théorème A.2.2.2, nous déduisons le théorème suivant.

Théorème A.2.2.4. *Soit $\boldsymbol{v} \in (F^\times)^{p'}$ comme en §A.2.2.1. Nous avons un isomorphisme (explicite) de F-algèbres*

$$\mathrm{H}_n^{\mathbf{\Lambda}}(q,\boldsymbol{v}) \simeq \bigoplus_{\lambda \models_{J'} n} \mathrm{Mat}_{m_\lambda}\left(\mathrm{H}_{\lambda_0}^{\mathbf{\Lambda}^0}(q) \otimes \cdots \otimes \mathrm{H}_{\lambda_{p'-1}}^{\mathbf{\Lambda}^{p'-1}}(q)\right).$$

En particulier, les algèbres $\mathrm{H}_n^{\mathbf{\Lambda}}(q,\boldsymbol{v})$ et $\oplus_{\lambda\models_{J'} n} \mathrm{H}_{\lambda_0}^{\mathbf{\Lambda}^0}(q) \otimes \cdots \otimes \mathrm{H}_{\lambda_{p'-1}}^{\mathbf{\Lambda}^{p'-1}}(q)$ sont Morita-équivalentes.

Nous retrouvons ainsi un cas particulier de l'équivalence de Morita de [DiMa].

Remarque A.2.2.5. Si $\mathbf{\Lambda}^0 = \cdots = \mathbf{\Lambda}^{p'-1}$, par [PA, Corollary 3.2] ou la Section A.3 nous savons que l'algèbre du Théorème A.2.2.4 est une algèbre de Yokonuma–Hecke cyclotomique de type A.

A.2.3 Restriction de la graduation

Notre but ici est d'utiliser les résultats de §A.1.3.2 pour donner une présentation de type Hecke carquois cyclotomique de la sous-algèbre $\mathrm{H}^{\mathbf{\Lambda}}_{p,n}(q)$.

Pour chaque $j \in J'$, définissons $v_j \coloneqq \zeta^j$. Il découle de la définition de p' que l'ensemble des sommets de $\Gamma_{e,p'}$ peut être identifié avec $V = \{\zeta^j q^i\}_{i\in I, j\in J'}$. Considérons un poids $\mathbf{\Lambda} = (\Lambda_i)_{i\in I}$ satisfaisant la condition de stabilité (A.2.1.6). Définissons $\sigma : V \to V$ par

$$\sigma(v) \coloneqq \zeta v,$$

pour tout $v \in V$. D'après la condition de stabilité (A.2.1.6) vérifiée par $\mathbf{\Lambda}$, l'application σ induit un automorphisme de l'algèbre de Hecke carquois cyclotomique $\mathrm{R}^{\mathbf{\Lambda}}_n(\Gamma_{e,p'})$. Désignons par $\widetilde{\sigma} : \mathrm{H}^{\mathbf{\Lambda}}_n(q,\zeta) \to \mathrm{H}^{\mathbf{\Lambda}}_n(q,\zeta)$ l'automorphisme de décalage de $\mathrm{H}^{\mathbf{\Lambda}}_n(q,\zeta)$. En utilisant une remarque de Stroppel et Webster [StWe], nous pouvons montrer le théorème suivant.

Théorème A.2.3.1. *On peut choisir un isomorphisme* $f : \mathrm{H}^{\mathbf{\Lambda}}_n(q,\zeta) \to \mathrm{R}^{\mathbf{\Lambda}}_n(\Gamma_{e,p'})$ *comme dans le Théorème A.2.2.2 tel que* $\sigma^{-1} \circ f = f \circ \widetilde{\sigma}$.

Pour chaque $\alpha \models_K n$, l'algèbre $\mathrm{H}^{\mathbf{\Lambda}}_{[\alpha]}(q) \coloneqq \oplus_{\beta\in[\alpha]} \mathrm{H}^{\mathbf{\Lambda}}_{\beta}(q)$ est stable sous $\widetilde{\sigma}$. Désignons par $\mathrm{H}^{\mathbf{\Lambda}}_{p,[\alpha]}(q)$ la sous-algèbre des points fixes.

Corollaire A.2.3.2. *L'isomorphisme de F-algèbres* $f : \mathrm{H}^{\mathbf{\Lambda}}_n(q,\zeta) \to \mathrm{R}^{\mathbf{\Lambda}}_n(\Gamma_{e,p'})$ *du Théorème A.2.3.1 induit un isomorphisme entre* $\mathrm{H}^{\mathbf{\Lambda}}_{p,n}(q)$ *et* $\mathrm{R}^{\mathbf{\Lambda}}_n(\Gamma_{e,p'})^{\sigma}$. *Par conséquent, l'algèbre* $\mathrm{H}^{\mathbf{\Lambda}}_{p,[\alpha]}(q)$ *a une présentation donnée par les générateurs*

$$\{e(\gamma)\}_{\gamma\in K^{[\alpha]}_{\sigma}} \cup \{y_1,\dots,y_n\} \cup \{\psi_1,\dots,\psi_{n-1}\},$$

et les relations (A.1.3.7), (A.1.3.9) *et* (A.1.3.13).

Corollaire A.2.3.3. *L'automorphisme de décalage* $\widetilde{\sigma} : \mathrm{H}^{\mathbf{\Lambda}}_n(q,\zeta) \to \mathrm{H}^{\mathbf{\Lambda}}_n(q,\zeta)$ *est homogène et la sous-algèbre* $\mathrm{H}^{\mathbf{\Lambda}}_{p,n}(q)$ *est une sous-algèbre graduée de* $\mathrm{H}^{\mathbf{\Lambda}}_n(q,\zeta)$.

Donnons maintenant un analogue d'un corollaire classique de [BrKl-a, Theorem 1.1].

Corollaire A.2.3.4. *Si* $\tilde{q} \in F \setminus \{0,1\}$ *a le même ordre* $e \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ *que* q *alors*

$$\mathrm{H}^{\mathbf{\Lambda}}_{p,n}(\tilde{q}) \simeq \mathrm{H}^{\mathbf{\Lambda}}_{p,n}(q),$$

en tant que F-algèbres graduées.

La preuve repose sur le fait suivant : il existe une racine primitive p-ième de l'unité $\tilde{\zeta} \in F^{\times}$ telle que $\tilde{q}^{\eta} = \tilde{\zeta}^{p'}$.

A.3 Algèbres de Yokonuma–Hecke cyclotomiques

Cette section est une adaptation de [Ro17-a]. Nous prouvons que les algèbres de Yokonuma–Hecke cyclotomiques de type A sont des algèbres de Hecke carquois cyclotomiques (Théorème A.3.2.3). Comme dans la section A.2, nous utilisons l'isomorphisme de Brundan et Kleshchev [BrKl-a] et le carquois est donné par des copies disjointe du même carquois cyclique. Cependant, la généralisation n'est maintenant plus immédiate. Finalement, nous faisons un lien avec un isomorphisme de Lusztig (§A.3.3).

A.3.1 Cadre

Soit $d \in \mathbb{N}^*$ et supposons que le corps F contient une racine primitive d-ième de l'unité ξ. Sauf mention du contraire, l'élement q sera toujours pris différent de 1. Définissons $J \coloneqq \mathbb{Z}/d\mathbb{Z} \simeq \{1, \dots, d\}$ ainsi que $K \coloneqq I \times J$. Nous utiliserons la *caractéristique quantique* de F, donnée par

$$\operatorname{char}_q(F) \coloneqq \begin{cases} e, & \text{si } e < \infty, \\ 0, & \text{si } e = \infty. \end{cases}$$

En particulier, nous avons $I = \mathbb{Z}/\operatorname{char}_q(F)\mathbb{Z}$ et $\operatorname{char}_1(F)$ est exactement la caractéristique usuelle de F.

Soit $\boldsymbol{\Lambda} = (\Lambda_i)_{i \in I} \in \mathbb{N}^{(I)}$ un poids avec $\ell(\boldsymbol{\Lambda}) = \sum_{i \in I} \Lambda_i > 0$. L'*algèbre de Yokonuma–Hecke cyclotomique de type A*, notée $\mathrm{Y}^{\boldsymbol{\Lambda}}_{d,n}(q)$, est la F-algèbre associative unitaire engendrée par les éléments

$$g_1, \dots, g_{n-1}, t_1, \dots, t_n, X_1, \tag{A.3.1.1}$$

soumis aux relations suivantes :

$$\begin{aligned}
t_a^d &= 1, && && \text{(A.3.1.2a)}\\
t_a t_{a'} &= t_{a'} t_a, && && \text{(A.3.1.2b)}\\
t_a g_b &= g_b t_{s_b(a)}, && && \text{(A.3.1.2c)}\\
g_b^2 &= q + (q-1) g_b e_b, && && \text{(A.3.1.2d)}\\
g_b g_{b'} &= g_{b'} g_b, && \text{si } |b - b'| > 1, && \text{(A.3.1.2e)}\\
g_{c+1} g_c g_{c+1} &= g_c g_{c+1} g_c, && && \text{(A.3.1.2f)}\\
X_1 g_1 X_1 g_1 &= g_1 X_1 g_1 X_1, && && \text{(A.3.1.2g)}\\
X_1 g_b &= g_b X_1, && \text{si } b > 1, && \text{(A.3.1.2h)}\\
X_1 t_a &= t_a X_1, && && \text{(A.3.1.2i)}\\
\prod_{i \in I} (X_1 - q^i)^{\Lambda_i} &= 0. && && \text{(A.3.1.2j)}
\end{aligned}$$

pour tout $a, a' \in \{1, \dots, n\}$, $b, b' \in \{1, \dots, n-1\}$ et $c \in \{1, \dots, n-2\}$, où s_b est la transposition $(b, b+1) \in \mathfrak{S}_n$ et $e_b \coloneqq \frac{1}{d} \sum_{j \in J} t_b^j t_{b+1}^{-j}$, Quand $d = 1$, l'algèbre $\mathrm{Y}^{\boldsymbol{\Lambda}}_{1,n}(q)$ est l'algèbre d'Ariki–Koike $\mathrm{H}^{\boldsymbol{\Lambda}}_n(q) \coloneqq \mathrm{H}^{\boldsymbol{\Lambda}}_n(q, 1)$, définie en §A.2.1 et utilisée dans [BrKl-a]. Dans ce cas, l'élément e_a est réduit à 1. Suivant [ChPA15], définissons par récurrence X_{a+1} pour chaque $a \in \{1, \dots, n-1\}$ par

$$q X_{a+1} \coloneqq g_a X_a g_a.$$

A.3.2 Isomorphisme gradué

Soit M un $\mathrm{Y}^{\boldsymbol{\Lambda}}_{d,n}(q)$-module de dimension finie. Écrivons M comme la somme directe de ses sous-espaces propres communs

$$M(\boldsymbol{j}) \coloneqq \left\{ v \in M : (t_a - \xi^{j_a}) v = 0 \text{ pour tout } a \in \{1, \dots, n\} \right\} \tag{A.3.2.1}$$

pour $\boldsymbol{j} \in J^n$. Considérons la famille de projections $\{e(\boldsymbol{j})\}_{\boldsymbol{j} \in J^n}$ associée à la décomposition $M = \oplus_{\boldsymbol{J} \in J^n} M(\boldsymbol{j})$.

Lemme A.3.2.2. *Soit $a \in \{1, \dots, n-1\}$ et $\boldsymbol{j} \in J^n$. Nous avons*

$$g_a^2 e(\boldsymbol{j}) = \begin{cases} q e(\boldsymbol{j}), & \textit{si } j_a \neq j_{a+1}, \\ (q + (q-1) g_a) e(\boldsymbol{j}), & \textit{si } j_a = j_{a+1}. \end{cases}$$

L'idée est la suivante : si $j_a \neq j_{a+1}$ alors nous sommes dans le cas de l'algèbre du groupe, un cas facile, et si $j_a = j_{a+1}$ nous sommes dans le cas Ariki–Koike, donc celui de [BrKl-a]. Nous pouvons donc adapter la preuve de [BrKl-a], avec des ajouts non triviaux, pour prouver le théorème suivant.

Théorème A.3.2.3. *Il y a une famille d'isomorphismes (explicites) de F-algèbres*

$$\mathrm{Y}_{d,n}^{\mathbf{\Lambda}}(q) \simeq \mathrm{R}_n^{\mathbf{\Lambda}}(\Gamma_{e,d}).$$

Dans l'énoncé précédent, l'algèbre $\mathrm{R}_\alpha^{\mathbf{\Lambda}}(\Gamma_{e,d})$ est définie en étendant le poids $\mathbf{\Lambda} \in \mathbb{N}^{(I)}$ en un élément de $\mathbb{N}^{(K)}$ via

$$\Lambda_{i,j} := \Lambda_i,$$

pour tout $(i,j) \in K = I \times J$. Nous pouvons définir une algèbre de Yokonuma–Hecke cyclotomique de type A *dégénérée* $\mathrm{Y}_{d,n}^{\mathbf{\Lambda}}(1)$ et montrer une version analogue du Théorème A.3.2.3, avec le même isomorphisme $\mathrm{Y}_{d,n}^{\mathbf{\Lambda}}(1) \simeq \mathrm{R}_n^{\mathbf{\Lambda}}(\Gamma_{e,d})$.

Pour $\alpha \models_K n$, définissons l'algèbre $\mathrm{Y}_\alpha^{\mathbf{\Lambda}}(q) := e(\alpha)\mathrm{Y}_{d,n}^{\mathbf{\Lambda}}(q)$ où $e(\boldsymbol{k}) \in \mathrm{Y}_{d,n}^{\mathbf{\Lambda}}(q)$ désigne l'image de l'élément $e(\boldsymbol{k}) \in \mathrm{R}_n^{\mathbf{\Lambda}}(\Gamma_{e,d})$ par un isomorphisme du Théorème A.3.2.3.

Corollaire A.3.2.4. *Si q est un élément arbitraire de $F^\times$, il y a une présentation de l'algèbre $\mathrm{Y}_\alpha^{\mathbf{\Lambda}}(q) \simeq \mathrm{R}_\alpha^{\mathbf{\Lambda}}(\Gamma_{e,d})$ donnée par les générateurs (A.1.1.1) et les relations (A.1.1.2), (A.1.1.5) et (A.1.1.8).*

Comme dans [BrKl-a], nous avons toute une succession de corollaires.

Corollaire A.3.2.5. *L'algèbre de Yokonuma–Hecke cyclotomique (possiblement dégénérée) hérite de la $\mathbb{Z}$-graduation de l'algèbre de Hecke carquois cyclotomique.*

Corollaire A.3.2.6. *Si q et $\tilde{q}$ sont deux éléments arbitraires de $F^\times$ avec $\mathrm{char}_q(F) = \mathrm{char}_{\tilde{q}}(F)$ alors $\mathrm{Y}_{d,n}^{\mathbf{\Lambda}}(q)$ et $\mathrm{Y}_{d,n}^{\mathbf{\Lambda}}(\tilde{q})$ sont des F-algèbres isomorphes.*

Corollaire A.3.2.7. *Si F est de caractéristique $\mathrm{char}_q(F)$ alors l'algèbre de Yokonuma–Hecke cyclotomique $\mathrm{Y}_{d,n}^{\mathbf{\Lambda}}(q)$ est isomorphisme à sa dégénération rationnelle $\mathrm{Y}_{d,n}^{\mathbf{\Lambda}}(1)$. Cela s'applique en particulier quand F est de caractéristique 0 et q est générique.*

A.3.3 Un diagramme commutatif

Nous supposons ici que $F = \mathbb{C}$ et que $q \in F^\times$ (avec $q \neq 1$) est une racine de l'unité. Désignons par

$$\mathrm{BK} : \mathrm{H}_n^{\mathbf{\Lambda}}(q) \xrightarrow{\sim} \mathrm{R}_n^{\mathbf{\Lambda}}(\Gamma_e)$$

un $\mathbb{C}$-isomorphisme d'algèbres comme dans [BrKl-a]. Pour chaque $\lambda \models_d n$, définissons $\mathrm{H}_\lambda^{\mathbf{\Lambda}}(q) := \mathrm{H}_{\lambda_1}^{\mathbf{\Lambda}}(q) \otimes \cdots \otimes \mathrm{H}_{\lambda_d}^{\mathbf{\Lambda}}(q)$ et rappelons la définition de l'entier m_λ donnée en (A.1.2.2). Nous avons un isomorphisme d'algèbres

$$\mathrm{JPA} : \mathrm{Y}_{d,n}^{\mathbf{\Lambda}}(q) \xrightarrow{\sim} \bigoplus_{\lambda \models_d n} \mathrm{Mat}_{m_\lambda} \mathrm{H}_\lambda^{\mathbf{\Lambda}}(q),$$

démontré par Lusztig [Lu] dans le cas $\ell(\mathbf{\Lambda}) = 1$, et explicitement construit par Jacon–Poulain d'Andecy [JacPA] (pour $\ell(\mathbf{\Lambda}) = 1$) et Poulain d'Andecy [PA]. Désignons également par $\mathrm{BK} : \oplus_\lambda \mathrm{Mat}_{m_\lambda} \mathrm{H}_\lambda^{\mathbf{\Lambda}}(q) \to \oplus_\lambda \mathrm{Mat}_{m_\lambda} \mathrm{R}_\lambda^{\mathbf{\Lambda}}(\Gamma_{e,d})$ le morphisme qu'induit naturellement $\mathrm{BK} : \mathrm{H}_n^{\mathbf{\Lambda}}(q) \to \mathrm{R}_n^{\mathbf{\Lambda}}(\Gamma_e)$. Notons $\Phi_n^{\mathbf{\Lambda}} : \oplus_\lambda \mathrm{Mat}_{m_\lambda} \mathrm{R}_\lambda^{\mathbf{\Lambda}}(\Gamma_{e,d}) \to \mathrm{R}_n^{\mathbf{\Lambda}}(\Gamma_{e,d})$ l'isomorphisme de §A.1.2.2 et $\widetilde{\mathrm{BK}} : \mathrm{Y}_{d,n}^{\mathbf{\Lambda}}(q) \to \mathrm{R}_n^{\mathbf{\Lambda}}(\Gamma_{e,d})$ un isomorphisme comme dans le Théorème A.3.2.3 (où le choix est le « même » que pour BK).

Théorème A.3.3.1. *Le diagramme de la Figure A.1 commute.*

$$\begin{array}{ccc} \mathrm{Y}^{\boldsymbol{\Lambda}}_{d,n}(q) & \xrightarrow{\ \mathrm{JPA}\ } & \bigoplus_{\lambda \vDash_d n} \mathrm{Mat}_{m_\lambda} \mathrm{H}^{\boldsymbol{\Lambda}}_{\lambda}(q) \\ \Big\downarrow \widetilde{\mathrm{BK}} & & \Big\downarrow \mathrm{BK} \\ \mathrm{R}^{\boldsymbol{\Lambda}}_{n}(\Gamma_{e,d}) & \xleftarrow[\ \Phi^{\boldsymbol{\Lambda}}_{n}\ ]{} & \bigoplus_{\lambda \vDash_d n} \mathrm{Mat}_{m_\lambda} \mathrm{R}^{\boldsymbol{\Lambda}}_{\lambda}(\Gamma_{e,d}) \end{array}$$

FIGURE A.1 : Un diagramme commutatif

A.4 Blocs bégayants des algèbres d'Ariki–Koike

Cette section est une adaptation de [Ro17-b]. Après avoir rappelé quelques définitions de combinatoire comme les (multi-)partitions, leurs (multi-)ensembles de résidus et leurs abaques, nous énonçons les résultats principaux : le Théorème A.4.1.8 et le Corollaire A.4.1.9, qui relient le nombre d'éléments dans l'orbite d'un multi-ensemble sous l'action de décalage avec celui de l'orbite d'une multi-partition associée. La suite est principalement consacrée à la démonstration de ces résultats. En §A.4.2 nous énonçons la Proposition A.4.2.5, garantissant l'existence d'une certaine matrice binaire et en §A.4.3 nous montrons le théorème principal, via la résolution d'un problème d'optimisation sous contraintes (Lemme A.4.3.1). Finalement, nous présentons en §A.4.4 quelques applications à la théorie des représentations de $\mathrm{H}^{\boldsymbol{\Lambda}}_{p,n}(q)$.

A.4.1 Combinatoire

Dans cette section, après quelques définitions standards de combinatoire nous introduisons deux actions de *décalage* et énonçons notre résultat principal, le Théorème A.4.1.8. Nous identifions $\mathbb{Z}/e\mathbb{Z}$ avec $\{0, \dots, e-1\}$.

A.4.1.1 Partitions

Une *partition* de n est une suite décroissante (au sens large) d'entiers naturels $\lambda = (\lambda_0, \dots, \lambda_{h-1})$ de somme n. Nous écrirons $|\lambda| \coloneqq n$ et $h(\lambda) \coloneqq h$. Si λ est une partition, nous désignons par $\mathcal{Y}(\lambda)$ son diagramme de Young, défini par

$$\mathcal{Y}(\lambda) \coloneqq \Big\{(a,b) \in \mathbb{N}^2 : 0 \leq a \leq h(\lambda) - 1 \text{ et } 0 \leq b \leq \lambda_a - 1\Big\}.$$

Nous dirons que les éléments de $\mathcal{Y}(\lambda)$ sont des *nœuds*. Un *ruban* de λ est un sous-ensemble de $\mathcal{Y}(\lambda)$ de la forme suivante :

$$r^\lambda_{(a,b)} \coloneqq \{(a', b') \in \mathcal{Y}(\lambda) : a' \geq a,\ b' \geq b \text{ et } (a'+1, b'+1) \notin \mathcal{Y}(\lambda)\},$$

où $(a,b) \in \mathcal{Y}(\lambda)$. Nous dirons que $r^\lambda_{(a,b)}$ est un *h-ruban* s'il est de cardinalité h. La *main* du ruban $r^\lambda_{(a,b)}$ est le nœud $(a, b') \in r^\lambda_{(a,b)}$ avec b' maximal. Remarquons que l'ensemble $\mathcal{Y}(\lambda) \setminus r^\lambda_{(a,b)}$ est le diagramme de Young d'une certaine partition. Une partition λ est un *e-cœur* si λ n'a pas de e-ruban.

Soit λ une partition. Le *résidu* d'un nœud $\gamma = (a,b) \in \mathcal{Y}(\lambda)$ est $\mathrm{res}(\gamma) \coloneqq b - a \pmod e$. Nous notons $n^i(\lambda)$ la multiplicité de i dans le multi-ensemble des résidus des éléments de $\mathcal{Y}(\lambda)$. Remarquons que $\sum_{i=0}^{e-1} n^i(\lambda) = |\lambda|$.

Soit Q un $\mathbb{Z}$-module libre de rang e et soit $\{\alpha_i\}_{i\in\mathbb{Z}/e\mathbb{Z}}$ une base de Q. Nous avons $Q = \oplus_{i=0}^{e-1}\mathbb{Z}\alpha_i$ et nous définissons $Q^+ := \oplus_{i=0}^{e-1}\mathbb{N}\alpha_i$. Si λ est une partition, définissons

$$\alpha(\lambda) := \sum_{\gamma\in\mathcal{Y}(\lambda)} \alpha_{\mathrm{res}(\gamma)} = \sum_{i=0}^{e-1} n^i(\lambda)\alpha_i \in Q^+.$$

Le sous-ensemble suivant de Q^+,

$$Q^* := \{\alpha(\lambda) : \lambda \text{ est un } e\text{-cœur}\}, \tag{A.4.1.1}$$

est en bijection avec l'ensemble des e-cœurs (voir [JamKe, 2.7.41 Theorem] ou [LyMa]). Grâce à la représentation par *abaque* d'une partition, nous prouvons le théorème suivant (voir également [GKS, Ol]).

Théorème A.4.1.2. *Il y a une bijection*

$$x : \{e\text{-cœurs}\} \longrightarrow \{(x_0, \ldots, x_{e-1}) \in \mathbb{Z}^e : x_0 + \cdots + x_{e-1} = 0\} =: \mathbb{Z}_0^e,$$

qui vérifie

$$n^0(\lambda) = \frac{1}{2}\|x(\lambda)\|^2 = \frac{1}{2}\sum_{i=0}^{e-1} x_i(\lambda)^2$$

pour tout e-cœur λ.

Définition A.4.1.3. Si λ est un e-cœur, nous disons que le e-uplet $x(\lambda) \in \mathbb{Z}^e$ est la *variable de e-abaque* associée à λ.

A.4.1.2 Multi-partitions

Soient $d, \eta, p \in \mathbb{N}^*$ et supposons que $e = \eta p$. Définissons $r := dp$ et identifions $\mathbb{Z}/r\mathbb{Z}$ et $\{0, \ldots, r-1\}$. Soit $\kappa = (\kappa_0, \ldots, \kappa_{r-1}) \in (\mathbb{Z}/e\mathbb{Z})^r$ une multi-charge. Une *r-partition* (ou *multi-partition*) de n est un r-uplet $\boldsymbol{\lambda} = (\lambda^{(0)}, \ldots, \lambda^{(r-1)})$ de partitions tel que $|\boldsymbol{\lambda}| := |\lambda^{(0)}| + \cdots + |\lambda^{(r-1)}| = n$. Nous écrivons $\boldsymbol{\lambda} \in \mathcal{P}_n^\kappa$ si $\boldsymbol{\lambda}$ est une r-partition de n. Nous disons que κ est *compatible* avec (d, η, p) quand

$$\kappa_{k+d} = \kappa_k + \eta, \qquad \text{pour tout } k \in \mathbb{Z}/r\mathbb{Z}. \tag{A.4.1.4}$$

Le diagramme de Young d'une r-partition $\boldsymbol{\lambda} = (\lambda^{(0)}, \ldots, \lambda^{(r-1)})$ est la partie de $\mathbb{N}^3$ définie par

$$\mathcal{Y}(\boldsymbol{\lambda}) := \bigcup_{c=0}^{r-1} \left(\mathcal{Y}(\lambda^{(c)}) \times \{c\}\right).$$

Le *κ-résidu* d'un nœud $\gamma = (a, b, c) \in \mathcal{Y}(\boldsymbol{\lambda})$ est $\mathrm{res}_\kappa(\gamma) := b - a + \kappa_c \pmod e$. Pour chaque $i \in \mathbb{Z}/e\mathbb{Z}$, désignons par $n_\kappa^i(\boldsymbol{\lambda})$ sa multiplicité dans le multi-ensemble des κ-résidus des éléments de $\mathcal{Y}(\boldsymbol{\lambda})$. Nous définissons

$$\alpha_\kappa(\boldsymbol{\lambda}) := \sum_{\gamma\in\mathcal{Y}(\boldsymbol{\lambda})} \alpha_{\mathrm{res}_\kappa(\gamma)} = \sum_{i=0}^{e-1} n_\kappa^i(\boldsymbol{\lambda})\alpha_i \in Q^+.$$

D'après [LyMa], les blocs de $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q)$ partitionnent l'ensemble des r-partitions de n via l'application $\boldsymbol{\lambda} \mapsto \alpha_\kappa(\boldsymbol{\lambda})$. Finalement, soit $\boldsymbol{\lambda} = (\lambda^{(0)}, \ldots, \lambda^{(r-1)})$ une r-partition. On dit que $\boldsymbol{\lambda}$ est un *e-multi-cœur* si chaque $\lambda^{(k)}$ est un e-cœur pour $k \in \{0, \ldots, r-1\}$. Nous désignons alors par

$$x^{(k)}(\boldsymbol{\lambda}) := x(\lambda^{(k)}) \in \mathbb{Z}_0^e,$$

le paramètre du e-abaque associé au e-cœur $\lambda^{(k)}$.

A.4.1.3 Décalages

Nous sommes maintenant prêts pour définir nos deux applications de décalage.

Définition A.4.1.5. Rappelons que e est entièrement déterminé par η et p. Pour tout $i \in \mathbb{Z}/e\mathbb{Z}$ nous définissons $\sigma_{\eta,p} \cdot \alpha_i := \alpha_{i+\eta} \in Q^+$, et nous étendons $\sigma_{\eta,p}$ en une application $\mathbb{Z}$-linéaire $Q \to Q$.

Définition A.4.1.6. Rappelons que r est entièrement déterminé par d et p. Si $\boldsymbol{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(r-1)})$ est une r-partition, définissons

$$^{\sigma_{d,p}}\boldsymbol{\lambda} := \left(\lambda^{(r-d)}, \dots, \lambda^{(r-1)}, \lambda^{(0)}, \dots, \lambda^{(r-d-1)}\right).$$

Pour chaque $\alpha \in Q^+$, désignons par $\mathcal{P}^\kappa_\alpha$ le sous-ensemble de $\mathcal{P}^\kappa_n$ donné par les r-partitions $\boldsymbol{\lambda}$ telles que $\alpha_\kappa(\boldsymbol{\lambda}) = \alpha$. Les deux applications de décalage des Définitions A.4.1.5 et A.4.1.6 sont compatibles dans le sens suivant.

Lemme A.4.1.7. *Supposons que la multi-charge κ est compatible avec (d, η, p). Si $\boldsymbol{\lambda}$ est une r-partition alors $\alpha_\kappa(^{\sigma_{d,p}}\boldsymbol{\lambda}) = \sigma_{\eta,p} \cdot \alpha_\kappa(\boldsymbol{\lambda})$.*

Nous pouvons maintenant énoncer le théorème principal de cette section, dont la trame de la démonstration sera donnée en §A.4.3.

Théorème A.4.1.8. *Soit $\boldsymbol{\lambda}$ une r-partition et soit $\alpha := \alpha_\kappa(\boldsymbol{\lambda}) \in Q^+$. Supposons que κ est compatible avec (d, η, p). Si $\sigma_{\eta,p} \cdot \alpha = \alpha$ alors il existe une r-partition $\boldsymbol{\mu} \in \mathcal{P}^\kappa_\alpha$ avec $^{\sigma_{d,p}}\boldsymbol{\mu} = \boldsymbol{\mu}$.*

Nous disons qu'une r-partition $\boldsymbol{\mu}$ comme dans le Théorème A.4.1.8 est *bégayante*. Nous écrirons régulièrement σ pour désigner indifféremment $\sigma_{d,p}$ ou $\sigma_{\eta,p}$ quand le contexte ne porte pas à confusion.

Désignons par $[\boldsymbol{\lambda}]$ (respectivement par $[\alpha]$) l'orbite d'une r-partition $\boldsymbol{\lambda}$ (resp. de $\alpha \in Q^+$) sous l'action de σ.

Corollaire A.4.1.9. *Supposons que κ est compatible avec (d, η, p) et soit $\alpha \in Q^+$ tel que $\mathcal{P}^\kappa_\alpha$ est non vide. Alors $\#[\alpha]$ est le plus petit élément de l'ensemble $\{\#[\boldsymbol{\lambda}] : \boldsymbol{\lambda} \in \mathcal{P}^\kappa_\alpha\}$. En d'autres termes, si $\boldsymbol{\lambda}$ est une r-partition et $\alpha := \alpha_\kappa(\boldsymbol{\lambda})$, si $\sigma^j \cdot \alpha = \alpha$ pour un $j \in \{0, \dots, p-1\}$ alors il existe une r-partition $\boldsymbol{\mu}$ telle que $\alpha_\kappa(\boldsymbol{\mu}) = \alpha$ et $^{\sigma^j}\boldsymbol{\mu} = \boldsymbol{\mu}$.*

La proposition suivante est simple à montrer mais fondamentale pour ce qui va suivre.

Proposition A.4.1.10. *Il suffit de prouver le Théorème A.4.1.8 pour les e-multi-cœurs.*

A.4.2 Matrices binaires

Dans cette section, nous introduisons un outil technique, donné dans la Proposition A.4.2.5, dont nous avons besoin pour montrer le Théorème A.4.1.8. Nous désignons par $|\cdot| : \mathbb{R}^n \to \mathbb{R}^n$ la somme des coordonnées (nous avertissons le lecteur que nous ne prenons pas la somme des valeurs absolues).

Définition A.4.2.1. Une matrice est *binaire* si ses coefficients sont dans $\{0, 1\}$.

Le résultat suivant est bien connu.

Lemme A.4.2.2. *Soit $w_0, \dots, w_{n-1} \in \{0, \dots, p\}$. Pour chaque $i \in \{0, \dots, n-1\}$ nous définissons $v_i := \frac{w_i}{p}$ ainsi que $v := (v_0, \dots, v_{n-1}) \in [0,1]^n$. Il existe des vecteurs $\epsilon^0, \dots, \epsilon^{p-1} \in \{0,1\}^n$ tels que*

$$v = \frac{1}{p}\sum_{j=0}^{p-1} \epsilon^j.$$

En particulier,

$$\frac{1}{p}\sum_{j=0}^{p-1} |\epsilon^j| = |v|.$$

Si de plus $|v| \in \mathbb{N}$ alors pour tout $j \in \{0, \dots, p-1\}$ nous pouvons choisir ϵ^j tel que $|\epsilon^j| = |v|$.

Ce dernier résultat est équivalent à l'existence d'une matrice binaire $p \times n$ de sommes $(|v|, \dots, |v|)$ sur les lignes et $(w_0, \dots, w_{n-1})$ sur les colonnes. Une telle matrice existe d'après un résultat général de [Ga, Ry]. La preuve de Ryser [Ry] utilise des *interversions* entre deux matrices binaires : cela consiste à remplacer une sous-matrice $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ par $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ et vice versa. Cette même stratégie est utilisée pour donner dans la Proposition A.4.2.5 une version plus forte du Lemme A.4.2.2.

Introduisons d'abord quelques notations. Pour tout $\ell \in \{0, \dots, d-1\}$ et $i \in \{0, \dots, e-1\}$, soit $w_i^{(\ell)} \in \{0, \dots, p\}$ et posons $v_i^{(\ell)} := \frac{w_i^{(\ell)}}{p}$. Pour chaque $\ell \in \{0, \dots, d-1\}$, définissons

$$v^{(\ell)} := (v_0^{(\ell)}, \dots, v_{e-1}^{(\ell)}).$$

Nous obtenons une matrice $d \times e$

$$V := \begin{pmatrix} v^{(0)} \\ \vdots \\ v^{(d-1)} \end{pmatrix}.$$

Supposons que pour chaque $\ell \in \{0, \dots, d-1\}$ nous avons $|v^{(\ell)}| \in \mathbb{N}$. Ainsi, pour tout $\ell \in \{0, \dots, d-1\}$ nous pouvons appliquer le Lemme A.4.2.2. Nous obtenons des vecteurs $\epsilon^{j(\ell)} \in \{0,1\}^e$ pour chaque $j \in \{0, \dots, p-1\}$, tels que

$$v^{(\ell)} = \frac{1}{p}\sum_{j=0}^{p-1} \epsilon^{j(\ell)}, \tag{A.4.2.3}$$

et

$$|\epsilon^{j(\ell)}| = |v^{(\ell)}|. \tag{A.4.2.4}$$

Pour tout $j \in \{0, \dots, p-1\}$, définissons la matrice $d \times e$ suivante :

$$E^j := \begin{pmatrix} \epsilon^{j(0)} \\ \vdots \\ \epsilon^{j(d-1)} \end{pmatrix}.$$

Écrivons la matrice V avec η blocs de même taille $V = \begin{pmatrix} V^{[0]} & \cdots & V^{[\eta-1]} \end{pmatrix}$, et utilisons la même structure par blocs pour les matrices $E^j = \begin{pmatrix} E^{j[0]} & \cdots & E^{j[\eta-1]} \end{pmatrix}$.

Proposition A.4.2.5. *Nous conservons les notations précédentes. En plus de l'hypothèse $|v^{(\ell)}| \in \mathbb{N}$ pour chaque $\ell \in \{0, \dots, d-1\}$, supposons que pour tout $i \in \{0, \dots, \eta-1\}$ nous avons $|V^{[i]}| \in \mathbb{N}$. Alors nous pouvons choisir les vecteurs $\epsilon^{j(\ell)}$ pour tout $j \in \{0, \dots, p-1\}$ et $\ell \in \{0, \dots, d-1\}$ tels que les propriétés précédentes (A.4.2.3) et (A.4.2.4) restent vérifiées, en plus de*

$$|E^{j[i]}| = |V^{[i]}|,$$

pour tout $j \in \{0, \dots, p-1\}$ et $i \in \{0, \dots, \eta-1\}$.

A.4.3 Preuve du théorème principal

Nous sommes maintenant prêts pour prouver le Théorème A.4.1.8. Soit $\boldsymbol{\lambda}$ une r-partition et supposons que la multi-charge $\kappa \in (\mathbb{Z}/e\mathbb{Z})^r$ est compatible avec (d, η, p). Rappelons l'étape de réduction donnée à la Proposition A.4.1.10 et supposons que $\boldsymbol{\lambda}$ est un e-multi-cœur. Définissons

$$\begin{aligned} \alpha &:= \alpha_\kappa(\boldsymbol{\lambda}), & \\ x^{(k)} &:= x^{(k)}(\boldsymbol{\lambda}), & \text{pour tout } k \in \{0, \dots, r-1\}, \\ n^i &:= n^i_\kappa(\boldsymbol{\lambda}), & \text{pour tout } i \in \{0, \dots, e-1\}. \end{aligned}$$

Dans cette partie, nous supposons systématiquement que $\sigma \cdot \alpha = \alpha$. Grâce au Théorème A.4.1.2 et à l'invariance par décalage de α, nous pouvons écrire

$$n^0 =: f\big(x^{(0)}, \dots, x^{(r-1)}\big),$$

où l'application $f : (\mathbb{R}^e)^r \to \mathbb{R}$ est *fortement* convexe et symétrique. Définissons également

$$f^{\langle p \rangle}\big(x^{(0)}, \dots, x^{(d-1)}\big) := \frac{1}{p} f\big(x^{(0)}, \dots, x^{(d-1)}, \dots, x^{(0)}, \dots, x^{(d-1)}\big),$$

où dans l'expression $f\big(x^{(0)}, \dots, x^{(d-1)}, \dots, x^{(0)}, \dots, x^{(d-1)}\big)$ la séquence $x^{(0)}, \dots, x^{(d-1)}$ est répétée p fois. Pour chaque $i \in \{0, \dots, \eta - 1\}$, définissons

$$\delta_i := n^i - n^{i+1}.$$

Le point clé derrière la preuve du Théorème A.4.1.8 est le lemme suivant, qui nous ramène à un problème d'optimisation.

Lemme A.4.3.1. *Supposons que* $z^{(0)}, \dots, z^{(d-1)} \in \mathbb{Z}^e_0$ *sont tels que*

$$p f^{\langle p \rangle}\big(z^{(0)}, \dots, z^{(d-1)}\big) \leq f\big(x^{(0)}, \dots, x^{(r-1)}\big), \tag{A.4.3.2}$$

et

$$\sum_{\ell=0}^{d-1} \sum_{j=0}^{p-1} z^{(\ell)}_{i - j\eta - \kappa_\ell} = \delta_i, \tag{A.4.3.3}$$

pour tout $i \in \{0, \dots, \eta - 1\}$. *Alors le Théorème A.4.1.8 est vrai pour le e-multi-cœur* $\boldsymbol{\lambda}$ *: il existe une* r*-partition* $\boldsymbol{\mu}$ *telle que* $\alpha_\kappa(\boldsymbol{\mu}) = \alpha$ *et* ${}^\sigma\boldsymbol{\mu} = \boldsymbol{\mu}$.

Les éléments $\tilde{z}^{(0)}, \dots, \tilde{z}^{(d-1)}$ obtenus vérifient toutes les hypothèses du Lemme A.4.3.1 exceptée la suivante : ils sont en général dans $\frac{1}{p}\mathbb{Z}^e_0$ mais pas nécessairement dans $\mathbb{Z}^e_0$. Nous pouvons approcher ces points par des éléments $z^{(0)}, \dots, z^{(d-1)} \in \mathbb{Z}^e$ qui vérifient les contraintes (A.4.3.3) et qui sont dans $\mathbb{Z}^e_0$, grâce à la Proposition A.4.2.5 appliquée avec une matrice remplie des parties fractionnaires des coordonnées des vecteurs $\tilde{z}^{(\ell)}$. Nous montrons ensuite que (A.4.3.2) est préservée, grâce à la forte convexité de f.

A.4.4 Applications

Supposons que la multi-charge κ est compatible avec (d, η, p) (cf. (A.4.1.4)). Considérons le poids $\boldsymbol{\Lambda} \in \mathbb{N}^I$ donné par

$$\Lambda_i := \#\{k \in \{0, \dots, r-1\} : \kappa_k = i\},$$

pour tout $i \in I$. La condition de compatibilité pour κ donne

$$\Lambda_{i+\eta} = \Lambda_i,$$

pour tout $i \in I$. Ainsi, l'algèbre d'Ariki–Koike $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q) = \mathrm{H}_n^{\boldsymbol{\Lambda}}(q, \zeta)$ et sa sous-algèbre $\mathrm{H}_{p,n}^{\boldsymbol{\Lambda}}(q)$ sont bien définies (voir §A.2.1), où $\zeta := q^\eta$ est une racine primitive p-ième de l'unité. Remarquons que $p' = 1$ et que la relation cyclotomique (A.2.1.3) de $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q)$ est

$$\prod_{i \in I} (S - q^i)^{\Lambda_i} = \prod_{k=0}^{r-1} (S - q^{\kappa_k}) = 0.$$

Définissons

$$Q_n^\kappa := \left\{\alpha \in Q^+ : \text{il existe } \boldsymbol{\lambda} \in \mathcal{P}_n^\kappa \text{ tel que } \alpha_\kappa(\boldsymbol{\lambda}) = \alpha\right\}.$$

Soit $\alpha \in Q^+$. L'algèbre $\mathrm{H}_\alpha^{\boldsymbol{\Lambda}}(q)$ est un bloc de $\mathrm{H}_n^{\boldsymbol{\Lambda}}(q)$ si $\alpha \in Q_n^\kappa$ et est réduite à $\{0\}$ sinon. Soit $\mu : \mathrm{H}_n^{\boldsymbol{\Lambda}}(q) \to \mathrm{H}_n^{\boldsymbol{\Lambda}}(q)$ l'application linéaire définie par $\mu := \sum_{j=0}^{p-1} \sigma^j$. Nous avons $\mu(\mathrm{H}_n^{\boldsymbol{\Lambda}}(q)) = \mathrm{H}_{p,n}^{\boldsymbol{\Lambda}}(q)$. L'algèbre $\mathrm{H}_{[\alpha]}^{\boldsymbol{\Lambda}}(q) = \oplus_{\beta \in [\alpha]} \mathrm{H}_\beta^{\boldsymbol{\Lambda}}(q)$ est stable par σ, la sous-algèbre des points fixes étant

$$\mathrm{H}_{p,[\alpha]}^{\boldsymbol{\Lambda}}(q) := \mu\left(\mathrm{H}_{[\alpha]}^{\boldsymbol{\Lambda}}(q)\right).$$

L'algèbre $\mathrm{H}_{[\alpha]}^{\boldsymbol{\Lambda}}(q)$ est une *algèbre cellulaire graduée* (cf. [DJM, HuMa10]). En particulier, nous pouvons trouver une base homogène

$$\left\{c_{\mathfrak{st}}^{\boldsymbol{\lambda}} : \boldsymbol{\lambda} \in \mathcal{P}_{[\alpha]}^\kappa \text{ et } \mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\boldsymbol{\lambda})\right\},$$

avec la propriété $(c_{\mathfrak{st}}^{\boldsymbol{\lambda}})^* = c_{\mathfrak{ts}}^{\boldsymbol{\lambda}}$, où $\mathcal{P}_{[\alpha]}^\kappa := \cup_{\beta \in [\alpha]} \mathcal{P}_\beta^\kappa$ et $* : \mathrm{H}_{[\alpha]}^{\boldsymbol{\Lambda}}(q) \to \mathrm{H}_{[\alpha]}^{\boldsymbol{\Lambda}}(q)$ est l'unique anti-automorphisme d'algèbre qui est l'identité sur chaque générateur de l'algèbre de Hecke carquois cyclotomique associée (voir §A.2.2).

A.4.4.1 Cellularité de la sous-algèbre fixée

La proposition suivante est facile à montrer et ne requiert pas l'utilisation du Théorème A.4.1.8.

Proposition A.4.4.1. *Supposons $\#[\alpha] = p$. La famille*

$$\left\{\mu(c_{\mathfrak{st}}^{\boldsymbol{\lambda}}) : \boldsymbol{\lambda} \in \mathcal{P}_\alpha^\kappa \text{ et } \mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\boldsymbol{\lambda})\right\}. \tag{A.4.4.2}$$

est une base cellulaire graduée de $\mathrm{H}_{p,[\alpha]}^{\boldsymbol{\Lambda}}(q)$.

Corollary A.4.4.3. *Si p et n sont premiers entre eux alors l'algèbre $\mathrm{H}_{p,n}^{\boldsymbol{\Lambda}}(q)$ est cellulaire graduée.*

Nous voulons étudier la situation dans le cas où $\#[\alpha] < p$. Généralisant (A.4.4.2), nous pouvons donner une base de $\mathrm{H}_{p,[\alpha]}^{\boldsymbol{\Lambda}}(q)$ de la forme

$$\left\{\mu(c_{\mathfrak{st}}^{\boldsymbol{\lambda}}) : \boldsymbol{\lambda} \in \mathcal{P}_{[\alpha]}^\kappa, \mathfrak{s} \in \mathcal{T}(\boldsymbol{\lambda}), \mathfrak{t} \in \mathcal{T}_0(\boldsymbol{\lambda})\right\}, \tag{A.4.4.4}$$

où $\mathcal{T}_0(\boldsymbol{\lambda})$ est un certain sous-ensemble de $\mathcal{T}(\boldsymbol{\lambda})$. Nous obtenons

$$\dim \mathrm{H}_{p,[\alpha]}^{\boldsymbol{\Lambda}}(q) = \sum_{[\boldsymbol{\lambda}] \in \mathfrak{P}_{[\alpha]}^\kappa} \frac{p}{\#[\boldsymbol{\lambda}]} \left(\#\mathcal{T}_0[\boldsymbol{\lambda}]\right)^2, \tag{A.4.4.5}$$

où $\mathcal{T}_0[\boldsymbol{\lambda}] := \cup_{\boldsymbol{\mu} \in [\boldsymbol{\lambda}]} \mathcal{T}_0[\boldsymbol{\mu}]$ et $\mathfrak{P}_{[\alpha]}^\kappa$ est un système de représentants de $\mathcal{P}_{[\alpha]}^\kappa$ pour l'action de σ. Supposons maintenant que p est impair et que l'algèbre $\mathrm{H}_{p,[\alpha]}^{\boldsymbol{\Lambda}}(q)$ est cellulaire *adaptée*. Cela signifie que $\mathrm{H}_{p,[\alpha]}^{\boldsymbol{\Lambda}}(q)$ est cellulaire, et que (A.4.4.5) s'interprète comme la « façon naturelle » de calculer la dimension de $\mathrm{H}_{p,[\alpha]}^{\boldsymbol{\Lambda}}(q)$ en utilisant la structure cellulaire. En utilisant le Théorème A.4.1.8, nous déduisons le résultat suivant.

Proposition A.4.4.6. *Si $\#[\alpha] < p$ et p est impair alors la base (A.4.4.4) de $\mathrm{H}_{p,[\alpha]}^{\boldsymbol{\Lambda}}(q)$ n'est pas cellulaire adaptée.*

A.4.4.2 Restriction des modules de Specht

Il découle de la cellularité de $\mathrm{H}^{\boldsymbol{\Lambda}}_{[\alpha]}(q)$ que nous avons une collection de *modules cellulaires* $\{\mathcal{S}^{\boldsymbol{\lambda}} : \boldsymbol{\lambda} \in \mathcal{P}^{\kappa}_{[\alpha]}\}$, aussi appelés dans ce cas *modules de Specht*. La question de savoir si l'algèbre $\mathrm{H}^{\boldsymbol{\Lambda}}_{p,[\alpha]}(q)$ est cellulaire en général ou non est toujours ouverte, cependant Hu et Mathas [HuMa12] ont défini ce qu'ils ont appelé *modules de Specht* pour $\mathrm{H}^{\boldsymbol{\Lambda}}_{p,[\alpha]}(q)$. C'est une famille

$$\left\{\mathcal{S}^{\boldsymbol{\lambda}}_j : j \in \{0, \dots, \frac{p}{\#[\boldsymbol{\lambda}]} - 1\}\right\},$$

de $\mathrm{H}^{\boldsymbol{\Lambda}}_{p,n}(q)$-modules, la restriction de $\mathcal{S}^{\boldsymbol{\lambda}}$ en un $\mathrm{H}^{\boldsymbol{\Lambda}}_{p,[\alpha]}(q)$-module s'écrivant

$$\mathcal{S}^{\boldsymbol{\lambda}}_0 \oplus \dots \oplus \mathcal{S}^{\boldsymbol{\lambda}}_{\frac{p}{\#[\boldsymbol{\lambda}]}-1}, \tag{A.4.4.7}$$

pour tout $\boldsymbol{\lambda} \in \mathcal{P}^{\kappa}_{[\alpha]}$. Nous déduisons du Corollaire A.4.1.9 le résultat suivant.

Proposition A.4.4.8. *Le nombre maximal de facteurs dans (A.4.4.7) est $\frac{p}{\#[\alpha]}$ et cette borne est atteinte, lors de la restriction d'un module de Specht $\mathcal{S}^{\boldsymbol{\lambda}}$ avec $\boldsymbol{\lambda} \in \mathcal{P}^{\kappa}_{[\alpha]}$.*

A.5 Travaux en cours

Dans cette courte section, nous donnons un bref aperçu de nos travaux en cours.

A.5.1 Cellularité de l'algèbre de Hecke de type $G(r, p, n)$

Ce travail est en collaboration avec Jun Hu et Andrew Mathas. Nous avons entamé en §A.4.4.1 une rapide étude de la cellularité de l'algèbre $\mathrm{H}^{\boldsymbol{\Lambda}}_{p,[\alpha]}(q) = \mathrm{H}^{\boldsymbol{\Lambda}}_{[\alpha]}(q)^{\sigma}$. Cependant, exceptés certains cas simples, nous n'avons pas trouvé de base avec une « forme » cellulaire (c'est précisément notre « cellularité adaptée » de §A.4.4.1).

La raison principale est que σ se comporte mal en général vis-à-vis de la base cellulaire graduée $\{c^{\boldsymbol{\lambda}}_{\mathfrak{st}}\}$ de Hu et Mathas [HuMa10]. Remarquons que, dans le cas semi-simple, nous pouvons prouver que le morphisme σ permute les éléments de la base cellulaire précédente. Nous voudrions que ce soit toujours le cas.

Pour cela, il semblerait que l'une des bases cellulaires graduées introduites par Webster [We] et Bowman [Bow], construites à partir de l'*algèbre de Cherednik diagrammatique*, se comporte bien envers σ. Comme dans le cas semi-simple, le morphisme σ permute les éléments de la base. Nous obtenons alors une base de $\mathrm{H}^{\boldsymbol{\Lambda}}_{p,[\alpha]}(q)$ de la forme $\{c^{\lambda}_{\mathfrak{st}} : \lambda \in \Lambda, \mathfrak{s}, t \in \mathcal{T}(\lambda)\}$, qui vérifie presque les axiomes de cellularité : la condition $(c^{\lambda}_{\mathfrak{st}})^* = c^{\lambda}_{\mathfrak{ts}}$ doit être changée. Nous trouvons alors une notion un peu plus générale de cellularité, aux conséquences similaires sur la théorie des représentations.

A.5.2 Isomorphisme de décomposition pour des algèbres de Hecke carquois cyclotomiques de type B à carquois disjoints

Ce travail est en collaboration avec Loïc Poulain d'Andecy et Ruari Walker. Dans [PAWal], Poulain d'Andecy et Walker ont prouvé un analogue du Théorème A.2.2.2 pour les quotients cyclotomiques des algèbres de Hecke affines de type B et les quotients cyclotomiques des algèbres de Hecke carquois de type B, ces dernières algèbres étant une généralisation d'une famille d'algèbres introduites par Varagnolo et Vasserot [VaVa]. Le but est maintenant de donner un analogue de la version cyclotomique du Théorème A.1.2.6, concernant les algèbres de Hecke carquois à carquois non connexe, pour ces algèbres de Hecke carquois de type B. Comme avec le Théorème A.2.2.4, le but est d'obtenir un résultat d'équivalence de Morita, cette fois nouveau, pour les algèbres de Hecke cyclotomiques de type B.

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Titre : Algèbres de Hecke carquois et algèbres de Iwahori–Hecke généralisées

Mots-clés : algèbres de Hecke carquois, algèbres d'Ariki–Koike, algèbres de Yokonuma–Hecke, théorie des représentations, combinatoire algébrique

Résumé : Cette thèse est consacrée à l'étude des algèbres de Hecke carquois et de certaines généralisations des algèbres d'Iwahori–Hecke. Dans un premier temps, nous montrons deux résultats concernant les algèbres de Hecke carquois, dans le cas où le carquois possède plusieurs composantes connexes puis lorsqu'il possède un automorphisme d'ordre fini. Ensuite, nous rappelons un isomorphisme de Brundan–Kleshchev et Rouquier entre algèbres d'Ariki–Koike et certaines algèbres de Hecke carquois cyclotomiques. D'une part nous en déduisons qu'une équivalence de Morita importante bien connue entre algèbres d'Ariki–Koike provient d'un isomorphisme, d'autre part nous donnons une présentation de type Hecke carquois cyclotomique pour l'algèbre de Hecke de $G(r,p,n)$. Nous généralisons aussi l'isomorphisme de Brundan–Kleshchev pour montrer que les algèbres de Yokonuma–Hecke cyclotomiques sont des cas particuliers d'algèbres de Hecke carquois cyclotomiques. Finalement, nous nous intéressons à un problème de combinatoire algébrique, relié à la théorie des représentations des algèbres d'Ariki–Koike. En utilisant la représentation des partitions sous forme d'abaque et en résolvant, via un théorème d'existence de matrices binaires, un problème d'optimisation convexe sous contraintes à variables entières, nous montrons qu'un multi-ensemble de résidus qui est bégayant provient nécessairement d'une multi-partition bégayante.

Title: Quiver Hecke algebras and generalisations of Iwahori–Hecke algebras

Keywords: quiver Hecke algebras, Ariki–Koike algebras, Yokonuma–Hecke algebras, representation theory, algebraic combinatorics

Abstract: This thesis is devoted to the study of quiver Hecke algebras and some generalisations of Iwahori–Hecke algebras. We begin with two results concerning quiver Hecke algebras, first when the quiver has several connected components and second when the quiver has an automorphism of finite order. We then recall an isomorphism of Brundan–Kleshchev and Rouquier between Ariki–Koike algebras and certain cyclotomic quiver Hecke algebras. From this, on the one hand we deduce that a well-known important Morita equivalence between Ariki–Koike algebras comes from an isomorphism, on the other hand we give a cyclotomic quiver Hecke-like presentation for the Hecke algebra of type $G(r,p,n)$. We also generalise the isomorphism of Brundan–Kleshchev to prove that cyclotomic Yokonuma–Hecke algebras are particular cases of cyclotomic quiver Hecke algebras. Finally, we study a problem of algebraic combinatorics, related to the representation theory of Ariki–Koike algebras. Using the abacus representation of partitions and solving, via an existence theorem for binary matrices, a constrained optimisation problem with integer variables, we prove that a stuttering multiset of residues necessarily comes from a stuttering multipartition.