# Dispersive equations on manifolds

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*" Ci sono piu' connessioni possibili*
*nel cervello umano, che atomi*
*in tutto l'Universo"*
*(B. Russell),*

*"... à toi, cher Lecteur,*
*qui prêtes attention*
*à cette thèse "*

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# Abstract

The purpose of this thesis is to study the dispersive properties of the solutions of the Schrödinger, wave and heat equations and their perturbations with potential on Riemannian manifolds. Furthemore, we consider a few applications of these results to the corresponding nonlinear Cauchy problems.

A first main question studied in the present thesis is: what part of the dispersive properties is preserved if we perturb the equations with a potential term of the form $V(t,x)u$ or simply $V(x)u$? The importance of this question is clear both from the point of view of the applications, and as a first step for the general case of equations with variables coefficients.

In Chapter 2 we consider the perturbed wave equation

$$u_{tt} - \Delta u + V(x)u = 0, \qquad n = 3. \tag{0.0.1}$$

We show the dispersive estimates in the case of a small potential in the Kato class, [74], and then we extend these results under the weaker assumption that the potential belongs to a suitable Kato class (see Definition 2.2.1); the positive part of the potential can be large. This result is almost optimal results for the case of large potential [38].

We consider also the Schrödinger equation

$$\frac{1}{i}u_t - \Delta u + V(t,x)u = 0, \tag{0.0.2}$$

in arbitrary dimension $n \geq 1$. Instead of the stronger dispersive estimate, our goal here is to prove only the Strichartz estimates. We give two quite general results of this type.

In the first one, we deduce the complete Strichartz estimates for the solution of the Schrödinger equation (0.0.2) perturbed with a larger class of potentials satisfying $V \leq |x|^{-2}$, via interpolation between the endpoint and the energy estimate. These arguments are then extended to the case of a small time dependent potential $V(t,x)$.

We study also the heat equation

$$u_t - \Delta u + V(t,x)u = 0, \tag{0.0.3}$$

perturbed by a singular potential and we prove the existence of solutions, the maximum principle and the dispersive estimates.

In our second result concerning equation (0.0.2), we do not assume that the potential is small.

We study the dispersive properties of the linear Schrödinger equation with a time-dependent potential $V(t,x)$. We show that an appropriate integrability condition in space and time on $V$, i.e. the boundedness of a suitable $L_t^r L_x^s$ norm, is sufficient to prove the full set of Strichartz estimates. We also construct several counterexamples which show that our assumptions are optimal, both for local and for global Strichartz estimates, in the class of large unsigned potentials $V \in L_t^r L_x^s$.

The next chapters of the thesis are dedicated to the following question: do these techniques and ideas extend to more general equations on manifolds? We are interested in particular to investigate the extensions of these equations to more general Riemannian manifolds, and the influence of the curvature on the dispersive properties.

In Chapter 3, we deal with the case of noncompact manifolds of negative curvature. In particular, we study the Schrödinger equation perturbed with a potential $V \in L_t^r L_x^s$ on the hyperbolic spaces $\mathbb{H}^n$, obtaining suitable weighted Strichartz estimates with weights related to Banica's ([5]). As an application of these estimates, we prove the global existence of small solutions to the semilinear perturbed Schrödinger equation on $\mathbb{H}^n$; the nonlinear term may depend also on the space variables, and it is allowed to increase as $|x| \to \infty$.

In this paper, we prove Strichartz estimates for radial Schrödinger and wave equations on Damek-Ricci spaces and in particular on symmetric spaces of noncompact type and rank one, using the perturbative theory with potentials. It is natural to expect that the curvature of the manifold noncompact has some influence on the dispersive properties, indeed we obtain the weighted Strichartz estimates for the perturbed Cauchy problem.

Finally, the last Chapter 4 is devoted to the opposite situation of manifolds with positive curvature. We prove two new results about the Cauchy problem in the energy space for nonlinear Schrödinger equations on four-dimensional compact manifolds. The first one concerns global wellposedness for Hartree-type nonlinearities and includes approximations of cubic NLS on the sphere. The second one provides, in the case of zonal data on the sphere, local wellposedness for quadratic nonlinearities as well as global wellposedness for small energy data in the Hamiltonian case. Both results are based on new multilinear Strichartz-type estimates for the Schrödinger group.

# Contents

# Chapter 1

# Introduction

The main subject of this thesis is the study of the dispersive properties of some fundamental equations of mathematical physics, such as the *Schrödinger equation*

$$iu_t + \Delta u = 0,$$

the *heat equation*

$$u_t - \Delta u = 0$$

and the *wave equation*

$$u_{tt} - \Delta u = 0,$$

and their perturbations with a potential:

$$iu_t + \Delta u + V(t,x)u = 0, \qquad u_t - \Delta u + V(t,x)u = 0,$$

$$u_{tt} - \Delta u + V(t,x)u = 0.$$

Moreover, we shall study the extensions of these equations to more general Riemannian manifolds, and the influence of the curvature on the dispersive properties of the solutions. We shall also consider a few applications of these results to the corresponding nonlinear Cauchy problems.

The notion "dispersive properties" which we used above requires some explanation. It is well-known that some evolution equations of some classical waves have finite "speed of propagation". For instance, for the wave equation signals travel with speed equal to one; this means that if the initial data have support in a ball of radius $R$, the solution at time $T$ has support in a larger ball of radius $R+T$. Thus the energy of the solution spreads over a region that increases with time, and it is natural to expect that the size of the solution decreases accordingly. From a physical point of view, one can think of the waves spreading on the surface of a lake when we throw a stone: the circles become larger and larger, but the amplitude of the waves decreases until they disappear (this nice example is due to F.John). The traditional terminology for this phenomenon is the *decay of solutions* as $t \to \infty$.

But it is also well known that a similar phenomenon occurs also for other equations, even if the speed of propagation is not finite: the most important examples are the Schrödinger and the heat equation, mentioned above. For these equations it is very easy to prove the property, thanks to the explicit representation of the solutions; but it is also clear that the mechanism must be different from the wave equation. For instance, if the initial data have compact support, the solutions of these equations at time $T$ do not have compact support. In these cases, using the Fourier transform one can see that the components of the solution with different frequencies travel at different speeds. Then it is natural to think of a "cloud of particles" which have different energies, and for this reason travel at different speeds. This picture is probably at the origin of the modern terminology: in recent years, instead of *decay* of solutions, one speaks of *dispersion*, and the property is called *dispersive property*, in order to unify the cases of finite and infinite speed.

The study of these properties is of fundamental importance from several points of view. First of all, there is essential phisical importance of the study of asymptotic properties of the solutions: for instance, in scattering theory the most important problem is to determine the scattered amplitude of the waves *after* the interaction, but not the precise mechanism of the interaction. Moreover, dispersive estimates have been used as a very useful tool in many nonlinear problems; in particular, for the semilinear Schrödinger and wave equations, the modern theory of local and global well posedness is based essentially on these estimates. We mention among the others the results of global existence with small data for semilinear perturbations, and the local existence of solutions of low regularity (due to von Wahl, Strichartz, John, Pecher, Brenner, Klainerman, Kapitanski, Shatah, Struwe, Kenig, Ponce, Vega, Bourgain, Tao and many others; see the references [67], [68], [69], [112], [62], [12], [13], [90], [70], [92], [81]).

We must also mention that there is a very deep connection between dispersive estimates and some fundamental results of harmonic analysis known as *restriction properties*. The phenomenon can be described as follows: consider a function $f$ in $L^2(\mathbb{R}^n)$, and its Fourier transform $\widehat{f}$. Then we ask if it possible to restrict $\widehat{f}$ to a hypersurface $S$ of dimension smaller than $n$, and if we can estimate some norm of the restriction. In general $\widehat{f}$ is only $L^2$, and hence the restriction to $S$ has no meaning since $S$ has measure zero. But if we assume that $f$ is in $L^1$, then $\widehat{f}$ is bounded and continuous, and we can define the restriction of $\widehat{f}$ to $S$ and also estimate the maximum of $\widehat{f}\Big|_S$ with the $L^1$ norm of $f$. This argument can be extended to more general $L^p$ spaces and surfaces, and there are many deep open problems in this direction.

Now, consider for instance the solutions of the homogeneous wave or Schrödinger equation. If we take the Fourier transform of the solution with respect to space and time, we obtain a measure with support on a hypersurface (cone or hyperboloid). Then the dispersive estimates for the solution imply corresponding estimates for these measures. In other words, dispersive properties imply restriction properties, and viceversa. This connection has been used in both directions and has been intensively investigated in recent years.

We now describe our results in more details; first of all we recall some standard facts. Consider first the $n$-dimensional Schrödinger equation, with $n \geq 1$,

$$iu_t + \Delta u = 0, \qquad u(0,x) = f(x).$$

Since the solution can be represented as

$$u(t,x) = e^{it\Delta} f(x) = \frac{1}{(4\pi i t)^{n/2}} \int e^{i\frac{|x-y|^2}{4t}} f(y)dy,$$

one obtains directly the following decay estimate

$$|e^{it\Delta} f(x)| \leq C\ t^{-\frac{n}{2}} \|f\|_{L^1}. \tag{1.0.1}$$

Notice that the solution of the *heat* equation

$$u_t - \Delta u = 0, \qquad u(0,x) = f(x)$$

have a (formally) very similar representation, apart from an imaginary factor at the exponent:

$$u(t,x) = e^{-t\Delta} f(x) = \frac{1}{(4\pi t)^{n/2}} \int e^{-\frac{|x-y|^2}{4t}} f(y)dy.$$

Then by the same method we obtain

$$|e^{-t\Delta} f(x)| \leq C\ t^{-\frac{n}{2}} \|f\|_{L^1}. \tag{1.0.2}$$

The corresponding estimate for the wave equation is more delicate. Although already known in some special cases, the first complete analysis was the 1971 paper of von Wahl (see [112]), who proved that the solution to the $n$-dimensional wave equation, $n \geq 2$

$$\Box u \equiv (\partial_t^2 - \Delta)u = 0, \qquad u(0,x) = 0, \qquad u_t(0,x) = f$$

satisfies the decay estimate

$$|u(t,x)| \leq C\ (1+t)^{-\frac{n-1}{2}} \|f\|_{W^{N,1}}$$

for $N = N(n)$ large enough and where $W^{N,1}$ are the classical Sobolev spaces. This estimate was improved, extended and refined by Brenner (who introduced the use of Besov spaces), Pecher, Kapitanski, Ginibre and Velo, and others (see the references [62], [51]), and finally the following optimal estimate was obtained:

$$|u(t,x)| \leq C\, t^{-\frac{n-1}{2}} \|f\|_{\dot{B}_{1,1}^{\frac{n-1}{2}}(\mathbb{R}^n)}. \tag{1.0.3}$$

Here $\dot{B}^s_{p,q}(\mathbb{R}^n)$ is the *homogeneous Besov space* defined by

$$\|f\|^q_{\dot{B}^s_{p,q}(\mathbb{R}^n)} = \sum_{j\in\mathbb{Z}} 2^{jsq} \|\phi_j(\sqrt{-\Delta})f\|^q_{L^p} \tag{1.0.4}$$

where $\phi_j(r) = \phi_j(|x|)$ is a Paley-Littlewood partition of unity, i.e., $\phi_j(r) = \phi_0(2^{-j}r)$, $\phi_0(r) = \psi(r) - \psi(r/2)$, with $\psi(r)$ being a nonnegative function in $C_0^\infty$ such that $\psi(r) = 1$ for $r < 1$ and $\psi(r) = 0$ for $r > 2$.

These estimates are now called the $L^\infty - L^1$ *dispersive estimates.*

Starting from the dispersive estimates, it is possible to deduce several other space-time estimates which are generally called *Strichartz estimates.* Actually, the estimate originally proved by Strichartz was only a special case; his method of proof was based on techniques of harmonic analysis (e.g. Stein interpolation theorem). On the other hand, by refining the technique of Brenner and using some subtle functional analysis arguments, Ginibre and Velo [51] obtained the complete set of estimates, with the exclusion of some exceptional cases (the *endpoint* cases); the gap was finally closed by Keel and Tao [66] who gave the final form of the estimates.

For the Schrödinger equation on $\mathbb{R}^n$, the Strichartz estimates can be written in the following form:

$$\|e^{it\Delta}f\|_{L^p(I;L^q(\mathbb{R}^n))} \leq \|f\|_{L^2(\mathbb{R}^n)} \tag{1.0.5}$$

for any $f \in L^2$, any (bounded or unbounded) time interval $I \subseteq \mathbb{R}$, and for all sharp $\frac{n}{2}$-admissible couples $(p,q)$:

$$\frac{1}{p} + \frac{n}{2q} = \frac{n}{4}, \quad p,q \geq 2 \text{ and } (p,q) \neq (2,\infty). \tag{1.0.6}$$

The case $(p,q) = (2, \frac{2n}{n-2})$ is called the endpoint; estimate (1.0.5) is true also at the endpoint for $n \geq 3$. When $n = 2$ the endpoint is exactly $(p,q) = (2,\infty)$; in this case the estimate is false in general. The equivalent nonhomogeneous form of (1.0.5) is

$$\left\| \int_0^t e^{i(t-s)\Delta} F(s,x)ds \right\|_{L^p(I;L^q(\mathbb{R}^n))} \leq C\|F\|_{L^{\tilde{p}'}(I;L^{\tilde{q}'}(\mathbb{R}^n))} \tag{1.0.7}$$

for all $(p,q)$ and $(\tilde{p},\tilde{q})$ admissible, $\tilde{p}'$ and $\tilde{q}'$ being dual to $\tilde{p}$, $\tilde{q}$ respectively.

We consider now the case of the wave equation. The Strichartz estimates for the wave equation on $\mathbb{R}^n$

$$\partial_t^2 u - \Delta u = F(t,x), \qquad u(0,x) = u_0(x), \;\; \partial_t u(0,x) = u_1(x), \tag{1.0.8}$$

under the assumption that the dimensional analysis (or "gap") condition

$$\frac{1}{p}+\frac{n}{q}=\frac{n}{2}-\gamma=\frac{1}{\tilde{p}'}+\frac{n}{\tilde{q}'}-2, \tag{1.0.9}$$

holds, are the following

$$\|u\|_{L^p_t L^q_x} \leq C\left(\|u_0\|_{\dot{H}^\gamma}+\|u_1\|_{\dot{H}^{\gamma-1}}+\|F\|_{L^{\tilde{p}'}_t L^{\tilde{q}'}_x}\right), \tag{1.0.10}$$

for any data $u_0 \in \dot{H}^\gamma$, $u_1 \in \dot{H}^{\gamma-1}$, $F \in L^{\tilde{p}'}_I L^{\tilde{q}'}$, any (bounded or unbounded) time interval $I \subseteq \mathbb{R}$, and for all $\frac{n-1}{2}$-admissible couples $(p,q)$, $(\tilde{p},\tilde{q})$, i.e. such that

$$\frac{1}{p}+\frac{n-1}{2q} \leq \frac{n-1}{4}, \quad p \in ]2,\infty] \text{ and } q \in \left[2, \frac{2(n-1)}{n-3}\right[, \; n \geq 3. \tag{1.0.11}$$

Estimate (1.0.10) is true also at the endpoint $(p,q)=(2,\frac{2(n-1)}{n-3})$ for $n \geq 4$, but is false when $n=3$.

As mentioned above, one of the most important applications of these estimates is to nonlinear evolution equations, in particular semilinear equations of the form

$$(i\partial_t - H)u = F(u), \qquad u(0,x)=f(x)$$

(to fix the ideas, we consider the case of the Schrödinger equation). The usual way to prove local existence for this type of equations is a contraction mapping method. More precisely, one considers first the linear map $\Phi\colon G \mapsto u$, where $u$ is the solution of the linear equation

$$(i\partial_t - H)u = G, \qquad u(0,x)=f(x).$$

By suitable linear estimates, which in the classical results are energy estimates, one proves that $\Phi$ is bounded between two suitable Banach spaces, $\Phi : Y_T \to X_T$; the index $T$ refers to the fact that we consider solutions defined on a bounded interval of time $0 \leq t \leq T$. Since $\Phi$ is a linear mapping, it is actually Lipschitz continuous, and the Lipschitz constant (in many cases) depends on $T$ and is small when $T$ is small. In other words, $\Phi$ is a contraction for small times. Now consider the nonlinear term $F(u)$. If we can prove that the $F(u)$ takes $X_T$ to $Y_T$ and is also Lipschitz continuous between these spaces, in other words if $F(u)$ satisfies a nonlinear estimate of the form

$$\|F(u)-F(v)\|_{Y_T} \leq \phi(\|u\|_{X_T})\|u-v\|_{X_T}$$

then the composition $\Phi(F(u))$ is a contraction on $X_T$ for small times. The fixed point is a local solution of the Cauchy problem considered.

In many situation, the linear estimate can be improved using the Strichartz estimates; this can be used for instance to obtain the local well posedness for solutions with low regularity.

Moreover, using Strichartz or more general space-time estimates, this method can be applied also for large times; the contraction property of the nonlinear term is now obtained by assuming that the initial data are small.

We mention that these techniques are not sufficient to handle more general nonlinear terms, for instance containing derivatives. For the nonlinear Schrödinger equation, this more difficult problem was studied by Bourgain, Kenig-Ponce-Vega and others ([12], [13], [67], [68]), using more refined methods, including smoothing estimates, local Morawetz estimates, and suitable modified Sobolev spaces adapted to the structure of the equation (which are now called *Bourgain spaces*). For the nonlinear wave equation and related equations and systems of mathmatical physics, including Yang-Mills, Maxwell-Klein-Gordon and others, Klainerman and his group have applied analogous method to prove delicate results of local well posedness in low regularity spaces.

We must also mention the beautiful theory developed by Burq, Gérard and Tzvetkov (see [22], [24], [25]), concerning the nonlinear Schrödinger equation on compact manifolds.

A first main question studied in the present thesis is: what part of the dispersive properties is preserved if we perturb the equations with a potential term of the form $V(t,x)u$ or simply $V(x)u$? The importance of this question is clear both from the point of view of the applications, and as a first step to the general case of equations with variables coefficients.

Notice that it is easy to destroy the dispersive properties by a potential perturbation. For instance, if we add to $-\Delta$ a *negative* potential term $V(x)u$, $V<0$, it is well known that the operator $-\Delta+V(x)$ has eigenfunctions $u(x)$ for positive eigenvalues, provided $V$ is large enough; then it is sufficient to consider the corresponding standing wave, of the form $e^{i\lambda t}u(x)$, to produce a solution of the evolution equation with a norm constant in time. Thus we see that the potential $V$ must satisfy suitable assumptions.

In particular for the Schrödinger equation perturbed with a potential independent of time, this problem has been studied by many authors. A basic general results was obtained by Journé, Soffer and Sogge [60] who proved that the dispersive estimate is still true provided the potential is nonnegative and belongs to the Schwartz class. This assumption has been relaxed and the result refined by many authors, in particular we mentions Yajima, Rodnianski, Schlag and Goldberg ([108], [88], [52]). Notice that the main problem here is to find minimal assumptions on the potential $V(x)$ which guarantee that the dispersive estimate is true; in dimension $n=1,2,3$ this program has almost been completed, while in higher dimension it is still not clear what are the minimal assumptions.

Much less is known for potentials $V(t,x)$ which depend also on time. In general one must assume that the potential is small in a suitable norm.

Rodnianski and Schlag proved the dispersive estimate for the equation

$$\frac{1}{i}u_t - \Delta u + V(t,x)u = 0, \tag{1.0.12}$$

provided the space dimension is $n = 3$ and $V$ satisfies

$$\sup_t \|V(t,\cdot)\|_{L^{3/2}(\mathbb{R}^3)} + \sup_x \int_{\mathbb{R}^3}\int_{\mathbb{R}} \frac{|V(\hat{\tau},x)|}{|x-y|} d\tau dy < \epsilon,$$

$\epsilon$ small enough. Here $V(\hat{\tau},x)$ is the Fourier transform of $V$ with respect to time.

In Chapter 2 we consider equation (1.0.12) in general dimension $n \geq 1$. Instead of the stronger dispersive estimate, our goal here is to prove only the Strichartz estimates. We give two quite general results of this type.

In the first one, we prove that the Strichartz estimates hold for (1.0.12), $n \geq 1$, under the assumption that the norm

$$\sup_{t\in\mathbb{R}} \|V(t,\cdot)\|_{L^{(\frac{n}{2},\infty)}} < \epsilon$$

is small enough. Here $L^{(\frac{n}{2},\infty)}$ is the weak Lebesgue (or Lorentz) space. Notice that, even in the special case $n = 3$, this assumption is much weaker that Rodniaski and Schlag's; indeed, the Lorentz space $L^{(\frac{3}{2},\infty)}$ contains the Lebesgue space $L^{\frac{3}{2}}$ strictly, and we make no assumption concerning the norm of the Fourier transform of $V$.

In our second result concerning equation (1.0.12), we do not assume that the potential is small. Instead, we replace this by a condition of "smallness at infinity", i.e., integrability, of the following form

$$\|V\|_{L^r(\mathbb{R};L^s(\mathbb{R}^n))} < \infty$$

where the indices satisfy

$$\frac{1}{r} + \frac{n}{2s} = 1. \tag{1.0.13}$$

We further stress that the potential $V$ can be large and also negative. Under these conditions, we prove that the Strichartz estimates are valid for any dimension $n \geq 1$. Moreover, by a suitable class of counterexamples, we prove that our assumption (1.0.13) is necessary for the Strichartz estimates to hold, at least in the class of potentials $V \in L^rL^s$.

In Chapter 2 we consider also the perturbed wave equation

$$u_{tt} - \Delta u + V(x)u = 0, \qquad n \geq 2. \tag{1.0.14}$$

For this equation, Beals and Strauss proved the dispersive estimate provided the potential is nonnegative (or small) and in the Schwarz class ([7], [8]). As for the Schrödinger equation, also in this case many authors have tried to relax the assumptions on $V$, including Yajima, Cuccagna, Georgiev and Visciglia ([108], [32], [44]). We consider the special case of dimension $n = 3$, for which we have obtained a first result in the case of a small potential in the Kato class in [74], and then we extended the results to the case of a large potential, an almost optimal result in [38]. Indeed, we can prove the dispersive estimate under the quite weak assumption that the potential belongs to a suitable Kato class (see Definition 2.2.1); the positive part of the potential can be large. When the potential is large we have the additional problem of resonances and eigenvalues, and this makes the proof of the decay properties much harder.

The next chapters of the thesis are dedicated to the following question: do these techniques and ideas extend to more general equations on manifolds? We are interested in particular to the study of the dispersive properties of some evolution equations on curved manifolds.

We begin by studying, in Chapter 3, the case of noncompact manifolds of negative curvature. In this case it is natural to expect that the dispersive properties should be better than the ones in the flat case, since the solutions have more "room" to disperse.

We recall that the asymptotic properties of evolution equations on noncompact manifolds have been studied only very recently. Banica [5] considered the constant negative curvature case and studied the Schrödinger equation on the hyperbolic space $\mathbb{H}^n$. In dimension $n = 3$ she obtained a dispersive estimate with the same rate of decay $t^{-1}$ as in the flat case; however the $L^\infty$ and $L^1$ norms are replaced by suitable weighted norms, and this shows that the curvature improves the dispersion at space infinity.

In the first part of Chapter 3 we apply this result to the Schrödinger equation on $\mathbb{H}^n$ perturbed with a potential $V \in L^r_t L^s_x$; as expected, we obtain suitable weighted Strichartz estimates with weights related to Banica's. As an application of these estimates, we prove the global existence of small solutions to the semilinear perturbed Schrödinger equation on $\mathbb{H}^n$; the nonlinear term may depend also on the space variables, and it is allowed to increase as $|x| \to \infty$.

In the second part of Chapter 3 we consider also a more general class of noncompact manifolds, which are frequently called the *Damek-Ricci* spaces, also known as Harmonic AN groups; these spaces have been studied by several authors in the past 15 years ([4], [89], [11], [10], [29], [30], [33], [35], [36], [87], [100] and others). As Riemannian manifolds, these solvable Lie groups include all symmetric spaces of noncompact type and rank one, namely the hyperbolic spaces $\mathbb{H}^n(\mathbb{R})$, $\mathbb{H}^n(\mathbb{C})$, $\mathbb{H}^n(\mathbb{H})$, $\mathbb{H}^2(\mathbb{O})$, but most of them are not symmetric, thus providing numerous counterexemples to the Linchnerowicz conjecture [35]. This was implicitely formulated in 1944 by Linchnerowicz, who showed that every harmonic manifold of dimension at most 4 is a symmetric space, leaving open the question, if this assertion remains true in every dimension. Though in 1990, Szabo proved it is true for any simply connected compact harmonic manifold ([99]), in 1992, Ewa Damek and Fulvio Ricci found a large class of non-compact harmonic manifolds which are not symmetric spaces. More details on Damek-Ricci spaces are contained in the section 3.4.1.

We restrict to the radial case, in which the Laplace operator admits a explicit description and can be reduce to the Jacobi operator. Then we prove, both for the Schrödinger and for the wave equation, suitable weighted Strichartz estimates with weights depending on the parameters of the manifold. In the special case of the three-dimensional hyperbolic space $\mathbb{H}^3$ our method allows us to reobtain Banica's dispersive estimate by a very simple proof.

The idea of our proof is to transform the equation into a new perturbed one with a suitable potential $V$ on $R^n$; then, using the results of the perturbative theory of Burq, Planchon, Stalker and Tahvildar-Zadeh [19], we can obtain the Strichartz estimates. More precisely, the radial operator $-\Delta_M$ can be reduced to an operator of the form $-\Delta + \widetilde{V}$, where the potential $\widetilde{V}$ has a critical decay $\sim |x|^{-2}$ and can be treated by the methods of [21].

It is interesting to note that we obtain the results on these noncompact manifolds as application of the perturbative theory on $R^n$, thus avoiding the difficulties caused by the geometry of these spaces.

Our first result concerns the Schrödinger equation on S; we can prove the following weighted Strichartz estimates

$$\|w_q u\|_{L^p(\mathbb{R},L^y(S))} \leq C\|w_2\, u_0\|_{L^2(S)} + C\|w_{\widetilde{q}'} F\|_{L^{\tilde{p}'}(\mathbb{R},L^{\tilde{q}'}(S))},$$

with the weight

$$w_q(r) = \left(\frac{\sinh r}{r}\right)^{\frac{(m+k)}{2}(1-\frac{2}{q})} (\cosh r)^{\frac{k}{2}(1-\frac{2}{q})}.$$

Also for the wave equation on $S$ we are able to prove the following weighted Strichartz estimates

$$\|w_q u\|_{L^p(\mathbb{R},L^q(S))} \leq C\left\|\frac{u_0}{\sigma}\right\|_{H^\gamma(S)} + C\left\|\frac{u_1}{\sigma}\right\|_{H^{\gamma-1}(S)} + C\|w_{\widetilde{q}'} F\|_{L^{\tilde{p}'}(\mathbb{R},L^{\tilde{q}'}(S))},$$

with the weights

$$w_q(r) = \left(\frac{\sinh r}{r}\right)^{\frac{(m+k)}{2}(1-\frac{2}{q})} (\cosh r)^{\frac{k}{2}(1-\frac{2}{q})},$$

and

$$\sigma(r) = r^{\alpha+\frac{1}{2}}(\sinh r)^{-(\alpha+\frac{1}{2})}(\cosh r)^{-(\beta+\frac{1}{2})}.$$

Finally, the last Chapter 4 is devoted to the opposite situation of manifolds with positive curvature. In contrast with the negative curvature case, the positive curvature tends to destroy the decay properties of the equation, and in general the results both from the point of view of decay and regularity are worse than in the flat case. More precisely, we study the nonlinear Schrödinger equation on the four dimensional sphere $S^4$, or, more generally, a four dimensional compact manifold $M$. In this situation, the cubic equation

$$iu_t + \Delta_M u = (|u|^2)u,$$

is critical, and well posedness barely fails. However, if we introduce a slightly regularizing operator as follows

$$iu_t + \Delta_M u = ((1-\Delta)^{-\alpha}|u|^2)u, \quad \alpha > 0, \tag{1.0.15}$$

then the situation is greatly improved. Notice that (1.0.15) can be regarded as a natural generalization of the classical Hartree equation

$$iu_t + \Delta u = \left(|x|^{-\gamma} * |u|^2\right) u.$$

We consider (1.0.15) both on a general four-dimensional compact manifold and on the sphere $S^4$. In both cases we obtain the global well posedness in the energy space, provided $\alpha > 1/2$ in the general case and $\alpha > 0$ in the case of the sphere. The main tool here is a careful application of suitable multilinear estimates, adapted to the case of a compact manifold. These estimates are new and they are close to the restriction method of Bourgain.

In order to go below the cubic powers, but using the same multilinear techniques we are led to deal with the following quadratic equations on the sphere $S^4$:

$$i\partial_t u + \Delta u = q(u),$$

where $q(u)$ is a homogeneous quadratic polynomial in $u, \overline{u}$, i.e.,

$$q(u) = au^2 + b\overline{u}^2 + c|u|^2.$$

Notice that a subclass of these equations consists of Hamiltonian equations

$$q(u) = \frac{\partial V}{\partial \overline{u}}$$

where $V$ is a real-valued homogeneous polynomial of degree 3 in $u, \overline{u}$; with the above notation, this corresponds to $c = 2\overline{a}$. The advantage of Hamiltonian equations is the conservation of energy

$$E = \frac{1}{2}\int_M |\nabla u|^2\, dx + \int_M |V(u)\, dx = const.$$

For instance we have

$$q(u) = |u|^2 + \frac{1}{2}u^2 \quad \Longrightarrow \quad V(u) = \frac{1}{2}|u|^2(u+\overline{u}).$$

Concerning the local existence, we are able to prove a well posendess result below the energy norm, and precisely in $H^s$ for any $s > 1/2$, provided we assume that the data are "radial", which in the case of the sphere becomes the assumtpion of *zonal* initial data.

On the other hand, a general global existence result with small data meets essential difficulties. Indeed, the conservation of energy is not sufficient to prevent the blow up; we construct explicit (and easy) examples of this phenomenon. However, the possibility of constructing these blow up solutions is connected with an algebraic condition on the quadratic polynomial $q$; we are able to characterize completely the terms which give rise to blow up, and for the other cases we can prove a result of global existence with small (zonal) data in the energy space $H^1$.

The results of my thesis are contained in the following papers ([74], [38], [75], [39],[76], [77], [47]):

V.PIERFELICE; Decay estimate for the wave equation with a small potential, to appear on *NoDEA*.

P. D'ANCONA, V. PIERFELICE; On the wave equation with a large rough potential to appear on *Journal of Funct. Anal.*

V. PIERFELICE; Strichartz estimates for the Schrödinger and heat equations perturbed with singular and time dependent potentials. *Preprint 2004.*

P. D'ANCONA, V. PIERFELICE, N. VISCIGLIA; Some remarks on the Schrödinger equation with a potential in $L^r_t L^s_x$ to appear to *Mathematische Annalen.*

V. PIERFELICE; Weighted Strichartz estimates for the radial perturbed Schrödinger equation on the hyperbolic space. *Preprint 2004.*

V. PIERFELICE; Weighted Strichartz estimates for the Schrödinger and wave equations on Damek-Ricci spaces. *Preprint 2005.*

P. GÉRARD, V. PIERFELICE; Nonlinear Schödinger equation on four-dimensional compact manifolds. *Preprint 2005.*

# Chapter 2

# Dispersive equations with potential perturbations on flat manifolds

## 2.1 Introduction

In this chapter we study the dispersive properties of several perturbed evolution equations (wave, Schrödinger, heat) in the absence of curvature, i.e., on $\mathbb{R}^n$. The perturbations we consider are of potential type, both depending and not depending on time.

For the three dimensional wave equation

$$\Box u + V(x) = 0, \qquad n = 3$$

the potential will be independent of time and very rough: more precisely $V(x)$ belongs to the Kato class (see Definition 2.2.1). We shall first consider the case of a small potential, for which the proofs are simpler, and then we shall extend the results to the case of a large potential in the Kato class. When the potential is large we have the additional problem of resonances and eigenvalues, and this makes the proof of the decay properties much harder. In both cases we shall prove the dispersive estimate

$$|u(t,x)| \leq \frac{C}{t}$$

for a suitable constant $C$ depending on the initial data. These results have been published in the papers [74] and [38].

Several works have investigated the Cauchy problem for the wave equation perturbed with a potential and the dispersive estimate for it. In [8] the potential satisfies (essentially) the following decay assumption:

$$|V(x)| \leq \frac{C}{|x|^{4+\delta}}, \ |x| \geq 1,$$

for some $C, \delta > 0$, moreover $V$ must be smooth. Under this condition the authors proved $L^p - L^{p'}$ decay estimates but not the dispersive estimate, which was obtained by the same methods and under similar assumptions in [7]. These works treat also the case of dimension $n \geq 3$.

In [32] (only for the case of space dimension 3) the previous assumption is weakened and the decay required at infinity for the $C^2$ potential $V$ is the following one:

$$|D^\alpha V(x)| \leq \frac{C}{|x|^{3+\delta}}, \qquad |\alpha| \leq 2.$$

For general dimension $n$, the best results are due to Yajima, who, in a series of papers (see e.g. [106], [107]), proved the $L^p$ boundedness of the wave operator intertwining the free with the perturbed operator; as a consequence he obtains dispersive estimates for a variety of equations, including the wave equation. We should also mention that the Strichartz estimates can be proved independently of the dispersive estimates, under quite general assumptions on the perturbed operator; for a nice proof see [21]; see also [20] and [27].

In the special case of dimension $n = 3$, Georgiev and Visciglia [44] were able to prove the dispersive estimate for potentials of Hölder class $V(x) \in C^\alpha(\mathbb{R}^3 \setminus 0)$, $\alpha \in ]0,1[$, satisfying for some $\varepsilon > 0$

$$0 \leq V(x) \leq \frac{C}{|x|^{2+\varepsilon} + |x|^{2-\varepsilon}}. \tag{2.1.1}$$

One sees that the potential $V(x)$ is bounded by

$$V(x) \leq \frac{C}{|x|^{2+\varepsilon}} \qquad \text{if } |x| \geq 1,$$

and by

$$V(x) \leq \frac{C}{|x|^{2-\varepsilon}} \qquad \text{if } |x| \leq 1.$$

The last estimate shows that $V$ admits a singularity such that it is not in $L^2_{loc}(\mathbb{R}^3)$ (when $\varepsilon < \frac{1}{2}$). In fact one has $V \in L^{3/2-\delta} \cap L^{3/2+\delta}$ for $\delta$ small $(0 < \delta < 3\varepsilon/4)$.

Notice that the space of functions with bounded Kato norm contains $L^{3/2,1}$ since

$$\|V\|_K \leq C\|V\|_{L^{3/2,1}}$$

by the Hardy-Sobolev inequality. Thus from the point of view of regularity assumption

$$\|V\|_K < \infty \tag{2.1.2}$$

is weaker than (2.1.1).

The critical behavior for the potential is clearly $V \sim |x|^{-2}$. The family of radial potentials

$$V(x) = \frac{a}{|x|^2}, \text{ where } a > -\frac{(n-2)^2}{4}, \ n \geq 2,$$

is studied in the papers [78] and [19]. More precisely, in the first paper one shows that in the radial case, i.e. when the initial data are radially symmetric, the solution to the perturbed wave equation satisfies the generalized space-time Strichartz estimates (1.0.10) but not the dispersive estimate (1.0.3), as it is shown by suitable counterexamples. Since their proof was based on estimates for the elliptic operator $P_a := -\Delta + \frac{a}{|x|^2}$, the corresponding Strichartz estimates hold also for the Schrödinger equation. In the second paper these results are extended to general non radial initial data. Notice that the inverse square potential belongs to the weak $L^{3/2}_w \simeq L^{3/2,\infty}$ Lorentz space.

Thus it is natural to ask what are the weakest assumptions on the potential that imply the dispersive estimate. In section 2.3 we prove that it is sufficient to assume that $V$ belongs to a suitable Kato class of potentials, and no smoothness at all is required. The proof of this result is quite lenghty and difficult. For this reason, we decided to treat in section 2.2 the special case of a small potential satisfying the condition

$$\|V\|_K < 4\pi. \tag{2.1.3}$$

In this case the proof is easier to follows since it is based on a Neumann development of the perturbed resolvent.

For the Schrödinger equation

$$iu_t - \Delta u + V(t,x)u = 0, \qquad n \geq 2 \tag{2.1.4}$$

we shall investigate the case of time dependent potentials. In this case, for large potentials it is known that in general there is no decay.

In a classical paper, Journé, Soffer and Sogge ([60]) proved the standard dispersive estimate

$$|u(t,x)| \leq Ct^{-\frac{n}{2}}\|u(0,\cdot)\|_{L^1} \tag{2.1.5}$$

provided the time independent potential $V(x)$ is sufficiently smooth and decaying at infinity, and 0 is not a resonance. This result was improved by several authors, in the direction of requiring less regularity and decay of $V(x)$. It appears that the limiting behaviour is $V \sim |x|^{-2}$, or more generally $V \in L^{n/2}$; in dimension three Goldberg [52] recently proved that (2.1.5) holds provided $V \in L^{3/2+} \cap L^1$, and this appears to be nearly optimal.

The situation when the potential $V(t,x)$ depends also on time is much more difficult, and almost completely open. In dimension three Rodnianski and Schlag [88] were able to prove the dispersive estimate for potentials $V(t,x)$ such that the norm

$$\sup_{t\in\mathbb{R}} \|V(t,\cdot)\|_{L^{3/2}} + \sup_{x\in\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{-\infty}^{\infty} \frac{|V(\hat{\tau},x)|}{|x-y|} d\tau\, dx$$

is small enough, where $V(\hat{\tau},x)$ is the Fourier transform with respect to $t$ of $V(t,x)$. The cases of higher dimensions or large potentials are still open.

From all the above results it appears that $V \in L^{n/2}$ or $V \sim |x|^{-2}$ are both reasonable candidates for the limiting behaviour of the potential. In section 2.4 we unify these conditions and we go one step further; indeed, we consider potentials belonging to the weak Lebesgue (Lorentz) space $L^{(\frac{n}{2},\infty)}$. Since our results are based on perturbative methods we need to impose a smallness condition, however with the advantage that we can treat also time dependent potentials $V(t,x)$.

More precisely, we can prove the complete Strichartz estimates for (2.1.4) when the real valued potential $V = V(t,x)$ satisfies

$$\sup_{t\in\mathbb{R}} \|V(t,\cdot)\|_{L^{(\frac{n}{2},\infty)}} \equiv C_0 \quad \text{is small enough} \tag{2.1.6}$$

(see Theorem 4.2.24 below; see also [9] for more details on Lorentz spaces). When the potential does not depend on time we can compute the constant more accurately: the same result holds provided

$$\|V(\cdot)\|_{L^{(\frac{n}{2},\infty)}} < \frac{2n}{C_s(n-2)}, \tag{2.1.7}$$

where $C_s$ is the Strichartz constant for the unperturbed equation (see Theorem **??** below) . We mention that the case of dimension $n = 3$ and of a potential $V = V(x)$ independent of time has been considered earlier by Georgiev and Ivanov in [43].

For the heat equation the results are stronger, as natural. Indeed, in Theorem **??** we consider a real valued potential $V(x) \in L^{(\frac{n}{2},\infty)}$, which we split into positive and negative part $V(x) = V_+(x) - V_-(x)$, $V_\pm \geq 0$, and we assume that the negative part satisfies

$$\|V_-\|_{L^{(\frac{n}{2},\infty)}} \equiv C_0 < \frac{2n}{C_s(n-2)}. \tag{2.1.8}$$

Under this condition we can prove that the maximum principle holds, and as a consequence we deduce the full Strichartz estimates. When the potential is nonnegative, we can also prove the stronger $L^\infty - L^1$ estimate (2.1.5) (Proposition 5).

Finally, we study equation (2.1.4) when the potential $V(t,x)$ is large but satisfies an integrability condition of the form

$$V \in L^r_t L^s_x, \qquad \frac{1}{r} + \frac{n}{2s} = 1$$

and again we prove the complete Strichartz estimates in all dimensions. We also show that if the potential is in $V \in L^r_t L^s_x$ but $\frac{1}{r} + \frac{n}{2s} \neq 1$, the Strichartz estimates are not true. These results have been published in the papers [75] and [39].

## 2.2 The wave equation with a small rough potential

In this section, we prove a dispersive $L^\infty$ decay estimate for the wave equation perturbed with a small non smooth potential belonging to Kato class in the case three dimensional. Notice that from this estimate, following [66], one can obtain the complete set of space-time estimates as above. In order to introduce our assumption on the potential $V$ we recall the following classical definition:

**Definition 2.2.1.** The measurable function $V(x)$ on $\mathbb{R}^n$, $n \geq 3$, is said to belong to the *Kato class* if

$$\lim_{r \downarrow 0} \sup_{x \in \mathbb{R}^n} \int_{|x-y|<r} \frac{|V(y)|}{|x-y|^{n-2}} dy = 0. \tag{2.2.1}$$

Moreover, the *Kato norm* of $V(x)$ is defined as

$$\|V\|_K = \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|V(y)|}{|x-y|^{n-2}} dy. \tag{2.2.2}$$

For $n = 2$ the kernel $|x-y|^{2-n}$ is replaced by $\log(|x-y|^{-1})$.

The two notions are of course related (e.g., a compactly supported function of Kato class has a finite Kato norm, see Lemma 2.3.11 in Section 2.3).

*Remark* 2.2.1. The relevance of the Kato class in the study of Schrödinger operators is well known; full light on its importance was shed in Simon [91] and Aizenmann and Simon [2]. The stronger norm (2.2.2) was used by Rodnianski and Schlag [88] who proved the dispersive estimate for the three dimensional Schrödinger equation with a potential having both the Kato and the Rollnik norms small.

We can now state the main result of this section. Consider the Cauchy problem

$$\begin{cases} \Box u + V(x)u = 0, \ t \geq 0, \ x \in \mathbb{R}^3, \\ u(0,x) = 0, \\ u_t(0,x) = f(x), \end{cases} \tag{2.2.3}$$

then we have:

**Theorem 2.2.1.** *Assume that $V$ is a real-valued, measurable function on $\mathbb{R}^3$ such that*

$$\|V\|_K < 4\pi, \tag{2.2.4}$$

*then the solution $u(t,x)$ of (2.2.3) satisfies the dispersive estimate*

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^3)} \leq \frac{C}{t}\|f\|_{B^1_{1,1}(\mathbb{R}^3)}. \tag{2.2.5}$$

*Remark* 2.2.1. It is natural to expect that the estimate (2.2.5) holds with the homogeneous Besov spaces $\dot{B}^1_{1,1}(\mathbb{R}^3)$ instead of $B^1_{1,1}(\mathbb{R}^3)$. Indeed, in the next section we shall show that this can be obtained by a much more complex proof; the interest of (2.2.5) is mainly in the simplicity of the arguments used.

### 2.2.1 Properties of perturbed operator

We denote by $H_0$ the Laplace operator $-\Delta$ as a self-adjoint operator on $L^2(\mathbb{R}^3)$ with dense domain $H^2(\mathbb{R}^3)$. In this section we shall only consider the case of a small potential, since the proofs are simpler; but the following lemma can be extended also to potentials with a large positive part, as we shall show in the next section. Thus we have:

**Lemma 2.2.1.** *Let $V$ be a real-valued function on $\mathbb{R}^3$ such that*

$$\|V\|_K < 4\pi. \tag{2.2.6}$$

*Then there exists a unique non-negative self-adjoint operator $-\Delta_V = -\Delta + V$ with $\mathcal{D}(-\Delta_V) = H^2(\mathbb{R}^3)$ such that*

$$(\varphi, (-\Delta + V)\psi)_{L^2} = (\varphi, -\Delta\psi)_{L^2} + (V\varphi, \psi)_{L^2}\,, \quad \forall\, \varphi, \psi \in H^2(\mathbb{R}^3). \tag{2.2.7}$$

*Proof.* To prove this fact we can use the KLMN Theorem (see [83] Theorem 10.17), and it is sufficient to verify the following estimate

$$\int_{\mathbb{R}^3} |V(x)||\varphi(x)|^2 dx \leq a \int_{\mathbb{R}^3} |\nabla\varphi(x)|^2 dx + b\|\varphi\|^2_{L^2(\mathbb{R}^3)} \tag{2.2.8}$$

for some constants $a < 1, b > 0$. We can rewrite (2.2.8) as follows

$$|(V\varphi,\varphi)_{L^2}| \leq a(\varphi, -\Delta\varphi)_{L^2} + b\|\varphi\|^2_{L^2} = a\left\|\left(H_0 + \frac{b}{a}\right)^{\frac{1}{2}}\varphi\right\|^2_{L^2}.$$

Writing $g = \left(H_0 + \frac{b}{a}\right)^{\frac{1}{2}}\varphi$, we see that we need only to prove the following inequality

$$\left\||V|^{\frac{1}{2}}\left(H_0 + \frac{b}{a}\right)^{-\frac{1}{2}} g\right\|_{L^2} \leq a\|g\|_{L^2},$$

for some $1 > a > 0, b > 0$.

Now consider the operator $T = |V|^{\frac{1}{2}} \left(H_0 + \frac{b}{a}\right)^{-\frac{1}{2}}$ and its adjoint

$$T^* = \left(H_0 + \frac{b}{a}\right)^{-\frac{1}{2}} |V|^{\frac{1}{2}}.$$

We must prove that

$$\|TT^*\|_{L^2 \to L^2} = a < 1. \tag{2.2.9}$$

Using the explicit representation of resolvent in $\mathbb{R}^3$:

$$\left(H_0 + \frac{b}{a}\right)^{-1} \varphi = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{-\sqrt{\frac{b}{a}}|x-y|}}{|x-y|} f(y) dy, \tag{2.2.10}$$

we can write

$$\|TT^*\varphi\|_{L^2}^2 = \left\| |V|^{\frac{1}{2}} \left(H_0 + \frac{b}{a}\right)^{-1} |V|^{\frac{1}{2}} \varphi \right\|_{L^2}^2 =$$

$$= \frac{1}{(4\pi)^2} \int_{\mathbb{R}^3} |V(x)| \left| \int_{\mathbb{R}^3} \frac{e^{-\sqrt{\frac{b}{a}}|x-y|}}{|x-y|} |V(y)|^{\frac{1}{2}} |\varphi(y)| dy \right|^2 dx$$

and using the Cauchy-Schwartz inequality we have

$$\leq \frac{1}{(4\pi)^2} \int |V(x)| \left( \int \frac{e^{-\sqrt{\frac{b}{a}}|x-y|}}{|x-y|} |V(y)| dy \right) \left( \int \frac{e^{-\sqrt{\frac{b}{a}}|x-y|}}{|x-y|} |\varphi(y)|^2 dy \right) dx$$

$$\leq \frac{1}{(4\pi)^2} \int |V(x)| \left( \int \frac{|V(y)|}{|x-y|} dy \right) \left( \int \frac{|\varphi(y)|^2}{|x-y|} dy \right) dx$$

which by the definition of Kato norm $\|V\|_K$ we can estimate as follows

$$\leq \frac{\|V\|_K}{(4\pi)^2} \iint \frac{|V(x)|}{|x-y|} |\varphi(y)|^2 dy dx \leq \frac{\|V\|_K^2}{(4\pi)^2} \|\varphi\|_{L^2}^2.$$

Therefore we have

$$\|TT^*\|_{L^2} = \frac{\|V\|_K}{(4\pi)} \equiv a < 1 \tag{2.2.11}$$

by the assumption (2.2.6). Thus we have proved that $-\Delta + V$ is a self-adjoint operator with domain $H^2(\mathbb{R}^3)$. Notice that we have proved inequality (2.2.8) for all $b > 0$.

Now we prove that $-\Delta + V$ is a positive operator. Indeed

$$((-\Delta + V)\varphi, \varphi)_{L^2} = (-\Delta\varphi, \varphi)_{L^2} + (V\varphi, \varphi)_{L^2} \geq \|\nabla\varphi\|_{L^2}^2 - |(V\varphi, \varphi)_{L^2}|$$

using inequality (2.2.8) we have

$$\geq (1-a)\|\nabla\varphi\|_{L^2}^2 - b\|\varphi\|_{L^2}^2 \geq -b\|\varphi\|_{L^2}^2$$

for every $b > 0$, and this implies that

$$((-\Delta + V)\varphi, \varphi)_{L^2} \geq 0. \tag{2.2.12}$$

□

### 2.2.2 Proof of Theorem 2.2.1

The proof of Theorem 2.2.1 is based on the representation formula (see [110]) for functions of the self-adjoint operators $H$:

$$\phi(H)f = L^2 - \lim_{\varepsilon\to 0} \frac{1}{2\pi i}\int_0^\infty \phi(\lambda)[R_V(\lambda + i\varepsilon) - R_V(\lambda - i\varepsilon)]f d\lambda, \tag{2.2.13}$$

valid at least for all $f \in C_0^\infty(\mathbb{R}^3)$. Consider the following Cauchy problem

$$\begin{cases} \Box u + V(x)u = 0, \ t \geq 0, \ x \in \mathbb{R}^3, \\ u(0,x) = 0, \\ u_t(0,x) = \varphi_j(\sqrt{-\Delta_V})f(x). \end{cases} \tag{2.2.14}$$

Here $\varphi_j$, $j = 0, 1, \dots$ is a standard non homogeneous Paley-Littlewood partition of unity; we recall that $\varphi_j(\lambda) = \varphi_0(2^{-j}\lambda)$ and that

$$\psi_0 + \sum_{j\geq 0} \varphi_j = 1$$

for a suitable $\psi_0 \in C_0^\infty(\mathbb{R}^3)$.

Then the solution of (2.2.14) can be expressed as

$$u(t,x) = \mathcal{U}_V(t)\varphi_j(\sqrt{-\Delta_V})f, \tag{2.2.15}$$

where

$$\mathcal{U}_V(t) = \frac{\sin(t\sqrt{-\Delta_V})}{\sqrt{-\Delta_V}}.$$

Since $-\Delta_V$ is a self-adjoint operator we can write the solution using the spectral representation (2.2.13), i.e.

$$u(t,x) = L^2 - \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_0^\infty \varphi_j(\sqrt{\lambda})\frac{\sin t\sqrt{\lambda}}{\sqrt{\lambda}}[R_V(\lambda + i\varepsilon) - R_V(\lambda - i\varepsilon)]f d\lambda.$$

The main point in the proof of Theorem 2.2.1 are the following $L^\infty - L^1$ estimates of the resolvent $R_V(\lambda \pm i0)$ and its square:

**Proposition 2.2.2.** *Assume that the potential* $V$ *satisfies*

$$\frac{\|V\|_K}{4\pi} < 1. \tag{2.2.16}$$

*Then for any* $\lambda \in \mathbb{R}^+, \varepsilon > 0$ *we have the following estimates:*

$$\|[R_V(\lambda + i\varepsilon) - R_V(\lambda - i\varepsilon)]f\|_{L^\infty} \leq C_V \frac{\sqrt{\lambda_\varepsilon}}{2\pi} \|f\|_{L^1}, \tag{2.2.17}$$

$$\|[R_V(\lambda + i\varepsilon)^2 - R_V(\lambda - i\varepsilon)^2]f\|_{L^\infty} \leq \frac{C_V}{8\pi\sqrt{\lambda_\varepsilon}} \|f\|_{L^1}, \tag{2.2.18}$$

*where* $C_V = \left(1 - \frac{\|V\|_K}{4\pi}\right)^{-2}$ *and*

$$\lambda_\varepsilon = \frac{\lambda + (\lambda^2 + \varepsilon^2)^{1/2}}{2} > 0.$$

Before proving Proposition 2.2.2, we show how from it the dispersive estimate follows easily. Define

$$u_\varepsilon(t, x) = \int_0^\infty \varphi_j(\sqrt{\lambda}) \frac{(-\partial_\lambda \cos \sqrt{\lambda} t)}{t} [R_V(\lambda + i\varepsilon) - R_V(\lambda - i\varepsilon)] f d\lambda,$$

so that for all $t > 0$

$$u_\varepsilon(t, \cdot) \to u(t, \cdot) \ \text{ in } \ L^2;$$

integrating by parts we have

$$u_\varepsilon = \frac{1}{t} \int_0^\infty \partial_\lambda \left( \varphi_j(\sqrt{\lambda}) [R_V(\lambda + i\varepsilon) - R_V(\lambda - i\varepsilon)] f \right) (\cos \sqrt{\lambda} t) d\lambda.$$

By the properties of the Paley-Littlewood decomposition and using the following relation

$$\partial_\lambda \left[ R_V(\lambda + i\varepsilon) - R_V(\lambda - i\varepsilon) \right] = R_V(\lambda + i\varepsilon)^2 - R_V(\lambda - i\varepsilon)^2, \tag{2.2.19}$$

we obtain

$$|u_\varepsilon| \leq \frac{1}{t} \int_0^\infty |\partial_\lambda \varphi_j'(\sqrt{\lambda})| |[R_V(\lambda + i\varepsilon) - R_V(\lambda - i\varepsilon)] f| d\lambda +$$
$$+ \frac{1}{t} \int_0^\infty \varphi_j(\sqrt{\lambda}) |[R_V(\lambda + i\varepsilon)^2 - R_V(\lambda - i\varepsilon)^2] f| d\lambda.$$

Then applying Proposition 2.2.2 and the elementary inequalities

$$\sqrt{\lambda} \leq \sqrt{\lambda_\varepsilon} \leq \sqrt{\lambda} + \sqrt{\varepsilon}$$

we obtain, since $\varphi_j(\sqrt{\lambda}) = \varphi_0(2^{-j}\sqrt{\lambda})$,

$$|u_\varepsilon| \leq C_0 \frac{C_V}{t} \int_0^\infty \left[ 2^{-j} |\varphi_0'(2^{-j}\sqrt{\lambda})| (\sqrt{\lambda} + \sqrt{\varepsilon}) + |\varphi_0(2^{-j}\sqrt{\lambda})| \right] \frac{d\lambda}{\sqrt{\lambda}}$$

and after the change of variables $\mu = 2^{-j}\sqrt{\lambda}$ we obtain

$$\|u_\varepsilon(t,\cdot)\|_{L^\infty} \leq \left(1 - \frac{\|V\|_K}{4\pi}\right)^{-2} C_1 \frac{2^j + \sqrt{\varepsilon}}{t} \|f\|_{L^1}, \tag{2.2.20}$$

for some constant $C_1$ independent of $j$ and $\varepsilon$. If now we let $\varepsilon \to 0$, and we remark that $u_\varepsilon \to u$ in $L^2$ implies the convergence a.e. for a subsequence, we obtain

$$\|u(t,\cdot)\|_{L^\infty} \leq \left(1 - \frac{\|V\|_K}{4\pi}\right)^{-2} C_1 \frac{2^j}{t} \|f\|_{L^1}. \tag{2.2.21}$$

The estimate for the term corresponding to $\psi_0$ is identical.

Now we use a standard trick: writing for $j \geq 1$

$$\widetilde{\varphi}_j = \varphi_{j-1} + \varphi_j + \varphi_{j+1}$$

we have that $\widetilde{\varphi}_j$, $j = 0, 1, 2, \ldots$ is another Paley-Littlewood decomposition with the property that $\varphi_j \equiv \widetilde{\varphi}_j \cdot \varphi_j$. Hence the Cauchy problem (2.2.14) is identical to the problem

$$\begin{cases} \Box u + V(x)u = 0, \ t \geq 0, \ x \in \mathbb{R}^3, \\ u(0,x) = 0, \\ u_t(0,x) = \widetilde{\varphi}_j(\sqrt{-\Delta_V})\varphi_j(\sqrt{-\Delta_V}) f(x) \end{cases} \tag{2.2.22}$$

and estimate (2.2.21) gives also

$$\|u(t,\cdot)\|_{L^\infty} \leq C_V \frac{2^j}{t} \|\varphi_j(\sqrt{-\Delta_V}) f\|_{L^1}. \tag{2.2.23}$$

If we now consider the original Cauchy problem (2.2.3) we obtain by linearity, after summation over $j$,

$$\|u(t,\cdot)\|_{L^\infty} \leq \frac{C_V}{t} \|f\|_{B^1_{1,1}(V)}, \tag{2.2.24}$$

$$\|f\|_{B^1_{1,1}(V)} \equiv \left( \|\psi_0(\sqrt{-\Delta_V}) f\|_{L^1} + \sum_{j=0}^{\infty} 2^j \|\varphi_j(\sqrt{-\Delta_V}) f\|_{L^1} \right)$$

where the last equality is the definition of the perturbed Besov norm $B^1_{1,1}(V)$. The final step in the proof of Theorem 1 is the inequality

$$\|f\|_{B^1_{1,1}(V)} \leq C \|f\|_{B^1_{1,1}(\mathbb{R}^3)}$$

to estimate with the standard Besov norms. This step will be completed in Section 2.2.

We now go back to the proof of Proposition 2.2.2. We split the proof into a few lemmas. An essential tool will be the explicit representation of the free resolvent $R_0$ (see [83] p 58):

$$R_0(\xi^2)g = (-\Delta - \xi^2)^{-1}g = \begin{cases} \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i\xi|x-y|}}{|x-y|}g(y)dy, & Im\ \xi > 0, \\ \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{-i\xi|x-y|}}{|x-y|}g(y)dy, & Im\ \xi < 0. \end{cases} \tag{2.2.25}$$

By elementary computations we obtain that for any $\lambda \in \mathbb{R}$ and $\varepsilon > 0$

$$R_0(\lambda \pm i\varepsilon)g(x) = \frac{1}{4\pi}\int \frac{e^{\pm i\sqrt{\lambda_\varepsilon}|x-y|}}{|x-y|}e^{-\varepsilon|x-y|/2\sqrt{\lambda_\varepsilon}}g(y)dy, \tag{2.2.26}$$

where

$$\lambda_\varepsilon = \frac{\lambda + (\lambda^2+\varepsilon^2)^{1/2}}{2} > 0. \tag{2.2.27}$$

Moreover by the resolvent identity

$$\frac{\mathrm{d}}{\mathrm{dz}}R_0(z) = R_0^2(z),$$

we can represent also the square of the resolvent:

$$R_0(\lambda \pm i\varepsilon)^2 g = \frac{1}{8\pi}\left(\pm\sqrt{\lambda_\varepsilon} + i\frac{\varepsilon}{2\sqrt{\lambda_\varepsilon}}\right)^{-1}\int e^{\left(\pm i\sqrt{\lambda_\varepsilon} - \frac{\varepsilon}{2\sqrt{\lambda_\varepsilon}}\right)|x-y|}g(y)dy. \tag{2.2.28}$$

It is easy to derive from (2.2.26) the inequality

$$|R_0(\lambda \pm i\varepsilon)g(x)| \le \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{|g(y)|}{|x-y|}dy, \tag{2.2.29}$$

which is true for all $\lambda, \varepsilon$. On the positive real axis the following well known representation holds: for any $\lambda \ge 0$,

$$R_0(\lambda \pm i0)g(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{e^{\pm i\sqrt{\lambda}|x-y|}}{|x-y|}g(y)dy, \tag{2.2.30}$$

while on the negative real axis we have (now we are outside the spectrum)

$$R_0(-\lambda)g(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{e^{-\sqrt{\lambda}\,|x-y|}}{|x-y|}g(y)dy, \qquad \lambda \ge 0. \tag{2.2.31}$$

Then we have:

**Lemma 2.2.3.** *For any $\lambda \in \mathbb{R}^+$, $\varepsilon \ge 0$ the operators $R_0(\lambda+i\varepsilon) - R_0(\lambda-i\varepsilon)$ are bounded operators in $\mathcal{L}(L^1; L^\infty)$ satisfying*

$$\|[R_0(\lambda+i\varepsilon) - R_0(\lambda-i\varepsilon)]f\|_{L^\infty} \le \frac{\sqrt{\lambda_\varepsilon}}{2\pi}\|f\|_{L^1}. \tag{2.2.32}$$

*A similar property holds for the operators* $R_0(\lambda \pm i\varepsilon)^2$ *which satisfy for* $\lambda \in \mathbb{R}$, $\varepsilon \geq 0$ *the estimate*

$$\|R_0(\lambda \pm i\varepsilon)^2 f\|_{L^\infty} \leq \frac{1}{8\pi\sqrt{\lambda_\varepsilon}}\|f\|_{L^1}. \tag{2.2.33}$$

*Finally, for any measurable function* $V(x)$ *with* $\|V\|_K < \infty$, *the operators* $VR_0(\lambda \pm i\varepsilon)$ *are bounded on* $L^1$, *the operators* $R_0(\lambda \pm i\varepsilon)V$ *are bounded on* $L^\infty$, *and we have for all* $\lambda \in \mathbb{R}$, $\varepsilon \geq 0$

$$\|R_0(\lambda \pm i\varepsilon)Vf\|_{L^\infty} \leq \frac{\|V\|_K}{4\pi}\|f\|_{L^\infty} \tag{2.2.34}$$

*and*

$$\|VR_0(\lambda \pm i\varepsilon)f\|_{L^1} \leq \frac{\|V\|_K}{4\pi}\|f\|_{L^1} \tag{2.2.35}$$

*Proof.* The estimates (2.2.32) and (2.2.33) follow easily from (2.2.25), (2.2.26), (2.2.29).

Using (2.2.29) we obtain immediately

$$|R_0(z)V(x)f(x)| \leq \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{|V(y)|}{|x-y|}|f(y)|dy,$$

and hence

$$\|R_0(z)Vf\|_{L^\infty} \leq \frac{1}{4\pi}\sup_{x\in\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{V(y)}{|x-y|}dy\|f\|_{L^\infty} = \frac{\|V\|_K}{4\pi}\|f\|_{L^\infty}.$$

In a similar way, using the explicit representation of resolvent $R_0$ we have

$$\begin{aligned}\|VR_0 f\|_{L^1} \leq \frac{1}{4\pi}\int_{\mathbb{R}^3}\left|V(x)\int_{\mathbb{R}^3}\frac{|f(y)|}{|x-y|}dy\right|dx = \\ \frac{1}{4\pi}\int\int_{\mathbb{R}^3}\frac{|V(x)f(y)|}{|x-y|}dxdy \leq \frac{\|V\|_K}{4\pi}\|f\|_{L^1}.\end{aligned} \tag{2.2.36}$$

□

**Lemma 2.2.4.** *Let* $\lambda \in \mathbb{R}$, $\varepsilon \geq 0$. *Assume that the potential* $V$ *is a real-valued, measurable function on* $\mathbb{R}^3$ *such that*

$$\frac{\|V\|_K}{4\pi} < 1. \tag{2.2.37}$$

*Then the operator* $I + R_0(\lambda \pm i\varepsilon)V$ *belongs to* $\mathcal{L}(L^\infty; L^\infty)$ *and has an inverse satisfying*

$$\|(I + R_0(\lambda \pm i\varepsilon)V)^{-1}\|_{L^\infty \to L^\infty} \leq \left(1 - \frac{\|V\|_K}{4\pi}\right)^{-1}; \tag{2.2.38}$$

*analogously we have*

$$\|(I+VR_0(\lambda \pm i\varepsilon))^{-1}\|_{L^1\to L^1} \leq \left(1-\frac{\|V\|_K}{4\pi}\right)^{-1}. \tag{2.2.39}$$

*Moreover for $\lambda \in \mathbb{R}$, $\varepsilon > 0$ the operator $I - VR_V(\lambda \pm i\varepsilon)$ belongs to $\mathcal{L}(L^1; L^1)$ with norm bounded by*

$$\|I - VR_V(\lambda \pm i\varepsilon)\|_{L^1\to L^1} \leq \left(1-\frac{\|V\|_K}{4\pi}\right)^{-1}. \tag{2.2.40}$$

*Proof.* Since $\frac{\|V\|_K}{4\pi} < 1$ by assumption, by (2.2.34) the operator $(I + R_0V)$ is invertible and the Neumann series

$$(I+R_0V)^{-1} = \sum_{k=0}^{\infty}(-1)^k(R_0V)^k$$

converges in $\mathcal{L}(L^\infty; L^\infty)$. In conclusion we have

$$\|(I+R_0V)^{-1}\|_{L^\infty\to L^\infty} \leq \frac{1}{1-\frac{\|V\|_K}{4\pi}}.$$

In a similar way, since $\frac{\|V\|_K}{4\pi} < 1$ by assumption, by (2.2.35) the operator $(I + VR_0)$ is invertible and the Neumann series

$$(I+VR_0)^{-1} = \sum_{k=0}^{\infty}(-1)^k(VR_0)^k$$

converges in $\mathcal{L}(L^1; L^1)$. Then we have

$$\|(I+VR_0(z))^{-1}\|_{L^1\to L^1} \leq \frac{1}{1-\frac{\|V\|_K}{4\pi}}.$$

Finally recalling the resolvent identity

$$R_0(z) = R_V(I+VR_0),$$

since $(I + VR_0)$ is invertible in $L^1$ as proved above, we can write

$$(I-VR_V) = (I - VR_0(I+VR_0)^{-1}),$$

and (2.2.39) implies that $(I - VR_V) : L^1 \to L^1$ with norm

$$\|(I-VR_V)\|_{L^1\to L^1} \leq \frac{1}{1-\frac{\|V\|_K}{4\pi}}. \tag{2.2.41}$$

This concludes the proof of the Lemma. □

**Lemma 2.2.5.** *Assume $V$ satisfies* (2.2.37). *Then for all $z = \lambda + i\varepsilon$ with $\lambda \in \mathbb{R}$, $\varepsilon > 0$ the following identity holds:*

$$R_V(z) - R_V(\bar{z}) = (I + R_0(\bar{z})V)^{-1}[R_0(z) - R_0(\bar{z})](I - VR_V(z)) \quad (2.2.42)$$

*and defines a bounded operator in $\mathcal{L}(L^1; L^\infty)$. Moreover, we have the estimate*

$$\|[R_V(\lambda + i\varepsilon) - R_V(\lambda - i\varepsilon)]g\|_{L^\infty} \le C_V \sqrt{\lambda_\varepsilon} \|g\|_{L^1} \quad (2.2.43)$$

*where $C_V = (1 - \|V\|_K/(4\pi))^{-2}$.*

*Proof.* Thanks to Lemma 2.2.4, we can write the following identities for the resolvent operator $R_V$

$$R_V(z) = (I + R_0(z)V)^{-1} R_0(z), \quad (2.2.44)$$

$$R_V(z) = R_0(z)(I + VR_0(z))^{-1}, \quad (2.2.45)$$

$$R_V(z) = R_0(z)(I - VR_V(z)). \quad (2.2.46)$$

Then we can write

$$R_V(z) - R_V(\bar{z}) = R_0(z) - R_0(\bar{z}) - R_0(z)VR_V(z) + R_0(\bar{z})VR_V(\bar{z});$$

adding and subtracting $R_0(\bar{z})VR_V(z)$, and factorizing leads to

$$= (R_0(z) - R_0(\bar{z})) - (R_0(z) - R_0(\bar{z}))\, VR_V(z) - R_0(\bar{z})V\, (R_V(z) - R_V(\bar{z}))$$

whence (2.2.42) follows easily. The bound of this operator is an obvious consequence of Lemmas 2.2.3 and 2.2.4. □

We have proved the first half of Proposition 2.2.2. The second part is a consequence of the following Lemma:

**Lemma 2.2.6.** *Assume $V$ satisfies* (2.2.37). *Then for all $\lambda \in \mathbb{R}$, $\varepsilon > 0$ the following identity holds:*

$$R_V(\lambda \pm i\varepsilon)^2 = (I + R_0(\lambda \pm i\varepsilon)V)^{-1} R_0(\lambda \pm i\varepsilon)^2 (I + VR_0(\lambda \pm i\varepsilon))^{-1} \quad (2.2.47)$$

*and defines a bounded operator in $\mathcal{L}(L^1; L^\infty)$. Moreover, we have the estimate*

$$\|R_V(\lambda \pm i\varepsilon)^2 g\|_{L^\infty} \le \frac{C_V}{8\pi\sqrt{\lambda_\varepsilon}} \|g\|_{L^1} \quad (2.2.48)$$

*where $C_V = (1 - \|V\|_K/(4\pi))^{-2}$.*

*Proof.* The proof is analogous to the proof of the Lemma 2.2.5, and follows from the identities (2.2.44), (2.2.45), and from the properties proved in Lemma 2.2.3 □

### 2.2.3 The equivalence $B^1_{1,1}(V) \simeq B^1_{1,1}(\mathbb{R}^3)$

The main purpose of this section is to prove the equivalence between non homogeneous, perturbed Besov spaces and non homogeneous classic Besov spaces. This fact concludes the proof of Theorem 2.2.1, because it implies that from (2.2.24) we obtain the following dispersive estimate

$$\|u(t,\cdot)\|_{L^\infty} \leq C\frac{C_V}{t}\|f\|_{B^1_{1,1}(\mathbb{R}^3)}, \tag{2.2.49}$$

where $C_V = (1 - \frac{\|V\|_K}{4\pi})^{-2}$.

Now we recall the definition of the classical non homogeneous Besov spaces.

**Definition 2.2.1.** Let $\varphi_j$, $j = 0, 1, \ldots$ be a standard non homogeneous Paley-Littlewood partition of unity; we recall that $\varphi_j(\lambda) = \varphi_0(2^{-j}\lambda)$ and that

$$\text{supp}\,\varphi_0 = \{\lambda\colon 2^{-1} \leq |\lambda| \leq 2\} \text{ such that } \varphi_0(\lambda) > 0 \text{ for } 2^{-1} \leq |\lambda| \leq 2.$$

$$\psi_0 + \sum_{j\geq 0}\varphi_j = 1,$$

for a suitable $\psi_0 \in C_0^\infty(\mathbb{R}^3)$. The non homogeneous Besov spaces $B^s_{p,q}$ are defined by

$$B^s_{p,q} = \{u\colon u \in \mathcal{S}',\ \|u\|_{B^s_{p,q}} < \infty\}, \tag{2.2.50}$$

with the norm

$$\|u\|_{B^s_{p,q}} = \|\psi_0(D)u\|_{L^p} + \left(\sum_{j=0}^{\infty} 2^{sjq}\|\varphi_j(D)u\|^q_{L^p}\right)^{\frac{1}{q}}, \tag{2.2.51}$$

where $D = \sqrt{-\Delta}$, and $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$.

Clearly, $B^s_{p,q}$ are normed linear spaces with norms $\|\cdot\|_{B^s_{p,q}}$. Moreover, they are complete and therefore Banach spaces.

In a similar way, we can define non homogeneous perturbed Besov spaces as

$$B^s_{p,q}(V) = \{u\colon u \in \mathcal{S}',\ \|u\|_{B^s_{p,q}(V)} < \infty\}, \tag{2.2.52}$$

with the norm

$$\|u\|_{B^s_{p,q}(V)} = \|\psi_0(D_V)u\|_{L^p} + \left(\sum_{j=0}^{\infty} 2^{sjq}\|\varphi_j(D_V)u\|^q_{L^p}\right)^{\frac{1}{q}}, \tag{2.2.53}$$

where $D_V = \sqrt{-\Delta_V} \equiv \sqrt{-\Delta + V}$, and $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$.

Now we see the following

**Theorem 2.2.2.** *Assume that the potential* $V$ *satisfies (2.2.37). Then the following equivalence holds:*

$$B^1_{1,1}(V) \simeq B^1_{1,1}(\mathbb{R}^3), \tag{2.2.54}$$

*i.e.*

$$\|f\|_{B^1_{1,1}(V)} \simeq \|f\|_{B^1_{1,1}(\mathbb{R}^3)}. \tag{2.2.55}$$

To prove Theorem 2.2.2 we need some Lemmas. In the following we shall use the operator

$$(-\Delta)^{-1}f = R_0(0)f = \frac{1}{4\pi}\int \frac{f(y)}{|x-y|}dy$$

which satisfies the identity

$$I = (-\Delta)R_0(0) = R_0(0)(-\Delta)$$

(see standard references) and, writing $-\Delta_V = -\Delta + V$, the operator

$$(-\Delta_V)^{-1} = R_V(0) = R_0(0)(I + VR_0(0))^{-1} = (I + R_0(0)V)^{-1}R_0(0)$$

which satisfies the analogous identities

$$R_V(0)(-\Delta_V) = R_V(0)(-\Delta + V) = (I + R_0(0)V)^{-1}R_0(0)(-\Delta + V) = I$$

and

$$(-\Delta_V)R_V(0) = I.$$

Moreover we recall that the operator $VR_0(0)$ is bounded on $L^1$ since

$$\|VR_0(0)f\|_{L^1} \le \frac{1}{4\pi}\iint \frac{|V(y)|}{|x-y|}|f(y)|dydx \le \frac{1}{4\pi}\|V\|_K\|f\|_{L^1},$$

and its dual $R_0(0)V$ is bounded on $L^\infty$ with the same norm. Thus also $VR_V(0)$ and $R_V(0)V$ given by

$$VR_V(0) = VR_0(0)(I + VR_0(0))^{-1}, \quad R_V(0)V = (I + R_0(0)V)^{-1}R_0(0)V$$

are bounded on $L^1$ and $L^\infty$ respectively, with norms

$$\|VR_V(0)\| = \|R_V(0)V\| \le \frac{\|V\|_K}{4\pi}\left(1 - \frac{\|V\|_K}{4\pi}\right)^{-1}.$$

Now we proceed as Theorem 7.1 in [44].

**Lemma 2.2.7.** *Let $\varphi_j$, $j = 0, 1, \dots$ be a standard non homogeneous Paley-Littlewood partition of unity, and let $V$ satisfy (2.2.37). Then the following inequalities hold for all $p \in [1, \infty]$*

$$\|\varphi_j(\sqrt{-\Delta_V})(-\Delta_V)^{-1}\|_{L^p \to L^p} \le C 2^{-2j},\ j \ge 0, \tag{2.2.56}$$

$$\|\varphi_j(\sqrt{-\Delta_V})(-\Delta_V)\|_{L^p \to L^p} \le C 2^{2j},\ j \ge 0, \tag{2.2.57}$$

$$\|\psi_0(\sqrt{-\Delta_V})\|_{L^p \to L^p} + \|\psi_0(\sqrt{-\Delta_V})(-\Delta_V)\|_{L^p \to L^p} \le C, \tag{2.2.58}$$

$$\|\varphi_j(\sqrt{-\Delta_V})\|_{L^p \to L^p} \le C,\ j \ge 0 \tag{2.2.59}$$

We notice that (2.2.56), (2.2.57), (2.2.58) hold also if we consider the Laplace operator $-\Delta$ instead of $-\Delta_V$ (take $V = 0$).

*Proof.* Consider

$$g(\lambda\theta) = \varphi_0(\sqrt{\lambda\theta})\lambda^{-1}\theta^{-1},$$

where $\varphi_0(\sqrt{\lambda}) \in \mathcal{C}_c^\infty$. Since our potential belongs to the Kato class and $-\Delta_V$ is a non-negative operator we can apply Theorem 2.1 in [58] and obtain the following estimate

$$\|g((-\Delta_V)\theta)\|_{L^p \to L^p} \le C,$$

where $C$ is a constant independent of $\theta \in ]0, 1]$. Thus we have

$$\|\varphi_0(\theta\sqrt{(-\Delta_V)^{-1}})(-\Delta_V)^{-1}\|_{L^p \to L^p} \le C\theta,\quad \theta \in ]0, 1],$$

we can choose $\theta = 2^{-2j}$, $j \ge 0$, and we known that $\varphi_j(\sqrt{\lambda}) = \varphi_0(2^{-j}\sqrt{\lambda})$, so this proves the first inequality of the Lemma.

As above, we consider now

$$g(\lambda\theta) = \varphi_0(\sqrt{\lambda\theta})\lambda\theta,$$

and we apply to it again Theorem 2.1 in Nakamura-Jensen. If we choose $\theta = 2^{2j}$, $j \ge 0$ we obtain the second inequality

$$\|\varphi_j(\sqrt{-\Delta_V})(-\Delta_V)\|_{L^p \to L^p} \le C 2^{2j},\ j \ge 0.$$

Finally, in a similar way choosing

$$g(\lambda) = \psi_0(\sqrt{\lambda}) \quad \text{with} \quad \theta = 1$$

or

$$g(\lambda) = \psi_0(\sqrt{\lambda})\lambda \quad \text{with} \quad \theta = 1$$

or

$$g(\lambda\theta) = \varphi_0(\sqrt{\lambda\theta})\theta^{-1},$$

we prove the last two inequalities. $\square$

**Lemma 2.2.8.** *Under the same assumptions as in the preceding Lemma we have*

$$\|\varphi_j(\sqrt{-\Delta_V})\varphi_k(\sqrt{-\Delta})\|_{L^1\to L^1} \leq C2^{-2j+2k}, \quad \forall\, j,k\geq 0. \tag{2.2.60}$$

*Proof.* We can write $\varphi_j(\sqrt{-\Delta_V})\varphi_k(\sqrt{-\Delta})$ as

$$\begin{aligned}\varphi_j(\sqrt{-\Delta_V})(-\Delta_V)^{-1}(-\Delta_V)\varphi_k(\sqrt{-\Delta}) =&\\ = \varphi_j(\sqrt{-\Delta_V})(-\Delta_V)^{-1}(-\Delta)\varphi_k(\sqrt{-\Delta})+&\\ + \varphi_j(\sqrt{-\Delta_V})(-\Delta_V)^{-1}V\varphi_k(\sqrt{-\Delta}).&\end{aligned}$$

Using (2.2.56) and (2.2.57) it is easy to see that we have the following

$$\|\varphi_j(\sqrt{-\Delta_V})(-\Delta_V)^{-1}(-\Delta)\varphi_k(\sqrt{-\Delta})\|_{L^1\to L^1} \leq C2^{-2j+2k}, \; j,k\geq 0. \tag{2.2.61}$$

Moreover we can write

$$\begin{aligned}\varphi_j(\sqrt{-\Delta_V})(-\Delta_V)^{-1}V\varphi_k(\sqrt{-\Delta}) =&\\ = \varphi_j(\sqrt{-\Delta_V})(-\Delta_V)^{-1}VR_0(0)(-\Delta)\varphi_k(\sqrt{-\Delta}),&\end{aligned}$$

and we can apply to it (2.2.56):

$$\begin{aligned}\|\varphi_j(\sqrt{-\Delta_V})(-\Delta_V)^{-1}V\varphi_k(\sqrt{-\Delta})\|_{L^1\to L^1} \leq&\\ \leq C2^{-2j}\|VR_0(0)\|_{L^1\to L^1}\|(-\Delta)\varphi_k(\sqrt{-\Delta})\|_{L^1\to L^1}.&\end{aligned}$$

by (2.2.57) in Lemma 2.2.7 we obtain

$$\|\varphi_j(\sqrt{-\Delta_V})(-\Delta_V)^{-1}V\varphi_k(\sqrt{-\Delta})\|_{L^1\to L^1} \leq C2^{-2j}\frac{\|V\|_K}{4\pi}2^{2k}, \quad \forall\, j,k\geq 0$$

and this concludes the proof.

□

Now we see the proof of Theorem 2.2.2. The first step is to prove the following inequality

$$\|f\|_{B^1_{1,1}(V)} \leq C\|f\|_{B^1_{1,1}(\mathbb{R}^3)}. \tag{2.2.62}$$

By the definition of non homogeneous perturbed Besov spaces we have, writing for brevity

$$D_V = \sqrt{-\Delta_V}, \quad D = \sqrt{-\Delta}$$

$$\|f\|_{B^1_{1,1}(V)} = \|\psi_0(D_V)f\|_{L^1} + \sum_{j=0}^{\infty} 2^j\|\varphi_j(D_V)f\|_{L^1}. \tag{2.2.63}$$

We know that

$$\psi_0(D) + \sum_{k\geq 0} \varphi_k(D) = 1,$$

and thus we have

$$\|f\|_{B^1_{1,1}(V)} \leq \|\psi_0(D_V)\psi_0(D)f\|_{L^1} + \sum_{k=0}^{\infty} \|\psi_0(D_V)\varphi_k(D)f\|_{L^1} +$$

$$+ \sum_{j=0}^{\infty} 2^j \|\varphi_j(D_V)\psi_0(D)f\|_{L^1} + \sum_{j,k\geq 0} 2^j \|\varphi_j(D_V)\varphi_k(D)f\|_{L^1}.$$

Now we estimate separately the four terms.

Applying to the first term the (2.2.58) we obtain that $\psi_0(D_V)$ is bounded on $L^1$ so that

$$\|\psi_0(D_V)\psi_0(D)f\|_{L^1} \leq C\|f\|_{L^1} \tag{2.2.64}$$

and since

$$\|f\|_{L^1} \leq \|\psi_0(D)f\|_{L^1} + \sum_{j\geq 0} \|\varphi_j(D)f\|_{L^1},$$

this is smaller than $\|f\|_{B^1_{1,1}(\mathbb{R}^3)}$.

In the same way we have for the second term

$$\sum_{k=0}^{\infty} \|\psi_0(D_V)\varphi_k(D)f\|_{L^1} \leq C \sum_{k=0}^{\infty} \|\varphi_k(D)f\|_{L^1} \leq C\|f\|_{B^1_{1,1}(\mathbb{R}^3)}$$

For the third term we can write

$$\sum_{j=0}^{\infty} 2^j \|\varphi_j(D_V)\psi_0(D)f\|_{L^1} = \sum_{j=0}^{\infty} 2^j \|\varphi_j(D_V)(-\Delta_V)^{-1}(-\Delta_V)\psi_0(D)f\|_{L^1}$$

and from (2.2.56) in Lemma 2.2.2 we have

$$\leq C \sum_{j\geq 0} 2^{-j} \|(-\Delta_V)\psi_0(D)f\|_{L^1} = C\|(-\Delta_V)\psi_0(D)f\|_{L^1} \leq$$

$$\leq C\|(-\Delta)\psi_0(D)f\|_{L^1} + C\|V\psi_0(D)f\|_{L^1};$$

by our assumption on the potential we have

$$\|V\psi_0(D)f\|_{L^1} = \|VR_0(0)(-\Delta)\psi_0(D)f\|_{L^1} \leq \frac{\|V\|_K}{4\pi} \|(-\Delta)\psi_0(D)f\|_{L^1}$$

and since $(-\Delta)\psi_0(D)$ is bounded in $L^1$ by (2.2.58), the third term is bounded by

$$\sum_{j=0}^{\infty} 2^j \|\varphi_j(D_V)\psi_0(D)f\|_{L^1} \leq C_2\|f\|_{L^1}. \tag{2.2.65}$$

Finally, we divide the fourth term in the cases $j \leq k$ and $j > k$:

$$\sum_{j,k\geq 0} 2^j \|\varphi_j(D_V)\varphi_k(D)f\|_{L^1} = \sum_{j\leq k} + \sum_{j>k}$$

for $j \leq k$ we use the fact that $\varphi_j(D_V)$ are bounded on $L^1$ with uniform norm by (2.2.59) and we obtain

$$\sum_{j\leq k} \leq \sum_{k\geq 0} \|\varphi_k(D)f\|_{L^1} \sum_{0\leq j\leq k} 2^j = 2\sum_{k\geq 0} 2^k \|\varphi_k(D)f\|_{L^1}.$$

For $j > k$, we know that $\varphi_j = \varphi_j \widetilde{\varphi_j}$ and we have

$$\sum_{j>k} 2^j \|\varphi_j(D_V)\varphi_k(D)f\|_{L^1} = \sum_{j>k} 2^j \|\varphi_j(D_V)\varphi_k(D)\widetilde{\varphi_k(D)}f\|_{L^1};$$

now applying to the last term the Lemma 2.2.8 we have

$$\sum_{j>k} 2^j \|\varphi_j(D_V)\varphi_k(D)\widetilde{\varphi_k(D)}f\|_{L^1} \leq \sum_{j>k} C 2^{k-j} 2^k \|\widetilde{\varphi_k} f\|_{L^1}$$

and since $\sum_{j>k} 2^{k-j} < 1$ we have

$$\sum_{j,k\geq 0} 2^j \|\varphi_j(D_V)\varphi_k(D)f\|_{L^1} \leq C \sum_{k\geq 0} 2^k \|\widetilde{\varphi_k(D)}f\|_{L^1}. \tag{2.2.66}$$

In conclusion, we obtain

$$\|f\|_{B^1_{1,1}(V)} \leq C\|f\|_{L^1} + C\sum_{k\geq 0} 2^k \|\widetilde{\varphi_k(D)}f\|_{L^1} \leq \|f\|_{B^1_{1,1}(\mathbb{R}^3)}. \tag{2.2.67}$$

The second step is to prove the following inequality

$$\|f\|_{B^1_{1,1}(\mathbb{R}^3)} \leq \|f\|_{B^1_{1,1}(V)}; \tag{2.2.68}$$

this is completely analogous to first step, and so the proof is concluded.

## 2.3 The wave equation with a large rough potential

We consider now the case of the wave equation

$$\Box_{1+n} u + V(x)u = 0, \qquad u(0,x) = 0, \qquad u_t(0,x) = f(x), \tag{2.3.1}$$

perturbed by a *large* potential in the Kato class.

The main new difficulty is the possibility that the operator $-\Delta + V(x)$ has eigenvalues or resonances.

As it is well known, the presence of eigenvalues or resonances can influence the decay properties of the solutions. The standard way out of this difficulty is to assume that no resonances are present on the positive real axis, and in many cases this reduces to assuming that 0 is not a resonance. In our first result this assumption takes the following form. We denote as usual by $R_0(z) = (-z-\Delta)^{-1}$ the resolvent operator of $-\Delta$, and by $R_0(\lambda \pm i0)$ the limits $\lim_{\varepsilon \downarrow 0} R(\lambda \pm i\varepsilon)$ at a point $\lambda \geq 0$. Then we assume that

> *The integral equation $f + R_0(\lambda + i0)Vf = 0$ has no nontrivial bounded solution for any $\lambda \geq 0$,*

or, equivalently,

$$f + \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} V(y)f(y)dy = 0, \quad f \in L^\infty, \quad \lambda \geq 0 \quad \Longrightarrow \quad f \equiv 0. \tag{2.3.2}$$

In several cases this assumption can be drastically weakened, as discussed below.

We can now state the first result of the section:

**Theorem 2.3.1.** *Let $V = V_1 + V_2$ be a real valued potential of Kato class. Assume that:*

*i) $V_1$ is compactly supported and has a bounded Kato norm;*

*ii) $V_2$ has a small Kato norm and precisely*

$$\|V_2\|_K \cdot \left(1 + \frac{1}{4\pi}\|V_1\|_K\right) < 4\pi; \tag{2.3.3}$$

*iii) the negative part $V_- = \max\{-V, 0\}$ satisfies*

$$\|V_-\|_K < 2\pi; \tag{2.3.4}$$

*iv) the non resonant condition (2.3.2) holds for all $\lambda \geq 0$.*

*Then any solution $u(t,x)$ to problem (2.3.1) satisfies the dispersive estimate*

$$\|u(t,\cdot)\|_{L^\infty} \leq C\, t^{-1}\|f\|_{\dot{B}^1_{1,1}(\mathbb{R}^3)}. \tag{2.3.5}$$

We give some comments on the above assumptions.

*Remark* 2.3.1. Condition (2.3.3) can be intepreted as a smallness at infinity of $V$, and is satisfied by quite a large class of potentials. For instance, assume that $V$ belongs to the Lorentz space $L^{3/2,1}(\mathbb{R}^3)$. By the extended Young inequality we have

$$\|f\|_K \leq c_0\|f\|_{L^{3/2,1}}$$

for some universal constant $c_0$. Thus we see that $V$ has a bounded Kato norm, and a similar argument shows that $V$ also belongs to the Kato class.

Moreover, if $\chi(x)$ is the characteristic function of the ball $\{|x| < 1\}$, we can decompose $V$ as follows: for any $R > 0$,

$$V = V_1 + V_2, \quad V_1 = \chi(x/R)V, \quad V_2 = (1 - \chi(x/R))V.$$

Notice that

$$\|V_2\|_K \le c_0 \|V_2\|_{L^{3/2,1}} \to 0 \quad \text{as} \quad R \to +\infty;$$

on the other hand,

$$\|V_1\|_K \le c_0 \|V_1\|_{L^{3/2,1}} \le c_0 \|V\|_{L^{3/2,1}}$$

independently of $R$, and hence

$$\|V_2\|_K \cdot \left(1 + \frac{1}{4\pi}\|V_1\|_K\right) \to 0 \quad \text{as} \quad R \to +\infty.$$

In other words, assumptions (i) and (ii) are automatically satisfied by any potential in $L^{3/2,1}$. We can sum up this argument in the following Corollary:

*Corollary* 2.3.2. *Assume the real valued potential $V$ belongs to $L^{3/2,1}$ with $\|V_-\|_K < 2\pi$ and satisfies the non resonant condition* (2.3.2)*. Then the same conclusion of Theorem 2.3.1 holds.*

In particular, this applies to potentials belonging to $L^{3/2-\delta}(\mathbb{R}^3) \cap L^{3/2+\delta}(\mathbb{R}^3)$ for some $\delta > 0$, in view of the embedding

$$L^{3/2-\delta}(\mathbb{R}^3) \cap L^{3/2+\delta}(\mathbb{R}^3) \subseteq L^{3/2,1}(\mathbb{R}^3).$$

This covers the potentials satisfying (2.1.1), as remarked above.

It is interesting to compare this to the results of Burq et al. [20], [21] concerning the inverse square potential; in the scale of Lorentz spaces we can say that the dispersive estimate holds when $V \in L^{3/2,1}$ but not when $V \in L^{3/2,\infty}$. It is not clear what can be said for potentials of Lorentz class $L^{3/2,q}$ with $1 < q < \infty$, and in particular for $L^{3/2} = L^{3/2,3/2}$.

*Remark* 2.3.2. It is a problem of independent interest to find conditions on the potential $V$ which ensure that no resonances in the sense of (2.3.2) occur on the positive real axis. A well known result in this direction was proved in [3] (see in particular Appendices 2 and 3). We briefly recall two special cases which can be applied here ($V$ is always real valued):

*Proposition* 2.3.3. *(Alsholm-Schmidt) Let $n = 3$. Assume that $V \in L^2_{loc}$ and that, for some $C, R, \epsilon > 0$, one has $|V(x)| \le C|x|^{-2-\epsilon}$ for $|x| > R$. Then property* (2.3.2) *holds for all $\lambda > 0$.*

*Proposition* 2.3.4. *(Alsholm-Schmidt) Let $n = 3$. Assume that, for some $C, R, \epsilon > 0$, one has $|V(x)| \le C|x|^{-1-\epsilon}$ for $|x| > R$. Moreover, assume that either $V \in L^1 \cap L^2$ or $\langle x \rangle^{1/2+\epsilon} V \in L^2$. Then property* (2.3.2) *holds for all $\lambda > 0$.*

Notice that the results of [3] do not apply to the potentials like (2.1.1) since the singularity $|x|^{-2+\epsilon}$ is not $L^2_{loc}$; however, in order to apply e.g. Proposition 2.3.3, it is sufficient to assume that

$$|V(x)| \le \frac{C}{|x|^{2+\varepsilon} + |x|^{3/2-\varepsilon}}. \tag{2.3.6}$$

When $V$ satisfies (2.3.6), (iii) of Theorem 2.3.1, and $\lambda = 0$ is not a resonance (in the sense of (2.3.2)), then the dispersive estimate is true.

We further stress that the above propositions do not rule out the possibility of a resonance at $\lambda = 0$. This case can be excluded (at least in the sense of (2.3.2)) if one requires a stronger decay at infinity of the potential; as an example, we can prove the following

*Theorem* 2.3.5. *Let $V_1$ be a nonnegative $L^2$ function such that $V_1(x) \le C|x|^{-3-\delta}$ ($\delta > 0$) for large $x$. Then there exists a constant $\epsilon(V_1) > 0$ such that: for all real valued functions $V_2$ of Kato class with*

$$\|V_2\|_K < \epsilon(V_1) \tag{2.3.7}$$

*and for $V = V_1 + V_2$, the solution $u(t,x)$ of problem* (2.3.1) *satisfies the dispersive estimate* (2.3.5).

In essence, this result states that the dispersive estimate holds (without additional assumptions on the resonances) for all nonnegative potentials decaying faster than $|x|^{-3}$ and for all "small enough" perturbations thereof; however, it does not give a measure of the smallness of admissible perturbations. For this, we must use Theorem 2.3.1 which requires the additional assumption (2.3.2).

*Remark* 2.3.3. In Section 2.3.6 we prove the equivalence of the standard homogeneous Besov norms with the perturbed ones, i.e., generated by the operator $-\Delta + V$:

$$\dot{B}^s_{1,q}(\mathbb{R}^n) \cong \dot{B}^s_{1,q}(V), \qquad 0 < s < 2, \quad 1 \le q \le \infty, \quad n \ge 3$$

for all potentials $V = V_+ - V_-$ with $V_\pm \ge 0$ and

$$\|V_+\|_K < \infty, \qquad \|V_-\|_K < \pi^{n/2}/\Gamma\left(\frac{n}{2} - 1\right) \tag{2.3.8}$$

(see Theorem 2.3.23). For this result, a suitable extension of some lemmas in [58]-[59] was needed, which in turn required an improvement in Simon's estimates for the Schrödinger semigroup [91]. Indeed, in Proposition 2.3.18 we prove that the semigroup $e^{t(\Delta - V)}$ has an integral kernel $k(t,x,y)$ such that ($n \ge 3$)

$$|k(t,x,y)| \le \frac{(2\pi t)^{-n/2}}{1 - 2\|V_-\|_K/c_n} e^{-|x-y|^2/8t} \tag{2.3.9}$$

and satisfies the estimate

$$\|e^{-tH}\|_{\mathcal{L}(L^p;L^q)} \leq \frac{(2\pi t)^{-\gamma}}{(1-\|V_-\|_K/c_n)^2}, \qquad \gamma = \frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right). \tag{2.3.10}$$

Thus, as a byproduct of our proof we obtain the following parabolic dispersive estimate (see Proposition 2.3.18):

*Theorem* 2.3.6. *Let $n \geq 3$, assume the potential $V(x)$ is of Kato class, has a finite Kato norm and its negative part $V_-$ satisfies*

$$\|V_-\|_K < 2\pi^{n/2}/\Gamma\left(\frac{n}{2}-1\right) \tag{2.3.11}$$

*Then the solution $u(t,x)$ to the perturbed heat equation*

$$u_t - \Delta u + V(x)u = 0, \qquad u(0,x) = f(x) \tag{2.3.12}$$

*satisfies the dispersive estimate*

$$\|u(t,\cdot)\|_{L^q} \leq Ct^{\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|f\|_{L^p}, \qquad \frac{1}{p}+\frac{1}{q}=1, \quad q\in[2,\infty]. \tag{2.3.13}$$

*Remark* 2.3.4. As noticed in [44], in dimension $n = 3$ the spectral representation of the solution and an integration by parts are sufficient to prove the dispersive estimate, provided suitable $L^1 - L^\infty$ estimates for the spectral measure are available. Here we follow a similar line of proof; however, we prefer to apply the spectral theorem outside the real axis and to prove estimates which are uniform in the imaginary part of the parameter. This approach does not require to extend the limiting absorption principle to the perturbed operator, as it would be necessary when working on the real axis. See also the previous work [76] where the case of potentials with a small Kato norm was considered.

### 2.3.1 Properties of the free resolvent

We have already studied the properties of the free resolvent in the last section; here we review and expand those results in a more systematic way.

We start from the representation of $R_0(z) = (-\Delta - z)^{-1}$ in $\mathbb{R}^3$ (see e.g. [88]):

$$R_0(\xi^2)g(x) = (-\Delta-\xi^2)^{-1}g = \begin{cases} \dfrac{1}{4\pi}\displaystyle\int_{\mathbb{R}^3}\frac{e^{i\xi|x-y|}}{|x-y|}g(y)dy & \text{for } \operatorname{Im}\xi > 0 \\[2ex] \dfrac{1}{4\pi}\displaystyle\int_{\mathbb{R}^3}\frac{e^{-i\xi|x-y|}}{|x-y|}g(y)dy & \text{for } \operatorname{Im}\xi < 0. \end{cases} \tag{2.3.14}$$

By elementary computations we obtain that for any $\lambda \in \mathbb{R}$ and $\varepsilon > 0$

$$R_0(\lambda \pm i\varepsilon)g(x) = \frac{1}{4\pi}\int \frac{e^{\pm i\sqrt{\lambda_\varepsilon}|x-y|}}{|x-y|} e^{-\varepsilon|x-y|/2\sqrt{\lambda_\varepsilon}} g(y)dy \tag{2.3.15}$$

where

$$\lambda_\varepsilon = \frac{\lambda + (\lambda^2+\varepsilon^2)^{1/2}}{2} > 0. \tag{2.3.16}$$

These formulas define bounded operators on $L^2$, provided $\varepsilon > 0$ or $\lambda < 0$. When approaching the positive real axis, i.e., as $\varepsilon \downarrow 0$, this property fails; however if we consider the limit operators for $\lambda \geq 0$

$$R_0(\lambda \pm i0)g(x) = \frac{1}{4\pi}\int \frac{e^{\pm i\sqrt{\lambda}|x-y|}}{|x-y|} g(y)dy \tag{2.3.17}$$

then the *limiting absorption principle* ensures that $R_0(\lambda \pm i0)$ are bounded from the weighted space $L^2(\langle x\rangle^s dx)$ to $L^2(\langle x\rangle^{-s}dx)$ for any $s > 1$, and actually $R_0(\lambda \pm i\varepsilon) \to R_0(\lambda \pm i0)$ in the operator norm (see e.g. [1], [57]).

For *negative* $\lambda$ the estimates are of course much stronger since we are in the resolvent set of $-\Delta$. Using

$$0 < \lambda_\varepsilon < \frac{\varepsilon}{2}, \qquad \frac{\varepsilon}{2\sqrt{\lambda_\varepsilon}} \geq \sqrt{|\lambda|} \qquad \text{for all } \lambda < 0$$

we have from (2.3.15), for all $\lambda < 0$, $\varepsilon \geq 0$

$$|R_0(\lambda \pm i\varepsilon)g(x)| \leq \frac{1}{4\pi}\int \frac{e^{-\sqrt{|\lambda|}|x-y|}}{|x-y|}|g(y)|dy \tag{2.3.18}$$

and actually for $\lambda < 0$, $\varepsilon = 0$

$$R_0(\lambda \pm i0)g(x) = \frac{1}{4\pi}\int \frac{e^{-\sqrt{|\lambda|}|x-y|}}{|x-y|}g(y)dy.$$

We collect here some immediate consequences of the above representations which will be used in the following. Since

$$[R_0(\lambda + i\varepsilon) - R_0(\lambda - i\varepsilon)]g = \frac{i}{2\pi}\int \frac{\sin(\sqrt{\lambda_\varepsilon}|x-y|)}{|x-y|} e^{-\varepsilon|x-y|/2\sqrt{\lambda_\varepsilon}} g(y)dy \tag{2.3.19}$$

we can write for all $\lambda \in \mathbb{R}$ and $\varepsilon \geq 0$

$$\|[R_0(\lambda + i\varepsilon) - R_0(\lambda - i\varepsilon)]g\|_{L^\infty} \leq \frac{\sqrt{\lambda_\varepsilon}}{2\pi}\|g\|_{L^1}. \tag{2.3.20}$$

Recalling Definition 2.2.1, a straightforward computation shows that

$$\|R_0(\lambda \pm i\varepsilon)Vg\|_{L^\infty} \leq \frac{1}{4\pi}\|V\|_K\|g\|_{L^\infty} \qquad \forall \lambda \in \mathbb{R},\ \varepsilon \geq 0 \tag{2.3.21}$$

for any measurable function $V(x)$, and in a similar way

$$\|VR_0(\lambda \pm i\varepsilon)g\|_{L^1} \leq \frac{1}{4\pi}\|V\|_K\|g\|_{L^1} \qquad \forall\lambda\in\mathbb{R},\ \varepsilon\geq 0. \tag{2.3.22}$$

Of course for *negative* $\lambda$ we have better estimates:

**Lemma 2.3.7.** *Assume $V$ is of Kato class and has a finite Kato norm. Then for all $\delta>0$ there exists $C_\delta>0$ such that*

$$\|R_0(\lambda \pm i\varepsilon)Vg\|_{L^\infty} \leq \left(\delta + C_\delta\frac{\|V\|_K}{\sqrt{|\lambda|}}\right)\|g\|_{L^\infty} \qquad \forall\lambda<0,\ \varepsilon\geq 0 \tag{2.3.23}$$

*and*

$$\|VR_0(\lambda \pm i\varepsilon)g\|_{L^1} \leq \left(\delta + C_\delta\frac{\|V\|_K}{\sqrt{|\lambda|}}\right)\|g\|_{L^1} \qquad \forall\lambda<0,\ \varepsilon\geq 0. \tag{2.3.24}$$

*Proof.* By (2.3.18) we have

$$|R_0(\lambda \pm i\varepsilon)Vg(x)| \leq \frac{1}{4\pi}\int\frac{|V(y)|}{|x-y|}|g(y)|e^{-\sqrt{|\lambda|}|x-y|}dy.$$

Now for any $r>0$ we can split the integral in two zones $|x-y|<r$ and $\geq r$; for the first piece we have

$$\frac{1}{4\pi}\int_{|x-y|<r}\frac{|V(y)|}{|x-y|}|g(y)|e^{-\sqrt{|\lambda|}|x-y|}dy \leq \frac{1}{4\pi}\int_{|x-y|<r}\frac{|V(y)|}{|x-y|}dy\|g\|_{L^\infty}$$

and this can be made smaller than $\delta\|g\|_{L^\infty}$ by the definition of Kato class (2.2.1), provided we choose $r<r(\delta)$. With this choice we can estimate the second piece as follows

$$\frac{1}{4\pi}\int_{|x-y|\geq r(\delta)}\frac{|V(y)|}{|x-y|}|g(y)|e^{-\sqrt{|\lambda|}|x-y|}dy \leq \frac{\|g\|_{L^\infty}}{4\pi r(\delta)\sqrt{|\lambda|}}\int\frac{|V(y)|}{|x-y|}dy$$

where we have used the inequality $e^{-a}\leq 1/a$, and this proves (2.3.23). Estimate (2.3.24) follows by duality. □

We shall also need estimates for the square of the resolvent $R_0(\lambda\pm i\varepsilon)^2$. Since by the resolvent identity

$$\frac{\mathrm{d}}{\mathrm{d}z}R_0(z) = R_0^2(z),$$

we have the explicit representations

$$R_0(\lambda\pm i\varepsilon)^2 g = \frac{1}{8\pi}\left(\pm\sqrt{\lambda_\varepsilon} + i\frac{\varepsilon}{2\sqrt{\lambda_\varepsilon}}\right)^{-1}\int e^{\left(\pm i\sqrt{\lambda_\varepsilon}-\frac{\varepsilon}{2\sqrt{\lambda_\varepsilon}}\right)|x-y|}g(y)dy \tag{2.3.25}$$

and

$$R_0(\lambda \pm i0)^2 g = \pm \frac{1}{8\pi\sqrt{\lambda}} \int e^{\pm i\sqrt{\lambda}|x-y|} g(y)dy. \tag{2.3.26}$$

From these relations we obtain immediately the estimate, valid for all $\lambda \in \mathbb{R}$ and $\varepsilon \geq 0$ with $(\lambda, \varepsilon) \neq (0,0)$

$$\|R_0(\lambda \pm i\varepsilon)^2 g\|_{L^\infty} \leq \frac{1}{8\pi\sqrt{\lambda_\varepsilon}} \|g\|_{L^1}. \tag{2.3.27}$$

### 2.3.2 The perturbed operator for large potentials

In Section 2.2.1 we proved the selfadjointness of the operator $-\Delta + V(x)$ for a real valued small potential in the Kato class. We show here that the same result can be proved also when the positive part of the potential is large, by a slightly more involved argument. More precisely we have:

**Lemma 2.3.8.** *Let $V = V_+ - V_-$ with $V_\pm \geq 0$ be a measurable function on $\mathbb{R}^3$ satisfying*

$$V_+ \text{ is of Kato class}, \quad \|V_-\|_K < 4\pi. \tag{2.3.28}$$

*Then the operator $-\Delta + V$ defined on $C_0^\infty(\mathbb{R}^n)$ extends to a unique nonnegative self-adjoint operator $H = -\Delta + V$ with domain $\mathcal{D}(H) = H^2(\mathbb{R}^3)$ such that*

$$(\psi, H\psi)_{L^2} = (\psi, -\Delta\psi)_{L^2} + (\psi, V\psi)_{L^2} \geq 0 \qquad \forall\, \psi \in H^2(\mathbb{R}^3). \tag{2.3.29}$$

*Proof.* We shall use the KLMN Theorem (see [91], Vol.II, Theorem 10.17). Thus it is sufficient to verify the following inequality:

$$\int_{\mathbb{R}^3} |V(x)||\varphi(x)|^2 dx \leq a \int_{\mathbb{R}^3} |\nabla\varphi(x)|^2 dx + b\|\varphi\|^2_{L^2(\mathbb{R}^3)} \tag{2.3.30}$$

for some constants $a < 1$, $b \in \mathbb{R}$ and for all test functions $\varphi$ (whence the same inequality is true for all $\varphi \in H^1$ which is the domain of the form $-(\Delta\varphi, \varphi)$).

First of all we prove that for some $a \in ]0,1[$ and for *all* $b > 0$

$$\int_{\mathbb{R}^3} V_-(x)|\varphi(x)|^2 dx \leq a\|\nabla\varphi\|^2_{L^2(\mathbb{R}^3)} + b\|\varphi\|^2_{L^2(\mathbb{R}^3)}. \tag{2.3.31}$$

This is equivalent to

$$|(V_-\varphi, \varphi)_{L^2}| \leq a(\varphi, -\Delta\varphi)_{L^2} + b\|\varphi\|^2_{L^2} = a\left\|\left(H_0 + \frac{b}{a}\right)^{\frac{1}{2}} \varphi\right\|^2_{L^2},$$

where $H_0 = -\Delta$ is the selfadjoint operator with domain $H^2(\mathbb{R}^3)$. Thus, writing $g = \left(H_0 + \frac{b}{a}\right)^{\frac{1}{2}} \varphi$, the inequality to be proved takes the form

$$\left\||V_-|^{\frac{1}{2}} \left(H_0 + \frac{b}{a}\right)^{-\frac{1}{2}} g\right\|_{L^2} \leq a\|g\|_{L^2},$$

for some $1 > a > 0$ and all $b > 0$; and this is equivalent to prove that

$$\|TT^*\|_{L^2 \to L^2} = a^2 < 1 \tag{2.3.32}$$

where we introduced the operator $T = |V_-|^{\frac{1}{2}} \left(H_0 + \frac{b}{a}\right)^{-\frac{1}{2}}$ and its adjoint

$$T^* = \left(H_0 + \frac{b}{a}\right)^{-\frac{1}{2}} |V_-|^{\frac{1}{2}}.$$

Using the explicit representation

$$\left(H_0 + \frac{b}{a}\right)^{-1} \varphi = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{-\sqrt{\frac{b}{a}}|x-y|}}{|x-y|} \varphi(y) dy$$

we can write

$$\|TT^*\varphi\|_{L^2}^2 = \left\| |V_-|^{\frac{1}{2}} \left(H_0 + \frac{b}{a}\right)^{-1} |V_-|^{\frac{1}{2}} \varphi \right\|_{L^2}^2 =$$

$$= \frac{1}{(4\pi)^2} \int |V_-(x)| \left| \int \frac{e^{-\sqrt{\frac{b}{a}}|x-y|}}{|x-y|} |V_-(y)|^{\frac{1}{2}} |\varphi(y)| dy \right|^2 dx$$

and by the Cauchy-Schwartz inequality we have

$$\leq \frac{1}{(4\pi)^2} \int |V_-(x)| \left( \int \frac{e^{-\sqrt{\frac{b}{a}}|x-y|}}{|x-y|} |V_-(y)| dy \right) \left( \int \frac{e^{-\sqrt{\frac{b}{a}}|x-y|}}{|x-y|} |\varphi(y)|^2 dy \right) dx.$$

Now by definition of Kato norm we have (for all $x$ and any $a, b > 0$)

$$\int \frac{e^{-\sqrt{\frac{b}{a}}|x-y|}}{|x-y|} |V_-(y)| dy \leq \int \frac{|V_-(y)|}{|x-y|} dy \leq \|V_-\|_K \tag{2.3.33}$$

which implies

$$\|TT^*\varphi\|_{L^2}^2 \leq \frac{\|V_-\|_K}{(4\pi)^2} \int \int |V_-(x)| \frac{e^{-\sqrt{\frac{b}{a}}|x-y|}}{|x-y|} |\varphi(y)|^2 dy dx.$$

Using again (2.3.33) we obtain

$$\|TT^*\varphi\|_{L^2}^2 \leq \frac{\|V_-\|_K^2}{(4\pi)^2} \|\varphi\|_{L^2}^2$$

which means

$$\|TT^*\|_{L^2 \to L^2} \leq \frac{\|V_-\|_K}{4\pi} \equiv a < 1 \tag{2.3.34}$$

by assumption (2.3.28), and this proves (2.3.31)

To conclude the proof it is sufficient to show that for all test functions $\varphi$, for all $a > 0$ and for *some* $b = b(a) \in \mathbb{R}$

$$\int_{\mathbb{R}^3} V_+(x)|\varphi(x)|^2 dx \leq a\|\nabla\varphi\|^2_{L^2(\mathbb{R}^3)} + b\|\varphi\|^2_{L^2(\mathbb{R}^3)} \tag{2.3.35}$$

The proof is almost identical to the above one; the only difference appears in estimate (2.3.33) where we split the integral as follows

$$\int \frac{e^{-\sqrt{\frac{b}{a}}|x-y|}}{|x-y|}|V_+(y)|dy = \int_{|x-y|<r} + \int_{|x-y|\geq r}$$

for arbitrary $r > 0$. Fix now $\delta > 0$; if we choose $r > 0$ small enough, the first integral can be made smaller than $\delta$ by assumption (2.3.28); on the other hand, with $r$ chosen, the second integral can be made smaller than $\delta$ by choosing $b$ large enough. In conclusion we have

$$\int \frac{e^{-\sqrt{\frac{b}{a}}|x-y|}}{|x-y|}|V_+(y)|dy \leq 2\delta$$

provided $b$ in (2.3.35) is large enough.

Inequality (2.3.30) is now a trivial consequence of (2.3.31) and (2.3.35); thus the assumptions of the KLMN theorem are satisfied and we can construct $H = -\Delta + V$ as a selfadjoint operator on $H^2$. To check that it is positive, we write

$$((-\Delta + V)\varphi, \varphi)_{L^2} = (-\Delta\varphi, \varphi)_{L^2} + (V\varphi, \varphi)_{L^2} \geq \|\nabla\varphi\|^2_{L^2} - |(V_-\varphi, \varphi)_{L^2}|;$$

by inequality (2.3.31) we may continue

$$\geq (1-a)\|\nabla\varphi\|^2_{L^2} - b\|\varphi\|^2_{L^2} \geq -b\|\varphi\|^2_{L^2}$$

for every $b > 0$, and this implies

$$((-\Delta + V)\varphi, \varphi)_{L^2} \geq 0. \tag{2.3.36}$$

□

*Remark* 2.3.5. The above proof can be easily extended to general dimension $n \geq 3$. Indeed, the kernel $K_M(x)$ of $(-\Delta + M)^{-1}$ for $M > 0$ satisfies

$$|K(x)| \leq \frac{1}{\alpha_n |x|^{n-2}}, \qquad \lim_{M\to+\infty} \sup_{|x|>r} e^{|x|}K(x) = 0 \tag{2.3.37}$$

for each fixed $r > 0$ (see e.g. [91], p.454), and these are exactly the properties we used in the above proof. Moreover, the constant $\alpha_n$ is well known and is equal to

$$\alpha_n = 4\pi^{n/2}/\Gamma\left(\frac{n}{2} - 1\right).$$

Thus we see that the result of Lemma 2.3.8 is true for all $n \geq 3$, provided the negative part of $V$ satisfies

$$\|V_-\|_K < 4\pi^{n/2}/\Gamma\left(\frac{n}{2}-1\right). \tag{2.3.38}$$

### 2.3.3 Spectral calculus for the perturbed operator

Lemma 2.3.8 allows us to apply the spectral theorem and hence to use the functional calculus for $H = -\Delta + V$, i.e., given any function $\phi(\lambda)$ continuous and bounded on $\mathbb{R}$, we can define the operator $\phi(H)$ on $L^2$ as

$$\phi(H)f = \frac{1}{2\pi i} \cdot L^2 - \lim_{\varepsilon \downarrow 0} \int \phi(\lambda)[R_V(\lambda + i\varepsilon) - R_V(\lambda - i\varepsilon)] f d\lambda \tag{2.3.39}$$

where

$$R_V(z) = (-\Delta + V - z)^{-1}$$

is the resolvent operator for $H$ (see e.g. Vol. II of [101]). When the limit absorption principle is satisfied, one can define the limit operators $R_V(\lambda \pm i0)$ and take the limit in the spectral formula as $\varepsilon \to 0$. Instead, here we shall use formula (2.3.39) exclusively, since our estimates will always be uniform in the parameter $\varepsilon > 0$.

For $z$ outside the positive real axis we have the well known identities

$$R_0(z) = (I + R_0(z)V)\, R_V(z) = R_V(z)\,(I + VR_0(z))\,, \tag{2.3.40}$$

and a standard way to represent $R_V(z)$ in terms of $R_0(z)$ is to construct the inverse operators $(I + R_0(z)V)^{-1}$. This is the content of the following proposition, which is the crucial result of the paper. In the following we shall consider in detail the case of dimension 3 alone, but all the results in this section can be extended to general dimension $n \geq 2$ by suitable modifications in the proofs.

**Proposition 2.3.9.** *Under the assumptions of Theorem 2.3.1 (or Theorem 2.3.5) there exists $\varepsilon_0 > 0$ such that the bounded operators $I + R_0(\lambda \pm i\varepsilon)V\colon L^\infty \to L^\infty$ are invertible for all $\lambda \in \mathbb{R}$, $0 \leq \varepsilon \leq \varepsilon_0$ with a uniform bound*

$$\|(I + R_0(\lambda \pm i\varepsilon)V)^{-1}\|_{\mathcal{L}(L^\infty;L^\infty)} \leq C \quad \text{for all } \lambda \in \mathbb{R},\ 0 \leq \varepsilon \leq \varepsilon_0. \tag{2.3.41}$$

We need a few lemmas. First of all we recall the standard $L^2$ weighted estimate of the free resolvent (see e.g. [1] or Vol.II of [57]; see also [6]):

**Lemma 2.3.10.** *For all $\lambda > 0$ and $\varepsilon \geq 0$, the free resolvent $R_0(\lambda \pm i\varepsilon)$ is a bounded operator from the weighted $L^2(\langle x\rangle^{2s}dx)$ to the weighted $L^2(\langle x\rangle^{-2s}dx)$ space for any $s > 1/2$; moreover the following estimate holds with a constant $C = C(s)$ independent of $\varepsilon$, $\lambda$:*

$$\|\langle x\rangle^{-s} R_0(\lambda \pm i\varepsilon) f\|_{L^2} \leq \frac{C}{\sqrt{\lambda}} \|\langle x\rangle^s f\|_{L^2}. \tag{2.3.42}$$

The following is an elementary but useful property of Kato class functions:

**Lemma 2.3.11.** *A compactly supported function of Kato class has a finite Kato norm.*

*Proof.* Let $V(x)$ be of Kato class with support contained in a ball $B(0,R) \subseteq \mathbb{R}^3$. Then by definition we have the uniform bound

$$\int_{|x-y|\leq 1} |V(y)|dy \leq \int_{|x-y|\leq 1} \frac{|V(y)|}{|x-y|} dy \leq C_0$$

for some $C_0$ independent of $x$; thus, covering the support of $V$ with a finite number of balls of radius 1, we see that $V \in L^1$. Hence we can write

$$\int \frac{|V(y)|}{|x-y|} dy \leq \int_{|x-y|\leq 1} \frac{|V(y)|}{|x-y|} dy + \int_{|x-y|\geq 1} \frac{|V(y)|}{|x-y|} dy \leq C_0 + \|V\|_{L^1}$$

and this concludes the proof. $\square$

The next lemma is sligthly modified from [91]:

**Lemma 2.3.12.** *If $V(x)$ is a compactly supported function in the Kato class, then there exists a sequence of functions $V_\varepsilon \in C_0^\infty(\mathbb{R}^3)$ such that $\|V_\varepsilon - V\|_K \to 0$ and $\operatorname{supp} V_\varepsilon \downarrow \operatorname{supp} V$ as $\varepsilon \to 0$. When $V \geq 0$, the functions $V_\varepsilon$ can be taken nonnegative too.*

*Proof.* By the preceding lemma $V$ has a finite Kato norm, and clearly it belongs to $L^1$. Consider now a sequence of nonnegative radial mollifiers, i.e., let $\rho(x) \in C_0^\infty(\mathbb{R}^3)$ be a nonnegative radial function with support in the ball $\{|x| \leq 1\}$ such that $\int \rho(x)dx = 1$, and set $\rho_\varepsilon(x) = \varepsilon^{-3}\rho(x/\varepsilon)$. Then we have the following standard properties of the Newton potential $1/|x|$:

$$\frac{1}{|x|} * \rho_\varepsilon \equiv \frac{1}{|x|} \quad \text{for } |x| \geq \varepsilon, \tag{2.3.43}$$

$$\frac{1}{|x|} * \rho_\varepsilon \leq \frac{1}{|x|} \quad \text{for all } |x| \neq 0. \tag{2.3.44}$$

Define now $V_\varepsilon = V * \rho_\varepsilon$; for fixed $x$ we have

$$\left| \int \frac{V(y)}{|x-y|} dy - \int \frac{V_\varepsilon(z)}{|x-z|} dz \right| = \left| \int V(y) \left( \frac{1}{|x-y|} - \int \frac{\rho_\varepsilon(z-y)}{|y-z|} dz \right) dy \right|$$

and since by (2.3.44) the term in brackets is positive,

$$\leq \int |V(y)| \left( \frac{1}{|x-y|} - \int \frac{\rho_\varepsilon(z-y)}{|y-z|} dz \right) dy \leq \int_{|x-y|<\varepsilon} \frac{|V(y)|}{|x-y|} dy$$

where in the last step we used (2.3.43). Taking the supremum in $x$, we obtain

$$\|V_\varepsilon - V\|_K \le \sup_{x\in\mathbb{R}^3} \int_{|x-y|<\varepsilon} \frac{|V(y)|}{|x-y|} dy$$

and recalling Definition 2.2.1 we conclude that $\|V_\varepsilon - V\|_K \to 0$. Finally, the support of $V_\varepsilon$ is contained in the set of points at distance $\le \varepsilon$ from the support of $V$, and clearly $V \ge 0$ implies $V_\varepsilon \ge 0$. □

We prove now a property of the squared operator $(R_0 V)^2$:

**Lemma 2.3.13.** *Let $V$ be a compactly supported function in the Kato class. Then for all $\lambda > 0$, $\varepsilon \ge 0$ and $\delta > 0$ there exists a constant $C_\delta$ depending only on $\delta$ such that*

$$\|R_0(\lambda \pm i\varepsilon)VR_0(\lambda \pm i\varepsilon)Vf\|_{L^\infty} \le \left(\delta + \frac{C_\delta}{\sqrt{\lambda}}\right)\|f\|_{L^\infty}. \tag{2.3.45}$$

*Proof.* By the maximum (Phragmén-Lindelöf) principle, since $R_0(z)$ is holomorphic, it is sufficient to prove the estimate for $\varepsilon = 0$, i.e., for the operators $R_0(\lambda \pm i0)$. If we approximate $V$ by the sequence of test functions $V_\varepsilon$ constructed in Lemma 2.3.12, we can write

$$R_0(\lambda \pm i0)VR_0(\lambda \pm i0)V = R_0(V - V_\varepsilon)R_0V + R_0V_\varepsilon R_0(V - V_\varepsilon) + R_0V_\varepsilon R_0V_\varepsilon$$

and using estimate (2.3.21) we obtain

$$\|R_0VR_0Vf\|_{L^\infty} \le (2\pi)^{-1}\|V\|_K \cdot \|V - V_\varepsilon\|_K \cdot \|f\|_{L^\infty} + \|R_0V_\varepsilon R_0V_\varepsilon f\|_{L^\infty}. \tag{2.3.46}$$

We can choose $\varepsilon = \varepsilon(\delta)$ so small that

$$(2\pi)^{-1}\|V\|_K \cdot \|V - V_\varepsilon\|_K \le \frac{1}{2}\delta,$$

and hence it sufficient to prove (2.3.45) with $V$ replaced by $V_\varepsilon$. Now we have

$$|R_0V_\varepsilon R_0V_\varepsilon f(x)| \le \int_{|x-y|<r} \frac{|V_\varepsilon|}{|x-y|} dy \|R_0V_\varepsilon f\|_{L^\infty} + \int_{|x-y|\ge r} \frac{|V_\varepsilon R_0V_\varepsilon f|}{|x-y|} dy;$$

the first term clearly satisfies

$$\int_{|x-y|<r} \frac{|V_\varepsilon|}{|x-y|} dy \le C \int_{|x-y|<r} \frac{dy}{|x-y|} = \sigma(r) \to 0$$

since $V_\varepsilon$ is bounded, so that we find for all $r > 0$

$$|R_0V_\varepsilon R_0V_\varepsilon f(x)| \le \sigma(r)\|V\|_K\|f\|_{L^\infty} + \frac{1}{r}\|V_\varepsilon R_0V_\varepsilon f\|_{L^1} \tag{2.3.47}$$

where in the last step we used the property

$$\int \frac{|V_\varepsilon|}{|x-y|}dy \le \int \frac{|V|}{|x-y|}dy$$

already used in the course of the proof of Lemma 2.3.12. In order to estimate the second term in (2.3.47), we may write for some $s > 1/2$

$$\|V_\varepsilon R_0 V_\varepsilon f\|_{L^1} \le \|\langle x\rangle^s V_\varepsilon\|_{L^2} \|\langle x\rangle^{-s} R_0 V_\varepsilon f\|_{L^2}$$

and applying Lemma 2.3.10 we get

$$\le \frac{C}{\sqrt{\lambda}} \|\langle x\rangle^s V_\varepsilon\|_{L^2}^2 \|f\|_{L^\infty} \le \frac{C_1}{\sqrt{\lambda}} \|f\|_{L^\infty}$$

since $V_\varepsilon$ is in $C_0^\infty$. Coming back to (2.3.47), we obtain

$$|R_0 V_\varepsilon R_0 V_\varepsilon f(x)| \le \left(\sigma(r)\|V\|_K \|f\|_{L^\infty} + \frac{C_1}{r}\frac{1}{\sqrt{\lambda}}\right)\|f\|_{L^\infty}$$

whence (2.3.45) follows. □

We prove now a fundamental compactness property:

**Lemma 2.3.14.** *Let $V$ be a compactly supported function in the Kato class. Then for all $\lambda \in \mathbb{R}$, $\varepsilon \ge 0$ the operator $R_0(\lambda \pm i\varepsilon)V : L^\infty \to L^\infty$ and the operator $VR_0(\lambda \pm i\varepsilon): L^1 \to L^1$ are compact operators. Moreover, if $f \in L^\infty$ then the function $R_0(\lambda \pm i\varepsilon)Vf$ satisfies*

$$|R_0(\lambda \pm i\varepsilon)Vf| \le \frac{C}{\langle x\rangle} \tag{2.3.48}$$

*for some $C > 0$, and hence in particular $R_0(\lambda \pm i\varepsilon)Vf \in L^2(\langle x\rangle^{-2s}dx)$ for all $s > 1/2$ and $\lambda, \varepsilon \ge 0$.*

*Proof.* If the support of $V$ is contained in the ball $\{|x| \le M\}$, we see that, for all $|x| > 2M$ and $y$ in the support of $V$, we have $|x-y| \ge |x| - M \ge |x|/2$. Thus by the explicit representation of $R_0$ we get

$$|R_0 Vf(x)| \le \int \frac{|V(y)f(y)|}{|x-y|}dy \le \frac{2}{|x|}\int |Vf|dy \quad \text{for } |x| \ge 2M$$

and recalling that $V \in L^1$ we obtain the inequality

$$|R_0 Vf(x)| \le \frac{2}{|x|}\|V\|_{L^1}\|f\|_{L^\infty} \quad \text{for } |x| \ge 2M. \tag{2.3.49}$$

From (2.3.49) and the usual estimate

$$|R_0 Vf(x)| \le \frac{\|V\|_K}{4\pi}\|f\|_{L^\infty}.$$

we easily deduce the final statement (2.3.48) and that $R_0 V f \in L^2(\langle x\rangle^{-2s}dx)$ for all bounded $f$ and $s > 1/2$.

In order to prove the compactness property, we may assume that $V$ is a smooth function with compact support. Indeed, by Lemma 2.3.12, $V$ can be approximated in the Kato norm by test functions $V_\varepsilon$, so that $R_0 V$ is the limit of the sequence of operators $R_0 V_\varepsilon$ in the $\mathcal{L}(L^\infty; L^\infty)$ norm, since

$$\|R_0 V_\varepsilon - R_0 V\|_{\mathcal{L}(L^\infty;L^\infty)} \leq \frac{1}{4\pi}\|V_\varepsilon - V\|_K.$$

Thus the compactness of $R_0 V$ follows from the compactness of $R_0 V_\varepsilon$. A similar argument holds for $VR_0$. From now on, we shall assume that $V \in C_0^\infty$.

Let $f_j$ be a bounded sequence in $L^\infty$; writing

$$\nabla_x R_0 V f(x) = \frac{1}{4\pi}\int V(y)f(y)\nabla_x\left(\frac{e^{\pm i\sqrt{\lambda_\varepsilon}|x-y|}}{|x-y|}e^{-\varepsilon|x-y|/2\sqrt{\lambda_\varepsilon}}\right)dy$$

we immediately obtain a bound for $\|\nabla R_0 V f_j\|_{L^\infty}$, uniform in $j$ (recall that $V$ now is smooth and compactly supported). Thus an application of the Ascoli-Arzelà theorem shows that the sequence $R_0 V f_j$ is precompact in the $L^\infty$ norm on any bounded set in $\mathbb{R}^3$. Using this compactness property for small $x$ and again inequality (2.3.49) for large $x$, by a diagonal procedure we obtain that $R_0 V f_j$ has a uniformly convergent subsequence on the whole $\mathbb{R}^3$.

To prove the compactness of $VR_0$ we write it as $VR_0 = A_r + B_r$ where

$$A_r g(x) = \frac{V(x)}{4\pi}\int \frac{e^{\pm i\sqrt{\lambda_\varepsilon}|x-y|}}{|x-y|}e^{-\varepsilon|x-y|/2\sqrt{\lambda_\varepsilon}}\chi_r(x-y)g(y)dy \qquad (2.3.50)$$

$$B_r g(x) = \frac{V(x)}{4\pi}\int \frac{e^{\pm i\sqrt{\lambda_\varepsilon}|x-y|}}{|x-y|}e^{-\varepsilon|x-y|/2\sqrt{\lambda_\varepsilon}}(1-\chi_r(x-y))g(y)dy; \quad (2.3.51)$$

here $\chi_r(y) = \chi(y/r)$ is a cutoff function equal to 1 for $x$ near the origin and vanishing for large $x$. It is easy to show that $B_r$ is a compact operator on $L^1$; indeed, it is a bounded operator from $L^1$ to $W^{1,1}(\Omega)$ for $\Omega$ any bounded open set containing the support of $V$, while $W^{1,1}(\Omega)$ is compactly embedded in $L^1(\mathbb{R}^3)$ by the Rellich-Kondrachov Theorem. Since $\|A_r\|_{\mathcal{L}(L^1;L^1)} \to 0$ as $r \to 0$, we regard as above $VR_0$ as the uniform limit of compact operators, and this concludes the proof. □

The following version of the same lemma will be useful later on:

**Lemma 2.3.15.** *Assume $V$ satisfies the inequality $|V(x)| \leq C\langle x\rangle^{-3-\delta}$ for some $C, \delta > 0$. Then all the conclusions of Lemma 2.3.14 remain true.*

*Proof.* The estimate follows immediately from the standard inequality

$$\int \frac{dy}{\langle y\rangle^{3+\delta}|x-y|} \leq \frac{C}{\langle x\rangle}$$

(see e.g. Appendix 2 of [2]). The compactness property is proved as above using the Ascoli-Arzelà Theorem. □

We are now ready to prove the main proposition of this section.

*Proof.* (of Proposition 2.3.9). The inversion of $I + R_0(z)V : L^\infty \to L^\infty$ is quite easy when $\Re z << 0$. Indeed, Lemma 2.3.7 states that for all $\delta > 0$ there exists a constant $C_\delta > 0$ such that

$$\|R_0(\lambda \pm i\varepsilon)V\|_{\mathcal{L}(L^\infty;L^\infty)} \leq \delta + C_\delta \frac{\|V\|_K}{\sqrt{|\lambda|}}, \qquad \forall \lambda < 0,\ \varepsilon \geq 0.$$

Hence, in particular, for $\lambda < -\delta^2(C_\delta\|V\|_K)^{-2}$ we have $\|R_0(\lambda \pm i\varepsilon)V\|_{\mathcal{L}(L^\infty;L^\infty)} < 2\delta$, and this means that the norm $\|R_0(\lambda \pm i\varepsilon)V\|_{\mathcal{L}(L^\infty;L^\infty)}$ tends to 0 for $\lambda \to -\infty$, uniformly in $\varepsilon$. Thus $I + R_0(\lambda \pm i\varepsilon)V$ can be inverted by expansion in Neumann series for any $\varepsilon \geq 0$ and any $\lambda < -M$ provided $M > 0$ is large enough, and the $\mathcal{L}(L^\infty; L^\infty)$ norm of the inverse operator is bounded by a constant depending only on $M$ (and $V$).

We now consider the case $\Re z >> 0$. Let $V = V_1 + V_2$ be as in Theorem 2.3.1, and write for brevity

$$T = R_0(z)V_1, \qquad S = R_0(z)V_2.$$

We first notice that $I + S$ can be inverted for all $z \in \mathbb{C}$, with bounded inverse; indeed, by (2.3.21) the norm of $S\colon L^\infty \to L^\infty$ is bounded by $\|V_2\|_K/(4\pi)$, which is strictly smaller than 1 by assumption (2.3.3), and the result follows again by a straightforward Neumann series expansion. We thus get for all $z$

$$\|(I+S)^{-1}\|_{\mathcal{L}(L^\infty;L^\infty)} \leq \left(1 - \|V_2\|_K/(4\pi)\right)^{-1}. \tag{2.3.52}$$

We then invert $I + T$ for large $\lambda = \Re z$. Lemma 2.3.13 ensures that $\|T^2\|_{\mathcal{L}(L^\infty;L^\infty)} \to 0$ as $\lambda \to \infty$. This implies that for any $\delta \in ]0, 1[$ we can find $\lambda_\delta$ such that for all $\Re z \geq \lambda_\delta$, $I - T^2$ is invertible with norm

$$\|(I - T^2)^{-1}\|_{\mathcal{L}(L^\infty;L^\infty)} \leq \frac{1}{1-\delta}. \tag{2.3.53}$$

Since $I - T$ has norm in $\mathcal{L}(L^\infty; L^\infty)$ bounded by $1 + (4\pi)^{-1}\|V_1\|_K$ independently of $z$ and

$$(I - T)(I - T^2)^{-1} = (I + T)^{-1},$$

we conclude that also $I+T$ is invertible for any $\Re z \geq \lambda_\delta$, with bound

$$\|(I+T)^{-1}\|_{\mathcal{L}(L^\infty;L^\infty)} \leq \frac{1}{1-\delta}(1+\|V_1\|_K/(4\pi)). \tag{2.3.54}$$

Consider now for $\Re z \geq \lambda_\delta$ the operator

$$S(I+T)^{-1};$$

by the usual bound $\|S\|_{\mathcal{L}(L^\infty;L^\infty)} \leq \|V_2\|_K/(4\pi)$ and by (2.3.54) we obtain

$$\|S(I+T)^{-1}\|_{\mathcal{L}(L^\infty;L^\infty)} \leq \frac{1}{4\pi}\|V_2\|_K \frac{1}{1-\delta}\left(1+\frac{\|V_1\|}{4\pi}\right) = \frac{\alpha}{1-\delta}$$

where the constant $\alpha$, recalling the main assumption (2.3.3), satisfies

$$\alpha \equiv \frac{1}{4\pi}\|V_2\|_K\left(1+\frac{\|V_1\|}{4\pi}\right) < 1.$$

Hence we see that

$$\|S(I+T)^{-1}\|_{\mathcal{L}(L^\infty;L^\infty)} \leq \frac{\alpha}{1-\delta} < 1$$

provided $\delta < 1-\alpha$, i.e., provided $\lambda_\delta$ is large enough. Thus, choosing a value of $\lambda_\delta$ large enough, we have that for $\Re z \geq \lambda_\delta$ the operator

$$I+S(I+T)^{-1}$$

is invertible. Finally, writing

$$(I+S+T)^{-1} = (I+T)^{-1}(I+S(I+T)^{-1})^{-1},$$

we see that $I+S+T = I+R_0V$ is invertible with the bound

$$\|(I+R_0(z)V)^{-1}\|_{\mathcal{L}(L^\infty;L^\infty)} \leq \left(1+\frac{\|V_1\|}{4\pi}\right)\frac{1}{1-\alpha-\delta} \tag{2.3.55}$$

for $\Re z \geq \lambda_\delta$.

It remains to invert $I+S+T$ for $-M \leq \Re z \leq \lambda_\delta$, $0 \leq \Im z \leq \varepsilon_0$ (or $0 \geq \Im z \geq -\varepsilon_0$), with a uniform bound. To this end we shall apply Fredholm theory; notice that the standard analytic Fredholm theory cannot be applied directly since we are not in the usual Hilbert framework but we are working in $L^\infty$ instead. We proceed in two slightly different ways according to the set of available assumptions.

### 2.3.4 Case A: assumptions of Theorem 2.3.1

The first step is to prove that $I+S+T : L^\infty \to L^\infty$ is injective. A general argument shows that this is always the case when $z$ is outside the positive real axis $[0,+\infty[$, provided $V = V_1 + V_2$ satisfies (i), (ii) of Theorem 2.3.1. To see this, we approximate $V_1$ with a sequence of nonnegative test functions $V_\delta$ in such a way that $\|V_1 - V_\delta\|_K \to 0$ (see Lemma 2.3.12); thus we can decompose $V$ as

$$V = V_\delta + W_\delta, \qquad 0 \le V_\delta \in C_0^\infty, \qquad \|W_\delta\|_K = \|V_2 + V_1 - V_\delta\|_K < 4\pi$$

for $\delta$ small enough. Assume now that the bounded function $g$ satisfies the integral equation

$$(I + R_0(z)V)g = 0, \qquad z \notin \mathbb{R}^+;$$

we shall prove that $g = 0$. Indeed, we can rewrite the equation as follows:

$$(I + R_0(z)W_\delta)g = -R_0(z)V_\delta g \in L^\infty.$$

Now, $R_0(z)W_\delta$ has norm $< 1$ as a bounded operator on $L^\infty$, hence we can invert $I + R_0(z)W_\delta$ and we obtain

$$g = -(I + R_0(z)W_\delta)^{-1}R_0(z)V_\delta g.$$

Note that

$$(I + R_0(z)W_\delta)^{-1}R_0(z) = (-z - \Delta + W_\delta)^{-1}$$

is exactly the resolvent operator of $-\Delta + W_\delta$, at a point $z$ outside the spectrum. Moreover, $V_\delta g$ is in $L^2$, hence $g = (-z - \Delta + W_\delta)^{-1}V_\delta g$ is in $H^2$; since

$$(-z - \Delta + V)g = 0, \qquad z \notin \mathbb{R}^+$$

we conclude that $g \equiv 0$ as claimed.

When $z \in [0,+\infty[$, assumption (iv) of Theorem 2.3.1 means exactly that $I+S+T$ is injective on $L^\infty$, thus we have nothing to prove in this case, and we obtain that $I+S+T$ is injective for all values of $z \in \mathbb{C}$.

The second step is to prove that $I+S+T$ is invertible. Recalling that $I+S$ is invertible for all $z$, we can write

$$I + S + T = (I + T(I+S)^{-1})(I+S)$$

which implies that $I + T(I+S)^{-1}$ is also injective for all $z$. But $T$, and hence $T(I+S)^{-1}$ are compact operators on $L^\infty$, thanks to Lemma 2.3.14. By Fredholm theory this implies that $I + T(I+S)^{-1}$ is invertible, and in conclusion $I+S+T$ is invertible too and the following identity holds:

$$(I+S+T)^{-1} = (I+S)^{-1}(I + T(I+S)^{-1})^{-1}. \tag{2.3.56}$$

The last step is to prove a uniform bound on $(I+S+T)^{-1}$. This is the content of the following lemma, which is our $L^\infty$ replacement for the usual analytic Fredholm theory in the Hilbert spaces $L^2(\langle x\rangle^s dx)$.

**Lemma 2.3.16.** *Assume $V = V_1 + V_2$, with $V_1$ compactly supported, $\|V_1\|_K < +\infty$, and $\|V_2\|_K < 4\pi$. If the operator $I + R_0(z)V : L^\infty \to L^\infty$ is invertible for all $z$ in a compact set $D \subset \mathbb{C}^+ = \{\Re z \geq 0\}$ (or $D \subset \mathbb{C}^-$), then*

$$\sup_{z \in D} \|(I + R_0(z)V)^{-1}\|_{\mathcal{L}(L^\infty;L^\infty)} < \infty.$$

*Proof.* We write as before

$$T = R_0(z)V_1, \qquad S = R_0(z)V_2 \tag{2.3.57}$$

and when $z_n$ is a sequence of points in $\mathbb{C}$ we shall also write

$$T_n = R_0(z_n)V_1, \qquad S_n = R_0(z_n)V_2 \tag{2.3.58}$$

Moreover, we shall denote by $L^\infty_K$ the space of bounded compactly supported functions, and by $L^\infty_0$ its closure in $L^\infty$; in other words $L^\infty_0$ is the space of bounded functions vanishing at infinity, with the uniform norm.

The proof consists in several steps.

Step 1: $S$ is a bounded operator from $L^\infty_0$ into itself. Indeed, given any $\phi \in L^\infty_0$, decompose it as

$$\phi = \phi_M + \psi_M, \qquad \phi_M = \phi \cdot \mathbf{1}_{\{|x|<M\}}$$

where $\mathbf{1}_{\{|x|<M\}}$ is the characteristic function of the ball $\{|x| < M\}$. As in the proof of Lemma 2.3.14, we have immediately

$$|S\phi_M(x)| \leq \frac{C}{|x|}\|V_2\|_{L^1(|y|\leq M)} \qquad \text{for } |x| > 2M. \tag{2.3.59}$$

On the other hand,

$$\|S\psi_M\|_{L^\infty} \leq C\|\psi_M\|_{L^\infty} \to 0 \qquad \text{for } M \to +\infty \tag{2.3.60}$$

since $\phi$ vanishes at infinity. Then, given any $\delta > 0$, we may choose $M = M_\delta$ such that $\|\psi_M\|_{L^\infty} < \delta$; from (2.3.59) we obtain

$$|S\phi(x)| \leq |S\phi_M(x)| + |S\psi_M(x)| \leq \frac{\|V_2\|_{L^1}}{|x|} + \delta \qquad \text{for } |x| > 2M_\delta$$

and this implies $S\phi \in L^\infty_0$.

Step 2: If $D \ni z_n \to z$ and $\phi \in L^\infty_0$, then $S_n\phi \to S\phi$ uniformly on $\mathbb{R}^n$ (with the notations (2.3.58)). To prove this, we notice that

$$\frac{|e^{iw_n|x-y|} - e^{iw|x-y|}|}{|x-y|}| \leq C|w_n - w|$$

provided $w_n, w$ stay in a compact subset of $\mathbb{C}$; from this, it easily follows that

$$|(R_0(z_n) - R_0(z)f| \leq C(D) \cdot |z^{1/2} - z_n^{1/2}| \cdot \|f\|_{L^1} \tag{2.3.61}$$

with the determination $(\rho e^{i\theta})^{1/2} = \sqrt{\rho} e^{i\theta/2}$. Now, let $\phi \in L_0^\infty$; to prove that $S_n\phi = R_0(z_n)V_2\phi$ converges to $S\phi = R_0(z)V_2\phi$ uniformly, we decompose $\phi = \phi_M + \psi_M$ as in Step 1 and write

$$|S_n\phi(x) - S\phi(x)| \leq |S_n\phi_M(x) - S\phi_M(x)| + |S_n\psi_M(x) - S\psi_M(x)|.$$

The second term is bounded by

$$|S_n\psi_M(x) - S\psi_M(x)| \leq \|V_2\|_K \|\psi_M\|_{L^\infty}$$

which can be made smaller than $\delta > 0$ provided $M > M_\delta$, as in the preceding step. To the first term we apply (2.3.61) and we obtain

$$|S_n\phi_M(x) - S\phi_M(x)| \leq C(D) \cdot |z_n^{1/2} - z^{1/2}| \cdot \|V_2\|_{L^1(|y|\leq M)} \|\phi_M\|_{L^\infty}$$

whence we see that this term tends uniformly to 0 for each fixed $M$, when $z_n \to z$, $z_n, z \in D$, and this proves the claim.

Note that in Steps 1 and 2 we did not use the assumption $\|V_2\|_K < 4\pi$; both properties are true for potentials of arbitrary (but bounded) Kato norm; in particular, they hold for $T, T_n$.

STEP 3: If $D \ni z_n \to z$, $\phi \in L_0^\infty$ and $k \geq 1$, then $S_n^k\phi \to S^k\phi$ uniformly on $\mathbb{R}^n$ (where $S_n^k, S^k$ are the $k$-th powers of the operators defined in (2.3.57), (2.3.58)). It is sufficient to write

$$S_n^k - S^k = \sum_{j=1}^{k} S_n^{j-1}(S_n - S)S^{k-j}$$

and prove the convergence of each term separately. Indeed, $S^{k-j}\phi$ is a fixed element of $L_0^\infty$ by Step 1, hence $(S_n - S)S^{k-j}\phi \to 0$ uniformly by Step 2, and remarking that $S_n^j$ are bounded operators on $L^\infty$ with norm $\|S_n^j\| \leq \|S_n\|^j < 1$, we conclude that $S_n^j(S_n - S)S^{k-j}\phi \to 0$ uniformly, as claimed.

STEP 4: If $D \ni z_n \to z$ and $\phi \in L_0^\infty$, then $(I+S_n)^{-1}\phi$ tends to $(I+S)^{-1}\phi$ uniformly on $\mathbb{R}^n$. To prove this, note that can write for any $N \geq 1$

$$(I + S_n)^{-1} - (I + S)^{-1} = \sum_{k=1}^{N}(-1)^k(S_n^k - S^k) + \sum_{k=N+1}^{\infty}(-1)^k(S_n^k - S^k);$$

the second sum can be estimated in the norm of bounded operators on $L^\infty$ as follows

$$\left\| \sum_{k=N+1}^{\infty}(-1)^k(S_n^k - S^k) \right\| \leq \frac{\|S_n\|^{N+1}}{1 - \|S_n\|} + \frac{\|S\|^{N+1}}{1 - \|S\|}$$

which is smaller than $\delta$ for $N \geq N_\delta$ large enough; on the other hand, we can apply Step 3 to the terms $S_n^k - S^k$ for $k = 1, \ldots, N$, and this concludes the proof of this step.

STEP 5: Conclusion of the proof. We know already that $(I+S)^{-1}$ is well defined with bounded operator norm for all $z$, hence by the identity

$$I+T+S=(I+S)(I+(I+S)^{-1}T)$$

we see that it is sufficient to bound the operator norm of $(I+(I+S)^{-1}T)^{-1}$ for $z\in D$. By the uniform boundedness principle, our claim reduces to the following: given any sequence $z_n$ in $D$, which can be assumed to converge to $z\in D$, we have that for all $\phi\in L^\infty$ there exists $c(\phi)>0$ such that, for all $n$,

$$\|(I+(I+S_n)^{-1}T_n)^{-1}\phi\|\le c(\phi) \tag{2.3.62}$$

(just take any sequence $z_n$ such that the norm in (2.3.62) converges to the supremum over $D$). We use again the notations (2.3.57), (2.3.58).

Indeed, assume by contradiction that there exists $\phi\in L^\infty$ such that

$$\|(I+(I+S_n)^{-1}T_n)^{-1}\phi\|\to\infty \quad \text{as } z_n\to z \tag{2.3.63}$$

and consider the renormalized functions

$$\psi_n=\frac{(I+(I+S_n)^{-1}T_n)^{-1}\phi}{\|(I+(I+S_n)^{-1}T_n)^{-1}\phi\|_{L^\infty}}.$$

Clearly we have

$$\|\psi_n\|_{L^\infty}=1, \qquad (I+(I+S_n)^{-1}T_n)\psi_n\to 0 \quad \text{in } L^\infty. \tag{2.3.64}$$

We have also $\|T_n-T\|\to 0$, since using again (2.3.61)

$$|(T_n-T)\phi|\le C(D)\cdot|z_n^{1/2}-z^{1/2}|\cdot\|V_1\|_{L^1}\|\phi\|_{L^\infty}.$$

This and (2.3.64) imply

$$\|\psi_n\|_{L^\infty}=1, \qquad (I+(I+S_n)^{-1}T)\psi_n\to 0 \quad \text{in } L^\infty. \tag{2.3.65}$$

Now, by Lemma 2.3.14, we know that $T$ is a compact operator on $L^\infty$ and the image of $T$ is contained in $L_0^\infty$ (see (2.3.48)), hence by possibly extracting a subsequence we obtain that $T\psi_n$ converges uniformly to some function $\zeta\in L_0^\infty$. Now we can write

$$(I+S_n)^{-1}T\psi_n=(I+S_n)^{-1}(T\psi_n-\zeta)+(I+S_n)^{-1}\zeta;$$

since $\|(I+S_n)^{-1}\|<C$ independent of $n$, the first term converges uniformly to 0, and by Step 4 we obtain that

$$(I+S_n)^{-1}T\psi_n\to(I+S)^{-1}\zeta$$

uniformly. By (2.3.65), this implies the uniform convergence

$$\psi_n\to-(I+S)^{-1}\zeta=:\psi;$$

notice in particular that $\|\psi\|_{L^\infty} = 1$. Summing up, we have proved that

$$\psi_n \to \psi \equiv -(I+S)^{-1}\zeta, \qquad T\psi_n \to \zeta \equiv T\psi$$

and this implies

$$\psi + (I+S)^{-1}T\psi = 0 \quad \text{i.e.} \quad (I+S+T)\psi = 0$$

which is absurd since $I+T+S$ is invertible and $\|\psi\|_{L^\infty} = 1$. □

### 2.3.5 Case B: assumptions of Theorem 2.3.5

We note that a potential $V$ satisfying the new assumptions can be split as $V = V_1' + V_2'$ with $V_1', V_2'$ as in (i), (ii) of Theorem 2.3.1 (take $V_1' = V$ for $|x| < R$ and 0 outside, with $R$ large enough). Thus, for $z \notin [0, \lambda_\delta]$ the same arguments as in Case A apply; also Lemma 2.3.16 can still be used. Hence it is sufficient to prove that $I + R_0(z)V$ is invertible for $z \in [0, \lambda_\delta]$ under the new assumptions.

Since $V_1$ fulfills the conditions of both Propositions 2.3.3 and 2.3.4, we see that the operators $I + R_0(\lambda \pm i0)V_1$ are injective on $L^\infty$ for all $\lambda > 0$.

We now prove injectivity also at $\lambda = 0$. Thus, let the bounded function $f$ satisfy

$$f(x) + \int \frac{V_1(y)f(y)}{|x-y|}dy = 0; \tag{2.3.66}$$

in particular, $f$ is a weak solution of

$$\Delta f = V_1 f \in L^2 \implies f \in H^2.$$

Now, if $V_1(x) < C\langle x\rangle^{-3-\delta}$ for $|x| > M$, we have immediately, for all $|x| > 2M$,

$$|f(x)| \leq \|V_1\|_{L^1(|x|<M)}\|f\|_{L^\infty}\frac{C}{|x|} + C\|f\|_{L^\infty}\int \frac{dy}{\langle y\rangle^{3+\delta}|x-y|} \leq \frac{C}{|x|}$$

(see Lemma 2.3.15 above). Differentiating (2.3.66) we see that $\nabla f$ satisfies an analogous integral equation

$$\nabla f(x) + \int V_1(y)f(y)\nabla_x \frac{1}{|x-y|}dy = 0$$

which implies

$$|\nabla f(x)| \leq C\|f\|_{L^\infty}\int \frac{|V_1(y)|}{|x-y|^2}dy.$$

Proceeding as above, we can write for $|x| > 2M$

$$|\nabla f(x)| \leq \|V_1\|_{L^1(|x|<M)}\|f\|_{L^\infty}\frac{C}{|x|^2} + C\|f\|_{L^\infty}\int \frac{dy}{\langle y\rangle^{3+\delta}|x-y|^2} \leq \frac{C}{|x|^2}$$

thanks to the standard inequality (see [2])

$$\int \frac{dy}{\langle y\rangle^{3+\delta}|x-y|^2} \leq \frac{C}{\langle x\rangle^2},$$

Thus we have proved that for all $|x| > 2M$

$$|f(x)| \leq \frac{C}{|x|}, \qquad |\nabla f(x)| \leq \frac{C}{|x|^2}. \tag{2.3.67}$$

Now a standard cutoff trick can be applied (see the Appendix of [51]): let $\phi \in C_0^\infty$ equal to 0 for $|x| > 2$ and equal to 1 for $|x| < 1$, consider the identity

$$\int \left(|\nabla f|^2 + V_1|f|^2\right) \phi\left(\frac{y}{R}\right) dy = -\frac{1}{R}\int_{R\leq|y|\leq 2R} \nabla\phi\left(\frac{y}{R}\right) \cdot \nabla f \cdot \overline{f} dy$$

and apply the estimates (2.3.67) to the right hand member, for $R$ large enough. We obtain

$$\int \left(|\nabla f|^2 + V_1|f|^2\right) \phi\left(\frac{y}{R}\right) dy \leq \frac{C}{R}$$

and taking the limit as $R \to \infty$ we conclude that $f \equiv 0$, i.e., 0 is not a resonance.

Writing as before $T = R_0(z)V_1$, we have just proved that $I+T$ is injective on $L^\infty$ for $z \in [0, \lambda_\delta]$. Now we remark that we can split $V_1 = V_1' + V_1''$ as the sum of a compactly supported function $V_1' \in L^2$, hence with bounded Kato norm, and a function $V_1'' < C\langle x\rangle^{-3-\delta}$. The corresponding operators $T = T' + T''$ are compact on $L^\infty$ by Lemmas 2.3.14, 2.3.15 respectively, hence $T$ is compact and by Fredholm theory we can conclude that $I + T$ is invertible for all $z \in [0, \lambda_\delta]$. Then Lemma 2.3.16 ensures that the operator norm $(I + T)^{-1}$ is bounded by some constant $C_0$ uniform on $z \in [0, \lambda_\delta]$.

Now, writing

$$I + T + S = (I + T)(I + (I + T)^{-1}S)$$

we see that in order to invert $I+T+S$ it is sufficient to invert $I+(I+T)^{-1}S$; since

$$\|(I + T)^{-1}S\| \leq \|(I + T)^{-1}\| \cdot \frac{\|V_2\|_K}{4\pi} \leq C_0\frac{\|V_2\|_K}{4\pi}$$

this can be achieved by a Neumann expansion as soon as the Kato norm of $V_2$ is small enough, i.e.,

$$\|V_2\|_K < \frac{4\pi}{C_0} =: \epsilon(V_1).$$

This is exactly assumption (2.3.7).

Thus we have proved that $I + S + T$ is invertible for all complex $z$, and a last application of Lemma 2.3.16 concludes the proof of Case B. □

We can now draw some consequences which shall be used in the following.

**Corollary 2.3.17.** *Under the assumptions of Theorem 2.3.1 (or Theorem 2.3.5) there exists $\varepsilon_0 > 0$ such that the bounded operators $I + VR_0(\lambda \pm i\varepsilon)\colon L^1 \to L^1$ are invertible for all $\lambda \in \mathbb{R}$, $0 \le \varepsilon \le \varepsilon_0$ with uniform bound*

$$\|(I + VR_0(\lambda \pm i\varepsilon))^{-1}\|_{\mathcal{L}(L^1;L^1)} \le C \quad \textit{for all } \lambda \in \mathbb{R},\ 0 \le \varepsilon \le \varepsilon_0. \tag{2.3.68}$$

*Proof.* The operators $I + VR_0$ are one to one on $L^1$ by duality, since by Proposition 2.3.9 the operators $I+R_0V$ are onto. They are onto by Fredholm theory, since $VR_0$ are compact operators on $L^1$ by Lemma 2.3.14. Finally, the bound on the inverse also follows by duality and the bound (2.3.41); indeed, $(L^1)' = L^\infty$ and hence

$$\|(I + VR_0)f\|_{L^1} = \sup_{\|h\|_{L^\infty}=1} \int h(I + VR_0)f dx = \sup_{\|h\|_{L^\infty}=1} \int f(I + R_0V)hdx.$$

□

As a consequence of (2.3.40) and of Proposition 2.3.9, Corollary 2.3.17 we can write the standard representation formulas:

$$R_V(z) = (I + R_0V)^{-1}R_0(z) = R_0(z)(I + VR_0)^{-1}. \tag{2.3.69}$$

By combining these relations we easily obtain the identity

$$\begin{aligned} &R_V(\lambda + i\varepsilon) - R_V(\lambda - i\varepsilon) = \\ &\quad = (I + R_0(\lambda - i\varepsilon)V)^{-1}(R_0(\lambda + i\varepsilon) - R_0(\lambda - i\varepsilon))(I + VR_0(\lambda + i\varepsilon))^{-1} \end{aligned} \tag{2.3.70}$$

for all $\lambda \in \mathbb{R}$, $\varepsilon \in ]0, \varepsilon_0]$. Then by the bounds (2.3.20) and (2.3.41), (2.3.68) we obtain

$$\|[R_V(\lambda + i\varepsilon) - R_V(\lambda - i\varepsilon)]g\|_{L^\infty} \le C\sqrt{\lambda_\varepsilon}\|g\|_{L^1}. \tag{2.3.71}$$

for all $\lambda \in \mathbb{R}$, $\varepsilon \in ]0, \varepsilon_0]$.

Moreover from (2.3.69) we get

$$R_V(\lambda \pm i\varepsilon)^2 = (I + R_0(\lambda \pm i\varepsilon)V)^{-1}R_0(\lambda \pm i\varepsilon)^2(I + VR_0(\lambda \pm i\varepsilon))^{-1} \tag{2.3.72}$$

and recalling (2.3.27) we obtain

$$\|R_V(\lambda \pm i\varepsilon)^2 g\|_{L^\infty} \le \frac{C}{\sqrt{\lambda_\varepsilon}}\|g\|_{L^1} \tag{2.3.73}$$

for all $\lambda \in \mathbb{R}$, $\varepsilon \in ]0, \varepsilon_0]$.

### 2.3.6 Equivalence of Besov norms

This section is devoted to prove the equivalence of perturbed and standard Besov spaces

$$\dot{B}^s_{1,q}(\mathbb{R}^3) \cong \dot{B}^s_{1,q}(V) \tag{2.3.74}$$

which holds for $0 < s < 2$ and $1 \le q \le \infty$ under our assumptions. An analogous property holds also for non homogeneous spaces.

We begin by adapting to our situation a result of Simon [91] (whose proof we follow closely). Hoping that estimates (2.3.77) and (2.3.79) may be of independent interest, we shall give the proof for general dimension $n$. If the negative part of the potential is in the Kato class but not small, by Theorem B.1.1 of [91] the semigroup is still bounded, but its norm may increase exponentially as $t \to \infty$.

**Proposition 2.3.18.** *Assume the potential $V = V_+ - V_-$ on $\mathbb{R}^n$, $n \ge 3$, $V_\pm \ge 0$, satisfies*

$$V_+ \text{ is of Kato class} \tag{2.3.75}$$

*and*

$$\|V_-\|_K < c_n \equiv 2\pi^{n/2}/\Gamma\left(\frac{n}{2} - 1\right) \tag{2.3.76}$$

*and consider the selfadjoint operator $H = -\Delta + V$. Then for all $t > 0$ and $1 \le p \le q \le \infty$ the semigroup $e^{-tH}$ is bounded from $L^p$ to $L^q$ with norm*

$$\|e^{-tH}\|_{\mathcal{L}(L^p;L^q)} \le \frac{(2\pi t)^{-\gamma}}{(1 - \|V_-\|_K/c_n)^2}, \qquad \gamma = \frac{n}{2}\left(\frac{1}{p} - \frac{1}{q}\right). \tag{2.3.77}$$

*Moreover, under the stronger assumption*

$$\|V_-\|_K < \frac{1}{2}c_n \tag{2.3.78}$$

*$e^{-tH}$ is an integral operator with kernel $k(t,x,y)$ satisfying*

$$|k(t,x,y)| \le \frac{(2\pi t)^{-n/2}}{1 - 2\|V_-\|_K/c_n} e^{-|x-y|^2/8t}. \tag{2.3.79}$$

*Proof.* In the following we shall use the more convenient notations

$$H = -\frac{1}{2}\Delta + V, \qquad H_0 = -\frac{1}{2}\Delta; \tag{2.3.80}$$

thus in the final step it will be necessary to substitute $t \to 2t$ and $V \to V/2$ in order to obtain the correct estimates.

The fundamental tool will be the *Feynman-Kač* formula

$$(e^{-tH}f)(x) = E_x\left(\exp\left(-\int_0^t V(b(s))ds\right) f(b(t))\right) \tag{2.3.81}$$

which is valid under much more general assumptions (see e.g. [111]). Here $E_x$ is the integral over the path space $\Omega$ with respect to the Wiener measure $\mu_x$, $x \in \mathbb{R}^n$, while $b(t)$ represents a generic path (brownian motion). We shall not need the full power of the theory but only a few basic facts:

i) Given a non negative function $G(x)$ on $\mathbb{R}^n$ we have the identity

$$E_x\left(\int_0^t G(b(s))ds\right) = \int Q_t(x-y)G(y)dy \tag{2.3.82}$$

where $Q_t(x)$ is the function

$$Q_t(x) = \int_0^t (2\pi s)^{-n/2} e^{-|x|^2/2s} ds. \tag{2.3.83}$$

It is easy to see by rescaling that

$$\int_0^\infty (2\pi s)^{-n/2} e^{-|x|^2/2s} ds = \int_0^\infty \tau^{\frac{n}{2}-2} e^{-\tau} d\tau \frac{|x|^{2-n}}{2\pi^{n/2}} = \Gamma\left(\frac{n}{2}-1\right)\frac{|x|^{2-n}}{2\pi^{n/2}}$$

so that by definition of $c_n$ (see (2.3.76))

$$Q_t(x) \leq \frac{1}{c_n|x|^{n-2}} \tag{2.3.84}$$

and by (2.3.82)

$$E_x\left(\int_0^t G(b(s))ds\right) \leq \frac{1}{c_n}\|G\|_K. \tag{2.3.85}$$

ii) Khasminskii's lemma ([65]; B.1.2 in [91]): *if $G(x)$ is a non negative function on $\mathbb{R}^n$ such that for some $t$*

$$\alpha \equiv \sup_x E_x\left(\int_0^t G(b(s))ds\right) < 1, \tag{2.3.86}$$

*then*

$$\sup_x E_x\left(\exp\left(\int_0^t G(b(s))ds\right)\right) \leq \frac{1}{1-\alpha}. \tag{2.3.87}$$

An immediate application is the following: if $V_-$ satisfies

$$\|V_-\|_K < c_n$$

we have

$$\alpha \equiv \sup_x E_x\left(\int_0^t V_-(b(s))ds\right) \leq \frac{1}{c_n}\|V_-\|_K < 1$$

by (2.3.85), so that

$$\sup_x E_x\left(\exp\left(\int_0^t V_-(b(s))ds\right)\right) \leq \frac{1}{1-\|V_-\|_K/c_n}. \tag{2.3.88}$$

These simple facts gives us the first $L^\infty - L^\infty$ estimate for the semigroup. Indeed, by the Feynman-Kač formula we have

$$\|e^{-tH}f\|_{L^\infty} = \sup_{x\in\mathbb{R}^n} E_x\left(\exp\left(-\int_0^t V(b(s))ds\right)f(b(t))\right) \leq$$

$$\leq \|f\|_{L^\infty} E_x\left(\exp\left(-\int_0^t |V_-(b(s))|ds\right)\right) \leq \frac{\|f\|_{L^\infty}}{1-\|V_-\|_K/c_n}. \quad (2.3.89)$$

The second step is a $L^2 - L^\infty$ estimate. By the Feynman-Kač formula and the Schwarz inequality

$$|e^{-tH}f(x)| \leq E_x\left(\exp\left(-2\int_0^t V_-(b(s))ds\right)\right)^{1/2} E_x\left(|f(b(t))|\right)^{1/2} \equiv$$

$$\equiv \left[(e^{-t(H_0+2V)}1)(x)\right]^{1/2}\left[e^{-tH_0}|f|^2\right]^{1/2} \quad (2.3.90)$$

where in the last step we used again the formula; now $e^{-tH_0}$ is the standard heat kernel which has norm $(2\pi t)^{-n/2}$ as an $L^1 - L^\infty$ operator, while we can apply estimate (2.3.89) to the operator $e^{-t(H_0+2V)}$. We thus obtain

$$|e^{-tH}f(x)| \leq \frac{\|1\|_{L^\infty}}{1-2\|V_-\|_K/c_n}(2\pi t)^{-n/4}\|f\|_{L^2}$$

which implies

$$\|e^{-tH}f\|_{L^\infty} \leq \frac{(2\pi t)^{-n/4}}{1-2\|V_-\|_K/c_n}\|f\|_{L^2}, \quad (2.3.91)$$

provided

$$\|V_-\|_K < \frac{c_n}{2}.$$

By duality, since $e^{-tH}$ is selfadjoint, we obtain the $L^2 - L^\infty$ estimate

$$\|e^{-tH}f\|_{L^2} \leq \frac{(2\pi t)^{-n/4}}{1-2\|V_-\|_K/c_n}\|f\|_{L^1}; \quad (2.3.92)$$

using the semigroup property we can write

$$e^{-tH}f = e^{-\frac{t}{2}H}e^{-\frac{t}{2}H}f$$

and applying (2.3.91) first, then (2.3.92) we obtain

$$\|e^{-tH}f\|_{L^\infty} \leq \frac{(\pi t)^{-n/2}}{(1-2\|V_-\|_K/c_n)^2}\|f\|_{L^1}. \quad (2.3.93)$$

Now recalling (2.3.89), by duality and interpolation we obtain

$$\|e^{-tH}f\|_{L^p} \leq \frac{(\pi t)^{-\gamma}}{(1-2\|V_-\|_K/c_n)^2}\|f\|_{L^q}$$

(the constant could be slightly but not essentially improved) with $\gamma$ as in the statement. The change $t \to 2t$, $V \to V/2$ gives (2.3.77).

Let now $g(x), h(x)$ be bounded functions; the same argument as in (2.3.90) gives

$$|e^{-tH}h(x)| \leq \left[(e^{-t(H_0+2V)}|h|)(x)\right]^{1/2} \left[e^{-tH_0}|h|(x)\right]^{1/2}$$

and multiplying by $g(x)$ and taking the sup we get

$$\|ge^{-tH}h\|_{L^\infty} \leq \|ge^{-t(H_0+2V)}|h|\|_{L^\infty}^{1/2} \|ge^{-tH_0}|h|\|_{L^\infty}^{1/2}. \tag{2.3.94}$$

We choose

$$g = \chi_{K_1}, \;\; h = f\chi_{K_2}$$

where $f(x)$ is a bounded function while $\chi_{K_1}, \chi_{K_2}$ are the characteristic functions of two disjoint compact sets $K_1, K_2$. We may estimate the first factor in (2.3.94) using (2.3.93) as follows

$$\|ge^{-t(H_0+2V)}|h|\|_{L^\infty} \leq \|e^{-t(H_0+2V)}|h|\|_{L^\infty} \leq \frac{(\pi t)^{-n/2}}{(1-4\|V_-\|_K/c_n)^2}\|f\chi_{K_2}\|_{L^1}$$

while for the second we may use the explicit kernel of $e^{-tH_0}$ i.e.,

$$(2\pi t)^{-n/2}\exp(-|x-y|^2/2t)$$

and we obtain

$$\|ge^{-tH_0}|h|\|_{L^\infty} \leq (2\pi t)^{-n/2}\exp(-d^2/2t)\|f\chi_{K_2}\|_{L^1}, \qquad d = \mathrm{dist}(K_1, K_2).$$

In conclusion we have

$$\|\chi_{K_1}e^{-tH}f\chi_{K_2}\|_{L^\infty} \leq \frac{(\pi t)^{-n/2}e^{-d^2/4t}}{1-4\|V_-\|_K/c_n}\|f\chi_{K_2}\|_{L^1}, \qquad d = \mathrm{dist}(K_1, K_2). \tag{2.3.95}$$

By the Dunford-Pettis Theorem (see Trèves [105] and A.1.1-A.1.2 in [91]), this implies at once that $e^{-tH}$ has an integral kernel representation, with kernel

$$k(t,x,y) = \frac{(\pi t)^{-n/2}}{1-4\|V_-\|_K/c_n}e^{-|x-y|^2/4t}$$

and this concludes the proof (after rescaling back $t \to 2t$, $V \to V/2$). □

We shall now use the above kernel representation of the semigroup to improve a result due to Jensen and Nakamura (Theorem 2.1 in [58]):

**Proposition 2.3.19.** *Assume the Kato class potential* $V = V_+ - V_-$ *on* $\mathbb{R}^n$, $n \geq 3$, $V_\pm \geq 0$, *satisfies*

$$\|V_+\|_K < \infty \tag{2.3.96}$$

*and*

$$\|V_-\|_K < \frac{1}{2}c_n \equiv \pi^{n/2}/\Gamma\left(\frac{n}{2}-1\right) \tag{2.3.97}$$

*and consider the selfadjoint operator* $H = -\Delta + V$. *Then for any* $g \in C_0^\infty(\mathbb{R})$ *and any* $\theta > 0$ *the operator* $g(\theta H)$ *is bounded on* $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, *with norm independent of* $\theta$*:*

$$\|g(\theta H)\|_{\mathcal{L}(L^p;L^p)} \leq C(p,n,g,V). \tag{2.3.98}$$

*The same property holds for the rescaled operators*

$$\|g(H_\theta)\|_{\mathcal{L}(L^p;L^p)} \leq C(p,n,g,V), \tag{2.3.99}$$

*where* $H_\theta = -\Delta + \theta V(\sqrt{\theta}x)$.

*Proof.* The proof for fixed $\theta$ is contained in [59]. In [58], Theorem 2.1, the result was extended to the uniform estimate (2.3.98) for $0 < \theta \leq 1$, under assumptions on the potential weaker than ours. Following that proof, in order to extend the result to $\theta \geq 1$ it will be sufficient to prove that a few estimates are uniform in $\theta \geq 1$. More precisely, consider the rescaled potential

$$V_\theta(x) = \theta V(\sqrt{\theta}x); \tag{2.3.100}$$

notice that the Kato norm is invariant under this transformation:

$$\|V_\theta\|_K \equiv \|V\|_K. \tag{2.3.101}$$

Consider the operator

$$H_\theta = -\Delta + V_\theta. \tag{2.3.102}$$

We proceed exactly as in the proof of Theorem 2.1 in [58]; as remarked there, (2.3.98) is a consequence of (2.3.99). Thus we are reduced to prove that

$$\|g(H_\theta)\|_{\mathcal{L}(L^p;L^p)} \leq C \tag{2.3.103}$$

uniformly in $\theta$, and this amounts to prove three estimates uniformly in $\theta$:

i) a pointwise estimate for the kernel of $e^{-tH_\theta}$,

ii) an $L^2 - L^2$ estimate for the operator $(H_\theta + M)^{-1/2}$, $M > 0$ a fixed constant (we can take $M = 1$ here),

iii) an $L^2 - L^2$ estimate for the operator $\partial_x(H_\theta + M)^{-1/2}$.

Step i) follows directly from estimate (2.3.79)

$$|k_\theta(t,x,y)| \leq \frac{(2\pi t)^{-n/2}}{1 - 2\|V_{\theta-}\|_K/c_n} e^{-|x-y|^2/4t}. \tag{2.3.104}$$

which is uniform in $\theta > 0$ since by (2.3.100)

$$\|V_{\theta -}\|_K \equiv \|V_-\|_K$$

does not depend on $\theta$.

Step ii) is trivial since $\|(H_\theta + M)^{-1/2}\|_{\mathcal{L}(L^2;L^2)} \leq M^{-1/2}$. To get iii), we must prove that

$$\|\partial_x (H_\theta + M)^{-1/2} f\|_{L^2} \leq C\|f\|_{L^2}$$

or equivalently

$$\|g\|_{\dot{H}^1} \leq C\|(H_\theta + M)^{1/2} g\|_{L^2} \tag{2.3.105}$$

for some $C$ independent of $\theta > 0$. We rewrite (2.3.105) as

$$C^{-1}\|g\|_{\dot{H}^1} \leq (-\Delta g, g) + (V_\theta g, g) + M\|g\|^2_{L^2}. \tag{2.3.106}$$

Clearly (2.3.106) is implied by

$$|(V_{\theta -} g, g)| \leq \alpha\|g\|_{\dot{H}^1} + M\|g\|^2_{L^2}, \quad \alpha < 1, \quad \alpha \text{ independent of } \theta. \tag{2.3.107}$$

Now recall (2.3.31), where we proved the inequality in dimension $n = 3$: for all $b > 0$

$$|(V_2\varphi, \varphi)| \leq a(-\Delta\varphi, \varphi) + b\|\varphi\|_{L^2} \tag{2.3.108}$$

where by (2.3.34)

$$a^2 = \frac{\|V_2\|_K}{4\pi}. \tag{2.3.109}$$

We can now apply (2.3.108), (2.3.109) to $V_{\theta -}$ whose Kato norm is independent of $\theta$:

$$a^2 = \frac{\|V_{\theta -}\|_K}{4\pi} = \frac{\|V_-\|_K}{4\pi} < \frac{c_3}{8\pi} = \frac{1}{4}$$

by (2.3.97), and this concludes the proof of iii) in dimension $n = 3$.

The proof for $n \geq 3$ is identical; it is sufficient to use again (2.3.31), (2.3.34) which are still true for general dimension $n$, as noticed in Remark 2.3.5. □

The following consequence will be useful:

**Corollary 2.3.20.** *Assume $V$ satisfies the assumptions of Proposition 2.3.19, let $H_\theta = -\Delta + \theta V(\sqrt{\theta} x)$, $H_0 = -\Delta$, and let $\varphi_j(s) = \varphi_0(2^{-j}s)$, $\psi_j(s) = \psi_0(2^{-j}s)$ be two homogeneous Paley-Littlewood partitions of unity, $j \in \mathbb{Z}$. Then we have the estimates: for all $j, k \in \mathbb{Z}$,*

$$\|\varphi_j(\sqrt{H_\theta})\psi_k(\sqrt{H_0})\|_{\mathcal{L}(L^1;L^1)} \leq C2^{-2j+2k} \tag{2.3.110}$$

*with a constant $C$ independent of $j, k$ and of $\theta > 0$. The same estimates hold interchanging $H_0$ and $H_\theta$.*

*Proof.* We first note two consequences of (2.3.98): for all $j$, with a constant independent of $j$,

$$\|\varphi_j(\sqrt{H_\theta})H_\theta\|_{\mathcal{L}(L^p;L^p)} \leq C2^{2j}, \qquad \|\varphi_j(\sqrt{H_\theta})H_\theta^{-1}\|_{\mathcal{L}(L^p;L^p)} \leq C2^{-2j} \tag{2.3.111}$$

and the analogous ones for $H_0$ instead of $H$ (indeed, the case $V = 0$ is a special case of (2.3.111)). The first one follows by choosing

$$g(s) = \varphi_0(\sqrt{s})s \quad \Longrightarrow \quad g(2^{-2j}H_\theta) = \varphi_j(\sqrt{H_\theta})2^{-2j}H_\theta;$$

the second one follows by

$$g(s) = \varphi_0(\sqrt{s})s^{-1} \quad \Longrightarrow \quad g(2^{-2j}H_\theta) = \varphi_j(\sqrt{H_\theta})2^{2j}H_\theta^{-1}.$$

Then we can write

$$\varphi_j(\sqrt{H_\theta})\psi_k(\sqrt{H_0}) = \varphi_j(\sqrt{H_\theta})H_\theta^{-1}H_\theta\psi_k(\sqrt{H_0}) = \\ = \varphi_j(\sqrt{H_\theta})H_\theta^{-1}H_0\psi_k(\sqrt{H_0}) + \varphi_j(\sqrt{H_\theta})H_\theta^{-1}V_\theta\psi_k(\sqrt{H_0}).$$

The first term can be estimated immediately using (2.3.111):

$$\|\varphi_j(\sqrt{H_\theta})H_\theta^{-1}H_0\psi_k(\sqrt{H_0})\|_{\mathcal{L}(L^p;L^p)} \leq C2^{-2j+2k};$$

for the second one we may write

$$\|\varphi_j(\sqrt{H_\theta})H_\theta^{-1}V_\theta\psi_k(\sqrt{H_0})\|_{\mathcal{L}(L^p;L^p)} \leq C2^{-2j}\|V_\theta\psi_k(\sqrt{H_0})\|_{\mathcal{L}(L^p;L^p)}$$

and since

$$V_\theta\psi_k(\sqrt{H_0}) = V_\theta R_0(0)H_0\psi_k(\sqrt{H_0}),$$

recalling that $V_\theta R_0$ is a bounded operator on $L^1$ (with norm proportional to the Kato norm of $V_\theta$ which does not depend on $\theta$) and applying again (2.3.111) we obtain (2.3.110).

For higher dimension $n > 3$ the proof is identical; only in the last step we need the estimate

$$\|VR_0(0)f\|_{L^1} \leq C\|V\|_K\|f\|_{L^1}$$

which is true for any $n$. Indeed, $R_0(0)$ apart from a constant is the convolution with the kernel $|x|^{2-n}$, and this gives immediately that $R_0(0)V$ is bounded on $L^\infty$ with norm $C\|V\|_K$. By duality we deduce that $VR_0(0)$ is a bounded operator on $L^1$ with the same norm. $\square$

Using Corollary 2.3.20 we can show the equivalence of non homogeneous Besov spaces $B^s_{1,q}(V)$ with the standard ones, and later on we shall prove the more delicate result concerning the homogeneous case. We recall the precise definition: given a homogeneous Paley-Littlewood partition of unity $\varphi_j(s) = \varphi_0(2^{-j}s)$, $j \in \mathbb{Z}$, we set for $p \in [1, \infty]$, $q \in [1, \infty[$, $s \in \mathbb{R}$

$$\|f\|_{\dot{B}^s_{p,q}(V)} = \left( \sum_{j \in \mathbb{Z}} 2^{jsq} \|\varphi_j(\sqrt{H})f\|^q_{L^p} \right)^{1/q}$$

with obvious modification when $q = \infty$. On the other hand, if we consider a non homogeneous Paley-Littlewood partition of unity, i.e., $\varphi_j$ as above for $j \geq 0$, and we set

$$\psi_0 = 1 - \sum_{j \geq 0} \varphi_j$$

we have $\psi_0 \in C_0^\infty(\mathbb{R}^n)$, and we can define the non homogeneous Besov norm as

$$\|f\|_{B^s_{p,q}(V)} = \left( \|\psi_0(\sqrt{H})f\|^q_{L^p} + \sum_{j \geq 0} 2^{jsq} \|\varphi_j(\sqrt{H})f\|^q_{L^p} \right)^{1/q}$$

When $V = 0$ we obtain the classical Besov spaces, which we denote simply by $\dot{B}^s_{p,q}$ and $B^s_{p,q}$.

**Theorem 2.3.21.** *Assume the Kato class potential $V = V_+ - V_-$ on $\mathbb{R}^n$, $n \geq 3$, $V_\pm \geq 0$, satisfies*

$$\|V_+\|_K < \infty \tag{2.3.112}$$

*and*

$$\|V_-\|_K < \frac{1}{2} c_n \equiv \pi^{n/2} / \Gamma\left(\frac{n}{2} - 1\right) \tag{2.3.113}$$

*Then we have the equivalence of norms*

$$\|f\|_{B^s_{1,q}(V)} \cong \|f\|_{B^s_{1,q}} \tag{2.3.114}$$

*for all $q \in [1, \infty]$, $0 \leq s < 2$. Moreover, for the rescaled potentials*

$$V_\theta(x) = \theta V(\sqrt{\theta} x) \tag{2.3.115}$$

*we have the uniform estimates*

$$C^{-1} \|f\|_{B^s_{1,q}} \leq \|f\|_{B^s_{1,q}(V_\theta)} \leq C \|f\|_{B^s_{1,q}} \tag{2.3.116}$$

*with a constant $C$ independent of $\theta > 0$.*

*Remark* 2.3.6. In order to improve the result and consider higher values of $s \geq 2$ stronger smoothness assumptions on the of the potential $V$ are necessary; we shall not pursue this problem here. Also, to prove the equivalence of Besov spaces $B^s_{p,q}$ for $p \neq 1$, one should prove different bounds for the operator $VR_0$ on $L^p$; this is possible but quite technical and we limit ourselves to the case $p = 1$ which is our main interest here.

*Proof.* We shall limit ourselves to the case $q = 1$ and we shall only prove the inequality

$$\|f\|_{B^s_{1,1}(V_\theta)} \leq C\|f\|_{B^s_{1,1}}; \tag{2.3.117}$$

the proof of the reverse inequality and of the cases $1 < q \leq \infty$ are completely analogous.

In the following we shall drop the index $\theta$ since all the estimates we use (from Proposition 2.3.19 and Corollary 2.3.20) have constants independent of $\theta > 0$.

Using the notations

$$D_V = \sqrt{H}, \quad D = \sqrt{H_0}$$

we have

$$\|f\|_{B^s_{1,1}(V)} = \|\psi_0(D_V)f\|_{L^1} + \sum_{j=0}^{\infty} 2^{js}\|\varphi_j(D_V)f\|_{L^1}. \tag{2.3.118}$$

Using

$$1 = \psi_0(D) + \sum_{k\geq 0} \varphi_k(D),$$

we have

$$\begin{aligned}\|f\|_{B^s_{1,1}(V)} \leq \|\psi_0(D_V)\psi_0(D)f\|_{L^1} + \sum_{k=0}^{\infty} \|\psi_0(D_V)\varphi_k(D)f\|_{L^1}+ \\ + \sum_{j=0}^{\infty} 2^{js}\|\varphi_j(D_V)\psi_0(D)f\|_{L^1} + \sum_{j,k\geq 0} 2^{js}\|\varphi_j(D_V)\varphi_k(D)f\|_{L^1} = \\ = I + II + III + IV.\end{aligned}$$

We estimate separately the four terms.

Since by (2.3.99) $\psi_0(D_V)$ is bounded on $L^1$, we have for the first term

$$I = \|\psi_0(D_V)\psi_0(D)f\|_{L^1} \leq C\|f\|_{L^1} \tag{2.3.119}$$

and since

$$\|f\|_{L^1} \leq \|\psi_0(D)f\|_{L^1} + \sum_{j\geq 0} \|\varphi_j(D)f\|_{L^1}$$

this is smaller than $C\|f\|_{B^s_{1,1}}$.

The same argument gives for the second term

$$II = \sum_{k=0}^{\infty} \|\psi_0(D_V)\varphi_k(D)f\|_{L^1} \leq C\sum_{k=0}^{\infty} \|\varphi_k(D)f\|_{L^1} \leq C\|f\|_{B^s_{1,1}}$$

As to the third term, we can write

$$\sum_{j=0}^{\infty} 2^{js}\|\varphi_j(D_V)\psi_0(D)f\|_{L^1} = \sum_{j=0}^{\infty} 2^{js}\|\varphi_j(D_V)(-\Delta_V)^{-1}(-\Delta_V)\psi_0(D)f\|_{L^1}$$

and recalling (2.3.111) used in the proof of the corollary we have (for $s<2$)

$$III \le C\sum_{j\ge 0} 2^{-j(2-s)}\|(-\Delta_V)\psi_0(D)f\|_{L^1} = C\|(-\Delta_V)\psi_0(D)f\|_{L^1} \le$$
$$\le C\|(-\Delta)\psi_0(D)f\|_{L^1} + C\|V\psi_0(D)f\|_{L^1}.$$

Now we have

$$\|V\psi_0(D)f\|_{L^1} = \|VR_0(0)(-\Delta)\psi_0(D)f\|_{L^1} \le C\|V\|_K\|(-\Delta)\psi_0(D)f\|_{L^1}$$

and since $(-\Delta)\psi_0(D)$ is bounded in $L^1$ by (2.3.99), we conclude that

$$III \le C_2\|f\|_{L^1} \le C_3\|f\|_{B^s_{1,1}} \tag{2.3.120}$$

as for the first term.

Finally, we split the fourth term in the two sums for $j\le k$ and $j>k$:

$$IV = \sum_{j,k\ge 0} 2^{js}\|\varphi_j(D_V)\varphi_k(D)f\|_{L^1} = \sum_{j\le k} + \sum_{j>k}.$$

For $j\le k$ we use the fact that $\varphi_j(D_V)$ are bounded on $L^1$ with uniform norm by (2.3.99) and hence

$$\sum_{j\le k} \le C\sum_{k\ge 0}\|\varphi_k(D)f\|_{L^1}\sum_{0\le j\le k} 2^{js} = 2C\sum_{k\ge 0} 2^{ks}\|\varphi_k(D)f\|_{L^1}.$$

For $j>k$, we write $\varphi_j = \varphi_j(\varphi_{j-1}+\varphi_j+\varphi_{j+1}) = \varphi_j\widetilde{\varphi_j}$ and we have

$$\sum_{j>k} 2^{js}\|\varphi_j(D_V)\varphi_k(D)f\|_{L^1} = \sum_{j>k} 2^{js}\|\varphi_j(D_V)\varphi_k(D)\widetilde{\varphi_k(D)}f\|_{L^1};$$

now by the corollary we obtain

$$\sum_{j>k} 2^{js}\|\varphi_j(D_V)\varphi_k(D)\widetilde{\varphi_k(D)}f\|_{L^1} \le \sum_{j>k} C2^{(k-j)(2-s)}2^{ks}\|\widetilde{\varphi_k}f\|_{L^1}$$

and since $\sum_{j>k} 2^{(k-j)(2-s)} < 1$ we have

$$IV = \sum_{j,k\ge 0} 2^{js}\|\varphi_j(D_V)\varphi_k(D)f\|_{L^1} \le C\sum_{k\ge 0} 2^k\|\widetilde{\varphi_k(D)}f\|_{L^1} \le C\|f\|_{B^1_{1,1}(\mathbb{R}^3)}. \tag{2.3.121}$$

and this concludes the proof. □

We shall finally show that the preceding result implies the equivalence also for homogeneous Besov spaces. Indeed, the uniformity of estimates (2.3.116) makes it possible to apply a rescaling argument, using the following lemma:

**Lemma 2.3.22.** *Let $s \in \mathbb{R}$, $p, q, \in [1, \infty]$. The homogeneous $\dot{B}^s_{p,q}(V)$ norm has the following rescaling property with respect to scaling $(S_\lambda f)(x) = f(\lambda x)$:*

$$\|S_\lambda f\|_{\dot{B}^s_{p,q}(V)} = \lambda^{s-\frac{n}{p}} \|f\|_{\dot{B}^s_{p,q}(V_{\lambda^{-2}})} \tag{2.3.122}$$

*provided $\lambda = 2^k$ for some $k \in \mathbb{Z}$.*

*Remark* 2.3.7. A similar property holds also for any positive $\lambda$, with equality replaced by equivalence of norms, however (2.3.122) will be sufficient for our purposes.

*Proof.* From the identity

$$(-\Delta + V(x)) S_\lambda f(x) = \lambda^2 S_\lambda(-\Delta + \lambda^{-2} V(x/\lambda)) f(x)$$

we obtain the rule

$$\Delta_V S_\lambda = \lambda^2 S_\lambda \Delta_{V_{\lambda^{-2}}}$$

with the usual notations

$$\Delta_V = \Delta + V, \qquad V_\theta = \theta V(\sqrt{\theta} x).$$

This implies

$$g(-\Delta_V) S_\lambda = S_\lambda \; g(-\lambda^2 \Delta_{V_{\lambda^{-2}}})$$

and in particular for the functions $\phi_j(s) = \phi_0(2^{-j} s)$, writing as usual $D_V = \sqrt{-\Delta_V}$,

$$\phi_j(D_V) S_\lambda = \phi_0(2^{-j} D_V) S_\lambda = S_\lambda \phi_0(2^{-j} \lambda D_{V_{\lambda^{-2}}}).$$

With the special choice $\lambda = 2^k$ this can be written

$$\phi_j(D_V) S_{2^k} = S_{2^k} \phi_{j-k}(D_{V_{2^{-2k}}}).$$

Hence we have the identity, for $\lambda = 2^k$,

$$\|S_\lambda\|^q_{\dot{B}^s_{p,q}} = \sum_{j \in \mathbb{Z}} 2^{jsq} \|\phi_j(D_V) S_\lambda f\|^q_{L^p} = \sum_{j \in \mathbb{Z}} 2^{jsq} 2^{-knq/p} \|S_\lambda \phi_{j-k}(D_{V_{2^{-2k}}}) f\|^q_{L^p}$$

since $L^p$ rescales as $\lambda^{-n/p}$; writing $2^{jsq} 2^{knq/p} = 2^{k(s-n/p)q} 2^{(j+k)sq}$ and shifting the sum $j + k \to j$ we conclude the proof. □

Thus we arrive at the final result of this section:

**Theorem 2.3.23.** *Assume the Kato class potential* $V = V_+ - V_-$ *on* $\mathbb{R}^n$, $n \geq 3$, $V_\pm \geq 0$, *satisfies*

$$\|V_+\|_K < \infty \tag{2.3.123}$$

*and*

$$\|V_-\|_K < \frac{1}{2}c_n \equiv \pi^{n/2}/\Gamma\left(\frac{n}{2} - 1\right) \tag{2.3.124}$$

*Then we have the equivalence of norms*

$$\|f\|_{\dot{B}^s_{1,q}(V)} \cong \|f\|_{\dot{B}^s_{1,q}} \tag{2.3.125}$$

*for all* $q \in [1, \infty]$, $0 < s < 2$. *Moreover, for the rescaled potentials*

$$V_\theta(x) = \theta V(\sqrt{\theta}x)$$

*we have the uniform estimates*

$$C^{-1}\|f\|_{\dot{B}^s_{1,q}} \leq \|f\|_{\dot{B}^s_{1,q}(V_\theta)} \leq C\|f\|_{\dot{B}^s_{1,q}} \tag{2.3.126}$$

*with a constant* $C$ *independent of* $\theta > 0$.

*Proof.* We shall consider in detail the case $q = 1$ only, the remaining cases being completely analogous.

We already know that (2.3.126) holds for dotless Besov spaces. Now we need to prove the following inequalities

$$C^{-1}\|f\|_{\dot{B}^s_{1,1}(V_\theta)} \leq \|f\|_{B^s_{1,1}(V_\theta)} \leq C\|f\|_{\dot{B}^s_{1,1}(V_\theta)} + C\|f\|_{\dot{B}^0_{1,1}(V_\theta)} \tag{2.3.127}$$

with a constant $C$ independent of $\theta > 0$.

First of all we prove that ($D = \sqrt{-\Delta}$, $D_{V_\theta} = \sqrt{-\Delta_{V_\theta}}$)

$$\sum_{j<-1} 2^{js}\|\varphi_j(D_{V_\theta})f\|_{L^1} \leq C\|\psi_0(D_{V_\theta})f\|_{L^1}. \tag{2.3.128}$$

We notice that $\psi_0$ is equal to 1 on the support of $\varphi_j$ for $j < -1$. Hence $\varphi_j = \varphi_j\psi_0$ for $j < -1$ and we can write

$$\|\varphi_j(D_{V_\theta})f\|_{L^1} = \|\varphi_j(D_{V_\theta})\psi_0(D_{V_\theta})f\|_{L^1} \leq C\|\psi_0(D_{V_\theta})f\|_{L^1}.$$

(we have used the uniform estimates (2.3.98)-(2.3.99)). Thus (2.3.128) follows, provided $s > 0$ so that $\sum_{j<-1} 2^{js}$ is convergent.

The term for $j = -1$ is estimated in a simple way ($\varphi_{-1} = \varphi_{-1}(\psi_0 + \varphi_1)$)

$$\|\varphi_{-1}(D_{V_\theta})f\|_{L^1} \leq \|\varphi_{-1}(D_{V_\theta})\psi_0(D_{V_\theta})f\|_{L^1} + \|\varphi_{-1}(D_{V_\theta})\varphi_1(D_{V_\theta})f\|_{L^1} \leq$$
$$\leq C\|\psi_0(D_{V_\theta})f\|_{L^1} + C\|\varphi_1(D_{V_\theta})f\|_{L^1}. \tag{2.3.129}$$

Clearly, (2.3.128) and (2.3.129) imply immediately the first inequality (2.3.127).

The second inequality in (2.3.127) is easier: it is sufficient to prove that

$$\|\psi_0(D_{V_\theta})f\|_{L^1} \leq C \sum_{j\leq 1} \|\varphi_j(D_{V_\theta})f\|_{L^1}$$

which follows from $\psi_0 = \psi_0 \cdot \sum_{j\leq 1} \varphi_j$, the triangle inequality, and the boundedness of $\psi_0(D_{V_\theta})$ on $L^1$ with uniform norm. This give (2.3.127). Notice that all the constants appearing in the above inequalities are uniform in $\theta > 0$.

By (2.3.127) and the equivalence (2.3.116) we can write for $0 < s < 2$

$$\|f\|_{\dot{B}^s_{1,1}} \leq C\|f\|_{B^s_{1,1}} \leq C\|f\|_{B^s_{1,1}(V_\theta)} \leq C\|f\|_{\dot{B}^s_{1,1}(V_\theta)} + C\|f\|_{\dot{B}^0_{1,1}(V_\theta)}.$$

If we apply this inequality to a rescaled function $S_{2^k}f$ and recall Lemma 2.3.22, we obtain for all $k \in \mathbb{Z}$

$$2^{k(s-n)}\|f\|_{\dot{B}^s_{1,1}} \leq C2^{k(s-n)}\|f\|_{\dot{B}^s_{1,1}(V_{\theta 2^{-2k}})} + C2^{-kn}\|f\|_{\dot{B}^0_{1,1}(V_{\theta 2^{-2k}})}$$

with constants independent of $k, \theta$; we can now choose $\theta = 2^{2k}\gamma$, divide by $2^{k(s-n)}$ and let $k \to +\infty$ to obtain

$$\|f\|_{\dot{B}^s_{1,1}} \leq C\|f\|_{\dot{B}^s_{1,1}(V_\gamma)}$$

which is the first part of the thesis. The reverse inequality is proved in the same way. □

### 2.3.7 Conclusion of the proof

By the spectral calculus for $H = -\Delta + V$, given any bounded continuous function $\phi(s)$ on $\mathbb{R}$, we can represent the operator $\phi(H)$ on $L^2$ as

$$\phi(H)f = \frac{1}{2\pi i} \cdot L^2 - \lim_{\varepsilon\to 0} \int \phi(\lambda)[R_V(\lambda + i\varepsilon) - R_V(\lambda - i\varepsilon)]f d\lambda. \quad (2.3.130)$$

If $\phi = \psi'$ is the derivative of a $C^1$ compactly supported function we can integrate by parts obtaining the equivalent form

$$\phi(H)f = \frac{i}{2\pi} \cdot L^2 - \lim_{\varepsilon\to 0} \int \psi(\lambda)[R_V(\lambda + i\varepsilon)^2 - R_V(\lambda - i\varepsilon)^2]f d\lambda. \quad (2.3.131)$$

Now, fix a smooth function $\psi(s)$ with compact support in $]0, +\infty[$ and consider the Cauchy problem

$$\begin{cases} \Box u + V(x)u = 0, & t \geq 0,\ x \in \mathbb{R}^3 \\ u(0,t) = 0, & u_t(0,x) = \psi(H)g \end{cases} \quad (2.3.132)$$

for some smooth $g$. Then the solution $u$ can be represented as

$$u(t,\cdot) = L^2 - \lim_{\varepsilon\to 0} u_\varepsilon(t,\cdot)$$

where

$$u_\varepsilon(t,x) = \frac{1}{2\pi i}\int_0^\infty \frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}\psi(\lambda)[R_V(\lambda+i\varepsilon) - R_V(\lambda-i\varepsilon)]gd\lambda \quad (2.3.133)$$

or equivalently, after integration by parts,

$$u_\varepsilon(t,x) = \frac{1}{\pi i t}\int_0^\infty \cos(t\sqrt{\lambda})\psi'(\lambda)[R_V(\lambda+i\varepsilon) - R_V(\lambda-i\varepsilon)]gd\lambda + \\ + \frac{1}{\pi i t}\int_0^\infty \cos(t\sqrt{\lambda})\psi(\lambda)[R_V(\lambda+i\varepsilon)^2 - R_V(\lambda-i\varepsilon)^2]gd\lambda. \quad (2.3.134)$$

Estimates (2.3.71) and (2.3.73) applied to (2.3.134) give

$$\|u_\varepsilon(t,\cdot)\|_{L^\infty} \le \|g\|_{L^1}\frac{C}{t}\int_0^\infty \left(|\psi'(\lambda)|\sqrt{\lambda_\varepsilon} + \frac{|\psi(\lambda)|}{\sqrt{\lambda_\varepsilon}}\right)d\lambda$$

and recalling that

$$\lambda \le \lambda_\varepsilon \le \lambda + \frac{\varepsilon}{2}$$

we obtain

$$\|u_\varepsilon(t,\cdot)\|_{L^\infty} \le \|g\|_{L^1}\frac{C}{t}\int_0^\infty \left(|\psi'(\lambda)|(\sqrt{\lambda}+\sqrt{\varepsilon}) + \frac{|\psi(\sqrt{\lambda})|}{\sqrt{\lambda}}\right)d\lambda. \quad (2.3.135)$$

Let now $\varphi_j(s)$, $j \in \mathbb{Z}$ be the homogeneous Paley-Littlewood partition of unity defined in the Introduction, with

$$\varphi_j(s) = \phi_0(2^{-j}s),$$

define

$$\widetilde{\varphi}_j(s) = \varphi_{j-1}(s) + \varphi_j(s) + \varphi_{j+1}(s) \quad (2.3.136)$$

and choose in (2.3.132)

$$\psi(\lambda) = \widetilde{\varphi}_j(\sqrt{\lambda}) \equiv \widetilde{\varphi}_0(2^{-j}\sqrt{\lambda}).$$

We thus obtain

$$\|u_\varepsilon(t,\cdot)\|_{L^\infty} \le \|g\|_{L^1}\frac{C}{t}\int_0^\infty \left(2^{-j}|\widetilde{\varphi}_0'(2^{-j}\sqrt{\lambda})|\frac{\sqrt{\lambda}+\sqrt{\varepsilon}}{2\sqrt{\lambda}} + \frac{|\widetilde{\varphi}_0(2^{-j}\sqrt{\lambda})|}{\sqrt{\lambda}}\right)d\lambda$$

which after the change of variables $\mu = 2^{-j}\sqrt{\lambda}$ gives

$$\|u_\varepsilon(t,\cdot)\|_{L^\infty} \le \frac{C}{t}(2^j + \sqrt{\varepsilon})\|g\|_{L^1}. \quad (2.3.137)$$

for some constant $C$ independent of $j, t$ and $g$. If we let $\varepsilon \to 0$, for fixed $t$ the functions $u_\varepsilon(t, \cdot)$ converge in $L^2$ to the solution $u(t,x)$; hence a subsequence converges a.e. and we obtain the estimate

$$\|u(t,\cdot)\|_{L^\infty} \leq C\frac{2^j}{t}\|g\|_{L^1} \tag{2.3.138}$$

for the solution $u(t,x)$ of the Cauchy problem

$$\begin{cases} \Box u + V(x)u = 0, & t \geq 0,\ x \in \mathbb{R}^3 \\ u(0,t) = 0, & u_t(0,x) = \widetilde{\varphi}_j(\sqrt{H})g \end{cases} \tag{2.3.139}$$

If we now choose

$$g = \varphi_j(\sqrt{H})f$$

and notice that $\widetilde{\varphi}_j g \equiv \widetilde{\varphi}_j \varphi_j f \equiv \varphi_j f$ since $\widetilde{\varphi}_j = 1$ on the support of $\varphi_j$, we conclude that: the solution $u(t,x)$ of the Cauchy problem

$$\begin{cases} \Box u + V(x)u = 0, & t \geq 0,\ x \in \mathbb{R}^3 \\ u(0,t) = 0, & u_t(0,x) = \varphi_j(\sqrt{H})f \end{cases} \tag{2.3.140}$$

satisfies the estimate

$$\|u(t,\cdot)\|_{L^\infty} \leq C\frac{2^j}{t}\|\varphi_j(\sqrt{H})f\|_{L^1} \tag{2.3.141}$$

Consider now the original Cauchy problem (2.3.1); decomposing the initial datum $f$ as

$$f = \sum_{j\in\mathbb{Z}} \varphi_j(\sqrt{H})f$$

applying estimate (2.3.141) and summing over $j$, we obtain by linearity that the solution $u(t,x)$ to (2.3.1) satisfies the estimate

$$\|u(t,\cdot)\|_{L^\infty} \leq \frac{C}{t}\|f\|_{\dot{B}^1_{1,1}(V)}. \tag{2.3.142}$$

Since by Theorem 2.3.23 this norm is equivalent to the standard one, we see that the proof of Theorem 2.3.1 is concluded.

## 2.4 The Schrödinger and heat equation perturbed with a small rough potential

In this section we consider perturbed Schrödinger and heat equations

$$\frac{1}{i}\partial_t u - \Delta u + Vu = 0, \quad u(0,x) = u_0(x), \tag{2.4.1}$$

$$\partial_t u - \Delta u + Vu = 0, \quad u(0,x) = u_0(x) \tag{2.4.2}$$

in dimension $n \geq 3$. The importance of these equations in quantum mechanics (see [61]), in the theory of combustion (see [109]) and in many other applications is well known.

In this Section we deduce the complete Strichartz estimates for the solution of the Schrödinger equation (2.4.1) perturbed with a larger class of potentials satisfying $V \leq |x|^{-2}$, via interpolation between the endpoint and the energy estimate. The arguments of the previous sections are then extended to the case of a small time dependent potential $V(t,x)$.

We study also the heat equation (2.4.2) perturbed by a singular potential and we prove the existence of solutions, the maximum principle and the dispersive estimates.

### 2.4.1 Selfadjointness of $H = -\Delta + V$

In this subsection we check that the sum $H = -\Delta + V$ can be realized as a selfadjoint operator on $L^2$ by a standard Friedrichs extension. This will allow us to consider the Schrödinger flow $e^{-itH}$ and the heat flow $e^{-tH}$ in the following of the section. Notice that here we assume that the potential is in the weak Lebesgue space $L^{(\frac{n}{2},\infty)}$, which is not comparable to the Kato class considered in the last sections.

Consider the bilinear form

$$B(f,f) = (\nabla f, \nabla f)_{L^2(\mathbb{R}^n)} + \int_{\mathbb{R}^n} V(x)|f(x)|^2 dx, \;\; x \in \mathbb{R}^n, n \geq 3.$$

It is not difficult to see that

$$f \to Vf$$

is a self adjoint operator with dense domain $\dot{H}^2(\mathbb{R}^n)$. In this case we can use the KLMN- theorem (see theorem 10.17 in [83]). Due to this theorem it is sufficient to verify the estimate

$$\left| \int_{\mathbb{R}^n} V(x)|f(x)|^2 dx \right| \leq a \int_{\mathbb{R}^n} |\nabla f(x)|^2 dx - b\|f\|^2_{L^2(\mathbb{R}^n)},$$

with $a < 1$. Indeed, our assumption

$$\|V(\cdot)\|_{L^{(\frac{n}{2},\infty)}} < \frac{2n}{C_s(n-2)},$$

implies that

$$\sqrt{|V|} \in L^{(n,\infty)},$$

so that, by the Hölder inequality for Lorentz spaces,

$$\|\sqrt{|V|}f\|_{L^2} \leq C\|\sqrt{|V|}\|_{L^{(n,\infty)}}\|f\|_{L^{(q,2)}} \leq CC_0\|f\|_{L^{(q,2)}},$$

where

$$\frac{1}{q} = \frac{1}{2} - \frac{1}{n}, \quad \text{i.e. } q = \frac{2n}{n-2}.$$

Using the Sobolev embedding (see [9]) $\dot{H}^1(\mathbb{R}^n) \hookrightarrow L^{(q,2)}(\mathbb{R}^n)$, we get

$$\|f\|_{L^{(q,2)}} \leq C_1\|f\|_{\dot{H}^1}$$

and

$$\left|\int_{\mathbb{R}^n} V(x)|f(x)|^2 dx\right| \leq \|\sqrt{|V|}f\|^2_{L^2(\mathbb{R}^n)} \leq C_0^2C^2C_1^2\|\nabla f\|^2_{L^2(\mathbb{R}^n)}.$$

If $C_0$ is such that $CC_0C_1 < 1$ i.e. $C_0 < \frac{1}{CC_1}$, where $C$ is the constant from the Hölder inequality (for Lorentz spaces) and $C_1$ is the constant from Sobolev embedding, then we can conclude, using the KLMN theorem, that there exists a self-adjoint operator $H = -\Delta + V$ such that

$$((-\Delta + V)f, f)_{L^2} = \|\nabla f\|^2_{L^2} + \int_{\mathbb{R}^n} V(x)|f(x)|^2 dx.$$

### 2.4.2 Strichartz estimates for the Schrödinger flow $e^{-itH}$

In this subsection we study the decay properties of the Schrödinger flow for the operator $H$ constructed above. More precisely, we can represent the solution to the Schrödinger equation (2.4.1) as

$$u(t) = U(t)u_0, \quad U(t) = e^{-itH}.$$

Our starting point will be the following Strichartz estimate, essentially proved in the paper [66]:

**Proposition 2.4.1.** *Let $n \geq 3$ and consider the Cauchy Problem for the Schrödinger equation*

$$\begin{cases} \frac{1}{i}\partial_t u - \Delta u = F(t,x), \\ u(0,x) = 0, \quad x \in \mathbb{R}^n, \end{cases} \tag{2.4.3}$$

*then the following estimates hold:*

$$\|u\|_{L^p_t L^{(q,2)}_x} \leq C\|F\|_{L^{\tilde{p}'}_t L^{(\tilde{q}',2)}_x}, \tag{2.4.4}$$

$$\|u\|_{L_t^p L_x^q} \leq C\|F\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}}, \tag{2.4.5}$$

*for all $p, \tilde{p} \in [2, \infty]$, and $q, \tilde{q} \in [2, \frac{2n}{n-2}]$, such that*

$$\frac{1}{p} + \frac{n}{2q} = \frac{n}{4}, \qquad \frac{1}{\tilde{p}} + \frac{n}{2\tilde{q}} = \frac{n}{4}.$$

*Remark* 2.4.1. Note that for the Schrödinger equation $(p, q) = (2, \frac{2n}{n-2})$ it is the end-point Schrödinger-admissible for $n \geq 3$.

*Proof.* The second estimate (2.4.5) is the standard Strichartz estimate, proved in [66]; notice that it follows from the stronger estimate (2.4.4) by embedding of Lorentz spaces.

Estimate (2.4.4) in the endpoint $p = \tilde{p} = 2$, $q = \tilde{q} = \frac{2n}{n-2}$ is proved in section 6 of [66]. On the other hand, the point $p = \tilde{p} = \infty$, $q = \tilde{q} = 2$ reduces to the standard conservation of energy since $L^{(2,2)} = L^2$. Thus by interpolation we obtain (2.4.4) in the dual case $p = \tilde{p}$, $q = \tilde{q}$. We conclude the proof applying as usual the $TT^*$ method. □

Our next step is to establish the end-point estimate for the perturbed Schrödinger equation:

**Proposition 2.4.2.** *Let $n \geq 3$ and consider the Cauchy Problem*

$$\begin{cases} \frac{1}{i}\partial_t u - \Delta u + Vu = F, \\ u(0, x) = 0, \quad x \in \mathbb{R}^n, \end{cases} \tag{2.4.6}$$

*where $V = V(x)$ is a real-valued potential such that*

$$\|V\|_{L^{(\frac{n}{2},\infty)}} \equiv C_0 < \frac{2n}{C_s(n-2)}, \tag{2.4.7}$$

*(here $C_s$ is the constant appearing in the Strichartz estimates for the unperturbed equation). Then the following estimate holds*

$$\|u\|_{L_t^2 L_x^q} \leq C\|F\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}}, \tag{2.4.8}$$

*where*

$$q = \frac{2n}{n-2},$$

*and $\tilde{p} \in [2, \infty]$, and $\tilde{q} \in [2, \frac{2n}{n-2}]$ are such that*

$$\frac{1}{\tilde{p}} + \frac{n}{2\tilde{q}} = \frac{n}{4}.$$

*Proof.* Indeed we can consider the solution $u = u_1 + u_2$ as the sum of solutions to following Cauchy problems

$$\begin{cases} \frac{1}{i}\partial_t u_1 - \Delta u_1 = F, \\ u(0,x) = 0, \quad x \in \mathbb{R}^n, \ n \geq 3, \end{cases} \tag{2.4.9}$$

and

$$\begin{cases} \frac{1}{i}\partial_t u_2 - \Delta u_2 = -Vu, \\ u(0,x) = 0, \quad x \in \mathbb{R}^n, \ n \geq 3. \end{cases} \tag{2.4.10}$$

For (2.4.9) we have the classical Schrödinger equation, such that

$$\|u_1\|_{L_t^2 L_x^{(q,2)}} \leq C_s \|F\|_{L_t^{\tilde{p}'} L_x^{(\tilde{q}',2)}} \tag{2.4.11}$$

is satisfied for the Proposition 2.4.1 (see [66]).
Since for the Cauchy problem (2.4.10) we have

$$\|u_2\|_{L_t^2 L_x^{(q,2)}} \leq C_s \|Vu\|_{L_t^2 L_x^{(q',2)}}, \tag{2.4.12}$$

we are in position to apply the Hölder estimate (see Theorem 3.5 in [73])

$$\|Vu\|_{L^{(q',2)}} \leq C_2 \|V\|_{L^{(\frac{n}{2},\infty)}} \|u\|_{L^{(q,2)}} \leq C_2 C_0 \|u\|_{L^{(q,2)}} \tag{2.4.13}$$

where

$$C_2 = q \equiv \frac{2n}{n-2},$$

so if $C_0$ is such that $C_s C_0 C_2 < 1$, i.e.

$$C_0 < \frac{2n}{C_s(n-2)},$$

we see that from (2.4.11), (2.4.12) and (2.4.13) that

$$\|u\|_{L_t^2 L_x^{(q,2)}} \leq \frac{C_s}{1 - C_s C_0 C_2} \|F\|_{L_t^{\tilde{p}'} L_x^{(\tilde{q}',2)}},$$

where

$$\frac{1}{\tilde{p}} + \frac{n}{2\tilde{q}} = \frac{n}{4}.$$

So using the Theorem of Calderón (see Lemma 2.5 in [73])

$$\|u\|_{L^{(p,d)}} \leq \left(\frac{d_1}{p}\right)^{\frac{1}{d_1} - \frac{1}{d}} \|u\|_{L^{(p,d_1)}},$$

for $d > d_1, \ 1 < p < \infty$, we get

$$\|u\|_{L_t^2 L_x^q} = \|u\|_{L_t^2 L_x^{(q,q)}} \leq \left(\frac{2}{q}\right)^{\frac{1}{2} - \frac{1}{q}} \|u\|_{L_t^2 L_x^{(q,2)}}$$

and

$$\|u\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}} = \|u\|_{L_t^{\tilde{p}'} L_x^{(\tilde{q}',\tilde{q}')}} \geq \|u\|_{L_t^{\tilde{p}'} L_x^{(\tilde{q}',2)}},$$

so we arrive at

$$\|u\|_{L_t^2 L_x^q} \leq C\|F\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}}, \quad q = \frac{2n}{n-2}, \quad n \geq 3,$$

where

$$C = \left(\frac{n-2}{n}\right)^{\frac{1}{n}} \left(\frac{C_s}{1 - C_s C_0 C_2}\right),$$

and

$$\frac{1}{\tilde{p}} + \frac{n}{2\tilde{q}} = \frac{n}{4}.$$

$\square$

In the next step we consider the point $p = \infty$, $q = 2$:

**Proposition 2.4.3.** *Let $n \geq 3$ and consider the Cauchy Problem for the perturbed Schrödinger equation*

$$\begin{cases} \frac{1}{i}\partial_t u - \Delta u + Vu = F, \\ u(0,x) = 0, \quad x \in \mathbb{R}^n, \end{cases} \tag{2.4.14}$$

*where $V = V(x)$ is a real-valued potential such that*

$$\|V\|_{L^{(\frac{n}{2},\infty)}} < \infty. \tag{2.4.15}$$

*Then the following estimate holds*

$$\|u\|_{L_t^\infty L_x^2} \leq C\|F\|_{L_t^{\tilde{p}'}, L_x^{\tilde{q}'}}, \tag{2.4.16}$$

*where $\tilde{p} \in [2,\infty]$, and $\tilde{q} \in [2, \frac{2n}{n-2}]$ are such that*

$$\frac{1}{\tilde{p}} + \frac{n}{2\tilde{q}} = \frac{n}{4}.$$

*Proof.* Multipling the perturbed Schrödinger equation (2.4.14) by $\bar{u}$ and taking the Imaginary part of integral

$$\mathrm{Im}\left(\frac{1}{i}\int_{\mathbb{R}^n} \partial_t u \cdot \bar{u} dx\right) + \mathrm{Im}\left(\int_{\mathbb{R}^n} |\nabla u|^2 dx\right) + \mathrm{Im}\left(\int_{\mathbb{R}^n} V|u|^2 dx\right) = \mathrm{Im}\left(\int_{\mathbb{R}^n} F\bar{u} dx\right),$$

we notice that

$$\mathrm{Im}\left(\int_{\mathbb{R}^n} |\nabla u|^2 dx\right) = 0$$

and

$$\mathrm{Im}\left(\int_{\mathbb{R}^n} V|u|^2 dx\right) = 0,$$

thus we have

$$-\text{Re}\left(\frac{1}{i}\int_{\mathbb{R}^n}\partial_t u\cdot\bar{u}dx\right)=\text{Im}\left(\int_{\mathbb{R}^n}F\bar{u}dx\right).$$

The Cauchy-Scwhartz inequality implies

$$\partial_t\|u(t)\|_{L^2}^2\leq\|F\|_{L^2}\|u\|_{L^2},$$

and we obtain

$$\|u(t)\|_{L^2}\leq\int_0^t\|F\|_{L^2}dt$$

so we obtain the following estimate

$$\|u\|_{L^\infty L^2}\leq C\|F\|_{L^1L^2}. \tag{2.4.17}$$

The estimate (2.4.8) leads to

$$\|u\|_{L^2L^q}\leq C\|F\|_{L^1L^2},\ \ q=\frac{2n}{n-2},$$

by duality we have also

$$\|u\|_{L^\infty L^2}\leq C\|F\|_{L^2L^{q'}},\ \ q'=\frac{2n}{n+2}. \tag{2.4.18}$$

Interpolating between (2.4.17) and (2.4.18), we obtain

$$\|u\|_{L_t^\infty L_x^2}\leq C\|F\|_{L_t^{\tilde{p}'},L_x^{\tilde{q}'}},$$

where

$$\frac{1}{\tilde{p}}+\frac{n}{2\tilde{q}}=\frac{n}{4}.$$

□

We can now conclude the proof of the full Strichartz estimates for the problem:

**Theorem 2.4.1.** *Let $n\geq 3$, $p,\tilde{p}\in[2,\infty]$, and let $q,\tilde{q}\in[2,\frac{2n}{n-2}]$ be such that*

$$\frac{1}{p}+\frac{n}{2q}=\frac{n}{4},\qquad \frac{1}{\tilde{p}}+\frac{n}{2\tilde{q}}=\frac{n}{4}.$$

*Let $V=V(x)$ be a real-valued potential such that*

$$\|V\|_{L^{(\frac{n}{2},\infty)}}\equiv C_0<\frac{2n}{C_s(n-2)}, \tag{2.4.19}$$

*where $C_s$ is the universal Strichartz constant for the unperturbed equation. Then the solution to the Cauchy Problem*

$$\begin{cases} \frac{1}{i}\partial_t u - \Delta u + V(x)u = F(t,x), \\ u(0,x) = f, \end{cases} \tag{2.4.20}$$

*satisfies the estimates*

$$\|u\|_{L^p(\mathbb{R}_t;L_x^{(q,2)})} + \|u\|_{C(\mathbb{R}_t;L^2)} \leq C\|F\|_{L^{\tilde{p}'}(\mathbb{R}_t;L_x^{(\tilde{q}',2)})} + C\|f\|_{L^2}, \tag{2.4.21}$$

*and*

$$\|u\|_{L^p(\mathbb{R}_t;L_x^q)} + \|u\|_{C(\mathbb{R}_t;L^2)} \leq C\|F\|_{L^{\tilde{p}'}(\mathbb{R}_t;L_x^{\tilde{q}'})} + C\|f\|_{L^2}. \tag{2.4.22}$$

*Proof.* Assume first that $f = 0$. By interpolation between (2.4.8) and (2.4.16), we get

$$\|u\|_{L_t^p L_x^q} \leq C\|F\|_{L_t^{\tilde{p}'}, L_x^{\tilde{q}'}}$$

for all $(p,q)$, $(\tilde{p},\tilde{q})$ as in the statement of the Theorem.

Assume now that $F = 0$ and $f$ arbitrary. The previous estimate and the $TT^*$ argument of [51], yield the estimate

$$\|u\|_{L_t^p L_x^q} \leq C\|f\|_{L^2}.$$

Notice that the conservation of energy gives also

$$\|u\|_{L_t^p L_x^q} + \|u\|_{C_t L^2} \leq C\|f\|_{L^2}.$$

Summing up we obtain (2.4.22). The proof of (2.4.21) is similar (see also the proof of Proposition 2.4.1). □

If we start from the local Strichartz estimates instead of the global ones, in a similar way we can prove the following

**Theorem 2.4.2.** *Under the assumptions of Theorem 2.4.1 we have*

$$\|u\|_{L^p([0,T];L_x^{(q,2)})} + \|u\|_{C([0,T];L^2)} \leq C\|F\|_{L^{\tilde{p}'}([0,T];L_x^{(\tilde{q}',2)})} + C\|f\|_{L^2} \tag{2.4.23}$$

*for all $T > 0$ and with a constant $C$ independent of $T$.*

### 2.4.3 The case of time dependent potentials

The arguments of the previous sections can be extended to cover the case of a small, time dependent potential $V(t,x)$. Indeed, our method of proof is based on a perturbation of the standard Strichartz estimates for the Schrödinger and heat equations. However, we notice that in this case we cannot use the standard theory of selfadjoint operators to study the perturbed Hamiltonian $H = -\Delta + V(t,x)$. Thus in the following we shall consider the problem of existence and of the decay of solutions.

Our first result is the following:

**Theorem 2.4.3.** *Let $n \geq 3$, $p, \tilde{p} \in [2, \infty]$, and let $q, \tilde{q} \in [2, \frac{2n}{n-2}]$ be such that*

$$\frac{1}{p} + \frac{n}{2q} = \frac{n}{4}, \qquad \frac{1}{\tilde{p}} + \frac{n}{2\tilde{q}} = \frac{n}{4}.$$

*Let $V = V(t,x)$ be a real-valued potential such that*

$$\|V\|_{L_t^\infty L_x^{(\frac{n}{2},\infty)}} \equiv C_0 \tag{2.4.24}$$

*is small enough. Then for any $F(t,x) \in L^{\tilde{p}'} L^{\tilde{q}'}$ there exists a unique global solution $u(t,x)$ of the the Cauchy Problem*

$$\begin{cases} \frac{1}{i}\partial_t u - \Delta u + V(t,x)u = F(t,x), \\ u(0,x) = f. \end{cases} \tag{2.4.25}$$

*which satisfies the estimates*

$$\|u\|_{L^p(\mathbb{R}_t;L_x^{(q,2)})} + \|u\|_{C(\mathbb{R}_t;L^2)} \leq C\|F\|_{L^{\tilde{p}'}(\mathbb{R}_t;L_x^{(\tilde{q}',2)})} + C\|f\|_{L^2},$$

*and*

$$\|u\|_{L^p(\mathbb{R}_t;L_x^q)} + \|u\|_{C(\mathbb{R}_t;L^2)} \leq C\|F\|_{L^{\tilde{p}'}(\mathbb{R}_t;L_x^{\tilde{q}'})} + C\|f\|_{L^2}.$$

*Analogous estimates hold on finite time intervals $[0,T]$ with constants independent of $T$.*

*Proof.* The proof follows the lines of the proof of Theorem 2.4.1. We define $\Phi(v)$ as the solution $u$ of the linear problem

$$\begin{cases} \frac{1}{i}\partial_t u - \Delta u = F(t,x) - V(t,x)v, \\ u(0,x) = f. \end{cases} \tag{2.4.26}$$

By Proposition 2.4.1 and [66] we have

$$\begin{aligned} \|u\|_{L^\infty L^2} + \|u\|_{L^2 L^{(q,2)}} &\leq C\|F - Vv\|_{L^2 L^{(q',2)}} + \|f\|_{L^2} \\ &\leq \|F\|_{L^2 L^{(q',2)}} + \|Vv\|_{L^2 L^{(q',2)}} + \|f\|_{L^2}, \end{aligned}$$

where $q = \frac{2n}{n-2}$. Using the Hölder inequality for Lorentz spaces (see [73]) and the assumption (2.4.24), we get

$$\|u\|_{L^\infty L^2} + \|u\|_{L^2 L^{(q,2)}} \leq C\|F\|_{L^2 L^{(q',2)}} + C_0\|v\|_{L^2 L^{(q,2)}} + \|f\|_{L^2}.$$

Thus $\Phi : v \in L^2 L^{(q,2)} \mapsto u \in L^2 L^{(q,2)} \cap L^\infty L^2$. We show now that $\Phi$ is a contraction on the space $L^2 L^{(q,2)}$. Let $v_1, v_2 \in L^2 L^{(q,2)}$ such that $\Phi(v_i) = u_i, i = 1, 2$; then we have

$$\|u_1 - u_2\|_{L^\infty L^2} + \|u_1 - u_2\|_{L^2 L^{(q,2)}} \leq \|V(v_1 - v_2)\|_{L^2 L^{(q',2)}} \leq C_0\|v_1 - v_2\|_{L^2 L^{(q,2)}}.$$

If $C_0 < 1$ the map $\Phi$ is a contraction, and this implies that for any $F \in L^2L^{(q',2)}$ and $f \in L^2$ there exists a unique solution $u(t,x) \in L^2L^{(q,2)} \cap L^\infty L^2$ of the Cauchy problem (2.4.25).

In particular for all $F \in C_c^\infty$ and $f \in L^2$ there exists a unique solution. When $F \in C_c^\infty$, we can proceed as in Proposition 2.4.2 and we can prove the endpoint estimate

$$\|u\|_{L_t^2L_x^q} \leq C\|F\|_{L_t^{\tilde{p}'}L_x^{\tilde{q}'}} + \|f\|_{L^2}, \tag{2.4.27}$$

with

$$q = \frac{2n}{n-2},$$

and $\tilde{p} \in [2,\infty]$, and $\tilde{q} \in [2, \frac{2n}{n-2}]$ are such that

$$\frac{1}{\tilde{p}} + \frac{n}{2\tilde{q}} = \frac{n}{4}.$$

The only difference in the proof is to replace (2.4.13) with the following Hölder estimate

$$\|Vu\|_{L^2L^{(q',2)}} \leq C\|V\|_{L^\infty L^{(\frac{n}{2},\infty)}}\|u\|_{L^2L^{(q,2)}} \leq CC_0\|u\|_{L^2L^{(q,2)}}. \tag{2.4.28}$$

On the other hand, we can repeat the proof of Proposition 2.4.3 and we obtain

$$\|u\|_{L_t^\infty L_x^2} \leq C\|F\|_{L_t^{\tilde{p}'},L_x^{\tilde{q}'}} + \|f\|_{L^2}, \tag{2.4.29}$$

where

$$\frac{1}{\tilde{p}} + \frac{n}{2\tilde{q}} = \frac{n}{4}.$$

Then by interpolation we obtain the full Strichartz estimates

$$\|u\|_{L^p(\mathbb{R}_t;L_x^{(q,2)})} + \|u\|_{C(\mathbb{R}_t;L^2)} \leq C\|F\|_{L^{\tilde{p}'}(\mathbb{R}_t;L_x^{(\tilde{q}',2)})} + C\|f\|_{L^2} \tag{2.4.30}$$

for all $F \in C_c^\infty$.

Since we have proved that for all such $F$ there exists a unique solution $u(t,x)$, by a density argument we easily obtain that for all $F \in L_t^{\tilde{p}'}L_x^{\tilde{q}'}$ there exists a unique global solution $u(t,x) \in L_t^pL_x^q$, with $\frac{1}{\tilde{p}} + \frac{n}{2\tilde{q}} = \frac{n}{4}$.

□

### 2.4.4 Heat equation perturbed with a singular potential

This section is devoted to a study of the perturbed heat equation. The ideas of the preceding sections can be applied also in this case with some modifications. The main difference is the role of the positive part $V_+$ of the potential $V$; indeed, in order to prove the decay of the solution, weaker assumptions on $V_+$ are sufficient.

Our result is the following:

**Theorem 2.4.4.** *Let $n \geq 3$ and assume the potential $V \in L^{(\frac{n}{2},\infty)}$. Moreover, assume that the negative part $V_- = -(V \wedge 0)$ satisfies*

$$\|V_-\|_{L^{(\frac{n}{2},\infty)}} \equiv C_0 < \frac{2n}{C_s(n-2)}. \tag{2.4.31}$$

*Then any solution to the following Cauchy problem*

$$\begin{cases} \partial_t u - \Delta u + V(x)u = F(t,x), \\ u(0,x) = u_0 \in L^1 \cap L^\infty, \end{cases} \tag{2.4.32}$$

*satisfies the Strichartz estimate*

$$\|u\|_{L^p(\mathbb{R}_t;L^q_x)} + \|u\|_{C(\mathbb{R}_t;L^2)} \leq C\|F\|_{L^{\tilde{p}'}(\mathbb{R}_t;L^{\tilde{q}'}_x)} + C\|u_0\|_{L^2}.$$

*where $p, \tilde{p} \in [2,\infty]$, and $q, \tilde{q} \in [2, \frac{2n}{n-2}]$ are such that*

$$\frac{1}{p} + \frac{n}{2q} = \frac{n}{4}, \qquad \frac{1}{\tilde{p}} + \frac{n}{2\tilde{q}} = \frac{n}{4}.$$

We split the proof of Theorem 2.4.4 in several parts.

**Proposition 2.4.4.** *Let $n \geq 3$ and consider the following Cauchy problem*

$$\begin{cases} \partial_t u - \Delta u + V(x)u = 0, \\ u(0,x) = u_0 \geq 0, \end{cases} \tag{2.4.33}$$

*with initial data $u_0 \in L^1 \cap L^\infty$, and we assume that*

$$V(x) \geq 0 \qquad \text{and} \qquad V \in L^{(\frac{n}{2},\infty)}. \tag{2.4.34}$$

*Then there exists a unique solution to the Cauchy problem (2.4.33)*

$$u(t,x) = e^{-tH_0}u_0$$

*satisfying the maximum principle, i.e.*

$$u \geq 0.$$

*Proof.* Since we know that the maximum principle holds if the potential is positive and $V \in L^\infty$, we consider a sequence of truncated potentials $V_k = V \wedge k,\ k \geq 1$ so that $V_k \in L^\infty$. We consider then the respectively approximated Cauchy problem

$$\begin{cases} \partial_t u_k - \Delta u_k + V_k(x)u_k = 0, k \geq 1, \\ u_k(0,x) = u_0, u_0 \geq 0, \end{cases} \tag{2.4.35}$$

and by maximum principle $0 \leq u_{k+1} \leq u_k \leq u_0$. Since $\{u_k\}$ is a sequence decreasing and $u_0 \in L^1 \cap L^\infty$, then by monotone convergence Theorem we have that $\{u_k\}$ converge in strong sense to $u(t,x)$

$$u(t,x) = L^p - \lim_{k\to\infty} u_k(t,x), \qquad 1 \leq p < \infty.$$

Now it suffices to prove that $u(t,x)$ is a solution to (2.4.33), so we have that $0 \leq u \leq u_k \leq u_0$. Thus since $u(t,x)$ satisfies the Maximum principle (see [72]), we have the uniqueness of the solution to (2.4.33).

Since $u_0 \in L^1 \cap L^\infty$ and $\{u_k\}$ is a sequence decreasing such that $u_k \leq |u_0|$, by Theorem of Lebesgue we have the convergence $u_k \to u$ in $L^1$.
As consequence we have following convergences in the distributional sense $\mathcal{D}'$ $\forall k \to \infty$

$$\begin{aligned} &u_k \to u, \\ &\partial_t u_k \to \partial_t u, \\ &\Delta u_k \to \Delta u. \end{aligned}$$

Then it remains to prove that we have the following convergence

$$V_k u_k \to V u$$

in the distributional sense. Indeed, we shall use the identity

$$V_k u_k - V u = (V_k - V)u_k + V(u_k - u). \tag{2.4.36}$$

Consider the first term to (2.4.36) and since $L^{(\frac{n}{2},\infty)} \subset L^1_{\text{loc}}$ we can take

$$V \in L^1_{\text{loc}}(\mathbb{R}^n),$$

that implies

$$\int_K |V(x) - V_k(x)| dx \to 0 \quad \forall k \to \infty,$$

so that

$$\int_K |V(x) - V_k(x)||u_k(t,x)| dx \leq \sup_{x\in\mathbb{R}^n} |u_k(t,x)| \int_K |V(x) - V_k(x)| dx \to 0, \quad \forall k \to \infty.$$

Thus the first term converges

$$(V_k - V)u_k \to 0 \quad \forall k \to \infty$$

in the distributional sense $\mathcal{D}'$.

Now we are ready to estimate the second term to (2.4.36). We have

$$\|V(u_k - u)\|_{L^1} \leq \|V\|_{L^{(\frac{n}{2},\infty)}} \|u_k - u\|_{L^{(q,1)}} \leq C_0 \|u_k - u\|_{L^{(q,1)}}$$

where $\frac{1}{q} = 1 - \frac{2}{n}$, and using the real interpolation (see [73])

$$L^{(q,1)} = (L^1, L^\infty)_{(1-\frac{1}{q},1)},$$

we have the following

$$\|u_k - u\|_{L^{(q,1)}} \leq \|u_k - u\|_{L^1}^{\frac{2}{n}} \|u_k - u\|_{L^\infty}^{\frac{n-2}{n}}.$$

Since $\{u_k\}$ is decreasing and $u_k \leq u_0 \in L^1 \cap L^\infty$, by monotone convergence Theorem one obtains

$$\|u_k - u\|_{L^1} \to 0,$$

and

$$\|u_k - u\|_{L^\infty} \to 0.$$

Thus $V(u_k - u) \to 0$ in $L^1$, and so it converges in distributional sense, i.e.

$$V_k u_k - V u \to 0.$$

This concludes the proof.

□

**Proposition 2.4.5.** *Let $n \geq 3$ and assume that*

$$V_+(x) \geq 0, \qquad V_+ \in L^{(\frac{n}{2},\infty)}. \tag{2.4.37}$$

*Then any solution to the Cauchy problem*

$$\begin{cases} \partial_t u - \Delta u + V_+(x) u = 0, \\ u(0,x) = u_0, \end{cases} \tag{2.4.38}$$

*satisfies the dispersive estimate*

$$\|u(t,\cdot)\|_{L^\infty} \leq \frac{C}{t^{\frac{n}{2}}} \|u_0\|_{L^1}. \tag{2.4.39}$$

*Proof.* Consider the Cauchy problem for the heat equation with the same initial data to (2.4.38)

$$\begin{cases} \partial_t \tilde{u} - \Delta \tilde{u} = 0, \\ \tilde{u}(0,x) = u_0, u_0 \geq 0. \end{cases} \tag{2.4.40}$$

The dispersive estimate (2.4.39) is valid for this problem.

Let $w = \tilde{u} - u$. Then $w$ is a solution to the following Cauchy problem

$$\begin{cases} \partial_t w - \Delta w = V_+(x) u, \\ w(0,x) = 0. \end{cases} \tag{2.4.41}$$

Since $0 \leq V_+ \in L^{(\frac{n}{2},\infty)}$ we can apply it the previous Proposition and we obtain that

$$u \geq 0.$$

So applying one more the maximum principle for (2.4.41) we obtain

$$0 \leq w = \tilde{u} - u.$$

Thus we have

$$0 \leq u \leq \tilde{u}$$

and the dispersive estimate

$$\|u(t,\cdot)\|_{L^\infty} \leq \frac{C}{t^{\frac{n}{2}}} \|u_0\|_{L^1},$$

follows. This concludes the proof of this Proposition. □

Now we use the connection between self-adjointness and semibounded quadratic form, extending the notion of "closed" from operators to forms.

**Lemma 2.4.6.** *Let $n \geq 3$ and assume that*

$$V_+(x) \geq 0, \qquad V_+ \in L^{(\frac{n}{2},\infty)}. \tag{2.4.42}$$

*Then the operator $H_0 = -\Delta + V_+$ is self-adjoint in $H^2(\mathbb{R}^n)$.*

*Proof.* Consider the quadratic form

$$B(f,f) = (\nabla f, \nabla f)_{L^2(\mathbb{R}^n)} + \int_{\mathbb{R}^n} V(x)|f(x)|^2 dx, \;\; x \in \mathbb{R}^n, n \geq 3,$$

on the dense subspace $H^1(\mathbb{R}^n)$ of $L^2(\mathbb{R}^n)$.

To prove this Lemma it suffices to apply the standard theory of symmetric quadratic forms (see e.g. Theorem VIII.15 in the [82]). One can see easily that $B(f,f)$ is a positive quadratic form, thus it remains to see that it is closed in $H^1(\mathbb{R}^n)$, i.e. $H^1(\mathbb{R}^n)$ is complete under the norm

$$|||f|||^2 := B(f,f) + \|f\|^2_{L^2}. \tag{2.4.43}$$

Since $V_+(x) \geq 0$ one obtains

$$|||f|||^2 = \|\nabla f\|^2_{L^2} + (V_+ f, f)_{L^2} + \|f\|^2_{L^2} \geq C\|f\|^2_{H^1}. \tag{2.4.44}$$

The assumption on the potential implies that

$$\sqrt{V_+} \in L^{(n,\infty)},$$

so that, by the Hölder inequality for Lorentz spaces,

$$\|\sqrt{V_+} f\|_{L^2} \leq C\|\sqrt{V_+}\|_{L^{(n,\infty)}} \|f\|_{L^{(q,2)}} \leq CC_0\|f\|_{L^{(q,2)}},$$

where

$$\frac{1}{q} = \frac{1}{2} - \frac{1}{n}.$$

Using the Sobolev embedding (see [9]) $\dot{H}^1(\mathbb{R}^n) \hookrightarrow L^{(q,2)}(\mathbb{R}^n)$, we get

$$\|f\|_{L^{(q,2)}} \leq C_1 \|f\|_{\dot{H}^1}$$

and

$$(V_+ f, f)_{L^2} = \left| \int_{\mathbb{R}^n} V(x)|f(x)|^2 dx \right| \leq \|\sqrt{V_+} f\|^2_{L^2(\mathbb{R}^n)} \leq \tilde{C} \|f\|^2_{\dot{H}^1(\mathbb{R}^n)},$$

so that

$$|||f|||^2 \leq C\|f\|^2_{H^1}.$$

Thus we have the equivalence

$$|||f||| \simeq \|f\|_{H^1}, \tag{2.4.45}$$

and the conclusion follows at once. □

*Remark* 2.4.2. Since $H_0 = -\Delta + V_+$ is a self-adjoint operator, we can represent the solution to the Cauchy problem

$$\begin{cases} \partial_t u - \Delta u + V_+(x)u = 0, \\ u(0,x) = u_0, \end{cases} \tag{2.4.46}$$

as

$$u(t) = U(t)u_0, \qquad U(t) = e^{-tH_0},$$

and $U(t)$ is a continuous semigroup in $L^2$ and we have the energy inequality

$$\|U(t)u_0\|_{L^2} \leq \|u_0\|_{L^2}. \tag{2.4.47}$$

Notice that interpolating the dispersive estimate (2.4.39) with the energy inequality we obtain $L^p$-decay estimates, and using the $TT^*$ method of Ginibre and Velo (see [51], [66]) it is possible obtain the full Strichartz space-time estimates

$$\|u\|_{L^p(\mathbb{R}_t; L^q_x)} + \|u\|_{C(\mathbb{R}_t; L^2)} \leq C\|F\|_{L^{\tilde{p}'}(\mathbb{R}_t; L^{\tilde{q}'}_x)} + C\|u_0\|_{L^2},$$

with

$$\frac{1}{p} + \frac{n}{2q} = \frac{n}{4}, \qquad \frac{1}{\tilde{p}} + \frac{n}{2\tilde{q}} = \frac{n}{4}.$$

*Remark* 2.4.3. Consider the following perturbed Cauchy problem

$$\begin{cases} \partial_t u - H_0 u + V_-(x)u = F(t,x), \\ u(0,x) = u_0, \end{cases} \tag{2.4.48}$$

where $V_- \in L^{(\frac{n}{2},\infty)}$, $\|V_-\|_{L^{(\frac{n}{2},\infty)}} \leq C_0$. Using the same argument of subsection 2.4.1 we show that the operator $H = H_0 - V_-$ is selfadjoint, so the solution to (2.4.48) is $u(t,x) = e^{-tH}u_0$. Moreover, repeating the same steps of section 2.4.2, it is not difficult to show the full Strichartz estimates for the heat flow $e^{-tH}$ and this concludes the proof of Theorem 2.4.4.

## 2.5 The Schödinger equation with a large potential

In the last section of this chapter we shall consider the Schr̆odinger equation perturbed by a large unsigned time dependent potential $V(t,x)$

$$i\partial_t u - \Delta u + V(t,x)u = 0 \tag{2.5.1}$$

and its inhomogeneous version with a source term. Of course in general there is no hope to prove decay estimates in this case; thus we shall assume an integrability condition at infinity of the form $V(t,x) \in L^r_t L^s_x$ which in some sense replaces the smallness condition of the preceding section.

Our goal here is to show that, by purely elementary arguments based on integrability properties of the potential, it is possible to obtain a great deal of information on the behaviour of the solution, and to prove the Strichartz estimates for a wide class of large potentials with no definite sign. Moreover, the usual obstructions to decay are present also in this general situation: existence of standing waves, rescaling and pseudoconformal symmetry of the equation. Using these, we are able to show that our conditions are also necessary, at least in the class of potentials under consideration.

For the convenience of the reader we recall here the classical Strichartz estimates for the Schrödinger equation, and introduce some notations. We use a prime to denote conjugate indices; moreover, for any subinterval $I$ of $\mathbb{R}$ (bounded or unbounded) we define the mixed space-time norms

$$\|u\|_{L^p_I L^q} \equiv \left( \int_I \|u(t,\cdot)\|^p_{L^q(\mathbb{R}^n)} dt \right)^{1/p} \tag{2.5.2}$$

and when $I = [0, +\infty[$ we write simply $L^p L^q$ in place of $L^p_I L^q$. Similarly, we shall write

$$C_I L^p \equiv C(I; L^p), \qquad CL^p \equiv C([0,+\infty[; L^p) \tag{2.5.3}$$

for $1 \leq p \leq \infty$.

**Definition 2.5.1.** Let $n \geq 2$. The pair $(p,q)$ is said to be *(Schrödinger) admissible* if

$$\frac{1}{p} + \frac{n}{2q} = \frac{n}{4}, \qquad p,q \in [2,\infty], \qquad (n,p,q) \neq (2,2,\infty). \tag{2.5.4}$$

The Strichartz estimates can be stated as follows: for all admissible couples $(p,q)$ and $(\tilde{p},\tilde{q})$ there exists a constant $C(p,\tilde{p})$ such that, for all interval $I \subseteq \mathbb{R}$ (bounded or unbounded), for all functions $u_0(x) \in L^2(\mathbb{R}^n)$, and $F(t,x) \in L^{\tilde{p}'}_I L^{\tilde{q}'}$ the following inequalities hold:

$$\left\| e^{it\Delta} u_0 \right\|_{L^p_I L^q} \leq C(p,\tilde{p}) \, \|u_0\|_{L^2} \tag{2.5.5}$$

$$\left\| \int_0^t e^{i(t-s)\Delta} F(s)\, ds \right\|_{L^p_I L^q} \leq C(p, \tilde{p}) \, \|F\|_{L^{\tilde{p}'}_I L^{\tilde{q}'}} \tag{2.5.6}$$

Note that the constant is independent of the interval $I$.

Clearly, when $n \geq 3$ the constant can be taken also independent of $p$ and $\tilde{p}$: we shall denote this universal constant (which depends now only on the space dimension $n$) by $C_0$. When $n = 2$, the constant is unbounded as $p \downarrow 2$ or $\tilde{p} \downarrow 2$.

Here $e^{it\Delta}$ is the unitary operator

$$e^{it\Delta} f = \int_{\mathbb{R}^n} \frac{e^{-\frac{i|x-y|^2}{4t}}}{(4\pi i t)^{n/2}} \, f(y) \, dy, \tag{2.5.7}$$

$$\int_0^t e^{i(t-s)\Delta} F(s) \, ds = \int_0^t \int_{\mathbb{R}^n} \frac{e^{-\frac{i|x-y|^2}{4(t-s)}}}{(4\pi i(t-s))^{n/2}} \, F(s, y) \, dy \, ds,$$

which is properly defined on $L^2$ but can be extended to different $L^p$ spaces using e.g. these explicit expressions.

Consider now the differential equation

$$i\partial_t u - \Delta u + V(t, x) u = F(t, x), \qquad u(0, x) = u_0(x). \tag{2.5.8}$$

For low regularity solutions, it is customary to replace (2.5.8) with the integral equation

$$u(t, x) = e^{it\Delta} u_0(x) + \int_0^t e^{i(t-s)\Delta} [F(s) - V(s) u(s)] \, ds. \tag{2.5.9}$$

The two formulations are equivalent under very mild assumptions on the class of solutions; we shall not discuss this problem here, instead we shall use the integral formulation exclusively.

We can now state our first result:

**Theorem 2.5.1.** *Let $n \geq 2$, let $I$ be either the interval $[0, T]$ or $[0, +\infty[$, and assume $V(t, x)$ is a real valued potential belonging to*

$$V(t, x) \in L^r_I L^s, \qquad \frac{1}{r} + \frac{n}{2s} = 1 \tag{2.5.10}$$

*for some fixed $r \in [1, \infty[$ and $s \in ]n/2, \infty]$. Let $u_0 \in L^2$ and $F \in L^{\tilde{p}'}_I L^{\tilde{q}'}$ for some admissible pair $(\tilde{p}', \tilde{q}')$.*

*Then the integral equation (2.5.9) has a unique solution $u \in C_I L^2$ which belongs to $L^p_I L^q$ for all admissible pairs $(p, q)$ and satisfies the Strichartz estimates*

$$\|u\|_{L^p_I L^q} \leq C_V \, \|u_0\|_{L^2} + C_V \|F\|_{L^{\tilde{p}'}_I L^{\tilde{q}'}}. \tag{2.5.11}$$

*When* $n \geq 3$*, the constant* $C_V$ *can be estimated by* $k(1+2C_0)^k$*, where* $C_0$ *is the Strichartz constant for the free equation, while* $k$ *is an integer such that the interval* $I$ *can be partitioned in* $k$ *subintervals* $J$ *with the property* $\|V\|_{L^r_J L^s} \leq (2C_0)^{-1}$*. A similar statement holds when* $n = 2$*, provided we replace* $C_0$ *by* $C(p, \tilde{p})$*.*

*Finally, when* $F \equiv 0$ *the solution satisfies the conservation of energy*

$$\|u(t)\|_{L^2} \equiv \|u_0\|_{L^2}, \qquad t \in I. \tag{2.5.12}$$

*Remark* 2.5.1. We emphasize that the potentials $V(t,x)$ considered in Theorem 2.5.1 may be both large and change sign. The usual smallness assumption is replaced here by the integrability condition with respect to time, which ensures smallness of $V$ on sufficiently small time intervals, and for $t >> 1$.

*Remark* 2.5.2. By iterating the argument of the proof, it is easy to extend the above result to any potential of the form

$$V = V_1 + \cdots + V_k$$

where $V_1, \ldots, V_k$ satisfy assumption (2.5.10), with possibly different indices $r_j, s_j$.

*Remark* 2.5.3. Note that when $I$ is a bounded interval, assumption (2.5.10) can be relaxed to

$$V(t,x) \in L^r_I L^s, \qquad \frac{1}{r} + \frac{n}{2s} \leq 1; \tag{2.5.13}$$

indeed, from (2.5.13), using Hölder's inequality in time we can easily show that also (2.5.10) holds, for a smaller value of $r$ and the same value of $s$.

Thus in particular we see that the existence part of our theorem extends a result of Yajima [108], who proved that the equation (2.5.9) (or (2.5.8)) is well posed in $L^2$ with conservation of energy, provided the potential $V$ satisfies

$$V = V_1 + V_2, \qquad V_1 \in L^r_I L^s, \quad V_2 \in L^\infty_I L^\beta \tag{2.5.14}$$

with $\beta > 1$ and

$$\frac{1}{r} + \frac{n}{2s} < 1 \tag{2.5.15}$$

(see also the preceding remark).

When the potential $V(t,x)$ belongs to $L^\infty_I L^{n/2}$, i.e., in the limiting case of Theorem 2.5.1, the result can not be true; indeed, this case includes the static potentials $V(x) \in L^{n/2}$ without any positivity or smallness assumption. We mention that even for a nonnegative potential in $L^{n/2}$ it is not known if the Strichartz estimates are valid in general. The best result in this direction is due to Rodnianski and Schlag [88] who considered bounded potentials defined on $\mathbb{R}^n$ satisfying the estimate $|V(x)| \leq C|x|^{-2-\epsilon}$ for $|x|$ large enough. However, in the limiting case we can prove a partial substitute of Theorem 2.5.1. To simplify our statement we introduce the following definition:

**Definition 2.5.2.** Let $V(x)$ be a real valued function such that

$$H = \Delta - V(x)$$

has a selfadjoint extension. We say that the potential $V(x)$ is of *Strichartz type* if for all bounded time interval $I = [0, T]$, for all $u_0 \in L^2$ and $F \in L_I^{\tilde{p}'} L^{\tilde{q}'}$ with $(\tilde{p}, \tilde{q})$ admissible, the integral equation

$$u(t,x) = e^{itH} u_0 + \int_0^t e^{i(t-s)H} F(s)\, ds \tag{2.5.16}$$

has a unique solution $u(t,x) \in C_I L^2$ satisfying the Strichartz estimates

$$\|u\|_{L_I^a L^b} \leq C(I,V)\, \|u_0\|_{L^2} + C(I,V)\, \|F\|_{L_I^{\tilde{p}'} L^{\tilde{q}'}} \tag{2.5.17}$$

for all admissible pairs $(a, b)$.

Then we have:

**Theorem 2.5.2.** *Let $n \geq 3$, let $I$ be a bounded interval $[0, T]$ and let $V(t,x) \in C_I L^{n/2}$. Assume that for each $t \in I$, $V(t, \cdot)$ is of Strichartz type, while the functions $u_0$ and $F(t,x)$ are as in Theorem 2.5.1. Then the conclusion of Theorem 2.5.1 holds true (local Strichartz estimates).*

*The result holds also in the case $I = [0, \infty[$ (global Strichartz estimates) under the additional assumption: there exists $T_0 > 0$ such that $\|V(t,\cdot)\|_{L^{n/2}} \leq (2C_0)^{-1}$ for $t > T_0$.*

*Remark* 2.5.4. By simple modifications in the proof, Theorem 2.5.2 can be extended to any potential of the form

$$V(t,x) = V_1(t,x) + V_2(t,x),$$

with $V_1$ as in the theorem while $V_2 \in L_I^\infty L^{n/2}$ satisfies

$$\|V_2\|_{L_I^\infty L^{n/2}} \leq \varepsilon(V_1)$$

for a suitable small constant $\epsilon(V_1)$ depending only on $V_1$.

*Example* 2.5.1. To illustrate a possible use of Theorem 2.5.1, consider the semilinear Schrödinger equation

$$i\partial_t u - \Delta u + f(u)u = 0, \qquad |f(u)| \leq C|u|^\gamma, \quad \gamma > 1, \tag{2.5.18}$$

$f$ real valued, which includes both focusing and defocusing equations with a power nonlinearity. Then we may regard (2.5.18) as a Schrödinger equation with a time dependent potential

$$V(t,x) = f(u(t,x)).$$

We see that $V$ satisfies the assumptions of Theorem 2.5.1 provided

$$u \in L^a L^b, \qquad \frac{1}{a} + \frac{n}{2b} = \frac{1}{\gamma}, \quad a < \infty. \tag{2.5.19}$$

Thus any solution satisfying (2.5.19) satsfies the full set of Strichartz estimates.

For instance, in the case of the (focusing or defocusing) quintic Schrödinger equation in three dimensions, any solution $u \in L^{10}L^{10}$ satisfies the Strichartz estimates; this was the first step in the proof of the global well posedness for the radial defocusing three dimensional quintic in [18].

*Example* 2.5.2. We give a simple application of Theorem 2.5.2. Consider a real valued potential $V \in CL^{\frac{n}{2}}$ and assume it satisfies the bounds

$$0 \leq V(t,x) \leq \frac{C}{(1+|x|)^{2+\delta}}, \qquad x \in \mathbb{R}^n, \quad n \geq 3 \tag{2.5.20}$$

for some $C, \delta > 0$. Then we can prove that $V(t,x)$ satisfies the assumptions of Theorem 2.5.2 and hence the local Strichartz estimates hold (and also the global ones, under the additional assumption of smallness at infinity).

Indeed, let $W(x) = V(t_0, x)$ for an arbitrary fixed $t_0$; we must show that $W(x)$ is of Strichartz type. The existence part of the definition is trivial; let us prove the estimates. Consider the operator $H = -\Delta + W(x)$; $H$ has a unique selfadjoint extension by standard results, with spectrum contained in $[0, +\infty[$; by Theorem XIII.58 in [85] $H$ has no strictly positive eigenvalues, since $W$ is bounded and decays as $|x|^{-2-\delta}$ at infinity; 0 is certainly not an eigenvalue since $Hf = 0$ implies $f = 0$ easily. This implies that the operator $H$ has a purely continuous spectrum. Now Theorem 1.4 in [88] states that $P_c e^{itH}$ satisfies the full set of Strichartz estimates when the potential is bounded and decays faster than $|x|^{-2}$ at infinity; here $P_c$ is the projection on the continuous subspace of $L^2$ for $H$, which coincides with all of $L^2$ as we have just proved. In conclusion, $W(x) = V(t_0, x)$ is of Strichartz type as claimed.

*Remark* 2.5.5. Condition (2.5.10) is quite natural, in view of the following argument: the standard rescaling $u_\epsilon(t,x) = u(\epsilon^2 t, \epsilon x)$ takes equation (2.5.1) into the equation

$$i\partial_t u_\epsilon - \Delta u_\epsilon + V_\epsilon(t,x) u_\epsilon = 0, \qquad V_\epsilon(t,x) = \epsilon^2 V(\epsilon^2 t, \epsilon x), \tag{2.5.21}$$

and we have

$$\|V_\epsilon\|_{L^r L^s} = \epsilon^{2\left(1 - \frac{1}{r} - \frac{n}{2s}\right)} \|V\|_{L^r L^s} \tag{2.5.22}$$

so that the $L^r L^s$ norm of $V_\epsilon$ is independent of $\epsilon$ precisely when $r, s$ satisfy (2.5.10).

Indeed, by a suitable use of rescaling arguments, it is possible to show that the condition $1/r + n/(2s) = 1$ is *necessary* in order that the global

Strichartz estimates be true for any potential belonging to the classes $L^r L^s$ (see Theorem 2.5.3 below).

Concerning the *local* Strichartz estimates, the situation is more interesting. When $1/r + n/(2s) < 1$, as already observed in Remark 2.5.3, the local Strichartz estimates are an elementary consequence of Theorem 2.5.1. On the other hand, when $1/r + n/(2s) > 1$, it is possible to show that the *local* Strichartz estimates fail. This case is more delicate; actually it is not even clear if equation (2.5.1) is well posed in $L^2$ under this assumption on $V$.

We collect our counterexamples in the following theorem, concerning the homogeneous equation

$$iu_t - \Delta u + V(t,x)u = 0. \tag{2.5.23}$$

Note that the case $(r,s) = (\infty, n/2)$ is almost trivial since it is based on the construction of a standing wave for (2.5.23); we state it in some length both for completeness, and because the remaining counterexamples are based on it. Thus, in the proof of Theorem 2.5.3 it is essential to use potentials which change sign.

**Theorem 2.5.3.** *Let $n \geq 2$. Then we have the following.*

*(i) (Case $r = \infty$) We can construct a potential $W(x) \in C_0^\infty(\mathbb{R}^n)$ and a function $u_0 \in H^s$ for all $s > 0$ such that*

$$-\Delta u_0 + W(x)u_0 + u_0 = 0. \tag{2.5.24}$$

*Hence the function $u(t,x) = e^{-it}u_0(x) \in CL^2$ solves (2.5.23) with*

$$V(t,x) \equiv W(x) \in L^\infty([0,+\infty[; L^{n/2}(\mathbb{R}^n)),$$

*and does not belong to the space $L^p([0,+\infty[; L^q)$ for all admissible pairs $(p,q) \neq (\infty, 2)$. In other words, there exists a potential $V(t,x)$ belonging to $L^\infty L^s$ for all $s \in [1,\infty]$ such that the global Strichartz estimates (2.5.11) on $I = [0,+\infty[$ do not hold for equation (2.5.23).*

*(ii) (Counterexamples to global estimates) For every pair $(r,s)$ with $r \in [1,\infty[$, $s \in ]n/2,\infty]$ and*

$$\frac{1}{r} + \frac{n}{2s} \neq 1, \tag{2.5.25}$$

*we can construct a potential $V(t,x) \in L^r([0,+\infty[; L^s)$ and a sequence of solutions $u_k(t,x) \in C([0,+\infty[; L^2)$ to equation (2.5.23) such that*

$$\lim_{k\to\infty} \frac{\|u_k\|_{L^pL^q}}{\|u_k(0)\|_{L^2}} = \infty \quad \textit{for every admissible pair} \quad (p,q) \neq (\infty,2). \tag{2.5.26}$$

*(iii) (Counterexamples to local estimates) For every pair $(r,s)$ with $r \in [1,\infty[$, $s \in ]n/2,\infty]$ and*

$$\frac{1}{r} + \frac{n}{2s} > 1, \tag{2.5.27}$$

*we can construct, on any given bounded time interval* $I = [0,T]$, *a potential* $V(t,x) \in L^r([0,T]; L^s)$ *and a sequence of solutions* $u_k(t,x) \in C([0,T]; L^2)$ *to equation* (2.5.23) *such that*

$$\lim_{k\to\infty} \frac{\|u_k\|_{L^p_I L^q}}{\|u_k(0)\|_{L^2}} = \infty \quad \textit{for every admissible pair} \quad (p,q) \neq (\infty, 2). \quad (2.5.28)$$

We conclude the paper with a result showing that, at least for a restricted range of indices $r, s$, the conclusion of Theorem 2.5.3, part (iii), can be improved in an essential way. While the above theorem was based on suitable rescaling arguments, Proposition 2.5.4 exploits the *pseudoconformal* invariance of the Schrödinger equation.

**Proposition 2.5.4.** *Let* $n \geq 2$, *and assume* $r \in [1, \infty[$ *and* $s \in ]n/2, n[$ *satisfy*

$$\frac{1}{2r} + \frac{n}{2s} > 1. \quad (2.5.29)$$

*Then we can construct a potential* $V(t,x) \in L^r(0,1; L^s(\mathbb{R}^n))$ *and a solution* $u(t,x) \in C([0,1]; L^2)$ *to equation* (2.5.23) *such that, for all admissible pairs* $(p,q)$ *with* $p < \infty$, *and for any* $0 < T < 1$, *we have*

$$u \in L^p(0,T; L^q(\mathbb{R}^n)) \qquad \textit{but} \qquad u \notin L^p(0,1; L^q(\mathbb{R}^n)).$$

### 2.5.1 Proof of Theorem 2.5.1

We shall consider in detail only the case $n \geq 3$; in the case $n = 2$, when the endpoint fails, it is sufficient to replace in the following arguments the space $L^2_J L^{\frac{2n}{n-2}}$ with any $L^p_J L^q$ with $q$ arbitrarily large.

We distinguish two cases, according to the value of $r \in [1, \infty[$.

### 2.5.2 Case A: $r \in [2, \infty[$

Consider a small interval $J = [0, \delta]$, and let $Z$ be the Banach space

$$Z = C_J L^2 \cap L^2_J L^{\frac{2n}{n-2}}, \qquad \|v\|_Z := \max\left\{ \|v\|_{L^\infty_J L^2},\ \|v\|_{L^2_J L^{\frac{2n}{n-2}}} \right\}.$$

Notice that, by interpolation, $Z$ is embedded in all admissible spaces $L^p_J L^q$.

For any $v(t,x) \in Z$ we define the mapping

$$\Phi(v) = e^{it\Delta} u_0 + \int_0^t e^{i(t-s)\Delta} [F(s) - V(s)v(s)]\, ds. \quad (2.5.30)$$

A direct application of (2.5.5), (2.5.6) gives

$$\|\Phi(v)\|_{L^p_J L^q} \leq C_0 \|u_0\|_{L^2} + C_0 \|Vv\|_{L^{p_0'}_J L^{q_0'}} + C_0 \|F\|_{L^{\tilde{p}'}_J L^{\tilde{q}'}} \quad (2.5.31)$$

for all admissible $(p,q)$, $(p_0,q_0)$, $(\tilde p,\tilde q)$, and by Hölder's inequality we can write

$$\|\Phi(v)\|_{L^p_J L^q} \leq C_0\|u_0\|_{L^2} + C_0\|V\|_{L^r_J L^s}\|v\|_{L^2_J L^{\frac{2n}{n-2}}} + C_0\|F\|_{L^{\tilde p'}_J L^{\tilde q'}} \quad (2.5.32)$$

provided we choose $p_0, q_0$ such that

$$\frac{1}{p_0} = \frac{1}{2} - \frac{1}{r}, \qquad \frac{1}{q_0} = \frac{n+2}{2n} - \frac{1}{s}.$$

Note that

$$\frac{1}{p_0} + \frac{n}{2q_0} = \frac{1}{2} + \frac{n+2}{4} - \left(\frac{1}{r} + \frac{n}{2s}\right) \equiv \frac{1}{2} + \frac{n+2}{4} - 1 \equiv \frac{n}{4}$$

by our assumptions on $r, s$, and moreover

$$r \in [2,\infty[ \quad \Longrightarrow \quad p_0 \in [2,\infty[$$

so that our choice of $p_0, q_0$ always gives an admissible pair in the case under consideration.

In particular, choosing $(p,q) = (\infty, 2)$ or $(2, 2n/(n-2))$, we obtain

$$\|\Phi(v)\|_Z \leq C_0\|u_0\|_{L^2} + C_0\|V\|_{L^r_J L^s}\|v\|_Z + C_0\|F\|_{L^{\tilde p'}_J L^{\tilde q'}} \quad (2.5.33)$$

Thus $\Phi(v)$ belongs to all the admissible spaces $L^p_J L^q$, and to prove that $\Phi(v)$ belongs to $Z$ it remains only to show that $u$ is continuous with values in $L^2$. But this is an immediate consequence of the following simple remark:

*Remark* 2.5.6. Let $G(t,x) \in L^{a'}_J L^{b'}$ with $(a,b)$ admissible. Then the function

$$w(t,x) = \int_0^t e^{i(t-s)\Delta} G(s)\, ds$$

belongs to $C_J L^2$. Indeed, this is certainly true if we know in addition that $G$ is a smooth function, compactly supported in $x$ for each $t$. If we approximate $G$ by a sequence of such functions $G_j$ in the $L^{a'}_J L^{b'}$ norm, the Strichartz estimates imply that the corresponding functions $w_j$ converge in $L^\infty L^2$, whence the claim follows.

We have thus constructed a mapping $\Phi : Z \to Z$. Assume now the length $\delta$ of the interval $J$ is chosen so small that

$$C_0\|V\|_{L^r_J L^s} \leq \frac{1}{2}; \quad (2.5.34)$$

this is certainly possible since $r < \infty$. With this choice we obtain immediately two consequences: first of all, the mapping $\Phi$ is a contraction on $Z$

and hence has a unique fixed point $v(t,x)$ which is the required solution; second, $v$ satisfies

$$\|v\|_{L^p_J L^q} \leq C_0\|u_0\|_{L^2} + \frac{1}{2}\|v\|_{L^p_J L^q} + C_0\|F\|_{L^{\tilde{p}'}_J L^{\tilde{q}'}} \tag{2.5.35}$$

whence we obtain

$$\|v\|_{L^p_J L^q} \leq 2C_0\|u_0\|_{L^2} + 2C_0\|F\|_{L^{\tilde{p}'}_J L^{\tilde{q}'}} \tag{2.5.36}$$

It is clear that the above argument applies on any subinterval $J = [t_0, t_1] \subseteq I$ on which a condition like (2.5.34) holds; of course, we will obtain an estimate of the form

$$\|v\|_{L^p_J L^q} \leq 2C_0\|v(t_0)\|_{L^2} + 2C_0\|F\|_{L^{\tilde{p}'}_J L^{\tilde{q}'}}. \tag{2.5.37}$$

Notice also that (2.5.37) implies in particular

$$\|v(t_1)\|_{L^2} \leq 2C_0\|v(t_0)\|_{L^2} + 2C_0\|F\|_{L^{\tilde{p}'}_J L^{\tilde{q}'}}. \tag{2.5.38}$$

Now we can partition the interval $I$ (bounded or unbounded) in a finite number of subintervals on which condition (2.5.34) holds. Applying inductively the estimates (2.5.37) and (2.5.38) we easily obtain (2.5.11) and the claimed estimate for the Strichartz constant.

The last remark (2.5.12) concerning the conservation of energy can be proved by approximation as follows: let $V_j(t,x)$ be a sequence of real valued smooth potentials, compactly supported in $x$, and let $v_j$ be the corresponding solutions; then the differences $w_j = v - v_j$ satisfy (in suitable integral sense)

$$i\partial_t w_j - \Delta w_j + V w_j = (V - V_j)v_j \equiv F_j.$$

Now we observe that the smooth solutions $v_j$ have a conserved energy; moreover, we can choose the approximating potentials $V_j$ in such a way that they converge to $V$ in $L^r_I L^s$ and their Strichartz constants do not exceed the above constructed constant for $V$. Indeed, if we can partition $I$ in a finite set of subintervals satisfying (2.5.34), we can choose exactly the same subintervals for each $V_j$ provided we construct $V_j$ by a convolution with standard mollifiers, so that their Lebesgue norm does not increase. In conclusion, the $v_j$ satisfy uniform Strichartz estimates, and this implies that the nonhomogeneous terms $F_j = (V - V_j)v_j$ tend to 0 in the (dual) admissible spaces, by estimates identical to the above ones. Thus in particular $w_j \to 0$ in $L^\infty L^2$ and this shows that also $v(t,x)$ satisfies the conservation of energy.

### 2.5.3 Case B: $r \in [1,2]$

The method in this case is quite similar to the above one, but instead of (2.5.31) we use the estimate

$$\|\Phi(v)\|_{L^p_J L^q} \leq C_0\|u_0\|_{L^2} + C_0\|Vv\|_{L^r_J L^{\frac{2s}{s+2}}} + C_0\|F\|_{L^{\tilde{p}'}_J L^{\tilde{q}'}} \tag{2.5.39}$$

where $(p,q)$ and $(\tilde{p},\tilde{q})$ are arbitrary admissible pairs, while the pair $(r, 2s/(s+2))$ is the dual of $(r', 2s/(s-2))$ and this last pair is admissible since

$$\frac{1}{r'} + \frac{n}{2}\cdot\frac{s-2}{2s} = \frac{n}{2s} + \frac{n}{2}\cdot\frac{s-2}{2s} = \frac{n}{4}$$

where we have used the assumption $1/r + n/(2s) = 1$; notice also that $r \in [1,2]$ and hence $2s/(s+2) \in [1,2]$ too.

Thus by Hölder's inequality we obtain

$$\|\Phi(v)\|_{L^p_J L^q} \leq C_0\|u_0\|_{L^2} + C_0\|V\|_{L^r_J L^s}\|v\|_{L^\infty_J L^2} + C_0\|F\|_{L^{\tilde{p}'}_J L^{\tilde{q}'}} \tag{2.5.40}$$

and choosing $(p,q) = (\infty, 2)$ or $(2, 2n/(n-2))$ and proceeding as above we arrive at

$$\|\Phi(v)\|_Z \leq C_0\|u_0\|_{L^2} + \frac{1}{2}\|v\|_Z + C_0\|F\|_{L^{\tilde{p}'}_J L^{\tilde{q}'}}. \tag{2.5.41}$$

From this point on, the proof is identical to the first case.

### 2.5.4 Proof of Theorem 2.5.2

The proof follows the same lines as the preceding one; indeed, the continuity in time of the potential allows to consider $V(t,x)$ as a small perturbation of $V(t_0,x)$ for $t$ near $t_0$.

Let $J = [0,\delta]$ be a small interval, and consider again the space

$$Z = C_J L^2 \cap L^2_J L^{\frac{2n}{n-2}}, \qquad \|v\|_Z := \max\left\{\|v\|_{L^\infty_J L^2},\ \|v\|_{L^2_J L^{\frac{2n}{n-2}}}\right\}.$$

On $Z$ we construct a map $\Phi$ defined as follows:

$$\Phi(v) = e^{itH}u_0 + \int_0^t e^{i(t-s)H}[F(s) - W(s)v(s)]\,ds, \tag{2.5.42}$$

where

$$H = \Delta - V(0,x), \qquad W(t,x) = V(t,x) - V(0,x). \tag{2.5.43}$$

We have used the assumption that $V(0,x)$ is of Strichartz type (Definition 2.5.2) to make meaningful the operators $e^{itH}$; on the other hand this implies also that the full Strichartz estimates (2.5.5), (2.5.6) hold for the group $e^{itH}$, hence we can write

$$\|\Phi(v)\|_{L^p_J L^q} \leq C\,\|u_0\|_{L^2} + C\,\|Wv\|_{L^2_J L^{\frac{2n}{n+2}}} + C\,\|F\|_{L^{\tilde{p}'}_J L^{\tilde{q}'}} \tag{2.5.44}$$

for all admissible pairs $(p,q)$ and $(\tilde{p},\tilde{q})$. Notice that here $C$ is a constant depending on $V$ and the interval $J$ only, and can be assumed to be non increasing when $\delta \downarrow 0$. This implies

$$\|\Phi(v)\|_{L^p_J L^q} \leq C\,\|u_0\|_{L^2} + C\,\|W\|_{L^\infty_J L^{n/2}}\|v\|_{L^2_J L^{\frac{2n}{n-2}}} + C\,\|F\|_{L^{\tilde{p}'}_J L^{\tilde{q}'}} \tag{2.5.45}$$

and if $\delta$ is so small that

$$C\,\|W\|_{L^\infty_J L^{n/2}} \le \frac{1}{2} \tag{2.5.46}$$

which is possible by the continuity of $V(t,x)$ as an $L^{n/2}$-valued function, we arrive at

$$\|\Phi(v)\|_Z \le C\,\|u_0\|_{L^2} + \frac{1}{2}\|v\|_Z + C\,\|F\|_{L^{\tilde p'}_J L^{\tilde q'}}. \tag{2.5.47}$$

This guarantees, as above, the existence of a unique local solution belonging to the space $Z$ and satisfying the Strichartz estimates with some constant $C(0)$ for some time interval $[0,\delta)$.

The same argument can be applied near each point $t_0 \in I$. More precisely, let $J = [t_0-\delta, t_0+\delta] \cap I$ and assume $\delta > 0$ is so small that the potential

$$W(t,x) = V(t,x) - V(t_0,x)$$

satisfies

$$\|W(t,\cdot)\|_{L^{n/2}} \le (2C(V(t_0,x)))^{-1} \qquad \text{for } t \in J, \tag{2.5.48}$$

where $C(V(t_0,x))$ is the Strichartz constant corresponding to the potential $V(t_0,x)$ and relative to the interval $[0,t_0+1]$. Then we may argue as above, and we obtain that for any given initial time $t_1 \in J$ and for any $f \in L^2$, the Cauchy problem

$$i\partial_t u - Hu = F(t,x) - W(t,x)u, \qquad u(t_1) = f, \qquad H = \Delta - V(t_0,x)$$

(interpreted as usual in integral form via the group $e^{itH}$) has a unique solution in $Z = C_J L^2 \cap L^2_J L^{\frac{2n}{n-2}}$, which satisfies the Strichartz estimates

$$\|\Phi(v)\|_Z \le 2C(t_0)\,\|u_0\|_{L^2} + 2C(t_0)\,\|F\|_{L^{\tilde p'}_J L^{\tilde q'}}. \tag{2.5.49}$$

for some constant $C(t_0)$ depending on the point $t_0$ but *not* on the initial time $t_1 \in J$.

Now we may proceed by a continuation argument as follows. Extend the local solution constructed on $[0,\delta]$ to a maximal interval $[0,T^*[$; i.e., consider the union of all intervals $[0,\delta]$ on which a solution $u \in C([0,\delta];L^2) \cap L^2(0,\delta;L^{\frac{2n}{n-2}})$ exists and satisfies (for all admissible pairs) the Strichartz estimates with some constant $C_\delta$. Assume by contradiction that $T^* < T$. Then the above local argument applied at $t_0 = T^*$ on a suitable interval of the form $J = [T^*-\varepsilon, T^*+\varepsilon]$ shows that we can patch the maximal solution and extend it to $[0,T^*+\varepsilon]$. Moreover, we claim that the extended solution satisfies the Strichartz estimates on $[0,T^*+\varepsilon]$: indeed, chosen any $t_1$ such that $T^* - \varepsilon < t_1 < T^*$, by construction we see that the estimates hold both on $I_1 = [0, t_1]$, with initial data at $t = 0$:

$$\|u\|_{L^p_{I_1} L^q} \leq C' \, \|u(t_0)\|_{L^2} + C' \, \|F\|_{L^{\tilde{p}'}_{I_1} L^{\tilde{q}'}}, \tag{2.5.50}$$

and on $J = [T^* - \varepsilon, T^* + \varepsilon]$, with initial data at $t = t_1$:

$$\|u\|_{L^p_J L^q} \leq C' \, \|u(t_1)\|_{L^2} + C' \, \|F\|_{L^{\tilde{p}'}_J L^{\tilde{q}'}}, \tag{2.5.51}$$

for a suitable constant $C'$. Since $\|u(t_1)\|_{L^2}$ can be estimated exactly by (2.5.50) ($p = 2, q = \infty$), we easily conclude the proof of our claim. This contradicts the assumption $T^* < T$ and we obtain that $T^* = T$.

The modifications required to prove the final remark concerning the case $I = [0, \infty[$, and also Remark 2.5.4, are obvious.

### 2.5.5 Proof of the counterexamples

**An eigenvalue problem.**

The first step in our construction requires to find a potential $V(x)$ such that the operator $-\Delta + V(x)$ has a negative eigenvalue, i.e., such that the equation

$$-\Delta u_0 + V(x) u_0 + \gamma^2 u_0 = 0 \tag{2.5.52}$$

admits a solution $u_0 \in H^1$ for some $\gamma > 0$. There are many results on this problem, and in general there is a clear connection between the number of such eigenvalues and the size of the negative part of $V$, in a suitable norm. This is true both in the negative sense (explicit bounds on the number of the eigenvalues) and in the positive sense, which is our main focus here. For instance, it is known that (see [85]) if $V(x) \in L^{n/2}(\mathbb{R}^n)$ satisfies the assumption

$$\text{the set } \{x \in \mathbb{R}^n : V(x) < 0\} \text{ has a positive measure,} \tag{2.5.53}$$

then there exists $\lambda_0 > 0$ such that, for all $\lambda > \lambda_0$, the equation

$$-\Delta u_0 + \lambda V(x) u_0 + \gamma^2 u_0 = 0 \tag{2.5.54}$$

admits at least a solution $f \in H^1$ for some $\gamma > 0$. It can also be proved that the dimension of the eigenspace grows to infinity as $\lambda$ tends to infinity.

However, for our purposes here we need only a much less precise result, which can be proved directly by an elementary variational argument. Both this result and the proof we give here are completely standard, but we prefer to include it here for the convenience of the reader. Indeed, take any smooth compactly supported function $w(x)$ such that $w(x_0) > 0$ at least in one point $x_0$. Then consider the minimization problem with a constraint

$$\min_{f \in M} \int_{\mathbb{R}^n} \left(|\nabla f|^2 + |f|^2\right) \, dx \qquad \text{on} \qquad M = \left\{ f \in H^1 \colon \int_{\mathbb{R}^n} w(x) \, |f|^2 dx = 1 \right\}. \tag{2.5.55}$$

Note that $M$ is not empty, thanks to the assumption $w(x_0) > 0$. The existence of a solution to problem (2.5.55) can be proved easily by a standard compactness argument, since we can work on the (bounded) support of $w(x)$. On the other hand, the Euler-Lagrange equation of the problem is

$$-\Delta f + f = \mu w(x) f \tag{2.5.56}$$

(where $\mu$ is a Lagrange multiplier); hence, choosing $W(x) = -\mu w(x)$ and $u_0 = f$, we see that $u_0$ solves the equation

$$-\Delta u_0 + W(x) u_0 + u_0 = 0 \tag{2.5.57}$$

and hence

$$u(t,x) = e^{-it} u_0(x) \quad \text{solves} \quad i u_t - \Delta u + W(x) u = 0. \tag{2.5.58}$$

Note also that a trivial bootstrapping argument gives $u_0 \in H^s$ for all $s > 0$.

This concludes the proof of Theorem 2.5.3, part (i).

**Proof of Theorem 2.5.3, case $1/r + n/(2s) < 1$, $r \neq \infty$**

We start from the function (2.5.58) and we apply the standard rescaling

$$u(t,x) \mapsto u_\epsilon(t,x) = u(\epsilon^2 t, \epsilon x) \equiv e^{-i\epsilon^2 t} u_0(\epsilon x). \tag{2.5.59}$$

Then the function $u_\epsilon$ solves globally

$$i\partial_t u_\epsilon - \Delta u_\epsilon + W_\epsilon(x) u_\epsilon = 0, \qquad W_\epsilon(x) = \epsilon^2 W(\epsilon x). \tag{2.5.60}$$

Consider now two monotone sequences of positive real numbers

$$0 = T_0 < T_1 < \cdots < T_k \uparrow +\infty, \qquad 0 < \epsilon_k \downarrow 0, \qquad k = 0, 1, 2, 3, \ldots \tag{2.5.61}$$

and define a potential $V(t,x)$ on $[0, +\infty[ \times \mathbb{R}^n$ by patching the potentials $V_\epsilon$ as follows:

$$V(t,x) = W_{\epsilon_k}(x) \qquad \text{for} \qquad t \in [T_k, T_{k+1}[, \quad k = 0, 1, 2, \ldots. \tag{2.5.62}$$

Thus $u_{\epsilon_k}$ solves the equation

$$i\partial_t u - \Delta u + V(t,x) u = 0 \tag{2.5.63}$$

on the interval $[T_k, T_{k+1}[$.

Choose now $r$ and $s$ satisfying

$$\frac{1}{r} + \frac{n}{2s} < 1, \qquad r \neq \infty, \tag{2.5.64}$$

and assume we can choose the parameters $T_k, \epsilon_k$ in such a way that

$$\|V\|_{L^r L^s} \leq \|W\|_{L^s} \sum_{k \geq 0} (T_{k+1} - T_k)^{1/r} \epsilon^{2-n/s} < \infty, \tag{2.5.65}$$

then $V(t,x) \in L^r([0, +\infty[; L^s)$. On the other hand by Theorem 2.5.1 we can extend (uniquely) $u_{\epsilon_k}$ to a global solution of (2.5.63) in $C([0, +\infty[; L^2(\mathbb{R}^n))$ which we shall denote by $u_k(t,x)$. Notice that, by the same theorem, we have

$$\|u_k(t,\cdot)\|_{L^2} \equiv \text{const.} \equiv \|u_{\epsilon_k}(T_k)\|_{L^2} \equiv \epsilon^{-n/2} \|u_0\|_{L^2} \tag{2.5.66}$$

recalling the explicit expression (2.5.59) of $u_\epsilon$. On the other hand, we can write

$$\|u_k\|_{L^p(\mathbb{R}; L^q)} \geq \|u_k\|_{L^p(T_k, T_{k+1}; L^q)} \equiv \|u_{\epsilon_k}\|_{L^p(T_k, T_{k+1}; L^q)} \equiv (T_k - T_{k+1})^{1/p} \epsilon_k^{-n/q} \|u_0\|_{L^q} \tag{2.5.67}$$

by an elementary calculation. The Strichartz esimates are violated when

$$\frac{\|u_k\|_{L^p(\mathbb{R}; L^q)}}{\|u_k(0)\|_{L^2}} \quad \text{is unbounded,} \tag{2.5.68}$$

and this holds provided the parameters $T_k, \epsilon_k$ satisfy the condition

$$\frac{\|u_k\|_{L^p(\mathbb{R}; L^q)}}{\|u_k(0)\|_{L^2}} \geq \frac{\|u_{\epsilon_k}\|_{L^p(T_k, T_{k+1}; L^q)}}{\|u_{\epsilon_k}(0)\|_{L^2}} \equiv \frac{\|u_0\|_{L^q}}{\|u_0\|_{L^2}} (T_k - T_{k+1})^{1/p} \epsilon_k^{\frac{n}{2} - \frac{n}{q}} \to \infty. \tag{2.5.69}$$

In conclusion, we only need to adjust the parameters (2.5.61) so to satisfy the two conditions (2.5.65) and (2.5.69):

$$\sum_{k \geq 0} (T_{k+1} - T_k)^{1/r} \epsilon_k^{2-n/s} < \infty, \qquad (T_k - T_{k+1})^{1/p} \epsilon_k^{\frac{n}{2} - \frac{n}{q}} \to \infty, \tag{2.5.70}$$

given an admissible pair $(p,q)$ with $p \neq \infty$ and $(r,s)$ as in (2.5.64). With the special choices

$$T_0 = 0, \quad T_{k+1} = T_k + k^\alpha, \quad \epsilon_0 = 1, \quad \epsilon_k = k^{-\beta/2}, \qquad k = 1, 2, 3, \ldots \tag{2.5.71}$$

for some $\alpha, \beta > 0$, the conditions reduce to

$$\frac{\alpha}{r} + \beta \frac{n}{2s} < \beta - 1, \qquad \frac{\alpha}{p} + \beta \frac{n}{2q} > \beta \frac{n}{4}. \tag{2.5.72}$$

Since $(p,q)$ is admissible, the second condition simplifies to $\alpha > \beta$, and rearranging the first one we are reduced to

$$\frac{\alpha-\beta}{r} + \beta\left(\frac{1}{r} + \frac{n}{2s}\right) < \beta - 1, \qquad \alpha > \beta. \tag{2.5.73}$$

The term in brackets is smaller then 1 by assumption, hence if we choose any

$$\alpha > \beta > \left[1 - \left(\frac{1}{r} + \frac{n}{2s}\right)\right]^{-1} \tag{2.5.74}$$

with $\alpha$ close enough to $\beta$, we conclude the proof of the first part of Theorem 2.5.2, (ii).

### 2.5.6 Proof of Theorem 2.5.3, case $1/r + n/(2s) > 1$, $r \neq \infty$

As in case 2.5.5 the proof is based on a rescaling argument. First of all we prove part (ii). Consider again the rescaled solution (2.5.59) which solves equation (2.5.60) globally with a smooth compactly supported potential $W_\epsilon(x) = \epsilon^2 W(\epsilon x)$. Now, take two monotone sequences of positive real numbers

$$1 = \epsilon_0 < \epsilon_1 < \cdots < \epsilon_k \uparrow +\infty, \qquad 0 < \delta_k \downarrow 0, \qquad k = 0,1,2,3,\ldots \tag{2.5.75}$$

and define a potential $V(t,x)$ on $[0,+\infty[\times\mathbb{R}^n$ as follows:

$$V(t,x) = \begin{cases} W_{\epsilon_k}(x) & \text{if } t \in [k, k+\delta_k],\ x \in \mathbb{R}^n, \\ 0 & \text{elsewhere.} \end{cases} \tag{2.5.76}$$

Note that $V(t,x) \in L_I^\infty L^\infty$ for any bounded time interval $I$, while globally

$$\|V\|_{L^r L^s} \leq \|W\|_{L^s} \sum_{k\geq 0} \delta_k^{1/r} \epsilon_k^{2-n/s}. \tag{2.5.77}$$

As above, the function $u_{\epsilon_k}$ solves the equation

$$i\partial_t u - \Delta u + V(t,x)u = 0 \tag{2.5.78}$$

on the interval $t \in [k, k+\delta_k]$, and can be extended to a global solution $u_k(t,x)$ of the same equation thanks to the existence part of Theorem 2.5.1 (recall that $V \in L_I^\infty L^\infty$). Moreover, $u_k$ has a conserved energy

$$\|u_k(t,\cdot)\|_{L^2} \equiv \|u_k(k,\cdot)\|_{L^2} \equiv \epsilon^{-n/2}\|u_0\|_{L^2}. \tag{2.5.79}$$

Then, as before, we can estimate

$$\frac{\|u_k\|_{L^p(\mathbb{R};L^q)}}{\|u_k(0)\|_{L^2}} \geq \frac{\|u_{\epsilon_k}\|_{L^p(k,k+\delta_k;L^q)}}{\|u_{\epsilon_k}(0)\|_{L^2}} \equiv \frac{\|u_0\|_{L^q}}{\|u_0\|_{L^2}} \delta_k^{1/p} \epsilon_k^{\frac{n}{2}-\frac{n}{q}}. \tag{2.5.80}$$

Again, in order to violate the Strichartz estimates for an admissible couple $(p,q)$ and the potential $V \in L^r L^s$, it is sufficient to satisfy the two conditions

$$\sum_{k\geq 0} \delta_k^{1/r} \epsilon_k^{2-n/s} < \infty, \qquad \delta_k^{1/p} \epsilon_k^{\frac{n}{2}-\frac{n}{q}} \to \infty. \tag{2.5.81}$$

With the special choices

$$\delta_k = k^{-\alpha}, \qquad \epsilon_k = k^{\beta/2}, \tag{2.5.82}$$

the parameters $\alpha, \beta > 0$ to be precised, we are reduced to

$$-\frac{\alpha}{r} + \left(1 - \frac{n}{2s}\right)\beta < -1, \qquad -\frac{\alpha}{p} + \left(\frac{n}{4} - \frac{n}{2q}\right)\beta > 0. \tag{2.5.83}$$

Since $(p,q)$ is an admissible pair, the second condition is equivalent to $\alpha < \beta$ and we can rewrite the conditions as

$$\frac{\alpha - \beta}{r} + \left(\frac{1}{r} + \frac{n}{2s}\right)\beta > \beta + 1, \qquad \alpha < \beta. \tag{2.5.84}$$

Recall now that we are considering the case

$$\frac{1}{r} + \frac{n}{2s} > 1, \tag{2.5.85}$$

hence we may choose any $\beta$ such that

$$\beta > \left[\left(\frac{1}{r} + \frac{n}{2s}\right) - 1\right]^{-1} \tag{2.5.86}$$

and choosing then any $\alpha < \beta$ close enough to $\beta$, we easily satisfy the conditions (2.5.84).

This concludes the proof of part (ii) of Theorem 2.5.3.

Part (iii) can be proved by a simple modification of the preceding proof. Indeed, consider again the sequence $\delta_k = k^{-\alpha}$ constructed above, and notice that it is not restrictive to assume that $\beta > \alpha > 1$. Thus the series $\sum \delta_k$ converges, and the sequence of partial sums

$$T_k = \sum_{j=0}^{k} \delta_k \tag{2.5.87}$$

is positive, strictly increasing, and converges to

$$\lim_{k\to\infty} T_k = T \equiv \sum_{k\geq 0} \delta_k. \tag{2.5.88}$$

We can now modify the definition (2.5.76) of the potential $V(t,x)$ as follows:

$$V(t,x)=\begin{cases} W_{\epsilon_k}(x) & \text{if } t\in[T_k,T_k+\delta_k],\ x\in\mathbb{R}^n, \\ 0 & \text{if } t\in[0,\delta_0[. \end{cases} \tag{2.5.89}$$

This defines a potential on $I\times\mathbb{R}^n=[0,T]\times\mathbb{R}^n$, whose $L^r_I L^s$ is given again by (2.5.77). The remaining arguments of the preceding proof apply without modification.

The proof of Theorem 2.5.3 is concluded.

### 2.5.7 Proof of Proposition 2.5.4

The main tool of the proof is the *pseudoconformal transform*

$$u(t,x) \quad\mapsto\quad U(T,X)=e^{-i\frac{|X|^2}{4T}}T^{-\frac{n}{2}}u\left(-\frac{1}{T},\frac{X}{T}\right) \tag{2.5.90}$$

which takes a solution $u(t,x)$ of the Schrödinger equation in the variables $t,x$ into another solution of the same equation, in the variables $T,X$. If we apply the transform to the solution (2.5.58), we obtain a function $U(T,X)$ which solves

$$i\partial_T U-\Delta_X U+V(T,X)U=0, \qquad U(1,X)=e^{i-i|X|^2}u_0(X), \tag{2.5.91}$$

where the potential $V(T,X)$ is given by

$$V(T,X)=\frac{1}{T^2}W\left(\frac{X}{T}\right). \tag{2.5.92}$$

It is easy to compute explicitly the norm of $V$ on the interval $[0,1]$:

$$\|V\|_{L^r(\delta,1;L^s)}=\left(\int_\delta^1 T^{r(n/s-2)}dT\right)^{1/r}\|W\|_{L^s}<\infty \tag{2.5.93}$$

and this integral converges since our assumption (2.5.29) on the pair $(r,s)$ is equivalent to

$$r\left(\frac{n}{s}-2\right)>-1.$$

On the other hand, the $L^p_I L^q$ norm of $U(T,X)$ on an interval of the form $[\delta,1]$ with $0<\delta<1$ is given by

$$\|U\|_{L^p_I L^q}=\left(\int_\delta^1 T^{p\left(\frac{n}{q}-\frac{n}{2}\right)}\right)^{1/p}\|W\|_{L^q}\equiv\left(\int_\delta^1 T^{-2}\right)^{1/p}\|W\|_{L^q} \tag{2.5.94}$$

since admissible pairs $(p,q)$ satisfy $p(n/q-n/2)\equiv-2$. This implies that $U(T,X)$ belongs to all $L^p_I L^q$ spaces for $I=[\delta,1]$ for all $0<\delta<1$, but not for $I=[0,1]$ where the integral diverges. Note also that

$$\|U(1,\cdot)\|_{L^2}\equiv\|u_0\|_{L^2}.$$

It is sufficient now to apply to $U(T,X)$ a reflection and a translation in time $T$ to obtain exactly the counterexample required in Theorem 2.5.4. The proof is concluded.

# Chapter 3

# Equations on noncompact manifolds with negative curvature

## 3.1 Introduction

This chapter is devoted to the study of the perturbed Schrödinger equation on some manifolds with constant negative curvature:

$$iu_t - \Delta_M u + V(t,x)u = F(t,x),$$

where $-\Delta_M$ denotes the Laplace-Beltrami operator of the manifold $M$. More precisely, we shall consider the special case $M = \mathbb{H}^n$, the hyperbolic space of dimension $n$, and the more general class of Damek-Ricci spaces.

Our first goal is to prove the analogous of Strichartz estimates on $\mathbb{H}^n$; the effect of negative curvature is that in the estimates new weights appears, increasing as $|x| \to \infty$. Thus in the presence of negative curvature the estimates are stronger than in the flat case. If a large time dependent potential $V(t,x)$ is present, we can extend the results of Section [lavoroconNicola] to this case, and we can prove the Strichartz estimates provided $V$ satisfies a suitable weighted $L^r_t L^s_x$ condition. We then apply these estimates to the semilinear Schrödinger equation with a power nonlinearity depending also on the space variables:

$$iu_t - \Delta_{\mathbb{H}^n} u + V(t,x)u = g(|x|,u).$$

We prove results of both local and global well-posedness for radial solutions in the energy class. The behaviour of the nonlinearity for which we have global existence is similar to the flat case, but here we can allow a growth of the nonlinear term as $|x| \to \infty$, which is more general than in the flat case.

In the next section we investigate the case of Damek-Ricci spaces S, and we consider the free Schrödinger and wave equations

$$iu_t - \Delta_S u = F(t, |x|), \qquad u_{tt} - \Delta_S u = F(t, |x|).$$

For these equations, in the radial case, we prove generalized Strichartz estimates with weights; again, these estimates are stronger than the corresponding ones on $\mathbb{R}^n$, as an effect of curvature. We notice also that in the case of the three dimensional hyperbolic space $\mathbb{H}^3$ we reobtain (with a simpler proof) a weighted dispersive estimate proved by Banica in [5].

The results of this chapter are contained in the papers [76] and [77].

## 3.2 Strichartz estimates

For the convenience of the reader, we collect here the Strichartz estimates for the Schrödinger and the wave equations on $\mathbb{R}^n$, which we shall extend to more general manifolds in the following sections. Standard references are [98], [51], and [66].

The Strichartz estimates for the Schrödinger equation on $\mathbb{R}^n$ can be written in the following form:

$$\|e^{it\Delta} f\|_{L^p(I;L^q(\mathbb{R}^n))} \leq \|f\|_{L^2(\mathbb{R}^n)} \tag{3.2.1}$$

for any $f \in L^2$, any (bounded or unbounded) time interval $I \subseteq \mathbb{R}$, and for all sharp $\frac{n}{2}$-admissible couples $(p, q)$:

$$\frac{1}{p} + \frac{n}{2q} = \frac{n}{4}, \quad p, q \geq 2 \text{ and } (p, q) \neq (2, \infty). \tag{3.2.2}$$

The case $(p, q) = (2, \frac{2n}{n-2})$ is called the endpoint; estimate (3.2.1) is true also at the endpoint for $n \geq 3$. When $n = 2$ the endpoint is exactly $(p, q) = (2, \infty)$; in this case the estimate is still true when $f$ is a radial function, but is known to be false in general.

The equivalent nonhomogeneous form of (3.2.1) is

$$\left\| \int_0^t e^{i(t-s)\Delta} F(s, x) ds \right\|_{L^p(I;L^q(\mathbb{R}^n))} \leq C \|F\|_{L^{\tilde{p}'}(I;L^{\tilde{q}'}(\mathbb{R}^n))} \tag{3.2.3}$$

for all $(p, q)$ and $(\tilde{p}, \tilde{q})$ admissible, $\tilde{p}'$ and $\tilde{q}'$ being dual to $p$, $q$ respectively.

The Strichartz estimates for the wave equation on $\mathbb{R}^n$

$$-\partial_t^2 u + \Delta u = F(t, x), \quad u(0, x) = u_0(x), \ \partial_t u(0, x) = u_1(x), \tag{3.2.4}$$

under the assumption that the dimensional analysis (or "gap") condition

$$\frac{1}{p} + \frac{n}{q} = \frac{n}{2} - \gamma = \frac{1}{\tilde{p}'} + \frac{n}{\tilde{q}'} - 2, \tag{3.2.5}$$

holds, are the following

$$\|u\|_{L_t^p L_x^q} \leq C\|u_0\|_{\dot{H}^\gamma} + C\|u_1\|_{\dot{H}^{\gamma-1}} + C\|F\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}}, \tag{3.2.6}$$

for any data $u_0 \in \dot{H}^\gamma$, $u_1 \in \dot{H}^{\gamma-1}$, $F \in L_I^{\tilde{p}'} L^{\tilde{q}'}$, any (bounded or unbounded) time interval $I \subseteq \mathbb{R}$, and for all $\frac{n-1}{2}$-admissible couples $(p,q)$, $(\tilde{p},\tilde{q})$, i.e. such that

$$\frac{1}{p} + \frac{n-1}{2q} \leq \frac{n-1}{4}, \quad p \in ]2,\infty] \text{ and } q \in \left[2, \frac{2(n-1)}{n-3}\right[, \; n \geq 3. \tag{3.2.7}$$

Estimate (3.2.6) is true also at the endpoint $(p,q) = (2, \frac{2(n-1)}{n-3})$ for $n \geq 4$, but is false when $n = 3$.

## 3.3 Hyperbolic spaces

We consider here the Schrödinger equation on the hyperbolic space

$$\begin{cases} i\partial_t u + \Delta_{\mathbb{H}^n} u = 0, \\ u(0,x) = f(\Omega), \; \Omega \in \mathbb{H}^n. \end{cases} \tag{3.3.1}$$

See the following section for the main properties of $\mathbb{H}^n$ and its Laplace-Beltrami operator $\Delta_{\mathbb{H}^n}$. The solution $u$ can be represented using the unitary operators $e^{it\Delta_{\mathbb{H}^n}}$ as

$$u(t,\Omega) = e^{it\Delta_{\mathbb{H}^n}} f. \tag{3.3.2}$$

It is natural to expect that the curvature of the manifold has some influence on the dispersive properties. Indeed, in [5] the following estimate was proved for $u(t,\Omega) = e^{it\Delta_{\mathbb{H}^n}} f$, $n \geq 3$ odd,

$$|u(t,\Omega)| \leq C\left(\frac{1}{|t|^{\frac{n}{2}}} + \frac{1}{|t|^{\frac{3}{2}}}\right) \int_{\mathbb{H}^3} |f(\Omega')| \left(\frac{\rho}{\sinh \rho}\right)^{\frac{n-1}{2}} d\Omega', \tag{3.3.3}$$

where by $\rho$ we denoted the hyperbolic distance between the points $\Omega$ and $\Omega'$. If we compare (3.3.3) with the standard dispersive estimate on $\mathbb{R}^n$, we see that the effect of the curvature is a weight in the right hand side of (3.3.3). If we restrict to radial data $f$, then (3.3.3) implies the weighted estimate

$$w(\Omega)|u(t,\Omega)| \leq \frac{C}{|t|^{\frac{n}{2}}} \int_{\mathbb{H}^n} |f(\Omega')| w^{-1}(\Omega') d\Omega' \tag{3.3.4}$$

where the weight function $w(\Omega)$ is given by

$$w(\Omega) = \frac{\sinh d(0,\Omega)}{d(0,\Omega)}. \tag{3.3.5}$$

Here we denote by 0 the origin of the hyperbolic space, $0 = (1, 0, \dots, 0)$, and the $L^p$ space on $\mathbb{H}^n$ as

$$L^p = L^p(\mathbb{H}^n) = L^p(d\Omega),$$

where $d\Omega$ is the measure on the hyperbolic space $\mathbb{H}^n$ (see the following section for the precise definitions).

Thus by using interpolation and the standard $TT^*$ argument of [51], [66], it is easy to obtain the weighted Strichartz estimates

$$\|e^{it\Delta_{\mathbb{H}^n}} f\|_{L^p(I;L^q(w^{q-2}))} \leq C\|f\|_{L^2}, \tag{3.3.6}$$

which can be written also

$$\|w^{1-\frac{2}{q}} e^{it\Delta_{\mathbb{H}^n}} f\|_{L^p(I;L^q)} \leq C\|f\|_{L^2}. \tag{3.3.7}$$

Moreover, the $TT^*$ argument gives the equivalent estimate

$$\left\| w^{1-\frac{2}{q}} \int_0^t e^{i(t-s)\Delta_{\mathbb{H}^n}} F(s,\Omega) ds \right\|_{L^p(I;L^q)} \leq C\|w^{1-\frac{2}{\tilde{q}'}} F\|_{L^{\tilde{p}'}(I;L^{\tilde{q}'})} \tag{3.3.8}$$

for all admissible couples $(p, q)$ and $(\tilde{p}, \tilde{q})$, for all radial functions $f(\Omega)$ and $F(t, \Omega)$, and for all unbounded interval $I \subset \mathbb{R}$ when $n = 3$ and bounded interval $I \subset \mathbb{R}$ when $n > 3$ odd.

Consider now a perturbed Schrödinger equation of the form

$$i\partial_t u + \Delta_{\mathbb{H}^n} u + V(t,\Omega)u = 0. \tag{3.3.9}$$

This can be regarded as a first step to the general equation with variable coefficients. As it was observed in [39], a perturbation of the form (3.3.9) can be treated if we assume that the potential $V$ satisfies suitable integrability properties in space and time.

The main result of this section is the following

**Theorem 3.3.1.** *Let $I$ be an interval of the form $[0, +\infty[$ in three dimension and $[0, T]$ bounded when $n > 3$ odd. Let $V : I \times \mathbb{H}^n \to \mathbb{C}$ be a function such that*

$$\|w(\Omega)^{-\frac{2}{s}} V\|_{L^r(I;L^s)} < +\infty \tag{3.3.10}$$

*and indices $r, s$ satisfying*

$$\frac{1}{r} + \frac{n}{2s} = 1, \quad r \in [1, \infty] \text{ and } s \in [\frac{n}{2}, \infty]. \tag{3.3.11}$$

*Moreover, assume that*
*i) $V$ is a radial function in $\Omega$;*
*ii) in the endpoint case $(r, s) = (\infty, \frac{n}{2})$, the norm $\|w^{-\frac{4}{n}} V\|_{L^\infty(I;L^{\frac{n}{2}})}$ is small enough.*

*Let $f \in L^2$ and $F$ such that $w^{1-\frac{2}{\tilde{q}'}} F \in L^{\tilde{p}'}(I; L^{\tilde{q}'})$ be two functions radial in $\Omega$, with $(\tilde{p}, \tilde{q})$ admissible. Then the Cauchy problem*

$$\begin{cases} i\partial_t u + \Delta_{\mathbb{H}^n} u + V(t,\Omega)u = F(t,\Omega), \\ u(0,\Omega) = f(\Omega), \end{cases} \tag{3.3.12}$$

*has a unique solution $u \in C(I; L^2)$ satisfying for all admissible couples $(p,q)$ the weighted Strichartz estimates*

$$\|w^{1-\frac{2}{q}} u\|_{L^p(I;L^q)} \leq C\|f\|_{L^2} + C\|w^{1-\frac{2}{\tilde{q}'}} F\|_{L^{\tilde{p}'}(I;L^{\tilde{q}'})} \tag{3.3.13}$$

*with $p, q, \tilde{p}, \tilde{q}$ as above.*

*When $F \equiv 0$, the norm $\|u\|_{L^2}$ is constant in time.*

*Remark* 3.3.1. Note that for a singular coefficient $V(t,\Omega)$ it is not clear in general if the Cauchy problem (3.3.12) is well posed. Thus in the proof of Theorem 3.3.1 we must also obtain the existence and uniqueness of the solution $u(t,\Omega)$.

*Remark* 3.3.2. By iterating the argument of the proof, one can treat easily the case of a general potential

$$V = V_1 + \ldots + V_k$$

such that each $V_1, \ldots, V_k$ satisfies the assumptions of the Theorem 3.3.1 (with possibly different values of $r$, $s$).

*Remark* 3.3.3. In the case of a bounded time interval $I = [0,T]$, we can easily extend the results of Theorem3.3.1 to any potential satisfying (ii) with

$$\frac{1}{r} + \frac{n}{2s} \leq 1.$$

Indeed, by Hölder inequality we see immediately that a such $V$ satisfies (ii) for a different couple $(\tilde{r}, \tilde{s})$.

In the second part of the paper we shall consider an application of Theorem3.3.1 to a nonlinear Schrödinger equation of the form

$$i\partial_t u + \Delta_{\mathbb{H}^n} u = g(\Omega, u). \tag{3.3.14}$$

Notice that our weighted estimate (3.4.20) makes it possible to consider coefficients $g(\Omega,u)$ which are unbounded as $|\Omega| \to \infty$. Our result is the following:

**Theorem 3.3.2.** *Let $n \geq 3$ odd. Let $V$ be as in Theorem 3.3.1. Assume $g : \mathbb{H}^n \times \mathbb{C} \to \mathbb{C}$ is such that:*

*(i) $Im(g(\Omega,u)) = 0$ (gauge invariance);*

*(ii) if* $1 \le \gamma < 1 + \frac{4}{n}$, *the following inequalities hold:*

$$|g(\Omega, u)| \le Cw(\Omega)^{\frac{4}{n}} |u|^{\gamma}, \tag{3.3.15}$$

$$|g(\Omega, v) - g(\Omega, w)| \le Cw(\Omega)^{\frac{4}{n}} (|v| + |w|)^{\gamma - 1} |v - w|; \tag{3.3.16}$$

*(iii)* $g$ *is a radial function of* $\Omega$.
*Then the Cauchy problem*

$$\begin{cases} i\partial_t u + \Delta_{\mathbb{H}^n} u + V(t, \Omega) u = g(\Omega, u), \\ u(0, \Omega) = f(\Omega) \ \ radial, \end{cases} \tag{3.3.17}$$

*has a unique global solution* $u \in C(\mathbb{R}, L^2)$ *such that* $w^{1-\frac{2}{q}} u \in L^p(\mathbb{R}; L^q)$ *for all admissible couples* $(p, q)$.

*Moreover, when* $\gamma = 1 + \frac{4}{n}$ *the result is still true provided the* $L^2$ *norm of data* $\|f\|_{L^2}$ *is sufficiently small and without hypothesis (i).*

### 3.3.1 Basic properties of $\mathbb{H}^n$

We recall briefly some properties of the hyperbolic space that we shall use in the following. We shall represent $\mathbb{H}^n$ as the upper branch of the hyperboloid:

$$\mathbb{H}^n = \{\Omega = (t, x) \in \mathbb{R}^{n+1}, \ (t, x) = (\cosh r, \omega \sinh r), \ r \ge 0, \ \omega \in \mathbb{S}^{n-1}\}.$$

This can be written in an equivalent way as follows:

$$\mathbb{H}^n = \{x = (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}, \ x_0 > 0, \ [x, x] = 1\}$$

where $[x, y]$ denotes the inner product on $\mathbb{R}^{n+1}$

$$[x, y] = x_0 y_0 - x_1 y_1 - \dots - x_n y_n.$$

If we restrict to $\mathbb{H}^n$ the Lorentz metric on $\mathbb{R}^{n+1}$

$$dl^2 = -dt^2 + dx^2$$

we obtain the following riemannian metric on the hyperbolic space

$$ds^2 = dr^2 + \sinh^2 r d\omega^2$$

as it follows immediately from the relations

$$dt = \sinh r dr, \quad dx = \cosh r \omega dr + \sinh r d\omega.$$

The distance between two points in this metric can be written explicitly, using the above defined inner product

$$d(\Omega, \Omega') = \cosh^{-1}([\Omega, \Omega']).$$

A useful special case is the distance of a point from the origin 0 which corresponds to $(x_0, x_1, \dots, x_n) = (1, 0, \dots, 0)$ or equivalently to $(t, r) = (1, 0)$:

$$d(\Omega, 0) = d((\cosh r, \sinh r\omega), (1, 0)) = \cosh^{-1}(\cosh r - 0) = r.$$

Finally, the corresponding measure can be written in the coordinates $r, \omega$ as follows:

$$\int_{\mathbb{H}^n} f(\Omega) d\Omega = \int_0^\infty \int_{\mathbb{S}^{n-1}} f(r, \omega) \sinh^{n-1} r dr d\omega.$$

The Laplace-Beltrami operator on the hyperboloid has a simple expression in terms of the laplace operator on the sphere:

$$\Delta_{\mathbb{H}^n} = \partial_r^2 + (n-1)\frac{\cosh r}{\sinh r}\partial_r + \frac{1}{\sinh^2 r}\Delta_{\mathbb{S}^{n-1}}.$$

### 3.3.2 Proof of Theorem 3.3.1

The proof of Theorem 3.3.1 follows closely the ideas of [39]. For the benefit of the reader we give here a complete proof, with the necessary modifications.

In the following for simplicity we write only $\Delta$ instead of $\Delta_{\mathbb{H}^n}$. We shall also introduce the notation

$$L^p_J L^q = L^p(J; L^q(d\Omega))$$

for the mixed spaces on the product $J \times \mathbb{H}^n$, where $J$ is any time interval $[0, \infty[$ when $n = 3$, and $[0, T]$ bounded when $n > 3$ odd.

We distinguish two cases, according to the value of $r \in [1, \infty[$.

### 3.3.3 Case A: $r \in [2, \infty[$

Consider a small interval $J = [0, \delta]$ and the norm

$$\|v\|_Z := \max\left\{ \|v\|_{L^\infty_J L^2}, \ \ \|w(\Omega)^{\frac{2}{n}} v\|_{L^2_J L^{\frac{2n}{n-2}}} \right\};$$

note that

$$1 - \frac{2}{r} = \frac{2}{n} \qquad \text{for} \qquad r = \frac{2n}{n-2}.$$

Let $Z$ be the Banach space

$$Z = \{f \in C_J L^2 \colon \|f\|_Z < \infty\}$$

with the norm $\|v\|_Z$. Then, by interpolation, $Z$ is embedded in all admissible spaces $L^p_J L^q$.

For any $v(t, \Omega) \in Z$ we define the mapping

$$\Phi(v) = e^{it\Delta_{\mathbb{H}^n}} f + \int_0^t e^{i(t-s)\Delta_{\mathbb{H}^n}} [F(s) - V(s)v(s)] \, ds. \qquad (3.3.18)$$

A direct application of the weighted Strichartz estimates (3.3.8) gives

$$\|w^{1-\frac{2}{q}}\Phi(v)\|_{L^p_J L^q} \leq C_0\|f\|_{L^2} + C_0\|w^{1-\frac{2}{q_0'}}Vv\|_{L^{p_0'}_J L^{q_0'}} + C_0\|w^{1-\frac{2}{\tilde{q}'}}F\|_{L^{\tilde{p}'}_J L^{\tilde{q}'}} \tag{3.3.19}$$

for all admissible $(p,q)$, $(p_0,q_0)$, $(\tilde{p},\tilde{q})$. Now, by Hölder's inequality we have

$$\|w^{1-\frac{2}{q_0'}}Vv\|_{L^{p_0'}_J L^{q_0'}} \leq \|w^{1-\frac{2}{q_0'}-\frac{2}{n}}V\|_{L^r_J L^s}\|w^{\frac{2}{n}}v\|_{L^2_J L^{\frac{2n}{n-2}}}$$

and this gives

$$\|w^{1-\frac{2}{q}}\Phi(v)\|_{L^p_J L^q} \leq C_0\|f\|_{L^2}+C_0\|w^{-\frac{2}{s}}V\|_{L^r_J L^s}\|w^{\frac{2}{n}}v\|_{L^2_J L^{\frac{2n}{n-2}}}+C_0\|w^{1-\frac{2}{\tilde{q}'}}F\|_{L^{\tilde{p}'}_J L^{\tilde{q}'}} \tag{3.3.20}$$

provided we choose $p_0, q_0$ such that

$$\frac{1}{p_0} = \frac{1}{2} - \frac{1}{r}, \qquad \frac{1}{q_0} = \frac{n+2}{2n} - \frac{1}{s}.$$

Indeed, our choice gives in particular (see the weight for $V$)

$$1 - \frac{2}{q_0'} - \frac{2}{n} = -\frac{2}{s}.$$

Note that

$$\frac{1}{p_0} + \frac{n}{2q_0} = \frac{1}{2} + \frac{n+2}{4} - \left(\frac{1}{r} + \frac{n}{2s}\right) \equiv \frac{1}{2} + \frac{n+2}{4} - 1 \equiv \frac{n}{4}$$

by our assumptions on $r, s$, and moreover

$$r \in [2,\infty[ \quad \Longrightarrow \quad p_0 \in [2,\infty[$$

so that our choice of $p_0, q_0$ always gives an admissible pair in the case under consideration.

In particular, choosing $(p,q) = (\infty, 2)$ or $(2, 2n/(n-2))$, we obtain

$$\|w^{1-\frac{2}{q}}\Phi(v)\|_Z \leq C_0\|f\|_{L^2} + C_0\|w^{-\frac{2}{s}}V\|_{L^r_J L^s}\|v\|_Z + C_0\|w^{1-\frac{2}{\tilde{q}'}}F\|_{L^{\tilde{p}'}_J L^{\tilde{q}'}} \tag{3.3.21}$$

Thus $\Phi(v)$ belongs to all the admissible weighted spaces $L^p_J L^q$, and to prove that $\Phi(v)$ belongs to $Z$ it remains only to show that $u$ is continuous with values in $L^2$. But this is an immediate consequence of the following simple remark:

*Remark* 3.3.4. Let $G$ be such that $w^{1-\frac{2}{b'}}G(t,\Omega) \in L^{a'}_J L^{b'}$ with $(a,b)$ admissible. Then the function

$$\tilde{w}(t,\Omega) = \int_0^t e^{i(t-s)\Delta_{\mathbb{H}^n}}G(s)\,ds$$

belongs to $C_J L^2$. Indeed, this is certainly true if we know in addition that $G$ is a smooth function, compactly supported in $\Omega$ for each $t$. If we approximate $G$ by a sequence of such functions $G_j$ so that $w^{1-\frac{2}{b'}} G_j$ converges to $w^{1-\frac{2}{b'}} G$ in the $L^{a'}_J L^{b'}$ norm, the Strichartz estimates imply that the corresponding functions $w_j$ converge in $L^\infty L^2$, whence the claim follows.

We have thus constructed a mapping $\Phi : Z \to Z$. Assume now the length $\delta$ of the interval $J$ is chosen so small that

$$C_0 \| w^{-\frac{2}{s}} V \|_{L^r_J L^s} \leq \frac{1}{2}; \tag{3.3.22}$$

this is certainly possible since $r < \infty$. With this choice we obtain immediately two consequences: first of all, the mapping $\Phi$ is a contraction on $Z$ and hence has a unique fixed point $v(t, \Omega)$ which is the required solution; second, $v$ satisfies

$$\| w^{1-\frac{2}{q}} v \|_{L^p_J L^q} \leq C_0 \| f \|_{L^2} + \frac{1}{2} \| w^{1-\frac{2}{q}} v \|_{L^p_J L^q} + C_0 \| w^{1-\frac{2}{\tilde{q}'}} F \|_{L^{\tilde{p}'}_J L^{\tilde{q}'}} \tag{3.3.23}$$

whence we obtain

$$\| w^{1-\frac{2}{q}} v \|_{L^p_J L^q} \leq 2C_0 \| f \|_{L^2} + 2C_0 \| w^{1-\frac{2}{\tilde{q}'}} F \|_{L^{\tilde{p}'}_J L^{\tilde{q}'}} \tag{3.3.24}$$

It is clear that the above argument applies on any subinterval $J = [t_0, t_1] \subseteq I$ on which a condition like (3.3.22) holds; of course, we will obtain an estimate of the form

$$\| w^{1-\frac{2}{q}} v \|_{L^p_J L^q} \leq 2C_0 \| v(t_0) \|_{L^2} + 2C_0 \| w^{1-\frac{2}{\tilde{q}'}} F \|_{L^{\tilde{p}'}_J L^{\tilde{q}'}}. \tag{3.3.25}$$

Notice also that (3.3.25) implies in particular

$$\| w^{1-\frac{2}{q}} v(t_1) \|_{L^2} \leq 2C_0 \| v(t_0) \|_{L^2} + 2C_0 \| w^{1-\frac{2}{\tilde{q}'}} F \|_{L^{\tilde{p}'}_J L^{\tilde{q}'}}. \tag{3.3.26}$$

Now we can partition the interval $I$ (bounded or unbounded) in a finite number of subintervals on which condition (3.3.22) holds. Applying inductively the estimates (3.3.25) and (3.3.26) we easily obtain (3.3.13).

The last remark concerning the conservation of energy can be proved by approximation as follows: let $V_j(t, \Omega)$ be a sequence of real valued smooth potentials, compactly supported in $\Omega$, and let $v_j$ be the corresponding solutions; then the differences $w_j = v - v_j$ satisfy (in suitable integral sense)

$$i\partial_t w_j - \Delta_{\mathbb{H}^n} w_j + V w_j = (V - V_j) v_j \equiv F_j.$$

Now we observe that the smooth solutions $v_j$ have a conserved energy; moreover, we can choose the approximating potentials $V_j$ in such a way that $w^{-\frac{2}{s}} V_j$ they converge to $w^{-\frac{2}{s}} V$ in $L^r_I L^s$ and their Strichartz constants do not exceed the above constructed constant for $V$. Indeed, if we can partition $I$ in a finite set of subintervals satisfying (3.3.22), we can choose exactly the same subintervals for each $V_j$ provided we construct $V_j$ by a convolution with standard mollifiers, so that their Lebesgue norm does not increase. In conclusion, the $v_j$ satisfy uniform Strichartz estimates, and this implies that the nonhomogeneous terms $F_j = (V - V_j)v_j$ tend to 0 in the (dual) admissible spaces, by estimates identical to the above ones. Thus in particular $w_j \to 0$ in $L^\infty L^2$ and this shows that also $v(t,\Omega)$ satisfies the conservation of energy.

### 3.3.4 Case B: $r \in [1,2]$

The method in this case is quite similar to the above one, but instead of (3.3.19) we use the estimate

$$\|w^{1-\frac{2}{q}}\Phi(v)\|_{L^p_J L^q} \leq C_0\|f\|_{L^2} + C_0\|w^{-\frac{2}{s}}Vv\|_{L^r_J L^{\frac{2s}{s+2}}} + C_0\|w^{1-\frac{2}{\tilde{q}'}}F\|_{L^{\tilde{p}'}_J L^{\tilde{q}'}} \tag{3.3.27}$$

where $(p,q)$ and $(\tilde{p},\tilde{q})$ are arbitrary admissible pairs, while the pair $(r, 2s/(s+2))$ is the dual of $(r', 2s/(s-2))$ and this last pair is admissible since

$$\frac{1}{r'} + \frac{n}{2}\cdot\frac{s-2}{2s} = \frac{n}{2s} + \frac{n}{2}\cdot\frac{s-2}{2s} = \frac{n}{4}$$

where we have used the assumption $1/r + n/(2s) = 1$; notice also that $r \in [1,2]$ and hence $2s/(s+2) \in [1,2]$ too.

Thus by Hölder's inequality we obtain

$$\|w^{1-\frac{2}{q}}\Phi(v)\|_{L^p_J L^q} \leq C_0\|f\|_{L^2} + C_0\|w^{-\frac{2}{s}}V\|_{L^r_J L^s}\|v\|_{L^\infty_J L^2} + C_0\|w^{1-\frac{2}{\tilde{q}'}}F\|_{L^{\tilde{p}'}_J L^{\tilde{q}'}} \tag{3.3.28}$$

and choosing $(p,q) = (\infty, 2)$ or $(2, 2n/(n-2))$ and proceeding as above we arrive at

$$\|\Phi(v)\|_Z \leq C_0\|f\|_{L^2} + \frac{1}{2}\|v\|_Z + C_0\|w^{1-\frac{2}{\tilde{q}'}}F\|_{L^{\tilde{p}'}_J L^{\tilde{q}'}}. \tag{3.3.29}$$

¿From this point on, the proof is identical to the first case.

### 3.3.5 Case C: $(r,s) = (\infty, n/2)$

In the last case we assume the potential to be small in the following sense:

$$\|w^{-\frac{4}{n}}V\|_{L^\infty L^{\frac{n}{2}}} < \epsilon.$$

The proof is similar to Case A, with the same choice of the indices; we obtain

$$\|w^{1-\frac{2}{q}}\Phi(v)\|_{L^p_J L^q} \leq C_0\|f\|_{L^2} + C_0\|w^{-\frac{4}{n}}V\|_{L^\infty_J L^{\frac{n}{2}}}\|w^{\frac{2}{n}}v\|_{L^2_J L^{\frac{2n}{n-2}}} + C_0\|w^{1-\frac{2}{q'}}F\|_{L^{\tilde{p}'}_J L^{\tilde{q}'}} \tag{3.3.30}$$

and by the smallness assumption we can write

$$\|w^{1-\frac{2}{q}}\Phi(v)\|_{L^p_J L^q} \leq C_0\|f\|_{L^2} + C_0\epsilon\|w^{\frac{2}{n}}v\|_{L^2_J L^{\frac{2n}{n-2}}} + C_0\|w^{1-\frac{2}{\tilde{q}'}}F\|_{L^{\tilde{p}'}_J L^{\tilde{q}'}}. \tag{3.3.31}$$

Choosing $(p, q) =$ endpoint we easily conclude the proof of the Theorem.

### 3.3.6 Proof of Theorem 3.3.2

We begin by the critical case $\gamma = 1 + 4/n$. We define $\Phi(v)$ as the solution $u$ of the Cauchy problem

$$\begin{cases} i\partial_t u + \Delta_{\mathbb{H}^n} u + V(t,\Omega)u = g(\Omega, v), \\ u(0,\Omega) = f(\Omega) \ \text{ radial.} \end{cases} \tag{3.3.32}$$

By Theorem 3.3.1 the following weighted Strichartz estimate holds

$$\|w^{1-\frac{2}{q}}u\|_{L^p(I;L^q)} \leq C\|f\|_{L^2} + C\|w^{1-\frac{2}{\tilde{q}'}}g(\Omega, v)\|_{L^{\tilde{p}'}(I;L^{\tilde{q}'})} \tag{3.3.33}$$

with $p, q, \tilde{p}, \tilde{q}$ as above. By (3.3.15) we have

$$\|w^{1-\frac{2}{q}}u\|_{L^p(I;L^q)} \leq C\|f\|_{L^2} + C\|w^{1-\frac{2}{\tilde{q}'}+\frac{4}{n}}|v|^\gamma\|_{L^{\tilde{p}'}(I;L^{\tilde{q}'})}$$

and we obtain that

$$\|w^{1-\frac{2}{q}}u\|_{L^p(I;L^q)} \leq C\|f\|_{L^2} + C\|w^\sigma v\|^\gamma_{L^{\tilde{p}'\gamma}(I;L^{\tilde{q}'\gamma})} \tag{3.3.34}$$

where

$$\sigma = \frac{1}{\gamma}\left(1 - \frac{2}{\tilde{q}'} + \frac{4}{n}\right).$$

We have to require the admissibility of couples $(p, q)$ and $(\tilde{p}, \tilde{q})$; moreover we must choose $\tilde{p}$, $\tilde{q}$ in such a way that the last norm in the above inequality is the same as the norm at the left hand side. We can express all these conditions by the following system:

$$\begin{cases} \tilde{p}'\gamma = p, \\ \tilde{q}'\gamma = q, \\ \frac{1}{p} + \frac{n}{2q} = \frac{n}{4}, \quad p, \tilde{p} \in [2, \infty] \\ \frac{1}{\tilde{p}} + \frac{n}{2\tilde{q}} = \frac{n}{4}, \quad q, \tilde{q} \in [2, \frac{2n}{n-2}], \end{cases} \tag{3.3.35}$$

i.e.

$$\frac{1}{\tilde{p}} + \frac{n}{2\tilde{q}} = 1 - \frac{\gamma}{p} + \frac{n}{2}(1 - \frac{\gamma}{q}) = \gamma\frac{n}{4} + 1 - \frac{n}{2} = \frac{n}{4}. \tag{3.3.36}$$

Now, if we know that

$$\gamma = 1 + \frac{4}{n}$$

we see that we can choose admissible couples $(p,q)$ and $(\tilde{p},\tilde{q})$ as above. Moreover, if we substitute in the definition of $\sigma$ the above relations, we obtain

$$\sigma = \frac{1}{\gamma}\left(1+\frac{4}{n}-\frac{2}{\tilde{q}'}\right) = 1-\frac{2}{q}$$

and thus we have proved that $\Phi$ maps the Banach space $X$ with norm

$$\|v\|_X := \|w^{1-\frac{2}{q}}v\|_{L^p(I;L^q)}$$

into itself.

We show now that $\Phi$ is a contraction on the space $X$. Let $v_1, v_2 \in X$ such that $\Phi(v_i) = u_i, i = 1,2$; then we can apply the weighted Strichartz estimate to the difference $v_1 - v_2$ and we get the following:

$$\|u_1 - u_2\|_X \leq \|w^{1-\frac{2}{\tilde{q}'}}(|v_1|^\gamma - |v_2|^\gamma)\|_{L^{\tilde{p}'}(I;L^{\tilde{q}'})}.$$

By (3.3.16) we have

$$\leq \|w^{1-\frac{2}{\tilde{q}'}+\frac{4}{n}}|v_1 - v_2|(|v_1|^{\gamma-1} + |v_2|^{\gamma-1})\|_{L^{\tilde{p}'}(I;L^{\tilde{q}'})}$$

and as before we obtain

$$\|u_1 - u_2\|_X \leq \|v_1 - v_2\|_X \|(|v_1| + |v_2|)\|_X^{\gamma-1}. \tag{3.3.37}$$

If we assume now that $v_i \in X$ such that $\|v_i\|_X < \varepsilon$, with $\varepsilon$ small enough, and also that $\|f\|_{L^2} < \delta$, by (3.3.34) we note that

$$\|u\|_X \leq C\delta + C\varepsilon^\gamma = C\delta + C\varepsilon(\varepsilon^{\gamma-1}) < \varepsilon;$$

provided $\varepsilon, \delta$ are such that $C\varepsilon^{\gamma-1} < \frac{1}{2}$ and $C\delta < \frac{\varepsilon}{2}$. We have also

$$\|u_1 - u_2\|_X \leq \|v_1 - v_2\|_X C 2\varepsilon^{\gamma-1} \leq \frac{1}{2}\|v_1 - v_2\|_X$$

provided $\varepsilon$ is so small that $2C\varepsilon^{\gamma-1} < \frac{1}{2}$. Thus, if initial data are small i.e. $\|f\|_{L^2} < \delta$, the map $\Phi$ is a contraction and this implies that there exists a unique solution $u(t,\Omega)$ of the Cauchy problem (3.3.2) such that $w^{1-\frac{2}{q}}u(t,\Omega) \in L^p(I;L^q)$ with a admissible couple $(p,q)$ when $\gamma = 1+\frac{4}{n}$. As observed above one see easily that this is the unique solution in $u(t,\Omega) \in C(\mathbb{R};L^2)$ with radial initial data in $L^2$.

In the subcritical case, i.e., when $\gamma < 1+\frac{4}{n}$, we proceed as above and using the Hölder inequality in time on $I = [0,T]$ we can prove that $\Phi$ is a map from $X_M := \{w^{1-\frac{2}{q}}v \in L^p(I;L^q) : \|w^{1-\frac{2}{q}}v\|_{L^p(I;L^q)} \leq M\}$ into itself, provided the time $T$ is small enough. Indeed, choosing the indices as above and applying Hölder's inequality in time we have

$$\|w^{1-\frac{2}{q}}u\|_{L^p(I;L^q)} \leq C\|f\|_{L^2} + CT^\lambda \|w^{1-\frac{2}{q}}v\|^\gamma_{L^p(I;L^q)}, \quad \lambda > 0 \tag{3.3.38}$$

for some $\lambda > 0$. We must to prove that $\Phi$ is a contraction on the space $X_M$. By hypothesis (ii) we obtain that

$$\|u_1 - u_2\|_{X_M} \leq C\|v_1 - v_2\|_{X_M} T^\lambda (\|v_1\|_{X_M}^{\gamma-1} + \|v_2\|_{X_M}^{\gamma-1}) \tag{3.3.39}$$

Let $v_i \in X_M$ and let $\|f\| \in L^2$, by (3.3.38) we note that

$$\|u\|_{X_M} \leq C\|f\|_{L^2} + CM^\gamma T^\lambda = C\|f\|_{L^2} + CM(M^{\gamma-1}) < M,$$

provided $M$ is so large that $C\frac{M}{2} \geq C\|f\|_L^2$ and $T$ is so small that $CT^\lambda M^{\gamma-1} < \frac{1}{2}$. Thus we have also

$$\|u_1 - u_2\|_{X_M} \leq \|v_1 - v_2\|_{X_M} C2T^\lambda M^{\gamma-1} \leq \frac{1}{2}\|v_1 - v_2\|_{X_M}$$

if $2CT^\lambda M^{\gamma-1} < \frac{1}{2}$. In conclusion, if $M \geq 2C\|f\|_{L^2}$ and $T \leq (\frac{1}{4CM^{\gamma-1}})^{\frac{1}{\lambda}}$, then the map $\Phi : X_M \to X_M$ is a contraction and as consequence there exists a unique solution $u \in X_M$ to Cauchy problem (3.3.17) when $\gamma < 1 + \frac{4}{n}$ for radial initial data large $f \in L^2$. Notice that $T$ depends only by $L^2$-norm of initial data i.e.

$$T = \left(\frac{1}{8C^2\|f\|_{L^2}^{\gamma-1}}\right)^{\frac{1}{\lambda}} = T(\|f\|_{L^2}),$$

and thanks to the conservation of charge, i.e., $\|u(t)\|_{L^2} \equiv \|f\|_{L^2}$ for all $t$, we can iterate the above argument starting at $t = T$ and we can solve up to time $2T$, then up to time $3T$, and so on. In other words, the solution exists for all times. Thus we have proved the global existence of a unique solution to Cauchy problem (3.3.17) for large radial initial data in $L^2$ when $\gamma < 1 + \frac{4}{n}$.

## 3.4 Damek-Ricci spaces

In this section we study the Schrödinger and wave equations in the more general context of Damek-Ricci spaces, also known as Harmonic AN groups; these spaces have been studied by several authors in the past 15 years ([4], [89], [11], [10], [29], [30], [33], [35], [36], [87], [100] and others). As Riemannian manifolds, these solvable Lie groups include all symmetric spaces of noncompact type and rank one, namely the hyperbolic spaces $\mathbb{H}^n(\mathbb{R})$, $\mathbb{H}^n(\mathbb{C})$, $\mathbb{H}^n(\mathbb{H})$, $\mathbb{H}^2(\mathbb{O})$, but most of them are not symmetric, thus providing numerous counterexemples to the Linchnerowicz conjecture [35]. This was implicitely formulated in 1944 by Linchnerowicz, who showed that every harmonic manifold of dimension at most 4 is a symmetric space, leaving open the question if this assertion remains true in every dimension. Though in 1990, Szabo proved it true for any simply connected compact harmonic manifold ([99]), in 1992, Ewa Damek and Fulvio Ricci found a large class of non-compact harmonic manifolds which are not symmetric spaces. More details on Damek-Ricci spaces are contained in the following section.

Our goal here is to extend the Strichartz estimates for the radial Schrödinger and wave equations on Damek-Ricci spaces.

The idea of the proof is to transform the equation into a new perturbed one with a suitable potential $V$ on $R^n$; then, using the results of the perturbative theory of Burq, Planchon, Stalker and Tahvildar-Zadeh [19], we can obtain the Strichartz estimates. More precisely, the radial operator $-\Delta_M$ can be reduced to an operator of the form $-\Delta + \widetilde{V}$, where the potential $\widetilde{V}$ has a critical decay $\sim |x|^{-2}$ and can be treated by the methods of [21].

It is interesting to note that we obtain the results on these noncompact manifolds as application of the perturbative theory on $R^n$, thus avoiding the difficulties caused by the geometry of these spaces.

Our first result concerns the Schrödinger equation on S; we can prove the following weighted Strichartz estimates

$$\|w_q u\|_{L^p(\mathbb{R},L^y(S))} \le C\|w_2\, u_0\|_{L^2(S)} + C\|w_{\widetilde{q}'} F\|_{L^{\tilde{p}'}(\mathbb{R},L^{\tilde{q}'}(S))},$$

with the weight

$$w_q(r) = \left(\frac{\sinh r}{r}\right)^{\frac{(m+k)}{2}(1-\frac{2}{q})} (\cosh r)^{\frac{k}{2}(1-\frac{2}{q})}.$$

Also for the wave equation on $S$ we are able to prove the following weighted Strichartz estimates

$$\|w_q u\|_{L^p(\mathbb{R},L^q(S))} \le C\left\|\frac{u_0}{\sigma}\right\|_{H^\gamma(S)} + C\left\|\frac{u_1}{\sigma}\right\|_{H^{\gamma-1}(S)} + C\|w_{\widetilde{q}'} F\|_{L^{\tilde{p}'}(\mathbb{R},L^{\tilde{q}'}(S))},$$

with the weights

$$w_q(r) = \left(\frac{\sinh r}{r}\right)^{\frac{(m+k)}{2}(1-\frac{2}{q})} (\cosh r)^{\frac{k}{2}(1-\frac{2}{q})},$$

and

$$\sigma(r) = r^{\alpha+\frac{1}{2}} (\sinh r)^{-(\alpha+\frac{1}{2})} (\cosh r)^{-(\beta+\frac{1}{2})}.$$

### 3.4.1 Harmonic analysis associated to $L_{\alpha,\beta}$ Jacobi operator

In this section we recall the spherical harmonic analysis on Damek-Ricci spaces $S = AN$, developed in [36] ([4], [89]), in accord with the general framework of Jacobi analysis [71].

First of all we recall briefly the structure of these spaces. Let $\mathfrak{n}$ be a two-step nilpotent Lie algebra equipped with an inner product $\langle\, ,\, \rangle$. Denote by $\mathfrak{z}$ the center of $\mathfrak{n}$ and by $\mathfrak{v}$ the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{n}$. So that

$$\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}, \quad [\mathfrak{v}, \mathfrak{v}] \subset \mathfrak{z}, \quad [\mathfrak{v}, \mathfrak{z}] = 0 \;\text{ and }\; [\mathfrak{z}, \mathfrak{z}] = 0.$$

For $Z \in \mathfrak{z}$ let $J_Z : \mathfrak{v} \to \mathfrak{v}$ be the linear map defined by

$$\langle J_Z X \, , \, Y \rangle = \langle Z \, , \, [X, Y] \rangle, \tag{3.4.1}$$

for every $X, Y \in \mathfrak{v}$. If, for every $Z \in \mathfrak{z}$, $X \in \mathfrak{v}$,

$$J_Z^2 X = -\|Z\|^2 X, \tag{3.4.2}$$

where $\| \cdot \|$ is the norm defined by an inner product, then $\mathfrak{n}$ is a algebra of Heisenberg type. Denoting by $m = \dim \mathfrak{v}$ and $k = \dim \mathfrak{z}$, for $k \geq 1$ there exists a algebra of Heisenberg type if and only if the possible dimensions $m, k$ are the values in the following table:

| $k$ | $8a+1$ | $8a+2$ | $8a+3$ | $8a+4$ | $8a+5$ | $8a+6$ | $8a+7$ | $8a+8$ |
|---|---|---|---|---|---|---|---|---|
| $m$ | $2^{4a+1}b$ | $2^{4a+2}b$ | $2^{4a+3}b$ | $2^{4a+4}b$ | $2^{4a+5}b$ | $2^{4a+6}b$ | $2^{4a+7}b$ | $2^{4a+8}b$ |

where $a \geq 0$ and $b \geq 1$ are arbitrary integers. In particular $m$ is always even.

The corresponding (connected) and simply connected Lie groups $N$ are called groups of Heisenberg type. We shall identify them with their Lie algebra $\mathfrak{n}$ via the exponential map $\exp : \mathfrak{n} \to N$. Thus multiplication in $N \equiv \mathfrak{n}$ reads

$$(X, Z) \cdot (X', Z') = (X + X', Z + Z' + \frac{1}{2}[X, X']). \tag{3.4.3}$$

We will not develop here the geometry and the analysis on $N$; see for example [10] chapter 2; [36] chapter 3. Consider ([11], [10], [29], [30], [33], [34], [35], [36], [100]) the semi-product $S = N \times \mathbb{R}_+^*$ defined by

$$(X, Z, a)(X', Z', a') = (X + a^{\frac{1}{2}} X', Z + aZ' + \frac{1}{2} a^{\frac{1}{2}} [X, X']. \tag{3.4.4}$$

$S$ is a solvable (connected and) simply connected Lie group, with Lie algebra $\mathfrak{s} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathbb{R}$ and Lie bracket

$$[(X, Z, \ell), (X', Z', \ell')] = (\frac{1}{2}\ell X' - \frac{1}{2}\ell' X, \ell Z' - \ell' Z + [X, X'], 0). \tag{3.4.5}$$

$S$ is equipped with left-invariant Riemnnian metric induced by

$$\langle (X, Z, \ell), (X', Z', \ell') \rangle = \langle X, X' \rangle + \langle Z, Z' \rangle + \ell \ell' \tag{3.4.6}$$

on $\mathfrak{s}$. The associated left-invariant (Riemannian-Haar) measure on $S$ is given by

$$a^{-Q} dX dZ \frac{da}{a}. \tag{3.4.7}$$

Here $Q = \frac{m}{2} + k$ is the homogeneous dimension of $N$. Thus we have the following definition.

**Definition 3.4.1.** *We call Damek-Ricci spaces the (connected and) simply connected Lie groups $S = AN$ for which Lie algebra is $\mathfrak{s} = \mathfrak{n} \oplus \mathbb{R}$ with the Lie bracket* (3.4.5)*, provided with left-invariant Riemnnian metric induced by inner product* (3.4.6) *on $\mathfrak{s}$.*

Most Riemannian symmetric spaces $G/K$ of noncompact type and rank one fit into this framework. According to the Iwasawa decomposition $G = NAK$, they can be realized indeed as $S = NA = AN$, with $A = \mathbb{R}$. $N$ is abelian for real hyperbolic spaces $G/K = H^n(\mathbb{R})$ and of Heisenberg type in the other cases $G/K = H^n(\mathbb{C}), H^n(\mathbb{H}), H^2(\mathbb{O})$. Notice that these classical examples form only a very small subclass of harmonic $AN$ group, as can be seen by looking at the dimension:

| | $H^n(\mathbb{R})$ | $H^n(\mathbb{C})$ | $H^n(\mathbb{H})$ | $H^2(\mathbb{O})$ |
|---|---|---|---|---|
| $k$ | [0] | 1 | 3 | 4 |
| $m$ | $[n-1]$ | $2(n-1)$ | $4(n-1)$ | 8 |

In the ball model $B(\mathfrak{s})$, the geodesics passing trough the origin are the diameters, the geodesic distance to the origin is given by

$$r = d(x', 0) = \log \frac{1 + \|x'\|}{1 - \|x'\|}, \text{ i.e. } \rho = \|x'\| = \tanh \frac{r}{2}, \tag{3.4.8}$$

and the Riemannian volume writes

$$dV = 2^{m+k} (\sinh \frac{r}{2})^{m+k} (\cosh \frac{r}{2})^k dr\, d\sigma, \tag{3.4.9}$$

where $d\sigma$ denotes the surface measure on the unit sphere $\partial B(\mathfrak{s})$ in $\mathfrak{s}$ and $n = \dim S = m + k + 1$. In particular, the volume density in normal coordinates at the origin, and by translation at any point, is a purely radial function, which means that $S$ is a harmonic manifold ([35], [99]). Like all harmonic manifolds, $S$ is an Einstein manifold. A lenghty computation yields the actual constant:

$$\textit{Ricci curvature} = -(\frac{m}{4} + k) \times \textit{Riemannian metric.} \tag{3.4.10}$$

The sectional curvature, as far as it is concerned, is nonpositive, with minimum $= -1$ ([10]). Notice that it may vanish, contrairly to the hyperbolic space case.

Now, we recall the principal techniques of harmonic analysis on these spaces. The commutativity of the convolution on bi-K-invariant objects on $G$ is basilar for the harmonic analysis on symmetric spaces $G/K$. If one replace bi-K-invariance by radiality, a similar phenomenum appears on general $S$. As established in [36], for the convolution on $S$:

$$(u * v)(x) = \int_S u(y)v(y^{-1}x)dy,$$

the radial integrable functions on $S$ form a commutative Banach algebra $L^1(S)^\sharp$. We note that for distribution, invariant differential operators, ... radiality is defined by means of an averaging operator over spheres, which can be written

$$f^\sharp(x') = \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}} \int_{\partial B(s)} d\sigma f(\rho\sigma)$$

in the ball model and generalizes $K$ averages on rank one symmetric spaces $G/K$. The algebra of invariant differential operators on $S$ which are radial is a polynomial algebra with a single generator, the Laplace-Beltrami operator $L$.

**Definition 3.4.2.** *We define a spherical function on $S$ as a radial eigenfunction $\varphi$ of $L$ (and thus automatically analytic), normalized by $\varphi(0) = 1$.*

The radial part (in geodesic polar coordinates) of the Laplace-Beltrami operator $L$ on $S$ writes

$$\mathrm{rad}L = \frac{\partial^2}{\partial s^2} + \left\{ \frac{m+k}{2} \coth \frac{s}{2} + \frac{k}{2} \tanh \frac{s}{2} \right\} \frac{\partial}{\partial s}. \tag{3.4.11}$$

By substituting $r = \frac{s}{2}$, 4rad$L$ becomes the Jacobi operator [71]

$$\mathrm{rad}L = \frac{\partial^2}{\partial r^2} + \{(2\alpha+1)\coth r + (2\beta+1)\tanh r\} \frac{\partial}{\partial r}, \tag{3.4.12}$$

with indices $\alpha = \frac{m+k+1}{2}$ and $\beta = \frac{k-1}{2}$, $\alpha > \beta > -\frac{1}{2}$. For every $\lambda \in \mathbb{C}$ there exists a unique radial $C^\infty$ function $\varphi_\lambda$ such that

$$L\varphi_\lambda = -(\lambda^2 + \rho^2)\varphi_\lambda \ \text{ and } \ \varphi(0) = 1. \tag{3.4.13}$$

Note that $\varphi_\lambda = \varphi_\mu$ if and only if $\lambda = \pm\mu$. Moreover

$$\varphi_\lambda(r) = {}_2F_1(\rho - i\lambda, \rho + i\lambda; \frac{n}{2}; -\sinh^2 \frac{r}{2}), \tag{3.4.14}$$

where ${}_2F_1$ is the hypergeometric function

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \tag{3.4.15}$$

with $(a)_0 = 1$, $(a)_k = a(a+1)\dot{(}a+k-1)$ if $k \geq 1$; the function ${}_2F_1$ is extended analytical to $\mathbb{C}\backslash[1, \infty]$.

For $Re(i\lambda) = -Im\lambda > 0$, we have the following asymptotic behaviour:

$$\varphi_\lambda(x) \sim c(\lambda) e^{i\lambda - \frac{Q}{2})r} \qquad \text{as } r = r(x) \to +\infty, \tag{3.4.16}$$

where $c(\lambda) = \frac{\Gamma(m+k)}{\Gamma(\frac{m+k}{2})} \frac{\Gamma(i2\lambda)}{\Gamma(i2\lambda+\frac{m}{2})} \frac{\Gamma(i\lambda+\frac{m}{4})}{\Gamma(i\lambda+\frac{m}{4}+\frac{k}{2})}$. Notice that spherical functions on $S$ are Jacobi functions:

$$\varphi_\lambda(r) = \phi_{2\lambda}^{(\alpha,\beta)}(\frac{r}{2}).$$

The spherical Fourier transform is defined by

$$\tilde{f}(\lambda) = \int_S dx \varphi_\lambda(x) f(x) = \frac{2^n \pi^{n/2}}{\Gamma(n/2)} \int_0^{+\infty} dr (\sinh \frac{r}{2})^{m+k} (\cosh \frac{r}{2})^k \varphi_\lambda(r) f(r), \tag{3.4.17}$$

for radial functions $f = f(x)$ on $S$, which we shall identify with functions $f = f(r)$ of the geodesic distance to the origin $r = d(x, 0) \in [0, +\infty)$. The spherical Fourier transform concides with the Jacobi transform:

$$\tilde{f}(\lambda) = 2^{2-k} \frac{\pi^{n/2}}{\Gamma(\frac{n}{2})} \widehat{f(2\cdot)}^{(\alpha,\beta)}(2\lambda).$$

### 3.4.2 Weighted Strichartz estimates for the Schrödinger equation on $S$

We obtain the following result.

**Theorem 3.4.1.** *Assume $n > 3$. Let $u_0$ and $F$ be two functions radial in $x \in S$, such that $w_2 u_0 \in L^2(S)$ and $w_{\tilde{q}'} F \in L^{\tilde{p}'}(\mathbb{R}; L^{\tilde{q}'}(S))$. Consider the Cauchy problem*

$$\begin{cases} i\partial_t u + L_{\alpha,\beta}\, u = F(t,x), \\ u(0,x) = u_0(r), \end{cases} \tag{3.4.18}$$

*then for all $\frac{n}{2}$-admissible couples $(p,q)$ and $(\tilde{p},\tilde{q})$, i.e. such that*

$$\frac{1}{p} + \frac{n}{2q} = \frac{n}{4}, \;\; p \in ]2, \infty], \text{and } q \in \left[2, \frac{2n}{n-2}\right[, \tag{3.4.19}$$

*the following weighted Strichartz estimates holds*

$$\|w_q u\|_{L^p(\mathbb{R}, L^q(S))} \leq C \|w_2\, u_0\|_{L^2(S)} + C \|w_{\tilde{q}'} F\|_{L^{\tilde{p}'}(\mathbb{R}, L^{\tilde{q}'}(S))}, \tag{3.4.20}$$

*with the weight*

$$w_q(r) = \left(\frac{\sinh r}{r}\right)^{\frac{(m+k)}{2}(1-\frac{2}{q})} (\cosh r)^{\frac{k}{2}(1-\frac{2}{q})}, \tag{3.4.21}$$

*and $\alpha = \frac{m+k-1}{2}, \quad \beta = \frac{k-1}{2}, \quad \alpha \geq \beta \geq -\frac{1}{2}$.*

*In the special case* $\alpha = \frac{1}{2}$, *the space* $S$ *is the three-dimensional real hyperbolic space* $\mathbb{H}^3(\mathbb{R})$, *the following weighted dispersive estimate holds*

$$\left(\frac{\sinh r}{r}\right)|u(t,x)| \le \frac{C}{t^{\frac{3}{2}}}\left\|u_0\left(\frac{r}{\sinh r}\right)\right\|_{L^1(\mathbb{H}^3(\mathbb{R}))}. \tag{3.4.22}$$

*Proof.* Let $L_{\alpha,\beta}$ be the Jacobi operator defined as

$$L_{\alpha,\beta} = \partial_r^2 + B(r)\partial_r + \rho^2, \tag{3.4.23}$$

where we have set

$$B(r) = (2\alpha+1)\coth r + (2\beta+1)\tanh r \tag{3.4.24}$$

and

$$\rho = (\alpha+\beta+1), \quad \alpha = \frac{m+k-1}{2}, \quad \beta = \frac{k-1}{2}, \quad \alpha \ge \beta \ge -\frac{1}{2}. \tag{3.4.25}$$

Notice that (3.4.23) includes the radial part of the Laplace-Beltrami operator on hyperbolic spaces and more generally on Damek-Ricci spaces $S$ defined above. Recall that the radial part of the Laplace operator in $\mathbb{R}^n$ is

$$\Delta = \partial_r^2 + \frac{n-1}{r}\partial_r.$$

The idea of the proof is to construct a transformation which maps the Jacobi operator on $S$ into the radial part of the Laplace operator defined on $\mathbb{R}^n$ by imposing the following

$$u(t,r) = \sigma(r)v(t,r). \tag{3.4.26}$$

We have then

$$\begin{aligned} &L_{\alpha,\beta}\, u(t,r) = \partial_r^2 u(t,r) + B(r)\partial_r u(t,r) + \rho^2 u(t,r) = \\ &\partial_r^2(\sigma(r)v(t,r)) + B(r)\partial_r(\sigma(r)v(t,r)) + \rho^2\sigma(r)v(t,r) = \\ &\sigma(r)\left[\partial_r^2 v(t,r) + \left(2\frac{\sigma'(r)}{\sigma(r)} + B(r)\right)\partial_r v(t,r) + \left(\frac{\sigma''(r)}{\sigma(r)} + B(r)\frac{\sigma'(r)}{\sigma(r)} + \rho^2\right)v(t,r)\right]. \end{aligned} \tag{3.4.27}$$

The crucial point is imposing the following condition

$$2\frac{\sigma'(r)}{\sigma(r)} + B(r) = \frac{2\alpha+1}{r}, \tag{3.4.28}$$

and solving this differential ordinary equation we obtain

$$\sigma(r) = r^{\alpha+\frac{1}{2}}(\sinh r)^{-(\alpha+\frac{1}{2})}(\cosh r)^{-(\beta+\frac{1}{2})}. \tag{3.4.29}$$

Replacing (3.4.29) in the coefficient of $v(t,r)$ in (3.4.27), after some computations, we obtain the potential

$$V(r) = \left(\alpha^2 - \frac{1}{4}\right)\frac{1}{r^2} - \frac{B'(r)}{2} - \frac{B^2(r)}{4} + \rho^2.$$

Thus, using formula (3.4.24) we achieve

$$V(r) = \left(\alpha^2 - \frac{1}{4}\right)\frac{1}{r^2} - \left(\alpha^2 - \frac{1}{4}\right)\coth^2 r - \left(\beta^2 - \frac{1}{4}\right)\tanh^2 r + \left(\alpha^2 + \beta^2 - \frac{1}{2}\right). \tag{3.4.30}$$

Notice that $V \in \mathcal{C}^\infty[0,\infty)$ and it tends to zero as $r \to \infty$. As a result, we have obtained the perturbed Schrödinger equation

$$i\partial_t v + \Delta v - \widetilde{V} v = 0 \tag{3.4.31}$$

on $\mathbb{R}^{2\alpha+2}$, where $\widetilde{V} = -V$. Now, we aim to study the behavior of $\widetilde{V}$. It is not difficult to check that our potential satisfies the inequality

$$\widetilde{V}(r) > -\frac{a}{r^2}, \tag{3.4.32}$$

where $a = \frac{(n-2)^2}{4}$. This allows us to apply the result of Burq, Planchon, Stalker and Tahvildar-Zadeh (see [19]), where they prove Strichartz estimates for the Schrödinger and wave equations perturbed with the potential satisfying inequality (3.4.32). Thus, if we consider the Cauchy problem

$$\begin{cases} i\partial_t v + \Delta v - \widetilde{V} v = \frac{F(t,r)}{\sigma(r)}, \\ v(0,x) = v_0, \end{cases} \tag{3.4.33}$$

with radial initial data, we obtain the following Strichartz estimates

$$\|v\|_{L^p(\mathbb{R},L^q(\mathbb{R}^{2\alpha+2}))} \leq C\|v_0\|_{L^2(\mathbb{R}^{2\alpha+2})} + C\left\|\frac{F}{\sigma}\right\|_{L^{\tilde{p}'}(\mathbb{R},L^{\tilde{q}'}(\mathbb{R}^{2\alpha+2}))} \tag{3.4.34}$$

If we put (3.4.26) we obtain the following inequality

$$\left\|\frac{u}{\sigma}\right\|_{L^p(\mathbb{R},L^q(\mathbb{R}^{2\alpha+2}))} \leq C\left\|\frac{u_0}{\sigma}\right\|_{L^2(\mathbb{R}^{2\alpha+2})} + C\left\|\frac{F}{\sigma}\right\|_{L^{\tilde{p}'}(\mathbb{R},L^{\tilde{q}'}(\mathbb{R}^{2\alpha+2}))}. \tag{3.4.35}$$

Writing explicitely the left hand one has

$$\left\|\frac{u}{\sigma}\right\|_{L^p(\mathbb{R},L^q(\mathbb{R}^{2\alpha+2}))} = \left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}^{2\alpha+2}} |u(t,x)\sigma(x)^{-1}|^q dx\right)^{\frac{p}{q}} dt\right)^{\frac{1}{p}} =$$

replacing (3.4.29) into the weight $\sigma$ in polar coordinates we have

$$= \left(\int_{\mathbb{R}}\left(\int_{\mathbb{S}^{n-1}}\int_{\mathbb{R}}\left|u(t,r,\omega)\left(\frac{\sinh r}{r}\right)^{\alpha+\frac{1}{2}}(\cosh r)^{\beta+\frac{1}{2}}\right|^q r^{n-1} dr d\omega\right)^{\frac{p}{q}} dt\right)^{\frac{1}{p}},$$

where $\alpha = \frac{m+k-1}{2}$, $\quad \beta = \frac{k-1}{2}$, $\quad \alpha \geq \beta \geq -\frac{1}{2}$; for simplicity let us consider only the radial component

$$= \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \left| u(t,r) \left( \frac{\sinh r}{r} \right)^{\frac{m+k}{2}(1-\frac{2}{q})} (\cosh r)^{\frac{k}{2}(1-\frac{2}{q})} \right|^q \sinh r^{m+k} \cosh r^k dr \right)^{\frac{p}{q}} dt \right)^{\frac{1}{p}},$$

since on the Damek-Ricci spaces $S$ the Riemannian volume is

$$dV = 2^{m+k} \sinh r^{m+k} \cosh^k dr d\widetilde{\omega},$$

where $d\widetilde{\omega}$ denotes the surface measure on the unit sphere $\partial B(\mathbf{s})$ in $\mathbf{s}$ and $n = dimS = m + k + 1$, we obtain

$$= C \left( \int_{\mathbb{R}} \left( \int_{\mathbb{S}} \left| u(t,r) \left( \frac{\sinh r}{r} \right)^{\frac{m+k}{2}(1-\frac{2}{q})} (\cosh r)^{\frac{k}{2}(1-\frac{2}{q})} \right|^q dr \right)^{\frac{p}{q}} dt \right)^{\frac{1}{p}},$$

thus denoting $w_q(r)$ our weight $\left(\frac{\sinh r}{r}\right)^{\frac{m+k}{2}(1-\frac{2}{q})} (\cosh r)^{\frac{k}{2}(1-\frac{2}{q})}$ we have

$$= C \|w_q\, u\|_{L^p(\mathbb{R},L^q(S))}.$$

In an analogous way, writing explicitly the right hand side of (3.4.35), by similar computations we conclude the proof of all weighted Strichartz estimates in Theorem 3.4.1.

In the special case $\alpha = \frac{1}{2}$ our Damek-Ricci space is the real hyperbolic space of dimension three $H^3(\mathbb{R})$ when $\beta = -\frac{1}{2}$. In this case $m = 2$ and $k = 0$. To prove the weighted dispersive estimate (3.4.42) we proceed as above; we notice that after our transformation (3.4.26) the potential (3.4.30) becomes

$$V(r) = 0,$$

thus we obtain a linear Cauchy problem

$$\begin{cases} i\partial_t v + \Delta v = 0, \\ v(0,x) = v_0, \end{cases} \tag{3.4.36}$$

which satisfies the dispersive estimate

$$\|v(t)\|_{L^\infty(\mathbb{R}^3)} \leq \frac{C}{t^{3/2}} \|v_0\|_{L^1(\mathbb{R}^3)}.$$

Using the inverse transformation and computing as before we prove the following

$$\left( \frac{\sinh r}{r} \right) |u(t,x)| \leq \frac{C}{t^{\frac{3}{2}}} \left\| \frac{r}{\sinh r} u_0 \right\|_{L^1(\mathbb{H}^3(\mathbb{R}))},$$

and this concludes the proof of Theorem 3.4.1. □

### 3.4.3 Weighted Strichartz estimates for the Wave equation on $S$

**Theorem 3.4.2.** *Assume $n > 3$. Let $u_0$ and $F$ be two functions radial in $x \in S$, such that $\frac{u_0}{\sigma} \in H^{\gamma}(S)$, $\frac{u_1}{\sigma} \in H^{\gamma-1}(S)$ and $w_{\tilde{q}'}F \in L^{\tilde{p}'}(\mathbb{R}; L^{\tilde{q}'}(S))$. Consider the Cauchy problem*

$$\begin{cases} -\partial_t^2 u + L_{\alpha,\beta}\, u = F(t,x), \\ u(0,x) = u_0(r), \\ u_t(0,x) = u_1(r), \end{cases} \tag{3.4.37}$$

*then for all $\frac{n-1}{2}$-admissible couples $(p,q)$ and $(\tilde{p},\tilde{q})$, i.e. such that*

$$\frac{1}{p} + \frac{n-1}{2q} \leq \frac{n-1}{4}, \quad p \in ]2,\infty], \text{and } q \in \left[2, \frac{2(n-1)}{n-3}\right[, \tag{3.4.38}$$

*the following weighted Strichartz estimates holds*

$$\|w_q u\|_{L^p(\mathbb{R},L^q(S))} \leq C \left\|\frac{u_0}{\sigma}\right\|_{H^{\gamma}(S)} + \left\|\frac{u_1}{\sigma}\right\|_{H^{\gamma-1}(S)} + C\|w_{\tilde{q}'}F\|_{L^{\tilde{p}'}(\mathbb{R},L^{\tilde{q}'}(S))}, \tag{3.4.39}$$

*with the weights*

$$w_q(r) = \left(\frac{\sinh r}{r}\right)^{\frac{(m+k)}{2}(1-\frac{2}{q})} (\cosh r)^{\frac{k}{2}(1-\frac{2}{q})}, \tag{3.4.40}$$

*and*

$$\sigma(r) = r^{\alpha+\frac{1}{2}} (\sinh r)^{-(\alpha+\frac{1}{2})} (\cosh r)^{-(\beta+\frac{1}{2})}. \tag{3.4.41}$$

*In the special case $\alpha = \frac{1}{2}$, the space $S$ is the three-dimensional real hyperbolic space $\mathbb{H}^3(\mathbb{R})$, the following weighted dispersive estimate holds*

$$\left(\frac{\sinh r}{r}\right) |u(t,x)| \leq \frac{C}{t} \left\|\frac{r}{\sinh r} u_1\right\|_{B^{1,1}(\mathbb{H}^3(\mathbb{R}))}. \tag{3.4.42}$$

The proof is based again on the change of variables (3.4.26), (3.4.29) which reduces Jacobi operator to a standard Laplace operator perturbed with a potential. Since the result of [19] are valid also for the wave equation, we can proceed exactly as in the proof of Theorem 3.4.1.

# Chapter 4

# Nonlinear Schrödinger equations on compact manifolds with positive curvature

## 4.1 Introduction

We have seen that, on a manifold, negative curvature has the effect of improving the dispersive properties of evolution equations. In this chapter we examine a model situation when the curvature is positive, by studying some nonlinear Schrödinger equations on the four dimensional sphere $S^4$; we also consider the more general case of compact four-dimensional manifolds. In contrast with the negative curvature case, the positive curvature tends to destroy the decay properties of the equation, and in general the results both from the point of view of decay and regularity are worse than in the flat case.

In particular, the situation for compact manifolds has been investigated in a recent series of papers ( [22], [24], [25], see also [26], [46]) by Burq-Gérard-Tzvetkov. They studied the Cauchy problem for nonlinear Schrödinger equations (NLS) on Riemannian compact manifolds, generalizing the work of Bourgain on tori ([14], [15]). In [22], Strichartz estimates with fractional loss of derivatives were established for the Schrödinger group. They led to global wellposedness of NLS on surfaces with any defocusing polynomial nonlinearity. On three-manifolds, these estimates also provided global existence and uniqueness for cubic defocusing NLS, but they failed to prove the Lipschitz continuity of the flow map on the energy space. These results were improved in [24], [25] for specific manifolds such as spheres, taking advantage of new multilinear Strichartz inequalities for the Schrödinger group (see also [23]). In particular, on such three-manifolds the Lipschitz continuity and the smoothness of the flow map on the energy space were established for cubic NLS, as well as global existence on the energy space for every defocusing subquintic NLS.

However, none of the above methods provided global wellposedness results in the energy space for NLS on four-dimensional manifolds. This is in strong contrast with the Euclidean case (see [50], [65], [27]). The only available global existence result on a compact four-manifold seems to be the one of Bourgain in [15], which concerns defocusing nonlinearities of the type $|u|u$ and Cauchy data in $H^2(\mathbb{T}^4)$. Let us discuss briefly the reasons of this difficulty. On the one hand, Strichartz estimates of [22] involve a too large loss of derivative in four space dimension ; typically, for cubic NLS, they lead to local wellposedness in $H^s$ for $s > 3/2$, which is not sufficient in view of the energy and $L^2$ conservation laws. Moreover, these estimates are restricted to $L_t^p L_x^q$ norms with $p \geq 2$ and the admissibility condition

$$\frac{1}{p} + \frac{2}{q} = 1 \ ,$$

so that the analysis does not improve when the nonlinearity becomes subcubic. On the other hand, the analysis based on bilinear Strichartz estimates is currently restricted to nonlinearities of cubic type, and on $S^4$ it only yields local wellposedness in $H^s$ for $s > 1$. In fact, this obstruction can be made more precise by combining two results from [22] and [24]. Indeed, from Theorem 4 in [22], we know that the estimate

$$\int_0^{2\pi} \int_{S^4} |e^{it\Delta} f\,(x)|^4 \, dt \, dx \lesssim \|f\|^4_{H^{1/2}(S^4)}$$

is wrong, which, by Remark 2.12 in [24], implies that the flow map of cubic NLS cannot be $C^3$ near the Cauchy data $u_0 = 0$ in $H^1(S^4)$.

The goal of this section is to provide further results on four-dimensional manifolds. We shall study two types of NLS equations. In section 4.2.1, we study NLS with the following nonlocal nonlinearity,

$$\left\{ \begin{array}{l} i\partial_t u + \Delta u = \left((1-\Delta)^{-\alpha}|u|^2\right) u, \\ u(0,x) = u_0(x) \end{array} \right. \tag{4.1.1}$$

where $\alpha > 0$. Notice that the homogeneous version of this nonlinearity on the Euclidean space $\mathbb{R}^d$ reads

$$\left( \frac{1}{|x|^{d-2\alpha}} * |u|^2 \right) u$$

so that (4.1.1) can be seen as a variant of Hartree's equation on a compact manifold. Combining the conservation laws for (4.1.1) with suitable bilinear estimates, we obtain the following result.

**Theorem 4.1.1.** *Let $(M,g)$ be a compact Riemannian manifold of dimension 4 and let $\alpha > \frac{1}{2}$. There exists a subspace $X$ of $\mathcal{C}(\mathbb{R}, H^1(M))$ such that, for every $u_0 \in H^1(M)$, the Cauchy problem (4.1.1) has a unique global solution $u \in X$. Moreover, in the special case $M$ is the four-dimensional standard sphere $M = \mathbb{S}^4$, the same result holds for all values $\alpha > 0$ of the parameter.*

The proof of Theorem 4.1.1 relies on the following quadrilinear estimates

$$\sup_{\tau\in\mathbb{R}} \left| \int_{\mathbb{R}} \int_M \chi(t)\, e^{it\tau} (1-\Delta)^{-\alpha} (u_1 \overline{u}_2) u_3 \overline{u}_4 dx dt \right| \\ \leq C(m(N_1, \cdots, N_4))^{s_0} \|f_1\|_{L^2(M)} \|f_2\|_{L^2(M)} \|f_3\|_{L^2(M)} \|f_4\|_{L^2(M)},$$

for every $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$, for every $s_0 < 1$ and for $f_1, f_2, f_3, f_4$ satisfying

$$\mathbf{1}_{\sqrt{1-\Delta}\in[N_j,2N_j]}(f_j) = f_j, \;\; j = 1,2,3,4.$$

Here and in the sequel $m(N_1, \cdots, N_4)$ denotes the product of the smallest two numbers among $N_1, N_2, N_3, N_4$. Moreover $u_j$ and $f_j$ are linked by

$$u_j(t,x) = S(t) f_j(x), \; j = 1,2,3,4,$$

where $S(t) = e^{it\Delta}$. Notice that, compared to the multilinear estimates used in [25], a frequency variable $\tau$ is added in the equation. It would be interesting to know if the smallest value of $\alpha$ for which these estimates (and hence Theorem 4.1.1) are valid depends or not on the geometry of $M$.

In Section 4.2.2, we come back to power nonlinearities. Since we want to go below the cubic powers and at the same time we want to use multilinear estimates, we are led to deal with quadratic nonlinearities. In other words, we study the following equations,

$$i\partial_t u + \Delta u = q(u), \tag{4.1.2}$$

where $q(u)$ is a homogeneous quadratic polynomial in $u, \overline{u}$

$$q(u) = au^2 + b\overline{u}^2 + c|u|^2.$$

Notice that a subclass of these equations consists of Hamiltonian equations

$$q(u) = \frac{\partial V}{\partial \overline{u}}$$

where $V$ is a real-valued homogeneous polynomial of degree 3 in $u, \overline{u}$; with the above notation, this corresponds to $c = 2\overline{a}$. In this case, the following energy is conserved,

$$E = \int_M |\nabla u|^2 + V(u)\, dx \; .$$

A typical example is

$$V(u) = \frac{1}{2}|u|^2(u+\overline{u}) \ , \ q(u) = |u|^2 + \frac{1}{2}u^2 \ .$$

Notice that this Hamiltonian structure does not prevent from blow up in general. In the above example, a purely imaginary constant as Cauchy data leads to a blow up solution ! Therefore we can only hope for local-in-time existence. Our results are the following.

**Theorem 4.1.2.** *If $(M,g)$ is the four-dimensional standard sphere , then the Cauchy problem (4.1.2) is (locally in time) uniformly well-posed in $H^s_{\mathrm{zonal}}(S^4)$ for every $s > \frac{1}{2}$, where $H^s_{\mathrm{zonal}}(S^4)$ denotes the $H^s$ space of zonal functions relative to some pole $\omega \in S^4$ : $f(x) = \tilde{f}(\langle x, \omega\rangle)$ .*

The main tool in the proof of Theorem 4.1.2 is the following trilinear estimate on linear solutions $u_j(t) = S(t)f_j$,

$$\begin{aligned} &\sup_{\tau\in\mathbb{R}}\left|\int_{\mathbb{R}}\int_{\mathbb{S}^4}\chi(t)\, e^{it\tau}\mathcal{T}(\ u_1(t,x),\ u_2(t,x),\ u_3(t,x))\ dx\, dt\right| \\ &\qquad \le C\,(\min(N_1,N_2,N_3))^{s_0}\|f_1\|_{L^2(\mathbb{S}^4)}\|f_2\|_{L^2(\mathbb{S}^4)}\|f_3\|_{L^2(\mathbb{S}^4)}, \end{aligned} \tag{4.1.3}$$

for every $\mathbb{R}$-trilinear expression $\mathcal{T}$ on $\mathbb{C}^3$, for every $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$, for every $s_0 > 1/2$ and for zonal functions $f_1, f_2, f_3$ satisfying

$$\mathbf{1}_{\sqrt{1-\Delta}\in[N_j,2N_j]}(f_j) = f_j, \ \ j = 1,2,3 \ .$$

It would be interesting to know whether the above estimate holds with non zonal functions for some $s_0 < 1$ ; this would extend the above theorem to any finite energy Cauchy data.

Moreover we give a classification for all the Hamiltonian quadratic nonlinearities for which the Cauchy problem associated to (4.1.2) has a unique global solution for suitable small initial data in $H^1_{\mathrm{zonal}}(S^4)$.

**Corollary 4.1.1.** *Assume $(M,g)$ is the four-dimensional standard sphere and $c = 2\overline{a}$. Then the following assertions are equivalent.*
*i) There exists a subspace $X$ of $\mathcal{C}(\mathbb{R}, H^1_{\mathrm{zonal}}(S^4))$ such that, for every small initial data $\|u_0\|_{H^1_{\mathrm{zonal}}(S^4)} \le \varepsilon$, the Cauchy problem (4.1.2) has a unique global solution $u \in X$.*
*ii) The parameters $a, b$ satisfy*

$$\frac{\overline{a}^2}{a} = b. \tag{4.1.4}$$

It would be interesting to know whether blowing up solutions exist for non small data under property (4.1.4).

When property (4.1.4) is not satisfied, our blowing up solutions are particularly simple, since they are solutions of the ordinary differential equation deduced from (4.1.2) for space-independent solutions. Another open problem is of course to find a wider variety of blowing up solutions for equation (4.1.2) in this case.

## 4.2 Wellposedness via multilinear estimates

The main step of this section is to prove a result of local existence in time for initial data in $H^1(M)$ using some multilinear estimates associated to the nonlinear Schrödinger equation, that we will establish in Section 4.2.2 with a special attention to the case of the sphere. For that purpose we follow closely the ideas of Burq, Gérard and Tzvetkov ([26], [24]). In those papers, the authors extended to general compact manifolds the nonlinear methods introduced by Bourgain ([14], [15], [17]) in the context of tori $\mathbb{R}^d/\mathbb{Z}^d$. Finally, we achieve the global wellposedness thanks to the conservation laws.

### 4.2.1 Well-posedness in Sobolev spaces for the Hartree nonlinearity

In this subsection we prove that the uniform wellposedness of (4.1.1) on $M$ can be deduced from quadrilinear estimates on solutions of the linear equation. Firstly, we recall the notion of wellposedness we are going to address.

**Definition 4.2.1.** Let $s \in \mathbb{R}$. We shall say that the nonlinear Schrödinger equation (4.1.1) is (locally in time) uniformly well-posed on $H^s(M)$ if, for any bounded subset $B$ of $H^s(M)$, there exists $T > 0$ and a Banach space $X_T$ continuously contained into $C([-T,T], H^s(M))$, such that

**i** For every Cauchy data $u_0 \in B$, (4.1.1) has a unique solution $u \in X_T$.

**ii** If $u_0 \in H^\sigma(M)$ for $\sigma > s$, then $u \in C([-T,T], H^\sigma(M))$.

**iii** The map $u_0 \in B \mapsto u \in X_T$ is uniformly continuous.

The following theorem stresses the general relationship between uniform wellposedness for equation (4.1.1) and a certain type of quadrilinear estimates.

**Theorem 4.2.1.** *Suppose that there exists $C > 0$ and $s_0 \geq 0$ such that for any $f_1, f_2, f_3, f_4 \in L^2(M)$ satisfying*

$$\mathbf{1}_{\sqrt{1-\Delta} \in [N_j, 2N_j]}(f_j) = f_j, \quad j = 1, 2, 3, 4, \tag{4.2.1}$$

*one has the following quadrilinear estimates*

$$\begin{aligned}
&\sup_{\tau\in\mathbb{R}}\left|\int_{\mathbb{R}}\int_M \chi(t)\, e^{it\tau}(1-\Delta)^{-\alpha}(u_1\overline{u}_2)u_3\overline{u}_4 dxdt\right| \\
&\leq C(m(N_1,\cdots,N_4))^{s_0}\|f_1\|_{L^2(M)}\|f_2\|_{L^2(M)}\|f_3\|_{L^2(M)}\|f_4\|_{L^2(M)}, \\
&u_j(t)=S(t)f_j,\ \ j=1,2,3,4,
\end{aligned} \tag{4.2.2}$$

*where $\chi\in\mathcal{C}_0^\infty(\mathbb{R})$ is arbitrary, and $m(N_1,\cdots,N_4)$ denotes the product of the smallest two numbers among $N_1,N_2,N_3,N_4$. Then the Cauchy problem (4.1.1) is uniformly well-posed in $H^s(M)$ for any $s>s_0$.*

*Proof.* The proof follows essentially the same lines as the one of Theorem 3 in [24] and relies on the use of a suitable class $X^{s,b}$ of Bourgain-type spaces. We shall sketch it for the commodity of the reader. We first show that (4.2.2) is equivalent to a quadrilinear estimate in the spaces $X^{s,b}$. We then prove the crucial nonlinear estimate, from which uniform wellposedness can be obtained by a contraction argument in $X_T^{s,b}$. Since this space is continuously embedded in $C([-T,T],H^s(M))$ provided $b>\frac{1}{2}$, this concludes the proof of the local well posedness result.

Following the definition in Bourgain [14] and Burq, Gérard and Tzvetkov [26], we introduce the family of Hilbert spaces

$$X^{s,b}(\mathbb{R}\times M)=\{v\in\mathcal{S}'(\mathbb{R}\times M):(1+|i\partial_t+\Delta|^2)^{\frac{b}{2}}(1-\Delta)^{\frac{s}{2}}v\in L^2(\mathbb{R}\times M)\} \tag{4.2.3}$$

for $s,b\in\mathbb{R}$. More precisely, with the notation

$$\langle x\rangle=\sqrt{1+|x|^2}\ ,$$

we have the following definition :

**Definition 4.2.2.** Let $(M,g)$ be a compact Riemannian manifold, and consider the Laplace operator $-\Delta$ on $M$. Denote by $(e_k)$ an $L^2$ orthonormal basis of eigenfunctions of $-\Delta$, with eigenvalues $\mu_k$, by $\Pi_k$ the orthogonal projector along $e_k$, and for $s\geq 0$ by $H^s(M)$ the natural Sobolev space generated by $(I-\Delta)^{\frac{1}{2}}$, equipped with the following norm

$$\|u\|^2_{H^s(M)}=\sum_k\langle\mu_k\rangle^s\|\Pi_k u\|^2_{L^2(M)}. \tag{4.2.4}$$

Then, the space $X^{s,b}(\mathbb{R}\times M)$ is defined as the completion of $C_0^\infty(\mathbb{R}_t;H^s(M))$ for the norm

$$\begin{aligned}
&\|u\|^2_{X^{s,b}(\mathbb{R}\times M)}=\sum_k\|\langle\tau+\mu_k\rangle^b\langle\mu_k\rangle^{\frac{s}{2}}\widehat{\Pi_k u}(\tau)\|^2_{L^2(\mathbb{R}_\tau;L^2(M))} \\
&=\|S(-t)\,u(t,\cdot)\|^2_{H^b(\mathbb{R}_t;H^s(M))},
\end{aligned} \tag{4.2.5}$$

where $\widehat{\Pi_k u}(\tau)$ denotes the Fourier transform of $\Pi_k u$ with respect to the time variable.

Denoting by $X_T^{s,b}$ the space of restrictions of elements of $X^{s,b}(\mathbb{R}\times M)$ to $]-T,T[\times M$, it is easy to prove the embedding

$$\forall b>\frac{1}{2},\quad X_T^{s,b}\subset C([-T,T],H^s(M)). \tag{4.2.6}$$

Moreover, we have the elementary property

$$\forall f\in H^s(M),\quad \forall b>0,\quad (t,x)\mapsto S(t)f(x)\in X_T^{s,b}. \tag{4.2.7}$$

We next reformulate the quadrilinear estimates (4.2.2) in the context of $X^{s,b}$ spaces.

**Lemma 4.2.1.** *Let $s\in\mathbb{R}$. The following two statements are equivalent:*
*i) For any $f_j\in L^2(M)$, $j=1,2,3,4$, satisfying* (4.2.1)*, estimate (4.2.2) holds;*
*ii) For any $b>\frac{1}{2}$ and any $u_j\in X^{0,b}(\mathbb{R}\times M)$, $j=1,2,3,4$, satisfying*

$$\mathbf{1}_{\sqrt{1-\Delta}\in[N_j,2N_j]}(u_j)=u_j,$$

*one has*

$$\left|\int_{\mathbb{R}}\int_M(1-\Delta)^{-\alpha}(u_1\overline{u}_2)u_3\overline{u}_4dxdt\right|\leq C(m(N_1,\cdots,N_4))^{s_0}\prod_{j=1}^4\|u_j\|_{X^{0,b}(\mathbb{R}\times M)}. \tag{4.2.8}$$

*Proof.* We sketch only the essential steps of the proof of ii) assuming i), since we follow closely the argument of Lemma 2.3 in [26]. The reverse implication is easier and will not be used in this paper.

Suppose first that $u_j$ are supported in time in the interval $(0,1)$ and we select $\chi\in\mathcal{C}_0^\infty(\mathbb{R})$ such that $\chi=1$ on $[0,1]$; then writing $u_j^\sharp(t)=S(-t)u_j(t)$ we have easily

$$\begin{aligned}\big((1-\Delta)^{-\alpha}(u_1\overline{u}_2)u_3\overline{u}_4\big)(t)&=\frac{1}{(2\pi)^4}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}e^{it(\tau_1-\tau_2+\tau_3-\tau_4)}\\ \times(1-\Delta)^{-\alpha}&(S(t)\widehat{u}_1^\sharp(\tau_1)\overline{S(t)\widehat{u}_2^\sharp(\tau_2)})S(t)\widehat{u}_3^\sharp(\tau_3)\overline{S(t)\widehat{u}_4^\sharp(\tau_4)}\,d\tau_1\,d\tau_2\,d\tau_3\,d\tau_4,\end{aligned}$$

where $\widehat{u}_j^\sharp$ denotes the Fourier transform of $u_j^\sharp$ with respect to time. Using i) and the Cauchy-Schwarz inequality in $(\tau_1,\tau_2,\tau_3,\tau_4)$ (here the assumption $b>\frac{1}{2}$ is used, in order to get the necessary integrability) yields

$$\begin{aligned}\left|\int_{\mathbb{R}\times M}(1-\Delta)^{-\alpha}(u_1\overline{u}_2)u_3\overline{u}_4dxdt\right|&\lesssim m(N_1,\cdots,N_4)^{s_0}\prod_{j=1}^4\|\langle\tau\rangle^b\widehat{u}_j^\sharp\|_{L^2(\mathbb{R}\times M)}\\ &\lesssim m(N_1,\cdots,N_4)^{s_0}\prod_{j=1}^4\|u_j\|_{X^{0,b}(\mathbb{R}\times M)}\ .\end{aligned}$$

Finally, by decomposing $u_j(t) = \sum_{n\in\mathbb{Z}} \psi(t-\frac{n}{2})u_j(t)$ with a suitable $\psi \in C_0^\infty(\mathbb{R})$ supported in $(0,1)$, the general case for $u_j$ follows from the special case of $u_j$ supported in the time interval $(0,1)$. □

Returning to the proof of Theorem 4.2.1, there is another way of estimating the $L^1$ norm of the product $((1-\Delta)^{-\alpha}(u_1\overline{u}_2)u_3\overline{u}_4)$.

**Lemma 4.2.2.** *Assume $\alpha$ as in Theorem 1 and that $u_1, u_2, u_3, u_4$ satisfy*

$$\mathbf{1}_{\sqrt{1-\Delta}\in[N,2N]}(u_j) = u_j. \tag{4.2.9}$$

*Then, for every $s' > s_0$ there exists $b' \in ]0, \frac{1}{2}[$ such that*

$$\left| \int_{\mathbb{R}} \int_M (1-\Delta)^{-\alpha}(u_1\overline{u}_2)u_3\overline{u}_4 dxdt \right| \leq Cm(N_1, \cdots, N_4)^{s'} \prod_{j=1}^4 \|u_j\|_{X^{0,b'}} . \tag{4.2.10}$$

*Proof.* We split the proof in several steps.

First of all we prove that, for $\alpha > 0$,

$$\left| \int_{\mathbb{R}} \int_M (1-\Delta)^{-\alpha}(u_1\overline{u}_2)u_3\overline{u}_4 dxdt \right| \leq Cm(N_1, \cdots, N_4)^2 \prod_{j=1}^4 \|u_j\|_{X^{0,1/4}}. \tag{4.2.11}$$

By symmetry we have to consider the following three cases:

$$m(N_1,\cdots,N_4) = N_1N_2 \;, m(N_1,\cdots,N_4) = N_3N_4 \;,\; m(N_1,\cdots,N_4) = N_1N_3.$$

In the first case, by a repeated use of Hölder's inequality, we obtain

$$\begin{aligned}
&\left| \int_{\mathbb{R}} \int_M (1-\Delta)^{-\alpha}(u_1\overline{u}_2)u_3\overline{u}_4 dxdt \right| \\
&\leq C\|(1-\Delta)^{-\alpha}(u_1\overline{u}_2)\|_{L^2(\mathbb{R},L^\infty(M))}\|u_3\overline{u}_4\|_{L^2(\mathbb{R},L^1(M))}, \\
&\leq C\|u_1\overline{u}_2\|_{L^2(\mathbb{R},L^\infty(M))}\|u_3\overline{u}_4\|_{L^2(\mathbb{R},L^1(M))} \\
&\leq C\|u_1\|_{L^4(\mathbb{R},L^\infty(M))}\|u_2\|_{L^4(\mathbb{R},L^\infty(M))}\|u_3\|_{L^4(\mathbb{R},L^2(M))}\|u_4\|_{L^4(\mathbb{R},L^2(M))},
\end{aligned}$$

where we also used that $(1-\Delta)^{-\alpha}$ is a pseudodifferential operator of negative order, hence acts on $L^\infty(M)$. By Sobolev inequality, we infer

$$\left| \int_{\mathbb{R}} \int_M (1-\Delta)^{-\alpha}(u_1\overline{u}_2)u_3\overline{u}_4 dxdt \right| \leq C(N_1N_2)^2 \prod_{j=1}^4 \|u_j\|_{L^4(\mathbb{R},L^2(M))} .$$

By the Sobolev embedding in the time variable for the function $v(t) = S(-t)u(t)$, we have $X^{0,1/4} \subset L^4(\mathbb{R}, L^2(M))$, and this conclude the proof of the first case.

In the second case $m(N_1,\cdots,N_4)=N_3N_4$ we can proceed in the same way by writing the integral in the form

$$\left|\int_{\mathbb{R}}\int_M u_1\overline{u}_2(1-\Delta)^{-\alpha}(u_3\overline{u}_4)dxdt\right|.$$

Finally, when $m(N_1,\cdots,N_4)=N_1N_3$, we write the integral as follows

$$\left|\int_{\mathbb{R}}\int_M (1-\Delta)^{-\frac{\alpha}{2}}(u_1\overline{u}_2)(1-\Delta)^{-\frac{\alpha}{2}}(u_3\overline{u}_4)dxdt\right|,$$

and by Cauchy-Schwarz and Hölder's inequalities we estimate it by

$$\begin{aligned}
&\leq \|(1-\Delta)^{-\frac{\alpha}{2}}(u_1\overline{u}_2)\|_{L^2(\mathbb{R},L^2(M))}\|(1-\Delta)^{-\frac{\alpha}{2}}(u_3\overline{u}_4)\|_{L^2(\mathbb{R},L^2(M))}\\
&\leq C\|u_1\overline{u}_2\|_{L^2(\mathbb{R},L^2(M))}\|u_3\overline{u}_4\|_{L^2(\mathbb{R},L^2(M))}\\
&\leq C\|u_1\|_{L^4(\mathbb{R},L^\infty(M))}\|u_2\|_{L^4(\mathbb{R},L^2(M))}\|u_3\|_{L^4(\mathbb{R},L^\infty(M))}\|u_4\|_{L^4(\mathbb{R},L^2(M))}\ .
\end{aligned}$$

Finally we conclude the proof of (4.2.11) by means of Sobolev's inequality in both space and time variables as above.

The second step consists in interpolating between (4.2.8) and (4.2.11) in order to get the estimate (4.2.10). To this end we decompose each $u_j$ as follows

$$u_j=\sum_{K_j}u_{j,K_j},\qquad u_{j,K_j}=\mathbf{1}_{K_j\leq\langle i\partial_t+\Delta\rangle<2K_j}(u_j),$$

where $K_j$ denotes the sequence of dyadic integers. Notice that

$$\|u_j\|^2_{X^{0,b}}\simeq\sum_{K_j}K_j^{2b}\|u_{j,K_j}\|^2_{L^2(\mathbb{R}\times M)}\simeq\sum_{K_j}\|u_{j,K_j}\|^2_{X^{0,b}}.$$

We then write the integral in the left hand side of (4.2.10) as a sum of the following elementary integrals,

$$I(K_1,\cdots,K_4)=\int_{\mathbb{R}}\int_M(1-\Delta)^{-\alpha}(u_{1,K_1}\overline{u}_{2,K_2})u_{3,K_3}\overline{u}_{4,K_4}dxdt\ .$$

Using successively (4.2.8) and (4.2.11), we estimate these integrals as

$$|I(K_1,\cdots,K_4)|\leq Cm(N_1,\cdots,N_4)^\sigma\sum_{K_1,K_2,K_3}(K_1K_2K_3K_4)^\beta\prod_{j=1}^4\|u_{j,K_j}\|_{L^2},\tag{4.2.12}$$

where either $(\sigma,\beta)=(s_0,b)$ for every $b>1/2$, or $(\sigma,\beta)=(2,1/4)$. Therefore, for every $s'>s_0$, there exists $b_1<1/2$ such that (4.2.12) holds for $(\sigma,\beta)=(s',b_1)$. Choosing $b'\in]b_1,1/2[$, this yields

$$\begin{aligned}
&\left|\int_{\mathbb{R}}\int_M(1-\Delta)^{-\alpha}(u_1\overline{u}_2)u_3\overline{u}_4dxdt\right|\\
&\leq C\,m(N_1,\cdots,N_4)^{s'}\sum_{K_1,\cdots,K_4}(K_1K_2K_3K_4)^{b_1-b'}\prod_{j=1}^4\|u_j\|_{X^{0,b'}}\ ,
\end{aligned}$$

which completes the proof, since the right hand side is a convergent series. □

We are finally in position to prove Theorem 4.2.1. We can write the solution of the Cauchy problem (4.1.1) using the Duhamel formula

$$u(t) = S(t)u_0 - i\int_0^t S(t-\tau)\left((1-\Delta)^{-\alpha}(|u(\tau)|^2)u(\tau)\right)d\tau \ . \tag{4.2.13}$$

The next lemma contains the basic linear estimate.

**Lemma 4.2.3.** *Let $b, b'$ such that $0 \leq b' < \frac{1}{2}$, $0 \leq b < 1 - b'$. There exists $C > 0$ such that, if $T \in [0,1]$, $w(t) = \int_0^t S(t-\tau)f(\tau)d\tau$, then*

$$\|w\|_{X_T^{s,b}} \leq CT^{1-b-b'}\|f\|_{X_T^{s,-b'}} . \tag{4.2.14}$$

We refer to [52] for a simple proof of this lemma.

The last integral equation (4.2.13) can be handled by means of these spaces $X_T^{s,b}$ using Lemma 4.2.3 as follows

$$\begin{aligned}\Big\|\int_0^t S(t-\tau)\ \left((1-\Delta)^{-\alpha}(|u(\tau)|^2)u(\tau)\right)d\tau\Big\|_{X_T^{s,b}}\\ \leq CT^{1-b-b'}\|\left((1-\Delta)^{-\alpha}(|u(\tau)|^2)u(\tau)\right)\|_{X_T^{s,-b'}} .\end{aligned} \tag{4.2.15}$$

Thus to construct the contraction $\Phi : X_T^{s,b} \to X_T^{s,b}$, $\Phi(v_i) = u_i, i = 1,2$ and to prove the propagation of regularity ii) in Definition 4.2.1, it is enough to prove the following result.

**Lemma 4.2.4.** *Let $s > s_0$. There exists $(b,b') \in \mathbb{R}^2$ satisfying*

$$0 < b' < \frac{1}{2} < b, \quad b + b' < 1, \tag{4.2.16}$$

*and $C > 0$ such that for every triple $(u_j)$, $j = 1,2,3$ in $X^{s,b}(\mathbb{R}\times M)$,*

$$\|(1-\Delta)^{-\alpha}(u_1\overline{u}_2)u_3\|_{X^{s,-b'}} \leq C\|u_1\|_{X^{s,b}}\|u_2\|_{X^{s,b}}\|u_3\|_{X^{s,b}}. \tag{4.2.17}$$

*Moreover, for every $\sigma > s$, there exists $C_\sigma$ such that*

$$\|(1-\Delta)^{-\alpha}(|u|^2)u\|_{X^{\sigma,-b'}} \leq C_\sigma\|u\|^2_{X^{s,b}}\|u\|_{X^{\sigma,b}} \ . \tag{4.2.18}$$

*Proof.* We only sketch the proof of (4.2.17). The proof of (4.2.18) is similar. Thanks to a duality argument it is sufficient to show the following

$$\left|\int_{\mathbb{R}}\int_M (1-\Delta)^{-\alpha}(u_1\overline{u}_2)u_3\overline{u}_4 dxdt\right| \leq C\left(\prod_{j=1}^3 \|u_j\|_{X^{s,b}}\right)\|u_4\|_{X^{-s,b'}} . \tag{4.2.19}$$

The next step is to perform a dyadic expansion in the integral of the left hand-side of (4.2.19), this time in the space variable. We decompose $u_1, u_2, u_3, u_4$ as follows:

$$u_j = \sum_{N_j} u_{j,N_j}, \qquad u_{j,N_j} = \mathbf{1}_{\sqrt{1-\Delta}\in[N_j,2N_j]}(u_j).$$

In this decomposition we have

$$\|u_j\|^2_{X^{s,b}} \simeq \sum_{N_j} N_j^{2s} \|u_{j,N_j}\|^2_{X^{0,b}} \simeq \sum_{N_j} \|u_{j,N_j}\|^2_{X^{s,b}}.$$

We introduce now this decomposition in the left hand side of (4.2.19), and we are left with estimating each term

$$J(N_1, \cdots, N_4) = \int_{\mathbb{R}} \int_M (1-\Delta)^{-\alpha}(u_{1,N_1}\overline{u}_{2,N_2}) u_{3,N_3} \overline{u}_{4,N_4} dxdt \ .$$

Consider the terms with $N_1 \leq N_2 \leq N_3$ (the other cases are completely similar by symmetry). Choose $s'$ such that $s > s' > s_0$. By Lemma 4.2.2 we can find $b'$ such that $0 < b' < \frac{1}{2}$ and

$$|J(N_1, \cdots, N_4)| \leq C \sum_{N_j} (N_1 N_2)^{s'} \prod_{j=1}^{4} \|u_{j,N_j}\|_{X^{0,b'}}. \tag{4.2.20}$$

This is equivalent to

$$|J(N_1, \cdots, N_4)| \leq C \sum_{N_j} (N_1 N_2)^{s'-s} \left(\frac{N_4}{N_3}\right)^s \prod_{j=1}^{3} \|u_{j,N_j}\|_{X^{s,b'}} \|u_{4,N_4}\|_{X^{-s,b'}} \ .$$

In this series we separate the terms in which $N_4 \leq CN_3$ from the others. For the first ones the series converges thanks to a simple argument of summation of geometric series and Cauchy-Schwarz inequality. To perform the summation of the other terms, it is sufficient to apply the following lemma, which is a simple variant of Lemma 2.6 in [24].

**Lemma 4.2.5.** *Let $\alpha$ a positive number. There exists $C > 0$ such that, if for any $j = 1, 2, 3$, $C\mu_{k_j} \leq \mu_{k_4}$, then for every $p > 0$ there exists $C_p > 0$ such that for every $w_j \in L^2(M)$, $j = 1, 2, 3, 4$,*

$$\int_M (1-\Delta)^{-\alpha}(\Pi_{k_1} w_1 \Pi_{k_2} w_2) \Pi_{k_3} w_3 \Pi_{k_4} w_4 dx \leq C_p \ \mu_{k_4}^{-p} \prod_{j=1}^{4} \|w_j\|_{L^2}.$$

*Remark* 4.2.1. Notice that if $M = \mathbb{S}^4$ the above lemma is trivial since in that case, by an elementary observation on the degree of the corresponding spherical harmonics, we obtain that if $k_4 > k_1 + k_2 + k_3$ then the integral (4.2.20) is zero.

Finally, the proof of Lemma 4.2.4 is achieved by choosing $b$ such that $\frac{1}{2} < b < 1 - b'$ and by merely observing that

$$\|u_j\|_{X^{s,b'}} \leq \|u_j\|_{X^{s,b}} \ , \ j = 1, 2, 3.$$

□

### 4.2.2 Local wellposedness for the quadratic nonlinearity

In this subsection, we study the wellposedness theory of the quadratic nonlinear Schrödinger equation posed on $S^4$

$$i\partial_t u + \Delta u = q(u), \qquad q(u) = au^2 + b\overline{u}^2 + c|u|^2, \tag{4.2.21}$$

with zonal initial data $u(0, x) = u_0(x)$.
In fact we shall prove Theorem 4.1.2 on every four-manifold satisfying the trilinear estimates (4.1.3). This is a result of independent interest that we state below.

**Theorem 4.2.2.** *Let $M$ be a Riemannian manifold, let $G$ be a subgroup of isometries of $M$. Assuming that there exists $C > 0$ and $s_0$ such that for any $u_1, u_2, u_3 \in L^2(\mathbb{S}^4)$ $G$-invariant functions on $M$ satisfying*

$$\mathbf{1}_{\sqrt{1-\Delta} \in [N_j, 2N_j]}(f_j) = f_j, \ \ j = 1, 2, 3, \tag{4.2.22}$$

*one has the trilinear estimates*

$$\sup_{\tau \in \mathbb{R}} \left| \int_{\mathbb{R}} \int_M \chi(t) \, e^{it\tau} \mathcal{T}(u_1, u_2, u_3) dx dt \right| \leq C (\min(N_1, N_2, N_3))^{s_0} \prod_{j=1}^{3} \|f_j\|_{L^2}, \tag{4.2.23}$$

*where $\mathcal{T}(u_1, u_2, u_3) = u_1 u_2 u_3$ or $\mathcal{T}(u_1, u_2, u_3) = u_1 u_2 \overline{u}_3$ and $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ is arbitrary. Then, for every $s > s_0$, the Cauchy problem (4.2.21) is uniformly well-posed on the subspace of $H^s(M)$ which consists of $G$-invariant functions.*

*Proof.* It is close to the one of Theorem 4.2.1 above, so we shall just survey it. We denote by $L^2_G(M)$, $H^s_G(M)$, $X^{s,b}_G(\mathbb{R} \times M)$ the subspaces of $L^2(M)$, $H^s(M)$, $X^{s,b}(\mathbb{R} \times M)$ which consist of $G$-invariant functions. For the sake of simplicity, we shall focus on the case

$$q(u) = |u|^2 + \frac{1}{2} u^2 \ .$$

The general case follows from straightforward modifications. As in the proof of Theorem 4.2.1, it is enough, for every $s > s_0$, to show that there exists $b, b'$ such that

$$0 < b' < \frac{1}{2} < b < 1 - b'$$

with the following estimates,

$$\|u_1u_2\|_{X^{s,-b'}} \le C\|u_1\|_{X^{s,b}}\|u_2\|_{X^{s,b}}\ ,\ \|u_1\overline{u}_2\|_{X^{s,-b'}} \le C\|u_1\|_{X^{s,b}}\|u_2\|_{X^{s,b}}\ ,$$
$$\|u^2\|_{X^{\sigma,-b'}} \le C_\sigma\|u\|_{X^{s,b}}\|u\|_{X^{\sigma,b}}\ ,\ \||u|^2\|_{X^{\sigma,-b'}} \le C_\sigma\|u\|_{X^{s,b}}\|u\|_{X^{\sigma,b}}\ ,\ \sigma > s\,,$$

where $u_1, u_2, u$ are $G$ -invariant. As before, we focus on the first set of estimates. Thanks to a duality argument, these estimates are equivalent to

$$\begin{aligned}\left|\int_{\mathbb{R}}\int_M u_1u_2\overline{u}_3dxdt\right| &\le C\|u_1\|_{X^{s,b}}\|u_2\|_{X^{s,b}}\|u_3\|_{X^{-s,b'}}\ ,\\ \left|\int_{\mathbb{R}}\int_M \overline{u}_1u_2u_3dxdt\right| &\le C\|u_1\|_{X^{s,b}}\|u_2\|_{X^{s,b}}\|u_3\|_{X^{-s,b'}}\ ,\end{aligned} \tag{4.2.24}$$

In this way, writing the solution of the Cauchy problem (4.2.21) using the Duhamel formula

$$u(t) = S(t)u_0 - i\int_0^t S(t-\tau)(|u(\tau)|^2 + \frac{1}{2}u^2(\tau))\,d\tau, \tag{4.2.25}$$

and applying Lemma 4.2.3, we obtain a contraction on $X_T^{s,b}$ proving a result of local existence of the solution to (4.2.21) on $H^s(M)$, $s > s_0$. Thus the proof of this theorem is reduced to establishing the trilinear estimates (4.2.24) for suitable $s, b, b'$. We just prove the first inequality in (4.2.24). The proof of the second one is similar.
First we reformulate trilinear estimates (4.2.23) in the context of Bourgain spaces.

**Lemma 4.2.6.** *Let $s_0 \in \mathbb{R}$. The following two statements are equivalent:*
*- For any $f_1, f_2, f_3 \in L^2_G(M)$ satisfying (4.2.22), estimate (4.2.23) holds.*
*- For any $b > \frac{1}{2}$ and any $u_1, u_2, u_3 \in X^{0,b}_G(\mathbb{R}\times M)$ satisfying*

$$\mathbf{1}_{\sqrt{1-\Delta}\in[N_j,2N_j]}(u_j) = u_j,\ \ j = 1,2,3, \tag{4.2.26}$$

*one has*

$$\left|\int_{\mathbb{R}}\int_M (u_1u_2\overline{u}_3)dxdt\right| \le C(\min(N_1,N_2,N_3))^{s_0}\prod_{j=1}^3\|u_j\|_{X^{0,b}}. \tag{4.2.27}$$

*Proof.* The proof of this lemma follows lines of Lemma 4.2.1 above. First we assume that $u_1, u_2, u_3$ are supported for $t \in [0,1]$, and we select $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $\chi = 1$ on $[0,1]$. We set $u_j^\sharp(t) = S(-t)u_j(t)$. Using the Fourier transform, we can write

$$\begin{aligned}&\left|\int_{\mathbb{R}}\int_M u_1u_2\overline{u}_3dxdt\right|\\ &\le C\int_{\tau_1}\int_{\tau_2}\int_{\tau_3}\left|\int_{\mathbb{R}}\int_M \chi(t)e^{it\tau}\prod_{j=1}^3 S(t)\widehat{u}_j^\sharp(\tau_j)dxdt\right|d\tau_1d\tau_2d\tau_3,\end{aligned}$$

where $\tau = (\tau_1 + \tau_2 - \tau_3)$. Supposing for instance $N_1 \le N_2 \le N_3$ and applying (4.2.23) we obtain that the right hand side is bounded by

$$\le CN_1^{s_0} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \|\widehat{u}_1^\sharp(\tau_1)\|_{L^2(M)} \|\widehat{u}_2^\sharp(\tau_2)\|_{L^2(M)} \|\widehat{u}_3^\sharp(\tau_3)\|_{L^2(M)} d\tau_1 d\tau_2 d\tau_3.$$

We conclude the proof as in the proof of Lemma 4.2.1 in section 2, using the Cauchy-Schwarz inequality in $(\tau_1, \tau_2, \tau_3)$, and finally decomposing each $u_j$ by means of the partition of unity

$$1 = \sum_{n\in\mathbb{Z}} \psi\left(t - \frac{n}{2}\right) ,$$

where $\psi \in \mathcal{C}_0^\infty([0, 1])$. □

**Lemma 4.2.7.** *For every $s' > s_0$ there exist $b'$ such that $0 < b' < \frac{1}{2}$ and, for every $G$-invariant functions $u_1, u_2, u_3$ satisfying (4.2.26),*

$$\left|\int_{\mathbb{R}}\int_M (u_1 u_2 \overline{u}_3) dxdt\right| \le C \min(N_1, N_2, N_3)^{s'} \prod_{j=1}^{3} \|u_j\|_{X^{0,b'}} . \tag{4.2.28}$$

*Proof.* Following the same lines of the proof of Lemma 4.2.2, it is enough to establish

$$\left|\int_{\mathbb{R}}\int_M (u_1 u_2 \overline{u}_3) dxdt\right| \le C \min(N_1, N_2, N_3)^2 \prod_{j=1}^{3} \|u_j\|_{X^{0,\frac{1}{6}}(\mathbb{R}\times M)}. \tag{4.2.29}$$

Then the lemma follows by interpolation with (4.2.27). Indeed, assuming for instance $N_1 \le N_2 \le N_3$, we apply the Hölder inequality as follows,

$$\left|\int_{\mathbb{R}}\int_M (u_1 u_2 \overline{u}_3) dxdt\right| \le C\|u_1\|_{L^3(\mathbb{R},L^\infty(M))} \|u_2\|_{L^3(\mathbb{R},L^2(M))} \|u_3\|_{L^3(\mathbb{R},L^2(M))}$$

and using the Sobolev embedding we obtain

$$\le C(N_1)^2 \|u_1\|_{L^3(\mathbb{R},L^2(M))} \|u_2\|_{L^3(\mathbb{R},L^2(M))} \|u_3\|_{L^3(\mathbb{R},L^2(M))}.$$

By the Sobolev embedding in the time variable for function $v(t) = S(-t)u(t)$, we know that

$$\|u\|_{L^3(\mathbb{R},L^2(M))} \le \|u\|_{X^{0,\frac{1}{6}}(\mathbb{R}\times M)}$$

and this completes the proof.

□

Let us sketch the last part of the proof of Theorem4.2.2. We decompose $u_1, u_2, u_3$ as follows:

$$u_j = \sum_{N_j} u_{j,N_j}, \qquad u_{j,N_j} = \mathbf{1}_{\sqrt{1-\Delta}\in[N_j, 2N_j]}(u_j).$$

We introduce this decomposition in the left hand side of (4.2.24) and we use Lemma 4.2.7. Supposing now for simplicity that $N_1 \leq N_2$, we obtain that for any $s' > s_0$ we can find $b'$ such that $0 < b' < \frac{1}{2}$ and

$$\left| \int_{\mathbb{R}} \int_M u_1 u_2 \overline{u}_3 dx dt \right| \leq C \sum_{N_j} (N_1)^{s'-s} \left( \frac{N_3}{N_2} \right)^s \|u_1\|_{X^{s,b'}} \|u_2\|_{X^{s,b'}} \|u_3\|_{X^{-s,b'}} \tag{4.2.30}$$

for any $s > s' > s_0$. Notice that the summation over $N_1$ can be performed via a crude argument of summation of geometric series. As for the summation over $N_2, N_3$, following the same proof as in Section 4.2.1, we conclude by observing that the main part of the series corresponds to the constraint $N_3 \lesssim N_2$. □

### 4.2.3 Conservation laws and global existence for the Hartree nonlinearity

Next we prove that for an initial datum $u_0 \in H^1(M)$, the local solution of the Cauchy problem (4.1.1) obtained above can be extended to a global solution $u \in C(\mathbb{R}, H^1(M))$.

By the definition of uniform wellposedness, the lifespan $T$ of the local solution $u \in C([0,T), H^1(M))$ depends only on the $H^1$ norm of the initial datum. Thus, in order to prove that the solution can be extended to a global one, it is sufficient to show that the $H^1$ norm of $u$ remains bounded on any finite interval $[0,T)$. This is a consequence of the following conservation laws, which can be proved by means of the multipliers $\overline{u}$ and $\overline{u}_t$,

$$\begin{aligned} &\int_M |u(t,x)|^2 \, dx = Q_0 \ ; \\ &\int_M |\nabla u(t,x)|_g^2 + \frac{1}{2} |(1-\Delta)^{-\alpha/2}(|u|^2)(t,x)|^2 \, dx = E_0 \ . \end{aligned} \tag{4.2.31}$$

*Remark* 4.2.2. Notice that a similar argument can be applied in the case of an attractive Hartree nonlinearity, at least when $\alpha > 1$. Indeed, consider the focusing Schrödinger equation

$$iu_t + \Delta u = -(1-\Delta)^{-\alpha}(|u|^2)u,$$

where the nonlinear term has the opposite sign. Computing as above, we obtain the conservation of energy

$$\|\nabla u\|_{L^2(M)}^2 - \frac{1}{2} \|(1-\Delta)^{-\alpha/2}(|u|^2)\|_{L^2}^2 = const,$$

but now the energy $E(t)$ does not control the $H^1$ norm of $u$. However, we can write

$$\|\nabla u\|_{L^2}^2 \leq C + C\|(1-\Delta)^{-\alpha/2}(|u|^2)\|_{L^2}^2,$$

and by Sobolev embedding we have

$$\|(1-\Delta)^{-\alpha/2}(|u|^2)\|_{L^2}^2 \leq C\| \, |u|^2\|_{L^q}^2 \equiv C\|u\|_{L^{2q}}^4, \qquad \frac{1}{q}=\frac{1}{2}+\frac{\alpha}{4},$$

so that we obtain, with $p=2q$,

$$\|\nabla u\|_{L^2} \leq C + C\|u\|_{L^p}^2, \qquad \frac{1}{p}=\frac{1}{4}+\frac{\alpha}{8}.$$

We now use the Gagliardo-Nirenberg inequality (for $d=4$)

$$\|w\|_{L^p}^p \leq C(\|w\|_{L^2}^{p-(p-2)\frac{d}{2}}\|\nabla w\|_{L^2}^{(p-2)\frac{d}{2}} + \|w\|_{L^2}^p)$$

and we obtain

$$\|\nabla u\|_{L^2} \leq C(1+\|u\|_{L^2}^2) + C\|u\|_{L^2}^{2-4(p-2)/p}\|\nabla u\|_{L^2}^{4(p-2)/p}.$$

Notice that, as in the defocusing case above, the $L^2$ norm of $u$ is a conserved quantity. If the power $4(p-2)/p$ is strictly smaller than 1, we infer that the $H^1$ norm of $u$ must remain bounded. In other words, we have proved global existence provided

$$4 \cdot \frac{p-2}{p} < 1 \quad \Longleftrightarrow \quad \alpha > 1.$$

□

### 4.2.4 Studying the global existence for the quadratic nonlinearity

**Proposition 4.2.8.** *Let $(M,g)$ be a four-dimensional Riemannian manifold satisfying the assumptions of Theorem 4.2.2. There exists $\varepsilon > 0$ and a subspace $X$ of $C(\mathbb{R}, H_G^1(M))$ such that, for every initial data $u_0 \in H_G^1(M)$ satisfying $\|u_0\|_{H^1} \leq \varepsilon$, the Cauchy problem (4.1.2), where $q(u) = (\mathrm{Re}\, u)^2$, has a unique global solution $u \in X$.*

*Proof.* By Theorem 4.2.2, we obtain that for an initial datum $u_0 \in H_G^1(M)$, there exists a local solution of the Cauchy problem

$$\begin{cases} i\partial_t u + \Delta u = (\mathrm{Re}\, u)^2, \\ u(0,x) = u_0(x). \end{cases}$$

By the definition of uniform wellposedness, the lifespan $T$ of the local solution $u \in C([0,T), H_G^1(M))$ only depends on a bound of the $H^1$ norm of the initial datum. Thus, in order to prove that the solution can be extended to a global one, it is sufficient to show that the $H^1$ norm of $u$ remains bounded on any finite interval $[0,T)$. This is a consequence of the following conservation laws and of a suitable assumption of smallness on the initial data. Notice that

$$\partial_t \left( \int_M u(t,x)\,dx \right) = -i \int_M (\mathrm{Re}\, u)^2\,dx,$$

from which

$$\int_M \mathrm{Re}\, u(t,x)\,dx = const. \tag{4.2.32}$$

Moreover the following energy is conserved,

$$\int_M |\nabla u(t,x)|^2 + \frac{2}{3}(\mathrm{Re}\, u(t,x))^3\,dx = E_0\ . \tag{4.2.33}$$

Consequently we can write

$$\|\nabla u\|_{L^2}^2 \le E_0 + C\left|\int_M (\mathrm{Re}\, u)^3\right|.$$

Since by Gagliardo-Nirenberg inequality we have

$$\left|\int_M (\mathrm{Re}\, u)^3 dx\right| \le C\|\mathrm{Re}\, u\|_{L^2}\|\nabla(\mathrm{Re}\, u)\|_{L^2}^2 + \|(\mathrm{Re}\, u)\|_{L^2}^3,$$

and by the following inequality

$$\|\mathrm{Re}\, u\|_{L^2} \le C\left|\int_M \mathrm{Re}\, u\,dx\right| + \|\nabla(\mathrm{Re}\, u)\|_{L^2},$$

we deduce that

$$\|\nabla u\|_{L^2}^2 \le E_0 + C\left(\left|\int_M \mathrm{Re}\, u\,dx\right| + \|\nabla u\|_{L^2}\right)\|\nabla u\|_{L^2}^2.$$

Thanks to (4.2.32) we know that

$$\left|\int_M \mathrm{Re}\, u\,dx\right| \le \|u_0\|_{L^1(M)} \le C\|u_0\|_{H^1(M)},$$

thus we obtain

$$\|\nabla u\|_{L^2}^2 \le E_0 + C\left(\|u_0\|_{H^1} + \|\nabla u\|_{L^2}\right)\|\nabla u\|_{L^2}^2.$$

Assuming that

$$\|u_0\|_{H^1} \le \varepsilon,$$

we infer, by a classical bootstrap argument, that $\|\nabla u\|$ cannot blow up, as well as $\|\mathrm{Re}\, u\|_{L^2}$. Using again the evolution law of the integral of $u$, this implies that this integral cannot blow up, and completes the proof of the proposition. $\square$

Notice that the proof above extends without difficulty to $q(u) = c(\text{Re } u)^2$, for any real number $c$. If $(M, g)$ satisfies the assumptions of Theorem 4.2.2, we can now prove that the conclusions of Corollary 4.1.1 hold on $M$.

*Proof.* Let $q(u) = au^2 + b\overline{u}^2 + 2\overline{a}|u|^2$ . The idea is to transform the equation into an equivalent one using the change of unknown $u = \omega v$, with $|\omega| = 1$, and then impose conditions on $a, b$ such that the transformed equation is of the special type corresponding to $q(u) = c(\text{Re } u)^2$ for which, thanks to Proposition 4.2.8, we know that the solution is global. Thus we try to impose

$$q(\omega v) = c\omega(\text{Re } v)^2$$

for some $c \in \mathbb{R}$ and some $\omega$ with $|\omega| = 1$, and we obtain the polynomial identity

$$a\omega^2 v^2 + +b\overline{\omega}^2\overline{v}^2 + 2\overline{a}|v|^2 = \frac{c\omega}{4}(v + \overline{v})^2.$$

Equating the coefficients of the two polynomials we obtain

$$a = c\frac{\overline{\omega}}{4}, \qquad b = c\frac{\omega^3}{4}$$

and this is equivalent to

$$\frac{\overline{a}^2}{a} = b \ .$$

Conversely, we prove that if this condition is not satisfied, it is always possible to construct small energy solutions which blow up in a finite time. We take as initial datum a constant in the form

$$u_0(x) = \omega y_0, \qquad y_0 \in \mathbb{R} \setminus \{0\} \ , \quad |\omega| = 1,$$

and then the equation reduces to the ordinary differential equation

$$iu_t = q(u), \qquad u(0) = \omega y_0.$$

Defining $y(t) = u(t)/\omega$, we see that $y(t)$ is a solution of the equation

$$i\omega y'(t) = q(u) = y^2 q(\omega)$$

which can be written

$$y'(t) = -iq(\omega)\overline{\omega}\, y^2, \qquad y(0) = y_0 \in \mathbb{R}$$

The solution can be written explicitly as

$$y(t) = \frac{1}{y_0^{-1} + iq(\omega)\overline{\omega}t}$$

and is not global if and only if $q(\omega)\overline{\omega}$ is purely imaginary. Thus to conclude the proof it is sufficient to show that we can find an $\omega$ such that

$$q(\omega)\overline{\omega} \equiv a\omega + b\overline{\omega}^3 + 2\overline{a\omega} \quad \text{is purely imaginary (and not 0).}$$

Writing $a = Ae^{i\alpha}$, $b = Be^{i\beta}$, $\omega = e^{i\theta}$ with $A, B \geq 0$, this is equivalent to finding a simple zero for the following function

$$f(\theta) = 3A\cos(\alpha+\theta) + B\cos(\beta - 3\theta).$$

Observe that the average of $f$ vanishes. A point where the sign of $f$ changes cannot be a double zero unless it is a triple zero, and a straightforward calculation shows that this corresponds exactly to the case $A = B$ and $3\alpha + \beta = 2k\pi$, namely $\frac{\overline{a}^2}{a} = b$. Hence, if this condition is not satisfied, $f$ has a simple zero. This completes the proof.

□

## 4.3 Multilinear estimates

In this section we establish multilinear estimates, which, combined with Theorems 4.2.1 and 4.2.2, yield Theorems 4.1.1 and 4.1.2. We recall that $S(t) = e^{it\Delta}$ .

### 4.3.1 Quadrilinear estimates

This subsection is devoted to the proof of quadrilinear estimates (4.2.2) with $s_0 < 1$ on arbitrary four-manifolds with $\alpha > 1/2$, and on the sphere $\mathbb{S}^4$ with $\alpha > 0$. In view of subsections 4.2.1 and 4.2.3, this will complete the proof of Theorem 4.1.1.

**Lemma 4.3.1.** *Let $\alpha > \frac{1}{2}$, $s_0 = \left(\frac{3}{2} - \alpha\right)$ and let $(M, g)$ a compact four-dimensional Riemannian manifold. Then there exists $C > 0$ such that for any $f_1, f_2 \in L^2(M)$ satisfying*

$$\mathbf{1}_{\sqrt{1-\Delta}\in[N,2N]}(f_1) = f_1, \quad \mathbf{1}_{\sqrt{1-\Delta}\in[L,2L]}(f_2) = f_2, \tag{4.3.1}$$

*one has the following bilinear estimate:*

$$\|(1-\Delta)^{-\frac{\alpha}{2}}(u_1u_2)\|_{L^2((0,1)\times M)} \leq C(\min(N,L))^{s_0}\|f_1\|_{L^2(M)}\|f_2\|_{L^2(M)} \ , \tag{4.3.2}$$

*with $u_j(t) = S(t)f_j$.*

*Proof.* By symmetry, it is not restrictive to assume that $N \leq L$. The Sobolev embedding implies

$$\|(1-\Delta)^{-\frac{\alpha}{2}}(u_1u_2)\|_{L^2((0,1)\times M)} \leq C\|u_1u_2\|_{L^2((0,1),L^q(M))}, \quad \frac{1}{q} = \frac{1}{2} + \frac{\alpha}{4},$$

and applying the Holdër inequality we obtain

$$\|(1-\Delta)^{-\frac{\alpha}{2}}(u_1u_2)\|_{L^2((0,1)\times M)} \le C\|u_1\|_{L^2((0,1),L^{\frac{4}{\alpha}}(M))}\|u_2\|_{L^\infty((0,1),L^2(M))}.$$

Thanks to the conservation of the $L^2$ norm we can bound the last factor with the $L^2$ norm of $f_2$; on the other hand, the $L^2L^{4/\alpha}$ term can be bounded using the Strichartz inequality on compact manifolds established by Burq, Gérard, Tzvetkov in [22] (see Theorem 1), which reads, in this particular case,

$$\|u_1\|_{L^2((0,1),L^4(M))} \le C\,N^{1/2}\|f_1\|_{L^2(M)}\ .$$

Combining this estimate with the Sobolev inequality, we obtain (4.3.2) as claimed. □

**Proposition 4.3.2.** *Let $\alpha > \frac{1}{2}$, $s_0 > \left(\frac{3}{2}-\alpha\right)$ and let $(M,g)$ a compact four dimensional Riemannian manifold. Then there exists $C>0$ such that for any $f_1, f_2, f_3, f_4 \in L^2(M)$ satisfying*

$$\mathbf{1}_{\sqrt{1-\Delta}\in[N_j,2N_j]}(f_j) = f_j,\ \ j=1,2,3,4,$$

*one has the following quadrilinear estimate for $u_j(t) = S(t)f_j$:*

$$\begin{aligned}&\sup_{\tau\in\mathbb{R}}\left|\int_{\mathbb{R}}\int_M \chi(t)\,e^{it\tau}(1-\Delta)^{-\alpha}(u_1\overline{u}_2)u_3\overline{u}_4dxdt\right| \\ &\le C(m(N_1,\cdots,N_4))^{s_0}\|f_1\|_{L^2(M)}\|f_2\|_{L^2(M)}\|f_3\|_{L^2(M)}\|f_4\|_{L^2(M)},\end{aligned}\tag{4.3.3}$$

*where $\chi\in\mathcal{C}_0^\infty(\mathbb{R})$ is arbitrary and $m(N_1,\cdots,N_4)$ is the product of the smallest two numbers among $N_1, N_2, N_3, N_4$.*

*Proof.* The proof of our quadrilinear estimate (4.3.3) when $m(N_1,\cdots,N_4) = N_1N_3$ follows directly by the Cauchy-Schwarz inequality and Lemma 4.3.1. In fact, assuming for instance that $\chi$ is supported into $[0,1]$, we have

$$\begin{aligned}I &\equiv \sup_{\tau\in\mathbb{R}}\left|\int_{\mathbb{R}}\int_M \chi(t)\,e^{it\tau}(1-\Delta)^{-\alpha}(u_1\overline{u}_2)u_3\overline{u}_4dxdt\right| \\ &\le C\|(1-\Delta)^{-\frac{\alpha}{2}}(u_1\overline{u}_2)\|_{L^2((0,1)\times M)}\|(1-\Delta)^{-\frac{\alpha}{2}}(u_3\overline{u}_4)\|_{L^2((0,1)\times M)} \\ &\le C(m(N_1,\cdots,N_4))^{s_0}\|f_1\|_{L^2(M)}\|f_2\|_{L^2(M)}\|f_3\|_{L^2(M)}\|f_4\|_{L^2(M)},\end{aligned}$$

by applying (4.3.2). By symmetry, it remains to consider only the case

$$m(N_1,\cdots,N_4) = N_1N_2\ .$$

By the self-adjointness of $(1-\Delta)$, Hölder's inequality and Sobolev's inequality we have

$$\begin{aligned}I &\le C\|u_1\overline{u}_2\|_{L^1((0,1),L^{q'}(M))}\|(1-\Delta)^{-\alpha}(u_3\overline{u}_4)\|_{L^\infty((0,1),L^q(M))} \\ &\le C\|u_1\overline{u}_2\|_{L^1((0,1),L^{q'}(M))}\|u_3\overline{u}_4\|_{L^\infty((0,1),L^1(M))},\end{aligned}$$

provided $\frac{1}{q} > 1 - \frac{\alpha}{2}$. Using again Hölder's inequality, we infer

$$I \le C \prod_{j=1,2} \|u_j\|_{L^2((0,1),L^{2q'}(M))} \prod_{k=3,4} \|u_k\|_{L^\infty((0,1),L^2(M))} \ .$$

Conservation of energy implies that $\|u_k\|_{L^\infty((0,1),L^2(M))} = \|f_k\|_{L^2(M)}$. On the other hand by Sobolev embedding we have

$$\|u_j\|_{L^2((0,1),L^{2q'}(M))} \le CN_j^{\frac{2}{q}-1}\|u_j\|_{L^2((0,1),L^4(M))}.$$

Now we can apply the above-mentioned Strichartz estimate of [22] to obtain

$$\|u_j\|_{L^2((0,1),L^{2q'}(M))} \le CN_j^{\frac{2}{q}-\frac{1}{2}}\|f_j\|_{L^2(M))} \ .$$

Since

$$s_0 = \frac{2}{q} - \frac{1}{2} > \frac{3}{2} - \alpha,$$

and $s_0$ can be arbitrarily close to $\frac{3}{2} - \alpha$, the proof is complete. □

*Remark* 4.3.1. In this case, an iteration scheme for solving can be performed as in [22], avoiding the use of Bourgain spaces, making in $X_T = C([0,T],H^1) \cap L^2([0,T],H_4^\sigma)$ .

On the four dimensional sphere, endowed with its standard metric, the precise knowledge of the spectrum $\mu_k = k(k+3)$, $k \in \mathbb{N}$ makes it possible to improve our quadrilinear estimate. We proceed in several steps, starting with an estimate on the product of two spherical harmonics.

**Lemma 4.3.3.** *Let $\alpha \in ]0, \frac{1}{2}]$ and let $s_0 = 1 - \frac{3\alpha}{4}$. There exists $C > 0$ such that for any $H_n, \widetilde{H}_l$ spherical harmonics on $\mathbb{S}^4$ of degree $n, l$ respectively, the following bilinear estimate holds:*

$$\|(1-\Delta)^{-\frac{\alpha}{2}}(H_n\widetilde{H}_l)\|_{L^2(\mathbb{S}^4)} \le C(1+\min((n,l))^{s_0}\|H_n\|_{L^2(\mathbb{S}^4)}\|\widetilde{H}_l\|_{L^2(\mathbb{S}^4)}. \tag{4.3.4}$$

*Proof.* It is not restrictive to assume that $1 \le n \le l$. We shall adapt the proof of multilinear estimates in [23],[25], using the approach described in [26].

Writing

$$h = (n(n+3))^{-1/2} \ , \ \widetilde{h} = (l(l+3)^{-1/2} \ ,$$

the equations satisfied by the eigenfunctions $H_n, \widetilde{H}_l$ read

$$h^2\Delta H_n + H_n = 0 \ , \ \tilde{h}^2\Delta\widetilde{H}_l + \widetilde{H}_l = 0 \ .$$

In local coordinates, these are semiclassical equations, with principal symbol

$$p(x,\xi) = 1 - g_x(\xi,\xi) \ .$$

We now decompose $H_n$ and $H_l$ using a microlocal partition of unity with semi-classical cut-off of the form $\chi(x, hD)$, $\widetilde{\chi}(x, \widetilde{h}D)$ respectively. When

$$\text{supp}\chi(x,\xi) \cap \{g_x(\xi,\xi) = 1\} = \emptyset,$$

i.e. in the "elliptic" case, the estimates are quite strong : we have, for all $s$, $p$,

$$\||D_x|^s \chi(x, hD_x) H_n\|_{L^2(S^4)} \leq C_{s,p} h^p \|H_n\|_{L^2(S^4)}, \tag{4.3.5}$$

with similar estimates for $\tilde{H}_l$. Consequently, it is sufficient to estimate

$$\|(1-\Delta)^{-\frac{\alpha}{2}}(\chi(x, hD_x) H_n \, \widetilde{\chi}(x, \widetilde{h}D_x) \widetilde{H}_l)\|_{L^2(S^4)} \tag{4.3.6}$$

when cut-off functions $\chi, \widetilde{\chi}$ are localized near the characteristic set

$$\{g_x(\xi,\xi) = 1\} \ .$$

Refining the partition of unity, we may assume that the supports of $\chi$, $\widetilde{\chi}$ are contained in small neighborhoods of $(m, \omega)$, $(m, \widetilde{\omega})$ where $m \in M$ and $\omega, \widetilde{\omega}$ are covectors such that

$$g_m(\omega, \omega) = g_m(\widetilde{\omega}, \widetilde{\omega}) = 1 \ .$$

Notice that functions $u = \chi(x, hD_x) H_n$ , $\widetilde{u} = \widetilde{\chi}(x, \widetilde{h}D_x) \widetilde{H}_l$ are compactly supported and satisfy

$$p^w(x, hD) u = hF \ , \ p^w(x, \widetilde{h}D) \widetilde{u} = \widetilde{h} \widetilde{F} \ ,$$

where $\|F\|_{L^2} \lesssim \|H_n\|_{L^2}$ and $\|\widetilde{F}\|_{L^2} \lesssim \|\widetilde{H}_l\|_{L^2}$.

Set $g_x(x, \xi) = \langle A(x)\xi, \xi \rangle$. Choose any system $(x_1, \ldots, x_4)$ of linear coordinates on $\mathbb{R}^4$ such that

$$\langle A(m)\omega, dx_1 \rangle \neq 0 \quad \text{and} \quad \langle A(m)\widetilde{\omega}, dx_1 \rangle \neq 0 \ .$$

Then, on the supports of $\chi$ and $\widetilde{\chi}$, one can factorize the symbol of the equation as

$$p(x,\xi) = e(x,\xi)(\xi_1 - q(x,\xi')) \ , \ p(x,\xi) = \widetilde{e}(x,\xi)(\xi_1 - \widetilde{q}(x,\xi')),$$

where $e, \widetilde{e}$ are elliptic symbol while $q, \widetilde{q}$ are real valued symbols. In other words, we can reduce the equations for $u, \widetilde{u}$ to *evolution* equations with respect to the variable $x_1$. Notice that $\xi' \in \mathbb{R}^{d-1} = \mathbb{R}^3$, i.e., the spatial dimension of these evolution equations is 3. Moreover, since the second fundamental form of the characteristic ellipsoid $\{\xi : g_m(\xi,\xi) = 1\}$ is non degenerate, the Hessian of $q, \widetilde{q}$ with respect to the $\xi'$ variables does not vanish on the supports of $\chi, \widetilde{\chi}$ respectively.

Therefore we can apply to this equation the (local) three-dimensional Strichartz estimates (see Corollary 2.2 of [26] for more details). We conclude that $u$ satisfies the 3-dimensional semiclassical Strichartz estimates in the following form:

$$\|u\|_{L^p_{x_1} L^q_{x'}} \le C h^{-\frac{1}{p}} \|H_n\|_{L^2} \lesssim n^{\frac{1}{p}} \|H_n\|_{L^2}, \tag{4.3.7}$$

for all $(p, q)$ satisfying the admissibility condition

$$\frac{2}{p} + \frac{3}{q} = \frac{3}{2}\ ,\ p \ge 2.$$

An identical argument is valid for $\widetilde{u}$. In fact, for $\widetilde{u}$ we shall only need the energy estimate

$$\|\widetilde{u}\|_{L^\infty_{x_1} L^2_{x'}} \le C \|\widetilde{H}_l\|_{L^2}\ . \tag{4.3.8}$$

Finally, we estimate the product $u\widetilde{u}$ as follows. By the Sobolev inequality,

$$\|(1-\Delta)^{-\frac{\alpha}{2}}(u\tilde{u})\|_{L^2} \le C\|u\tilde{u}\|_{L^q}, \quad \frac{1}{q} = \frac{1}{2} + \frac{\alpha}{4}.$$

Applying the Hölder inequality we obtain

$$\|(1-\Delta)^{-\frac{\alpha}{2}}(u\tilde{u})\|_{L^2} \le C\|u\|_{L^q_{x_1} L^{\frac{4}{\alpha}}_{x'}} \|\tilde{u}\|_{L^\infty_{x_1} L^2_{x'}},$$

Noticing that $q < 2$ and using the compactness of the support of $u$, we have

$$\|u\|_{L^q_{x_1} L^{\frac{4}{\alpha}}_{x'}} \le C\|u\|_{L^2_{x_1} L^{\frac{4}{\alpha}}_{x'}}.$$

Applying the Strichartz estimate (4.3.7) with $p = 2$ and the Sobolev embedding in the $x'$ variables, we obtain

$$\|u\|_{L^2_{x_1}(L^{\frac{4}{\alpha}}_{x'})} \le C n^{\frac{1}{2} - \frac{3\alpha}{4}} \|u\|_{L^2_{x_1} L^6_{x'}} \le C n^{1 - \frac{3\alpha}{4}} \|H_n\|_{L^2}. \tag{4.3.9}$$

Combining with the $L^\infty L^2$ estimate (4.3.8) on $\tilde{u}$, this completes the proof.

□

We now come to a quadrilinear estimate on spherical harmonics.

**Lemma 4.3.4.** *Let $\alpha \in ]0, \frac{1}{2}]$ and $s_0 = 1 - \frac{3\alpha}{4}$. There exists $C > 0$ such that for any $H^{(j)}_{n_j}, j = 1, \cdots, 4$, spherical harmonics on $\mathbb{S}^4$ of degree $n_j$ respectively, the following quadrilinear estimate holds:*

$$\int_{\mathbb{S}^4} (1-\Delta)^{-\alpha}(H^{(1)}_{n_1} H^{(2)}_{n_2}) H^{(3)}_{n_3} H^{(4)}_{n_4}\, dx \le C(1 + m((n_j)))^{s_0} \prod_{j=1}^{4} \|H^{(j)}_{n_j}\|_{L^2(\mathbb{S}^4)}. \tag{4.3.10}$$

*Proof.* By symmetry, it is sufficient to consider the two cases

$$m(n_1,\cdots,n_4)=n_1n_3 \quad ; \quad m(n_1,\cdots,n_4)=n_1n_2 \ .$$

In the first case, the proof follows directly by the Cauchy-Schwarz inequality and Lemma 4.3.3. It remains to consider only the case $m(n_1,\cdots,n_4)=n_1n_2$. We use the same idea as in Lemma 4.3.3 to decompose, if $n_j\geq 1$, each $H^{(j)}_{n_j}$ into a sum of terms of the form

$$u_j=\chi_j(x,h_jD_x)H^{(j)}_{n_j} \ , \ h_j=(n_j(n_j+3))^{-1/2} \ , \ j=1,2,3,4 \ .$$

As before, each $u_j$ may be microlocalized either into the elliptic zone, in which case we have much stronger semiclassical estimate (4.3.5), in particular an $L^\infty$ bound, or near the characteristic set, and for these terms we can use the Strichartz type estimate (4.3.7). Notice that the very special case $n_j=0$ can be included into the elliptic case. Thus we have several possibilities to consider.

If at least two $u_j$'s are microlocalized in the elliptic zone, then the quadrilinear estimate holds trivially (with $s_0=0$) by a simple application of the Cauchy-Schwarz inequality.

If $u_3$ or $u_4$ is microlocalized in the elliptic zone, then, again by the Cauchy- Schwarz inequality, the quadrilinear estimate is a consequence of estimate (4.3.4) of Lemma 4.3.3, with $\alpha$ replaced by $2\alpha$.

It remains to deal with the cases when only $u_1$ or $u_2$ is microlocalized in the elliptic zone, and when all the $u_j$ 's are microlocalized near the characterictic set. In both cases, we shall make use of the following variant of the Sobolev inequality.

**Lemma 4.3.5.** *Let $A$ be a pseudodifferential operator of order $-2\alpha$ on $\mathbb{R}^4$, and let $B$ be a bounded subset of $\mathbb{R}^4$. For any smooth function $F$ on $\mathbb{R}^4$ with support in $B$, we have the estimate*

$$\|A(F)\|_{L^\infty_{x_1}(L^q_{x'})}\leq C\|F\|_{L^\infty_{x_1}(L^1_{x'})} \tag{4.3.11}$$

*provided $\frac{1}{q}>1-2\alpha/3$.*

*Proof.* The kernel $K(x,y)$ of $A$ admits an estimate like

$$|K(x,y)|\leq\frac{C}{(|x-y|)^{4-2\alpha}}\leq\frac{C}{(|x_1-y_1|+|x'-y'|)^{4-2\alpha}}. \tag{4.3.12}$$

The claim is then a consequence of Young's inequality in variables $x'$. ☐

By the self-adjointness of $(1-\Delta)$ the terms to estimate can be written as follows:

$$I=\left|\int_{\mathbb{S}^4}(u_1u_2)\times(1-\Delta)^{-\alpha}(u_3u_4)\,dx\right| \ . \tag{4.3.13}$$

As in the proof of Lemma 4.3.3 we select a splitting $x = (x_1, x')$ of the local coordinates such that $u_2, u_3, u_4$ are solutions of semiclassical evolution equations, and therefore satisfy Strichartz estimates (4.3.7). Using the $L^\infty$ bound on $u_1$, we have

$$I \leq C\|H_{n_1}^{(1)}\|_{L^2(\mathbb{S}^4)} \|u_2\|_{L^1_{x_1}(L^{q'}_{x'})} \|(1-\Delta)^{-\alpha}(u_3u_4)\|_{L^\infty_{x_1}(L^q_{x'})},$$

and by Lemma 4.3.5 we obtain

$$I \leq C\|H_{n_1}^{(1)}\|_{L^2(\mathbb{S}^4)} \|u_2\|_{L^1_{x_1}(L^{q'}_{x'})} \|u_3u_4\|_{L^\infty_{x_1}(L^1_{x'})}$$

provided $\frac{1}{q} > 1 - \frac{2\alpha}{3}$. Hölder's inequality gives

$$I \leq C\|H_{n_1}^{(1)}\|_{L^2(\mathbb{S}^4)} \|u_2\|_{L^2_{x_1}(L^{q'}_{x'})} \|u_3\|_{L^\infty_{x_1}(L^2_{x'})} \|u_4\|_{L^\infty_{x_1}(L^2_{x'})},$$

and, applying estimate (4.3.8) on $u_3, u_4$ and estimate (4.3.7) with $p = 2$ on $u_2$, we obtain

$$I \leq C n_2^s \prod_{j=1}^{4} \|H_{n_j}^{(j)}\|_{L^2(\mathbb{S}^4)} ,$$

with

$$s = \max\left(\frac{1}{2}, 1 - \frac{3}{q'}\right) < s_0 ,$$

since $q'$ is arbitrary with $\frac{1}{q'} < \frac{2\alpha}{3}$.

Finally, we treat the case when all the factors are microlocalized near the characteristic set. Once again, we select a splitting $x = (x_1, x')$ of the local coordinates for which Strichartz estimates (4.3.7) are valid for each $u_j$. By Hölder's inequality and Lemma 4.3.5 we have

$$\begin{aligned} I &\leq C\|u_1u_2\|_{L^1_{x'}(L^{q'}_{x'})} \|u_3u_4)\|_{L^\infty_{x_1}(L^1_{x'})} \\ &\leq C\|u_1\|_{L^2_{x_1}(L^{2q'}_{x'})} \|u_2\|_{L^2_{x_1}(L^{2q'}_{x'})} \|u_3\|_{L^\infty_{x_1}(L^2_{x'})} \|u_4\|_{L^\infty_{x_1}(L^2_{x'})}. \end{aligned}$$

By estimates (4.3.7) with $p = 2$ on $u_1, u_2$ and (4.3.8) on $u_3, u_4$, we conclude

$$I \leq C(n_1n_2)^s \prod_{j=1}^{4} \|H_{n_j}^{(j)}\|_{L^2(\mathbb{S}^4)} ,$$

with

$$s = \max\left(\frac{1}{2}, 1 - \frac{3}{2q'}\right) < s_0 ,$$

since $q'$ is arbitrary with $\frac{1}{q'} < \frac{2\alpha}{3}$. This completes the proof. □

**Remark.** *It is clear that Lemma 4.3.3 and Lemma 4.3.4 extend to Laplace eigenfunctions on arbitrary compact four-manifolds. Moreover, a refinement of the study of the elliptic case shows that, as in [23], [25], eigenfunctions can be replaced by functions belonging to the range of spectral projectors of the type* $\mathbf{1}_{[n,n+1]}(\sqrt{-\Delta})$.

We now come to the main result of this subsection.

**Proposition 4.3.6.** *For every* $\alpha > 0$*, for every* $s_0 > 1 - \frac{3\alpha}{4}$*, the quadrilinear estimate* (4.2.2) *holds on* $\mathbb{S}^4$.

*Proof.* Let $f_1, \cdots, f_4$ be functions on $\mathbb{S}^4$ satisfying the spectral localization property

$$\mathbf{1}_{\sqrt{1-\Delta}\in[N_j,2N_j]}(f_j) = f_j, \quad j = 1,2,3,4 \ . \tag{4.3.14}$$

This implies that one can expand

$$f_j = \sum_{n_j} H_{n_j}^{(j)} \ ,$$

where $H_{n_j}^{(j)}$ are spherical harmonics of degree $n_j$, and where the sum on $n_j$ bears on the domain

$$N_j/2 \le 1 + n_j \le 2N_j \ . \tag{4.3.15}$$

Consequently, the corresponding solutions of the linear Schrödinger equation are given by

$$u_j(t) = S(t)f_j = \sum_{n_j} \mathrm{e}^{-itn_j(n_j+3)} H_{n_j}^{(j)}$$

and we have to estimate the expression

$$\begin{aligned} Q(f_1, \cdots, f_4, \tau) &= \int_{\mathbb{R}} \int_{\mathbb{S}^4} \chi(t)\, \mathrm{e}^{it\tau} (1-\Delta)^{-\alpha}(u_1\overline{u}_2)u_3\overline{u}_4 dx dt \\ &= \sum_{n_1,\cdots,n_4} \widehat{\chi}\Big(\sum_{j=1}^4 \varepsilon_j n_j(n_j+3) - \tau\Big)\, I(H_{n_1}^{(1)}, \cdots, H_{n_4}^{(4)}) \ , \end{aligned}$$

with $\varepsilon_j = (-1)^{j-1}$ and

$$I(H_{n_1}^{(1)}, \cdots, H_{n_4}^{(4)}) = \int_{\mathbb{S}^4} (1-\Delta)^{-\alpha}(H_{n_1}^{(1)}\overline{H}_{n_2}^{(2)})H_{n_3}^{(3)}\overline{H}_{n_4}^{(4)}\, dx \ .$$

Appealing to Lemma 4.3.4, we have, with $s = 1 - 3\alpha/4$,

$$|I(H_{n_1}^{(1)}, \cdots, H_{n_4}^{(4)})| \le C\, m(N_1, \cdots, N_4)^s \prod_{j=1}^4 \|H_{n_j}^{(j)}\|_{L^2} \ .$$

Using the fast decay of $\widehat{\chi}$ at infinity, we infer

$$|Q(f_1,\cdots,f_4,\tau)| \leq C\, m(N_1,\cdots,N_4)^s \sum_{\ell\in\mathbb{Z}}(1+|\ell|^2)^{-1} \sum_{\Lambda([\tau]+\ell)} \prod_{j=1}^{4} \|H_{n_j}^{(j)}\|_{L^2}$$

$$\lesssim m(N_1,\cdots,N_4)^s \sup_{k\in\mathbb{Z}} \sum_{\Lambda(k)} \prod_{j=1}^{4} \|H_{n_j}^{(j)}\|_{L^2} \ ,$$

where $\Lambda(k)$ denotes the set of $(n_1,\cdots,n_4)$ satisfying (4.3.15) for $j=1,2,3,4$ and

$$\sum_{j=1}^{4} \varepsilon_j n_j(n_j+3) = k \ .$$

Now we write

$$\{1,2,3,4\} = \{\alpha,\beta,\gamma,\delta\}$$

with $m(N_1,\cdots,N_4)) = N_\alpha N_\beta$, and we split the sum on $\Lambda(k)$ as

$$|Q(f_1,\cdots,f_4,\tau)| \lesssim m(N_1,\cdots,N_4)^s \sup_{k\in\mathbb{Z}} \sum_{a\in\mathbb{Z}} S(a)\, S'(k-a) \tag{4.3.16}$$

where

$$S(a) = \sum_{\Gamma(a)} \|H_{n_\alpha}^{(\alpha)}\|_{L^2} \|H_{n_\gamma}^{(\gamma)}\|_{L^2} \ ; \ S'(a') = \sum_{\Gamma'(a')} \|H_{n_\beta}^{(\beta)}\|_{L^2} \|H_{n_\delta}^{(\delta)}\|_{L^2} \ ,$$

$$\Gamma(a) = \{(n_\alpha,n_\gamma) : (4.3.15) \text{ holds for } j=\alpha,\gamma, \sum_{j\in\{\alpha,\gamma\}} \varepsilon_j n_j(n_j+3) = a\},$$

$$\Gamma'(a') = \{(n_\beta,n_\delta) : (4.3.15) \text{ holds for } j=\beta,\delta, \sum_{j\in\{\beta,\delta\}} \varepsilon_j n_j(n_j+3) = a'\}.$$

Now we appeal to the following elementary result of number theory (see e.g. Lemma 3.2 in [24]).

**Lemma 4.3.1.** *Let $\sigma \in \{\pm 1\}$. For every $\varepsilon > 0$, there exists $C_\varepsilon$ such that, given $M \in \mathbb{Z}$ and a positive integer $N$,*

$$\#\{(k_1,k_2)\in\mathbb{N}^2 : N \leq k_1 \leq 2N \ , \ k_1^2 + \sigma k_2^2 = M\} \leq C_\varepsilon N^\varepsilon \ .$$

A simple application of Lemma 4.3.1 implies, for every $\varepsilon > 0$,

$$\sup_a \#\Gamma(a) \leq C_\varepsilon N_\alpha^\varepsilon \ ; \ \sup_{a'} \#\Gamma'(a') \leq C_\varepsilon N_\beta^\varepsilon \ ,$$

and consequently, by a repeated use of the Cauchy-Schwarz inequality,

$$\sum_a S(a)\, S'(k-a) \leq C_\varepsilon\, (N_\alpha N_\beta)^\varepsilon \times$$

$$\left(\sum_a \sum_{\Gamma(a)} \|H^{(\alpha)}_{n_\alpha}\|^2_{L^2} \|H^{(\gamma)}_{n_\gamma}\|^2_{L^2}\right)^{1/2} \left(\sum_a \sum_{\Gamma'(k-a)} \|H^{(\beta)}_{n_\beta}\|^2_{L^2} \|H^{(\delta)}_{n_\delta}\|^2_{L^2}\right)^{1/2}$$

$$\leq C_\varepsilon\, (N_\alpha N_\beta)^\varepsilon \prod_{j=1}^{4} \|f_j\|_{L^2}\ ,$$

where, in the last estimate, we used the orthogonality of the $H^{(j)}_{n_j}$'s as $n_j$ varies. Coming back to (4.3.16), this completes the proof. □

**Remark.** *Using the remark before the statement of this proposition, the proof above extends easily to any compact four-dimensional Zoll manifold (see [24] for more details).*

### 4.3.2 Trilinear estimates on the sphere

In this subsection, we prove trilinear estimates (4.2.23) on $\mathbb{S}^4$, for every $s_0 > 1/2$, for zonal solutions of the Schrödinger equation. In view of subsections 2.2 and 2.4, this will complete the proof of Theorem 4.1.2 and of Corollary 4.1.1, by choosing for $G$ the group of rotations which leave invariant a given pole on $\mathbb{S}^4$.

First we recall the definition of zonal functions.

**Definition 4.3.1.** Let $d \geq 2$, and let us fix a pole on $\mathbb{S}^d$. We shall say that a function on $\mathbb{S}^d$ is a zonal function if it depends only on the geodesic distance to the pole.

The zonal functions can be expressed in terms of zonal spherical harmonics which in their turn can be expressed in terms of classical polynomials (see e.g. [92]). As in [25], we can represent the normalized zonal spherical harmonic $Z_p$ in the coordinate $\theta$ (the geodesic distance of the point $x$ to our fixed pole) as follows:

$$Z_p(x) = C(\sin\theta)^{-\frac{d-1}{2}} \left\{\cos[(p+\alpha)\theta+\beta] + \frac{\mathcal{O}(1)}{p\sin\theta}\right\}, \quad \frac{c}{p} \leq \theta \leq \pi - \frac{c}{p} \tag{4.3.17}$$

with $\alpha, \beta$ independent of $p$, and $C$ uniformly bounded in $p$. On the other hand, near the concentration points $\theta = 0, \pi$ we can write

$$|Z_p(x)| \leq C p^{\frac{d-1}{2}}, \quad \theta \notin [c/p, \pi - c/p]. \tag{4.3.18}$$

and $\|Z_p\|_{L^2(\mathbb{S}^d)} = 1$.

With this notation, we have the following trilinear eigenfunction estimates.

**Lemma 4.3.7.** *There exists a constant $C > 0$ such that the following trilinear estimate holds:*

$$\|Z_p Z_q Z_l\|_{L^1(\mathbb{S}^4)} \leq C(\min(p,q,l))^{1/2} \ . \tag{4.3.19}$$

*Proof.* It is not restrictive to assume that $p \leq q \leq l$. Moreover, by Cauchy-Schwarz inequality it is sufficient to prove (4.3.19) in the special case $q = l$. Then we have

$$\|Z_p Z_q^2\|_{L^1(S^4)} = c \int_0^\pi |Z_p(\theta)| Z_q(\theta)^2 (\sin\theta)^3 d\theta \ ,$$

where $c$ is some universal constant. We split the interval $[0,\pi]$ into the intervals $I_1 = [0, c/q]$, $I_2 = [c/q, c/p]$, $I_3 = [c/p, \pi/2]$ and $I_4 = [\pi, 2, \pi - c/p]$, $I_5 = [\pi - c/p, \pi - c/q]$, $I_6 = [\pi - c/p, \pi]$. Clearly, by symmetry, it is sufficient to estimate the integral on the first three intervals $I_1, I_2, I_3$.

On $I_1$ we can use (4.3.18) for both harmonics $Z_p, Z_q$ and the simple estimate $\sin\theta \leq \theta$, and we obtain

$$\int_0^{c/q} |Z_p| Z_q^2 (\sin\theta)^3 d\theta \leq C p^{3/2} q^3 \int_0^{c/q} \theta^3 d\theta \leq C p^{3/2} q^3 q^{-4} \leq C p^{1/2}$$

since $q \geq p$.

On the second interval $I_2$ we use (4.3.17) for $Z_p$ and (4.3.18) for $Z_q$:

$$\int_{c/q}^{c/p} |Z_p| Z_q^2 (\sin\theta)^3 d\theta \leq C p^{3/2} \int_{c/q}^{c/p} \left(1 + \frac{1}{q\sin\theta}\right)^2 d\theta$$

and by the elementary inequality

$$\left(1 + \frac{1}{q\sin\theta}\right)^2 \leq C + \frac{C}{q^2\theta^2} \tag{4.3.20}$$

we have immediately

$$\int_{c/q}^{c/p} |Z_p| Z_q^2 (\sin\theta)^3 d\theta \leq C p^{3/2} \left(\frac{c}{p} - \frac{c}{q} + \frac{C}{q^2}(q/c - p/c)\right) \leq C p^{1/2}.$$

Finally, in the interval $I_3$ we must use (4.3.17) for both harmonics:

$$\int_{c/p}^{\pi/2} |Z_p| Z_q^2 (\sin\theta)^3 d\theta \leq C \int_{c/p}^{\pi/2} \left(1 + \frac{1}{p\sin\theta}\right)\left(1 + \frac{1}{q\sin\theta}\right)^2 (\sin\theta)^{-3/2} d\theta.$$

Using again (4.3.20), the inequality $\sin\theta \geq C\theta$ on $[0, \pi/2]$, and the fact that $q \geq p$, we have easily

$$\left(1 + \frac{1}{p\sin\theta}\right)\left(1 + \frac{1}{q\sin\theta}\right)^2 (\sin\theta)^{-3/2} \leq C\theta^{-3/2} + Cp^{-3}\theta^{-9/2}.$$

Then integrating on $I_3$ we obtain

$$\int_{c/p}^{\pi/2} |Z_p| Z_q^2 (\sin\theta)^3 d\theta \leq C p^{1/2}$$

and this concludes the proof. □

We now come to the main result of this subsection, which asserts that trilinear estimates (4.2.23) hold for every $s_0 > 1/2$ on $M = \mathbb{S}^4$ in the particular case of zonal Cauchy data.

**Proposition 4.3.8.** *Let $s_0 > \frac{1}{2}$ and $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$. There exists $C > 0$ such that for any $f_1, f_2, f_3 \in L^2(\mathbb{S}^4)$ are zonal functions and satisfying*

$$\mathbf{1}_{\sqrt{1-\Delta} \in [N_j, 2N_j]}(f_j) = f_j, \quad j = 1, 2, 3, \tag{4.3.21}$$

*one has the following trilinear estimate for $u_j(t) = S(t) f_j$,*

$$\sup_{\tau \in \mathbb{R}} \left| \int_{\mathbb{R}} \int_{\mathbb{S}^4} \chi(t)\, e^{it\tau} u_1 u_2 \overline{u}_3 dx dt \right| \\ \leq C (\min(N_1, N_2, N_3))^{s_0} \|f_1\|_{L^2(\mathbb{S}^4)} \|f_2\|_{L^2(\mathbb{S}^4)} \|f_3\|_{L^2(\mathbb{S}^4)}. \tag{4.3.22}$$

*Proof.* The proof is very similar to the one of Proposition 4.3.6. We write

$$u_j(t) = \sum_{n_j} e^{-itn_j(n_j+3)} c_j(n_j) Z_{n_j},$$

where $n_j$ is subject to the condition (4.3.15) and

$$\sum_{n_j} |c_j(n_j)|^2 \sim \|f_j\|_{L^2}^2 .$$

Thus we can write the integral of the left hand-side of (4.3.22) as

$$J = \sum_{n_1, n_2, n_3} \widehat{\chi}\Big(\sum_{j=1}^{3} \varepsilon_j n_j (n_j + 3) - \tau\Big)\, c_1(n_1) c_2(n_2) \overline{c_3(n_3)} \int_{\mathbb{S}^4} Z_{n_1} Z_{n_2} Z_{n_3} dx \ ,$$

where $\varepsilon_1 = \varepsilon_2 = 1$ and $\varepsilon_3 = -1$. Using the fast decay of the Fourier transform $\widehat{\chi}$ and the estimate of Lemma 4.3.7, we obtain

$$\begin{aligned} |J| \leq & C\, (\min(N_1, N_2, N_3))^{\frac{1}{2}} \sum_{\ell \in \mathbb{Z}} \frac{1}{1 + \ell^2} \sum_{\Lambda_{[\tau]+\ell}} |c_1(n_1) c_2(n_2) c_3(n_3)| \ , \\ \lesssim & \ (\min(N_1, N_2, N_3))^{\frac{1}{2}} \sup_{k \in \mathbb{Z}} \sum_{\Lambda_k} |c_1(n_1) c_2(n_2) c_3(n_3)| \ , \end{aligned}$$

where

$$\Lambda_k = \{(n_1, n_2, n_3) : (4.3.15) \text{ holds for } j = 1, 2, 3 \ ; \ \sum_{j=1}^{3} \varepsilon_j n_j (n_j + 3) = k \ \} \ .$$

Suppose for instance that $\min(N_1, N_2, N_3)$ is $N_1$ or $N_2$. Introducing

$$\Lambda_k(n_3) = \{(n_1, n_2) : (n_1, n_2, n_3) \in \Lambda_k\},$$

we specialize index $n_3$ in the above sum as

$$\begin{aligned}
J \leq & C \sup_k \sum_{n_3} |c_3(n_3)| \left( \sum_{(n_1,n_2)\in\Lambda_k(n_3)} |c_1(n_1) c_2(n_2)| \right) \\
\leq & C \sup_k \left( \sum_{n_3} |c_3(n_3)|^2 \right)^{\frac{1}{2}} \left( \sum_{n_3} \Big( \sum_{(n_1,n_2)\in\Lambda_k(n_3)} |c_1(n_1) c_2(n_2)| \Big)^2 \right)^{\frac{1}{2}} \\
\leq & C \left( \sum_{n_3} |c_3(n_3)|^2 \right)^{\frac{1}{2}} \sup_k \left( \sum_{n_3} [\#\Lambda_k(n_3)] \sum_{(n_1,n_2)\in\Lambda_k(n_3)} |c_1(n_1)|^2 |c_2(n_2)|^2 \right)^{\frac{1}{2}} \ .
\end{aligned}$$

To complete the proof, it remains to appeal once again to Lemma 4.3.1, which yields the estimate

$$\#\Lambda_{\tau,\ell}(n_3) \leq C_\delta (\min(N_1, N_2))^\delta \ ,$$

for every $\delta > 0$. If $N_3$ is $\min(N_1, N_2, N_3)$, the proof is similar, by specializing the sum with respect to $n_1$, say. □

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