SHARP $L^p$-ESTIMATES FOR THE WAVE EQUATION ON HEISENBERG TYPE GROUPS

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Abstract. In these lectures, which are based on recent joint work with A. Seeger [23], I shall present sharp analogues of classical estimates by Peral and Miyachi for solutions of the standard wave equation on Euclidean space in the context of the wave equation associated to the sub-Laplacian on a Heisenberg type group. Some related questions, such as spectral multipliers for the sub-Laplacian or Strichartz-estimates, will be briefly addressed. Our results improve on earlier joint work of mine with E.M. Stein. The new approach that we use has the additional advantage of bringing out more clearly the connections of the problem with the underlying sub-Riemannian geometry.

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1. Introduction

Let 
\[ \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2, \]
with \( \dim \mathfrak{g}_1 = 2m \) and \( \dim \mathfrak{g}_2 = n \), be a Lie algebra of Heisenberg type, where
\[ [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}_2 \subset \mathfrak{g}(\mathfrak{g}), \]
\[ \mathfrak{z}(\mathfrak{g}) \] being the center of \( \mathfrak{g} \). This means that \( \mathfrak{g} \) is endowed with an inner product \( \langle \ , \ \rangle \) such that \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \) or orthogonal subspaces and the following holds true:

If we define for \( \mu \in \mathfrak{g}_2^* \setminus \{0\} \) the skew form \( \omega_{\mu} \) on \( \mathfrak{g}_1 \) by

\[
\omega_{\mu}(V, W) := \mu([V,W]),
\]

then there is a unique skew-symmetric linear endomorphism \( J_{\mu} \) of \( \mathfrak{g}_1 \) such that

\[
\omega_{\mu}(V, W) = \langle \mu, [V,W] \rangle = \langle J_{\mu}(V), W \rangle
\]

(here, we also used the natural identification of \( \mathfrak{g}_2^* \) with \( \mathfrak{g}_2 \) via the inner product). Then

\[
(1.1) \quad J_{\mu}^2 = -|\mu|^2 I
\]

for every \( \mu \in \mathfrak{g}_2^* \setminus \{0\} \).

Note that this implies in particular that \([\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2\).

As the corresponding connected, simply connected \textit{Heisenberg type Lie group} \( G \) we shall then choose the linear manifold \( \mathfrak{g} \), endowed with the Baker-Campbell-Hausdorff product

\[
(V_1, U_1) (V_2, U_2) := (V_1 + V_2, U_1 + U_2 + \frac{1}{2}[V_1, V_2])
\]

and identity element \( e = 0 \).

Note that the nilpotent part in the Iwasawa decomposition of a simple Lie group of real rank one is always of Heisenberg type or Euclidean.

As usual, we shall identify \( X \in \mathfrak{g} \) with the corresponding left-invariant vector field on \( G \) given by the Lie-derivative

\[
Xf(g) := \frac{d}{dt}f(g \exp(tX))|_{t=0},
\]

where \( \exp : \mathfrak{g} \to G \) denotes the exponential mapping, which agrees with the identity mapping in our case.

Let us next fix an orthonormal basis \( X_1, \ldots, X_{2m} \) of \( \mathfrak{g}_1 \), and let us define the non-elliptic \textit{sub-Laplacian}

\[
L := -\sum_{j=1}^{2m} X_j^2
\]
on $G$. Since the vector fields $X_j$ together with their commutators span the tangent space to $G$ at every point, $L$ is still hypoelliptic and provides an example of a non-elliptic “sum of squares operator” in the sense of Hörmander ([13]). Moreover, $L$ takes over in many respects of analysis on $G$ the role which the Laplacian plays on Euclidean space.

To simplify the notation, we shall also fix an orthonormal basis $U_1, \ldots, U_n$ of $\mathfrak{g}_2$, and shall in the sequel identify $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ and $G$ with $\mathbb{R}^{2n} \times \mathbb{R}^n$ by means of the basis $X_1, \ldots, X_{2m}, U_1, \ldots, U_n$ of $\mathfrak{g}$. Then our inner product on $\mathfrak{g}$ will agree with the canonical Euclidean product $z \cdot w = \sum_{j=1}^{2m+n} z_j w_j$ on $\mathbb{R}^{2m+n}$, and $J_\mu$ will be identified with a skew-symmetric $2m \times 2m$ matrix. Moreover, the Lebesgue measure $dx \, du$ on $\mathbb{R}^{2m+n}$ is a bi-invariant Haar measure on $G$. By

$$d := 2m + n$$

we shall denote the topological dimension of $G$. We also introduce the automorphic dilations

$$\delta_r(x, u) := (rx, r^2u), \quad r > 0,$$

on $G$, and the Koranyi norm

$$\|(x, u)\|_K := (|x|^4 + |u|^2)^{1/4}.$$ 

Notice that this is a homogeneous norm with respect to the dilations $\delta_r$, and that $L$ is homogeneous of degree 2 with respect to these dilations. Moreover, if we denote the corresponding balls by

$$Q_r(x, u) := \{(y, v) \in G : \|(y, v)^{-1}(x, u)\|_K < r\}, \quad (x, u) \in G, \quad r > 0,$$

then the volume $|Q_r(x, u)|$ is given by

$$|Q_r(x, u)| = |Q_1(0, 0)| r^D,$$

where

$$D := 2m + 2n$$

is the homogeneous dimension of $G$. We shall also have to work with the Euclidean balls

$$B_r(x, u) := \{(y, v) \in G : |(y - x, v - u)| < r\}, \quad (x, u) \in G, \quad r > 0,$$

with respect to the Euclidean norm

$$|(x, u)| := (|x|^2 + |u|^2)^{1/2}.$$

In the special case $n = 1$ we may assume that $J_\mu = \mu J, \mu \in \mathbb{R}$, where

$$J := \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$$
This is the case of the Heisenberg group $\mathbb{H}_m$, which is $\mathbb{R}^{2m} \times \mathbb{R}$, endowed with the product
$$(z, u) \cdot (z', u') = (z + z', u + u' + \frac{1}{2}(Jz, z')).$$
A basis of the Lie algebra $\mathfrak{h}_m$ of $\mathbb{H}_m$ is then given by $X_1, \ldots, X_{2m}, U$, where $U := \frac{\partial}{\partial u}$ is central.

Consider the following Cauchy problem for the wave equation on $G \times \mathbb{R}$ associated to $L$:

$$\frac{\partial^2 u}{\partial t^2} - (-L)u = 0, \quad u \big|_{t=0} = f, \quad \frac{\partial u}{\partial t} \big|_{t=0} = g,$$

where $t \in \mathbb{R}$ denotes time.

The solution to this problem is formally given by
$$u(x, t) = \left( \frac{\sin(t\sqrt{L})}{\sqrt{L}} g \right)(x) + (\cos(t\sqrt{L})f)(x),$$

$(x, t) \in G \times \mathbb{R}$.

In fact, the above expression for $u$ makes perfect sense at least for $f, g \in L^2(G)$, if one defines the functions of $L$ involved by the spectral theorem (notice that $L$ is essentially selfadjoint on $C_0^\infty(G)$ [33]) as follows: If $L = \int_0^\infty s \, dE_s$ denotes the spectral resolution of $L$ on $L^2(G)$, then every Borel function $m$ on $\mathbb{R}^+$ gives rise to a (possibly unbounded) operator
$$m(L) := \int_0^\infty m(s) \, dE_s$$
on $L^2(G)$. Let us also note that if $m$ is a bounded spectral multiplier, then, by left–invariance and the Schwartz kernel theorem, it is easy to see that $m(L)$ is a convolution operator $m(L)f = f \ast K_m = \int f(y)K_m(g^{-1} \cdot) \, dy$, where a priori the convolution kernel $K_m$ is a tempered distribution. We also write $K_m = m(L)\delta$.

If one decides to measure smoothness properties of the solution $u(x, t)$ for fixed time $t$ in terms of Sobolev norms of the form
$$\|f\|_{L^p_{\alpha}} := \|(1 + L)^{\alpha/2}f\|_{L^p},$$
one is naturally led to study the mapping properties of operators such as $\frac{\sin(t\sqrt{L})}{\sqrt{L}(1+L)^{\alpha/2}}$ or $\frac{\sin(t\sqrt{L})}{\sqrt{L}(1+L)^{\alpha/2}}$ as operators on $L^p(G)$ into itself.
For the classical wave equation on Euclidian space, sharp estimates for the corresponding operators have been established by Peral [35] and Miyachi [21].

In particular, if $\Delta$ denotes the Laplacian on $\mathbb{R}^d$, then $(1-\Delta)^{-\alpha/2}e^{it\sqrt{-\Delta}}$ is bounded on $L^p(\mathbb{R}^d)$, if $\alpha \geq (d-1)\left|\frac{1}{p} - \frac{1}{2}\right|$, for $1 < p < \infty$. Moreover, $(1 - \Delta)^{-\frac{(d-1)/2}{2}}e^{it\sqrt{-\Delta}}$ is bounded from the real Hardy space $H^1(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$.

Local analogues of these results hold true for solutions to strictly hyperbolic differential equations (see e.g. [4],[35],[2],[21],[37]).

Indeed, as has been shown in [2] and [37], the estimates in [35] and [21] locally hold true more generally for large classes of Fourier integral operators, and solutions to strictly hyperbolic equations can be expressed in terms of such operators.

The problem in studying the wave equation associated to the sub-Laplacian on the Heisenberg group is the lack of strict hyperbolicity, since $L$ is degenerate-elliptic, and classical Fourier integral operator technics do not seem to be available any more. However, as we shall see, it is still possible to represent solutions to (1.2) as infinite sums of Fourier integral operators with degenerate phases and amplitudes.

Interesting information about solutions to (1.2) have been obtained by Nachman [32]. Among other things, Nachman showed that the wave operator on $\mathbb{H}_m$ admits a fundamental solution supported in a “forward light cone”, whose singularities lie along the cone $\Gamma$ formed by the characteristics through the origin. Moreover, he computed the asymptotic behaviour of the fundamental solution as one approaches a generic singular point on $\Gamma$. His method does, however, not provide uniform estimates on these singularities, so that it cannot be used to prove $L^p$-estimates for solutions to (1.2). What his results do reveal, however, is that $\Gamma$ is by far more complex for $\mathbb{H}_m$ than the corresponding cone in the Euclidian case. This is related to the underlying, more complex sub-Riemannian geometry, which we shall briefly discuss.

Nevertheless, the following theorem is true, which improves on earlier work in [31] on the Heisenberg group, by also including the endpoint regularity, and which is a direct analogue of the results by Peral and Miyachi in the Euclidean setting.
Theorem 1.1. $e^{i\sqrt{L}(1+L)^{\alpha/2}}$ extends to a bounded operator on $L^p(G)$, for $1 < p < \infty$, when $\alpha \geq (d-1)(\frac{1}{p} - \frac{1}{2})$, and for $p = 1$, if $\alpha > \frac{d-1}{2}$.

The proof will be based on some “Hardy space type result” for the endpoint $p = 1$, which will be become evident later. The main difference compared to the Euclidean setting seems, however, that there is not just one single Hardy space, but a whole sequence of local Hardy spaces associated with the problem, whose definitions are based on some interesting interplay between two different homogeneities, namely those coming from the isotropic dilations of the underlying Euclidean structure of $G$ and the one coming from the non-isotropic automorphic dilations on $G$ (compare Remark 7.3).

Remark 1.2. The same result holds for $\sin \sqrt{L}(1+L)^{\alpha-1/2}$, or with the factors $(1+L)^{-\alpha/2}$ (respectively $(1+L)^{-(\alpha-1)/2}$) replaced by $(1+\sqrt{L})^{-\alpha}$, (respectively $(1+\sqrt{L})^{-(\alpha-1)}$).

Notice that the restriction to time $t = 1$ in our theorem is inessential, since $L$ is homogeneous of degree 2 with respect to the automorphic dilations $(z,t) \mapsto (rz, r^2 t)$, $r > 0$.

1.1. Connections with spectral multipliers and further facts about the wave equation. If $m$ is a bounded spectral multiplier, then clearly the operator $m(L)$ is bounded on $L^2(G)$. An important question is then under which additional conditions on the spectral multiplier $m$ the operator $m(L)$ extends from $L^2 \cap L^p(M)$ to an $L^p$-bounded operator, for a given $p \neq 2$. If so, $m$ is called an $L^p$-multiplier for $L$.

Fix a non-trivial cut-off function $\chi \in C_0^\infty(\mathbb{R})$ supported in the interval $[1, 2]$, and define for $\alpha > 0$

$$\|m\|_{sloc,\alpha} := \sup_{r > 0} \|\chi(r \cdot)\|_{H^\alpha},$$

where $H^\alpha = H^\alpha(\mathbb{R})$ denotes the classical Sobolev-space of order $\alpha$. Thus, $\|m\|_{sloc,\alpha} < \infty$, if $m$ is locally in $H^\alpha$, uniformly on every scale. Notice also that, up to equivalence, $\|\cdot\|_{sloc,\alpha}$ is independent of the choice of the cut-off function $\chi$. Then the following analogue of the classical Mihlin-Hörmander multiplier theorem holds true (see M. Christ [3], and also Mauceri-Meda [19]):
Theorem 1.3. If $G$ is a stratified Lie group of homogeneous dimension $D$, and if $\|m\|_{\text{stoc},\alpha} < \infty$ for some $\alpha > D/2$, then $m(L)$ is bounded on $L^p(G)$ for $1 < p < \infty$, and of weak type $(1,1)$.

Observe that, in comparison to the classical case $G = \mathbb{R}^d$, the homogeneous dimension $D$ takes over the role of the Euclidean dimension $d$. Because of this fact, which is an outgrowth of the homogeneity of $L$ with respect to the automorphic dilations, the condition $\alpha > D/2$ in Theorem 1.3 appeared natural and was expected to be sharp for a while. The following result, which was found in joint work with E.M. Stein [24], and independently also by W. Hebisch [12] a bit later, came therefore as a surprise:

Theorem 1.4. For the sub-Laplacian $L$ on a Heisenberg type group $G$, the statement in Theorem 1.3 remains valid under the weaker condition $\alpha > d/2$ instead of $\alpha > D/2$.

Theorem 1.1 can be used to give a new, short proof of this (sharp) multiplier theorem, based on the following simple Subordination Principle, respectively variants of it:

Proposition 1.5. Assume that $\frac{e^{i\sqrt{L}t}}{(1+L)^{\gamma}}$ extends to a bounded operator on $L^1(G)$, and let $\beta > \alpha + 1/2$. Then there is a constant $C > 0$, such that for any multiplier $\varphi \in H^\beta(\mathbb{R})$ supported in $[1, 2]$, the corresponding convolution kernel $K_\varphi = \varphi(L)\delta$ is in $L^1(G)$, and

$$\|K_\varphi\|_{L^1(G)} \leq C\|\varphi\|_{H^\beta(\mathbb{R})}.$$  

Proof. Observe first that $(1+L)^{-\varepsilon}\delta \in L^1(G)$ for any $\varepsilon > 0$. This follows from the formula

$$(1+L)^{-\varepsilon}\delta = \frac{1}{\Gamma(\varepsilon)} \int_0^\infty t^{\varepsilon-1}e^{-t(1+L)}\delta dt = \frac{1}{\Gamma(\varepsilon)} \int_0^\infty t^{\varepsilon-1}e^{-t}p_t dt$$

and the fact that the heat kernel $p_t := e^{-tL}\delta$ is a probability measure on $G$. Write

$$\varphi(s) = \psi(\sqrt{s})(1+s)^{-\gamma}, \text{ with } \gamma > \alpha/2,$$

and put $g := (1+L)^{-\gamma}\delta \in L^1(G)$. Then $\|\psi\|_{H^\beta} \simeq \|\varphi\|_{H^\beta}$, and

$$K_\varphi = \psi(\sqrt{L})((1+L)^{-\gamma}\delta) = \psi(\sqrt{L})g = c \int_{-\infty}^{\infty} \hat{\psi}(t)e^{i\sqrt{L}t}g dt.$$
And, our assumption in combination with some easy multiplier estimates (cf. [31]) and the homogeneity of $L$ yields

$$
\|e^{it\sqrt{L}}g\|_1 \lesssim \|(1 + t^2L)^{\alpha/2}g\|_1 \lesssim \|(1 + t^2)^{\alpha/2}(1 + L)^{\alpha/2}g\|_1,
$$

hence

$$
\|K\psi\|_1 \lesssim \int_{-\infty}^{\infty} |\hat{\psi}(t)|(1 + |t|)^{\alpha}\|(1 + L)^{\alpha/2}g\|_1 dt.
$$

But, $(1 + L)^{\alpha/2}g = (1 + L)^{\alpha/2-\gamma}\delta \in L^1$, hence, by Hölder’s inequality and Plancherel’s theorem,

$$
\|K\psi\|_1 \lesssim \left(\int_{-\infty}^{\infty} |\hat{\psi}(t)(1+|t|)^{\beta}|^2 dt\right)^{1/2} \cdot \left(\int_{-\infty}^{\infty} (1+|t|)^{2(\alpha-\beta)} dt\right)^{1/2} \lesssim \|\psi\|_{H^\beta}.
$$

Q.E.D.

Indeed, we may choose $\beta > \frac{d+1}{2} + \frac{1}{2} = \frac{d}{2}$ in this subordination principle. This is just the required regularity of the multiplier in Theorem 1.4, and one can in fact deduce this theorem from Theorem 1.1 by means of a refinement of the above subordination principle and standard arguments from Calderón-Zygmund theory.

In contrast to the close analogy between the wave equation on $G$ and on Euclidean space expressed in Theorem 1.1, there are other aspects where the solutions behave quite differently compared to the classical setting.

One instance of this is the “almost” failure of dispersive estimates for solutions to (1.2) on Heisenberg groups. Indeed, in [1], Bahouri, Gérard and Xu prove that solutions to (1.2) on $\mathbb{H}_m$ with, sufficiently smooth initial data in $L^1$, satisfy

$$
\|u(t)\|_\infty \leq C|t|^{-1/2},
$$

and that the exponent $-1/2$ is optimal, whereas for the Laplacian on $\mathbb{R}^d$ a dispersive estimate holds with exponent $-(d-1)/2$. Correspondingly, the Strichartz estimates proven in [1] are much weaker than the Euclidean analogues would suggest.

A related result is the “almost” failure of (spectral) restriction theorems for $\mathbb{H}_m$ proven in [25].
2. THE SUB-RIEMANNIAN GEOMETRY OF A HEISENBERG TYPE GROUP

Let $M$ be a smooth manifold, and let $\mathcal{D} \subset TM$ be a smooth distribution, i.e., a smooth subbundel of $TM$. A sub-Riemannian structure $(\mathcal{D}, g)$ on $M$ is given by a smooth distribution $\mathcal{D}$ and a Riemannian metric $g$ on $\mathcal{D}$. For $p \in M$ and $X, Y \in \mathcal{D}(p)$ we write $\langle X, Y \rangle_g := g_p(X, Y)$ and $\|X\| = \langle X, X \rangle_g^{1/2}$. $M$, endowed with such a structure, is called a sub-Riemannian manifold.

An absolutely continuous curve $\gamma : [0, T] \to M$ in $M$ is then called horizontal, if

$$\dot{\gamma}(t) = \frac{d}{dt} \gamma(t) \in \mathcal{D}(\gamma(t))$$

for almost every $t \in [0, T]$. As in Riemannian geometry, the length of a horizontal curve $\gamma$ is then defined by

$$L(\gamma) := \int_0^T \|\dot{\gamma}(t)\|_g dt.$$

The Carnot–Carathéodory distance (or sub-Riemannian distance) on $(M, \mathcal{D}, g)$ is then defined by

$$d_{CC}(p, q) := \inf \{ L(\gamma) : \gamma \text{ is a horizontal curve joining } p \text{ with } q \},$$

$p, q \in M$, where we use the convention that $\inf \emptyset := \infty$.

We shall here be interested in the special situation where $\mathcal{D} = \text{span} \{X_1, \ldots, X_k\}$, and where $X_1, \ldots, X_k$ are smooth vector fields on $M$ which form an orthonormal system with respect to $g$ at every point. Then a curve $\gamma$ is horizontal if there are functions $a_j(t)$ such that

$$\dot{\gamma}(t) = \sum_{j=1}^k a_j(t)X_j(\gamma(t))$$

for a.e. $t \in [0, T]$, and $L(\gamma) = \int_0^T \left( \sum_j a_j(t)^2 \right)^{1/2} dt$.

Moreover, if these vector fields are bracket generating, i.e., if they satisfy Hörmander’s condition, then by the Chow-Rashevskii theorem $d_{CC}(p, q) < \infty$ for every $p, q \in M$, and $d_{CC}$ is a metric on $M$ that induces the topology of $M$.

Let us now consider the sub-Riemannian structure on a group $G$ of Heisenberg type, where $\mathcal{D} = \text{span} \{X_1, \ldots, X_{2m}\}$. Here, the Carnot–Carathéodory distance is left-invariant, i.e.,

$$d_{CC}(x, y) = \|x^{-1}y\|_{CC},$$
where \( \|x\|_{CC} := d_{CC}(0,x) \) is again a homogeneous norm on \( G \). Observe that there exists a constant \( C \geq 1 \) such that
\[
C^{-1}\|x\|_{CC} \leq \|x\|_K \leq C\|x\|_{CC},
\]
(2.1) since any two homogeneous norms are equivalent \([7]\). In analogy with wave equations on Riemannian manifolds, one expects that singularities of solutions propagate along “geodesics” of the underlying geometry. However, it turns out that the notion of geodesic of a sub-Riemannian manifold is not as clear as one might expect. The perhaps best approach to the problem of determining length minimizing curves in this context is by means of control theory, as it is outlined in the article \([18]\) by Liu and Sussman, which I warmly recommend.

One type of geodesic can be constructed following classical geometrical optics ideas:

If \( x \in M \), we endow the dual space \( D(x)^* \) with the dual metric \( g_x^* \) to \( g_x \) on \( D(x) \subset T^*M \). This allows to define the length of a cotangent vector \( (x,\xi) \in T^*M \) as \( \|\xi\|_g := \|\xi\|_{D(x)}g_x^* \). The function \( H : T^*M \to \mathbb{R} \) given by
\[
H(x,\xi) := \frac{1}{2}\|\xi\|_g^2
\]
is called the Hamiltonian of the sub-Riemannian manifold \( M \). Observe that this is, apart from the factor \( \frac{1}{2} \), just the principal symbol of the associated sub-Laplacian. Since \( T^*M \) is a symplectic manifold in a canonical way with respect to the symplectic form \( \omega_M := \sum_j d\xi_j \wedge dx_j \) (in canonical coordinates), we may associate to \( H \) the Hamiltonian vector field \( \mathcal{H} \), i.e., \( \omega_M(X,\mathcal{H}) = dH(X) \) for every \( X \in TM \). The integral curves \((x(t),\xi(t))\) of \( \mathcal{H} \) are called bicharacteristics. They satisfy the Hamilton-Jacobi equations
\[
\dot{x}(t) = \frac{\partial H}{\partial \xi}(x(t),\xi(t)), \quad \dot{\xi} = -\frac{\partial H}{\partial x}(x(t),\xi(t)),
\]
and \( H \) is constant along every bicharacteristic. A normal geodesic of the sub-Riemannian manifold \( M \) will then be the space projection \( x(t) \) of a bicharacteristic \((x(t),\xi(t))\) along which the Hamiltonian \( H \) does not vanish. This notion is in accordance with the idea that the wave front set of a solution to our wave equation should evolve along the bicharacteristics of the wave equation.

It is known that every length minimizing curve in a sub-Riemannian manifold \( M \) is either such a normal geodesic, or an “abnormal geodesic” \([18]\). I won’t give a definition of the latter ones here, since
it is known \cite{11} that groups of Heisenberg type (and more generally, Metivier groups) do not admit non-trivial abnormal geodesics.

However, it should be warned here that in general abnormal geodesics do exist, even in nilpotent Lie groups (see \cite{11} for an example of a group of step 4, and \cite{34} for an example of step 2). This fact has been ignored in some cases in the literature, which had led to some confusion.

On a Heisenberg type group $G$, one easily finds that the Hamiltonian is given by

$$H = \frac{1}{2} |\xi - \frac{1}{2} J_\mu x|^2,$$

if we write coordinates for $T^*G$ as $(x, u, (\xi, \mu)) \in (\mathbb{R}^{2m} \times \mathbb{R}^n) \times (\mathbb{R}^{2m} \times \mathbb{R}^n)$. The Hamilton-Jacobi equations are then given by

$$\begin{align*}
\dot{x} &= \frac{\partial H}{\partial \xi}, & \dot{u} &= \frac{\partial H}{\partial \mu} \\
\dot{\xi} &= -\frac{\partial H}{\partial x}, & \dot{\mu} &= -\frac{\partial H}{\partial u}.
\end{align*}$$

By left-invariance, it suffices to consider bicharacteristics starting at the origin, i.e., $x(0) = 0, u(0) = 0$. We also put $\eta := \xi - \frac{1}{2} J_\mu x$. Then $H = \frac{1}{2} \langle \eta, \eta \rangle$, so that $|\eta|$ is constant along the bicharacteristic. Also $\mu = \text{const.}$, and

$$\begin{align*}
\dot{x} &= \eta, & \dot{\xi} &= -\frac{1}{2} J_\mu \eta \\
\dot{\eta} &= \frac{1}{2} J_\mu \dot{\eta} = -J_\mu \eta,
\end{align*}$$

hence

$$\eta(t) = e^{-t J_\mu} \alpha,$$

if we put $\xi(0) = \eta(0) := \alpha$. This implies $x(t) = \frac{1-e^{-t J_\mu}}{J_\mu} \alpha$, and since $J_\mu^2 = -|\mu|^2 I$, we have

$$x(t) = 2 \frac{\sin \left( \frac{t}{2} |\mu| \right)}{|\mu|} e^{-\frac{t}{2} J_\mu} \alpha. \quad (2.2)$$

For simplicity, let us now assume that $n = 1$, i.e., that we are dealing with a Heisenberg group. Then $J_\mu = \mu J$, with $\mu \in \mathbb{R}$, and we have

$$\begin{align*}
\dot{u} &= \langle \eta, -\frac{t}{2} Jx \rangle = -\frac{\sin \left( \frac{t}{2} |\mu| \right)}{|\mu|} \langle e^{-t J_\mu} \alpha, J e^{-\frac{t}{2} J_\mu} \alpha \rangle \\
&= -\frac{\sin \left( \frac{t}{2} |\mu| \right)}{|\mu|} \langle \alpha, J e^{\frac{t}{2} J_\mu} \alpha \rangle.
\end{align*}$$
Looking at the Taylor series of \(e^{\frac{it}{2}J_\mu}\) and using the fact that \(J\) is skew and satisfies \(J^2 = -I\), this yields

\[
\dot{u} = \frac{\mu}{|\mu|^2} |\alpha|^2 \sin^2\left(\frac{t}{2} |\mu|\right),
\]

if \(\mu \neq 0\). Consequently,

\[
(2.3) \quad u(t) = \begin{cases} 
\left(\frac{t}{2} - \frac{\sin\left(\frac{t}{2} |\mu|\right)}{|\mu|} \cos\left(\frac{t}{2} |\mu|\right)\right) |\alpha|^2 \frac{\mu}{|\mu|^2}, & \text{if } \mu \neq 0 \\
0, & \text{if } \mu = 0.
\end{cases}
\]

The same formula remains valid on an arbitrary Heisenberg type group [34]. Observe that the curve \(\gamma(t) := (x(t), u(t))\) is a normal geodesic iff \(\alpha \neq 0\), and it has constant speed \(\|\dot{\gamma}\|_g = |\dot{x}(t)| = |\eta| = |\alpha|\), since \(\dot{\gamma}(t) = \sum_j \dot{x}_j(t)X_j(\gamma(t))\).

Denote by \(\Sigma_1\) the set of endpoints of all geodesics of length 1 (see figure 1). These endpoints are given by (2.2),(2.3), with \(t = 1\), where \(\alpha\) can be chosen arbitrarily within the unit sphere in \(\mathbb{R}^{2m}\) and \(\mu\) in \(\mathbb{R}^n\). Therefore,

\[
(2.4) \quad \Sigma_1 = \{(x, u) \in G : |x| = \frac{\sin s}{s}, 4|u| = \left|\frac{s - \sin s \cos s}{s^2}\right| \text{ for some } s \in \mathbb{R}\}
\]

We then expect that \(\Sigma_1\) is exactly the singular support of any of the wave propagators

\[
\frac{\sin(t\sqrt{L})}{\sqrt{L}} \delta, \cos(t\sqrt{L})\delta \text{ or } e^{it\sqrt{L}}\delta,
\]

for time \(t = 1\), and this turns out to be true.
The “outer hull” of the set $\Sigma_1$ is just the unit sphere with respect to the Carnot-Carathéodory distance.

Let us finally mention the well-known fact that solutions to our wave equation have finite propagation speed (see [20], or [27] for an elementary proof for Lie groups), i.e., if $P_t$ denotes any of the wave propagators $rac{\sin(t\sqrt{L})}{\sqrt{L}}\delta$, $\cos(t\sqrt{L})\delta$, then, for $t \geq 0$,

$$\text{supp } P_t \subset \{(x,u) \in G : \| (x,u) \|_{CC} \leq t \}$$

3. The Schrödinger and the wave propagators on a Heisenberg type group

One principal problem in proving estimates for solutions to wave equations consists in finding sufficiently explicit expressions for the associated wave propagators. For strictly hyperbolic equations, such expressions can be given by means of Fourier integral operators, at least locally in space and time. These technics, however, do not apply to wave equations associated to non-elliptic Laplacians, and no sufficiently explicit expressions for the wave propagators are known to date in these more general situations, except for the case of Heisenberg type groups and a few related examples. What allows us to give such expressions on Heisenberg type groups is the existence of explicit formulas for the Schrödinger propagators $e^{itL}\delta$ on these groups.

There are several possible approaches to these formulas. One would consist in appealing to the Gaveau-Hulanicki formula [9],[16] for the heat kernels $e^{-tL}\delta$, which can be derived by means of representation theory and Mehler’s formula, and then trying to apply physicists approach and replacing time $t$ by complex time $-it$.

We shall here sketch another approach, which is more far-reaching and has turned out to be useful also in the study of questions of solvability of general second order left-invariant differential operators on 2-step nilpotent Lie groups (see [28], also for further references to this topic).

Let $G$ be a group of Heisenberg type as before. Given $f \in L^1(G)$ and $\mu \in g_2^* = \mathbb{R}^{2n}$, we define the partial Fourier transform $f^\mu$ of $f$ along
the center by
\[ f^\mu(x) := \int_{\mathbb{R}^n} f(x, u) e^{-2\pi i \mu \cdot u} \, du \quad (x \in \mathbb{R}^{2m}). \]

Moreover, if we define the $\mu$–twisted convolution of two suitable functions $\varphi$ and $\psi$ on $g_1 = \mathbb{R}^{2m}$ by
\[ (\varphi \ast_\mu \psi)(x) := \int_{\mathbb{R}^{2m}} \varphi(x - y) \psi(y) e^{-i\pi \omega_\mu(x,y)} \, dy, \]
then
\[ (f \ast g)^\mu = f^\mu \ast_\mu g^\mu, \]
where $f \ast g$ denotes the convolution product of the two functions $f, g \in L^1(G)$. Accordingly, the vector fields $X_j$ are transformed into the $\mu$-twisted first order differential operators $X_j^\mu$ such that $(X_j f)^\mu = X_j^\mu f^\mu$, and the sub-Laplacian is transformed into the $\mu$-twisted Laplacian $L^\mu$, i.e.,
\[ (Lf)^\mu = L^\mu f^\mu = -\sum_{j=1}^{2m} (X_j^\mu)^2 \]
(say for $f \in \mathcal{S}(G)$). An easy computation shows that explicitly
\[ X_j^\mu = \frac{\partial}{\partial x_j} + i\pi \omega_\mu(x, X_j). \]

What turns out to be important for us is that the one-parameter Schrödinger group $e^{it L^\mu}$, $t \in \mathbb{R}$, generated by $L^\mu$ can be computed explicitly:

**Proposition 3.1.** For $f \in \mathcal{S}(G)$,
\[ e^{it L^\mu} f = f \ast_\mu \gamma_t^\mu, \quad t \in \mathbb{R}, \]
where $\gamma_t^\mu \in \mathcal{S}'(\mathbb{R}^{2m})$ is a tempered distribution given by the imaginary Gaussian
\[ \gamma_t^\mu(x) = 2^{-m} \left( \frac{|\mu|}{\sin(2\pi t|\mu|)} \right)^m e^{-i\frac{\pi}{2} |\mu| \cos(2\pi t|\mu|)|x|^2}, \]
whenever $\sin(2\pi t|\mu|) \neq 0$. Moreover, by well-known formulas for the inverse Fourier transform of a generalized Gaussian [14], its Fourier transform is
\[ \widehat{\gamma_t^\mu}(\xi) = \frac{1}{(\cos(2\pi t|\mu|))^m} e^{i\frac{2\pi}{|\mu|} \tan(2\pi t|\mu|)|\xi|^2} \]
whenever $\cos(2\pi t|\mu|) \neq 0$. 

3.1. **Twisted convolution and the metaplectic group.** It is not difficult to reduce the proof of Proposition 3.1 to the case of the Heisenberg group $G = \mathbb{H}_m$, which we shall consider now. For any real, symmetric matrix $A := (a_{jk})_{j,k=1,\ldots,2m}$ let us put

$$S := -AJ,$$

where $J := \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$ is as in the definition of the product in $\mathbb{H}_m$.

It is easy to see that the mapping $A \mapsto S$ is a vector space isomorphism of the space of all real, symmetric $2m \times 2m$ matrices onto the Lie algebra

$$\mathfrak{sp}(m, \mathbb{R}) := \{ S : tSJ + JS = 0 \}.$$

Let us define

$$\Delta_S := 2m \sum_{j,k=1}^{2m} a_{jk}X_jX_k, \quad S \in \mathfrak{sp}(m, \mathbb{R}),$$

where $A$ is related to $S$ by (3.1). Then one verifies easily that for any $S_1, S_2 \in \mathfrak{sp}(m, \mathbb{R})$, we have

$$[\Delta_{S_1}, \Delta_{S_2}] = -2U[\Delta_{S_1}, S_2],$$

where $U$ denotes again the central basis element of $h_m$. Applying a partial Fourier transform along the (here one-dimensional) center, this leads to the following commutation relations among the $\mu$-twisted convolution operators $\Delta_S^\mu$ :

$$[\Delta_S^\mu_{S_1}, \Delta_S^\mu_{S_2}] = -4\pi i\mu \Delta^\mu_{[S_1, S_2]}.$$

Since $\Delta_S^\mu$ is formally self-adjoint, this means that the mapping

$$S \mapsto \frac{i}{4\pi \mu} \Delta_S^\mu$$

is a representation of $\mathfrak{sp}(m, \mathbb{R})$ by (formally) skew-adjoint operators on $L^2(\mathbb{R}^{2m})$.

Let us consider the case $\mu = 1$ (it is not difficult to reduce considerations to this case). In this case, we just speak of the twisted convolution, and write $\varphi \times \psi$ in place of $\varphi *_1 \psi$.

In [15], R. Howe has proved for this case that the map (3.3) can be exponentiated to a unitary representation of the metaplectic group $M_p(m, \mathbb{R})$, a two-fold covering of the symplectic group $\text{Sp}(m, \mathbb{R})$. The
group $M_p(m, \mathbb{R})$ can in fact be represented by twisted convolution operators of the form \( f \mapsto f \times \gamma \), where the \( \gamma \)'s are suitable measures which, generically, are multiples of purely imaginary Gaussians
\[
e_A(z) := e^{-i\pi t A z},
\]
with real, symmetric \( 2m \times 2m \) matrices \( A \). In particular, one has
\[
e^{i\frac{4}{\pi} \Delta} f = f \times \gamma_t, \quad t \in \mathbb{R}.
\]
(3.4)

The distributions \( \gamma_t \) have been determined explicitly in [22]. To indicate how this can be accomplished, let us argue on a completely formal basis:

If \( e_A, e_B \) are two Gaussians as before such that \( \det(A + B) \neq 0 \), one computes that
\[
e_A \times e_B = [\det(A + B)]^{-1/2} e_{A-(A-J/2)(A+B)^{-1}(A+J/2)},
\]
where a suitable determination of the root has to be chosen. Choosing \( A = \frac{1}{2} JS_1, \ B = \frac{1}{2} JS_2 \), with \( S_1, S_2 \in \mathfrak{sp}(m, \mathbb{R}) \), and assuming that \( S_1 \) and \( S_2 \) commute, one finds that
\[
e_{\frac{1}{2} JS_1} \times e_{\frac{1}{2} JS_2} = 2^n (\det(S_1 + S_2))^{-1/2} e_{\frac{1}{2} [S_1 S_2 + I] (S_1 + S_2)^{-1}}.
\]

This reminds of the addition law for the hyperbolic cotangent, namely
\[
\coth(x + y) = \frac{\coth x \coth y + 1}{\coth x + \coth y}.
\]

We are thus led to define, for non-singular \( S \),
\[
A(t) := \frac{1}{2} J \coth(t S/2),
\]
(3.5)

which is well-defined at least for \( |t| > 0 \) small.

Then
\[
e_{A(t_1)} \times e_{A(t_2)} = 2^n (\det(A(t_1) + A(t_2)))^{-1/2} e_{A(t_1 + t_2)}.
\]

And, from
\[
\coth x + \coth y = \frac{\sinh(x + y)}{\sinh x \sinh y},
\]
we obtain (ignoring again the determination of roots)
\[
[\det \sinh((t_1 + t_2) S/2)]^{1/2} [\det(A(t_1) + A(t_2))]^{-1/2} = [\det \sinh(t_1 S/2)]^{1/2} [\det \sinh(t_2 S/2)]^{1/2}.
\]
Together this shows that
\[
\gamma_{t,S} := 2^{-n} \left[ \det \sinh \left( tS / 2 \right) \right]^{-1/2} e^{A(t)}
\]
forms a (local) 1-parameter group under twisted convolution, and it is not hard to check that its infinitesimal generator is \( \frac{i}{4\pi} \Delta_S^{1/2} \).

Finally, if we choose \( S := J \), so that \( A = -I \) and \( \Delta_S = L \), we obtain Proposition 3.1, at least for \( |\mu| = 1 \); the general case follows by a scaling argument.

**Remark 3.2.** It is important to note that Howe’s construction of the metaplectic group by means of twisted convolution operators can be extended to the so-called oscillator semigroup. If we denote by \( \mathfrak{sp}^+(\mathbb{C}, m) \) the cone of all elements \( S \in \mathfrak{sp}(\mathbb{C}, m) \) such that the associated complex, symmetric matrix \( A = SJ \) has positive semi-definite real part, then accordingly explicit formulas for the one-parameter semigroups \( e^{t \Delta_S^k} \), \( t \geq 0 \), have been derived in [28].

### 3.2. A subordination formula

Fix a bump function \( \chi_0 \in C_0^\infty(\mathbb{R}) \) supported in the interval \( [1/2, 2] \). If \( a \in S^\nu \) is a symbol of order \( \nu \in \mathbb{R} \) on \( T^* \mathbb{R} = \mathbb{R}^2 \), i.e., if
\[
|\partial_s^\alpha \partial_\sigma^\beta a(s, \sigma)| \leq C_{\alpha,\beta} (1 + |\sigma|)^{\nu-\beta}, \quad (s, \sigma) \in \mathbb{R}^2,
\]
for every \( \alpha, \beta \in \mathbb{N} \), then we say that \( a \) is supported over the subset \( I \subset \mathbb{R} \) if \( \text{supp} \ a(\cdot, \sigma) \subset I \) for every \( \sigma \in \mathbb{R} \).

**Proposition 3.3.** There exist a symbol \( a_0 \in S^0 \) supported over the interval \( [1/16, 4] \) and a Schwartz function \( \rho \in S(\mathbb{R}^2) \) such that, for \( x \geq 0 \),
\[
\chi_0 \left( \frac{\sqrt{x}}{\lambda} \right) e^{it\sqrt{x}} = \rho \left( \frac{tx}{\lambda}, \lambda t \right) + \sqrt{\lambda t} \int a_0(s, \lambda t) e^{i\frac{\lambda s}{4x}} e^{i\frac{t}{\lambda}x} \, ds,
\]
for every \( \lambda \geq 1 \) and \( t > 0 \) such that \( \lambda t \geq 1 \).

**Proof.** This can easily be obtained by the method of stationary phase. \( \square \)

The explicit formulas in Proposition 3.1 for \( e^{itL^\mu} \) in combination with Proposition 3.3 allow us to describe spectrally localized parts of the wave propagators \( e^{\pm it\sqrt{\mathcal{L}_\delta}} \). Of course, there is no loss of generality in assuming that \( t \geq 0 \) and that the sign in the exponent is positive.
Moreover, since \( L \) is homogeneous of degree 1 with respect to the automorphic dilations \( \delta_r \), we may reduce ourselves to considering the case where \( t = 1 \).

Let us recall here that it is well-known that if \( \psi \in \mathcal{S}(\mathbb{R}) \), then the convolution kernel \( \psi(L)\delta \) will also be a Schwartz function (on \( G \)), whose \( L^1(G) \)-norm will be controlled by some Schwartz norm on \( \mathcal{S}(\mathbb{R}) \) (see, [17]; also [27] for a connection with finite propagation speed).

**Proposition 3.4.** Let \( \chi_0 \in C^\infty_0(\mathbb{R}) \) be a bump function supported in the interval \([1/2, 2]\), and assume that \( \lambda \geq 1 \). Then there exist Schwartz functions \( \psi_\lambda \in \mathcal{S}(G) \) and \( K_\lambda \in \mathcal{S}(G) \) such that

\[
\chi_0\left(\sqrt{\frac{L}{\lambda}}\right)e^{i\sqrt{\pi}t} = \psi_\lambda + K_\lambda,
\]

and so that the following hold true:

(i) For every \( N \in \mathbb{N} \) there is constant \( C_N \geq 0 \) so that

\[
||\psi_\lambda||_1 \leq C_N \lambda^{-N}
\]

for every \( \lambda \geq 1 \).

(ii) The partial Fourier transform \( K_\lambda^\mu \in \mathcal{S}(\mathbb{R}^{2m}), (\mu \neq 0) \), of \( K_\lambda \) is given by an oscillatory integral of the form

\[
\langle K_\lambda^\mu, \varphi \rangle = \sqrt{\lambda} \int a_0(t, \lambda) e^{i\frac{\pi}{4}t} \langle \gamma_\mu t/\lambda, \varphi \rangle dt, \quad \varphi \in \mathcal{S}(\mathbb{R}^{2m}),
\]

where \( a_0 \in S^0 \) is a symbol of order 0 supported over the interval \([1/16, 4]\) and where \( \gamma^\mu t \) is given by Proposition 3.1.

**Proof.** Apart from (i), all the claims follow easily from the subordination formula (3.7) in Proposition 3.3, the spectral resolution \( L = \int_{\mathbb{R}_+} s dE_s \) and Fubini’s theorem in combination with Proposition 3.1.

To prove (i), notice that we may choose \( \psi_\lambda := \rho\left(\frac{x}{\lambda}, \lambda\right)\delta \), with a Schwartz function \( \rho \in \mathcal{S}(\mathbb{R}^2) \). In particular, every Schwartz norm of \( x \mapsto \rho\left(\frac{x}{\lambda}, \lambda\right) \) decays like \( = O(\lambda^{-N}) \) for every \( N \in \mathbb{N} \). By the previous remark, it is then also clear that the convolution kernel \( \psi_\lambda \) of the spectral multiplier operator \( \rho\left(\frac{x}{\lambda}, \lambda\right) \) will satisfy the estimates in (i).

\( \square \)
In view of this proposition, it will in the sequel suffice to estimate the convolution operators defined by the kernels \( K_\lambda \). Moreover, for simplicity, we may and shall assume that

\[
\langle K^\mu_\lambda, \varphi \rangle = \sqrt{\lambda} \int \chi_0(t) e^{i\frac{\lambda}{2} t} \langle \gamma^\mu_{t/\lambda}, \varphi \rangle dt, \quad \varphi \in \mathcal{S}(\mathbb{R}^{2m}),
\]

where \( \chi_0 \in C^\infty_0(\mathbb{R}) \) is supported in the interval \([1/16, 4]\). Indeed, this will be justified since our estimates will only depend on some \( C^k \)-norm of \( \chi_0 \).

We next choose \( \eta_0 \in C^\infty_0(\mathbb{R}) \) such that \( \text{supp} \eta_0 \subset [\frac{-3\pi}{8}, \frac{3\pi}{8}] \) and \( \sum_{k \in \mathbb{Z}} \eta_0(s + k\frac{\pi}{2}) = 1 \) for every \( s \in \mathbb{R} \), and put

\[
\langle A^\mu_{\lambda,k}, \varphi \rangle := \lambda^{1/2} \int \chi_0(t) \eta_0(2\pi t|\mu|/\lambda - k\pi) e^{i\frac{\lambda}{2} t} \langle \gamma^\mu_{t/\lambda}, \varphi \rangle dt,
\]

\[
\langle B^\mu_{\lambda,k}, \varphi \rangle := \lambda^{1/2} \int \chi_0(t) \eta_0(2\pi t|\mu|/\lambda - k\pi - \frac{\pi}{2}) e^{i\frac{\lambda}{2} t} \langle \gamma^\mu_{t/\lambda}, \varphi \rangle dt.
\]

By \( A_{\lambda,k} \) and \( B_{\lambda,k} \) we shall denote the tempered distributions defined on \( G \) whose partial Fourier transforms along the center are given by \( A^\mu_{\lambda,k} \) and \( B^\mu_{\lambda,k} \), respectively. Then

\[
K_\lambda = \sum_{k=0}^{\infty} (A_{\lambda,k} + B_{\lambda,k})
\]

in the sense of distributions.

For the sake of this exposition, we shall restrict ourselves to considering the \( A_{\lambda,k} \). Since

\[
\langle \gamma^\mu_{t/\lambda}, \varphi \rangle = \langle \hat{\gamma}^\mu_{t/\lambda}, \hat{\varphi} \rangle,
\]

where \( \hat{\gamma}^\mu_{t/\lambda} \) is given by Proposition 3.1, it is clear that \( A^\mu_{\lambda,k} \) is a well-defined tempered distribution. However, since the integral kernel \( \gamma^\mu_{t/\lambda} \) has a singularity in \( t \) where \( 2\pi t|\mu|/\lambda = k\pi \) if \( k \neq 0 \), we shall here perform a priori a dyadic decomposition with respect to \( t \). More precisely, if \( k \geq 1 \), let us fix another bump function \( \chi_1 \in C^\infty_0(\mathbb{R}) \) supported in the interval \([1/2, 2]\) so that \( \sum_{l=0}^{\infty} \chi_1(2^l s) = 1 \) for every \( s \in [0, \frac{3\pi}{8}] \), and decompose the distribution \( A^\mu_{\lambda,k} \) as

\[
\langle A^\mu_{\lambda,k}, \varphi \rangle = \sum_{l=0}^{\infty} (\langle A^\mu_{\lambda,k,l,+}, \varphi \rangle + \langle A^\mu_{\lambda,k,l,-}, \varphi \rangle),
\]
where $A_{\lambda,k,l,\pm}$ is defined to be the smooth function

$$A_{\lambda,k,l,\pm}(x) := \lambda^{1/2} \int \chi_0(t) \chi_1 \left( \pm 2^l(2\pi t|\mu|/\lambda - k\pi) \right) \eta_0(2\pi t|\mu|/\lambda - k\pi)$$

$$- 2^{-m} \left( \frac{|\mu|}{\sin(2\pi|\mu|/\lambda)} \right)^m e^{i\frac{1}{2} 2^{-m} e^{-i\frac{1}{2} |\mu| \cot(2\pi|\mu|/\lambda)|x|^2} dt$$

By Fourier inversion in the central variable $u$, we have

$$A_{\lambda,k,l,\pm}(x,u) = \int_{\mathbb{R}^n \setminus \{0\}} A_{\lambda,k,l,\pm}^\mu(x) e^{2\pi i u \cdot \mu} d\mu,$$

In the integration w.r. to $\mu$, we introduce polar coordinates $\mu = r \omega$, $r > 0$, $\omega \in S^{n-1}$. Then $d\mu = r^{n-1} d\sigma_{n-1}(\omega)$, where $d\sigma_{n-1}$ denotes the surface measure on the sphere $S^{n-1}$. It is well-known [14] by the method of stationary phase that

$$\int_{S^{n-1}} e^{iu \cdot \omega} d\sigma_{n-1}(\omega) = e^{i|u| a_+ + (|u|)} + e^{-2\pi i |u| a_- (|u|)}, \quad u \in \mathbb{R}^n,$$

where $a_+, a_- \in S^{-(n-1)/2}$ are symbols of order $-(n-1)/2$.

Since the integral defining $A_{\lambda,k,l,\pm}^\mu$ is absolutely convergent in $t$, applying this and suitable changes of coordinates, we then find that

$$\lambda_{\lambda,k,l,\pm}(x,u) = A_{\lambda,k,l,\pm,a_+}(x,|u|) + A_{\lambda,k,l,\pm,a_-}(x,-|u|),$$

where for any symbol $a \in S^{-(n-1)/2}$ the function $A_{\lambda,k,l,\pm,a}(x,v)$ on $\mathbb{R}^{2m} \times \mathbb{R}$ is given by

$$A_{\lambda,k,l,\pm,a}(x,v) = \lambda^{m+n+1/2} \int \int \chi(t) \chi_1 \left( \pm 2^l(s - k\pi) \right) \eta(s - k\pi) s^{m+n-1}$$

$$\sin(s)^{-m} e^{i\lambda st (s - \cot(s)|x|^2 + 4v)} a(\lambda st |v|) ds dt.$$

Since $k \geq 1$, this can be re-written in the form

$$A_{\lambda,k,l,\pm,a}(x,v) = \lambda^{m+n+1/2} k^{m+n-1} \int \int \chi \left( \frac{t}{s/k + \pi} \right) \frac{\chi_1(\pm 2^l s)}{\sin s^{m+n} \eta(s)} ds dt,$$

where the functions $\chi$ and $\eta$ have similar properties as $\chi_1$ and $\eta_0$.

A similar expressions can be given for $k = 0$. 

(3.13)
4. Estimation of $A_{\lambda,k,l,\pm}$ if $k \geq 1$

We next need to establish estimates for the kernels $A_{\lambda,k,l,\pm}$, into which the kernels $A_{\lambda,k}$ decompose, and their derivatives. In some regions, the estimates that can be obtained for these kernels will be useful only if $\lambda \gtrsim k$. However, this problem can be fixed in an easy way, and we shall not go into details here but prefer to give a somewhat simplified account of the actual estimates that we can obtain, for the sake of clarity of exposition.

We shall only consider $A_{\lambda,k,l,+,a}$, since $A_{\lambda,k,l,-,a}$ can be estimated very much in the same way. Let us use the short-hand notation $F_l(x,v) := A_{\lambda,k,l,+,a}(x,v)$, so that, by (3.13), for $k \geq 1$ we have

$$F_l(x,v) = \frac{\lambda^{m+n+1/2}k^{m+n-1}2^{(m-1)l}}\int\int \chi_k,\ell(r,t) e^{i\lambda \ell t} \left( \psi(2^{-l}r) + 4v \right) a(\lambda \ell t|v|) dr dt,$$

where the function $\psi(s)$ is given by

$$\psi(s) := \frac{1}{s + k\pi} - |x|^2 \cot(s),$$

and where

$$\chi_l(r) := \frac{\chi_1(r)}{2^l \sin(2^{-l}r)^m}$$

is again supported where $r \sim 1$. Moreover, if $2^{-l}r \in \text{supp } \eta$ and $k \geq 1$, then $|2^{-l}r| \leq \frac{3\pi}{8} \leq \frac{k\pi}{2}$, so that

$$\frac{1}{\pm 2^{-l}r + k\pi} \sim k^{-1}$$

for every $k \geq 1, l \geq 0$ (the case of the negative sign will actually become relevant for the $A_{\lambda,k,l,-,a}$ only.)

Notice that all derivatives of $\chi_l(r)$ are uniformly bounded in $l$. We can therefore re-write

$$F_l(x,v) = \lambda^{m+n+1/2}k^{m+n-1}2^{(m-1)l}$$

$$(4.1) \quad \cdot \int\int \chi_k,\ell(r,t) e^{i\lambda \ell t} \left( \psi(2^{-l}r) + 4v \right) a(\lambda \ell t|v|) dr dt,$$

where $\chi_k,\ell(r,t)$ is a smooth function supported where $r \sim 1, t \sim 1,$
such that
\begin{equation}
|\partial_r^\alpha \partial_t^\beta \chi_{k,l}(r,t)| \leq C_{\alpha,\beta} \text{ for every } \alpha, \beta \in \mathbb{N},
\end{equation}
uniformly in \( k \) and \( l \).

Since
\begin{align}
\psi'(s) &= -\frac{1}{(s + k\pi)^2} + |x|^2 \sin(s)^{-2}, \\
\psi''(s) &= \frac{2}{(s + k\pi)^3} - 2|x|^2 \cos(s) \sin(s)^{-3},
\end{align}
we see that the critical points \( s = s_c \) of \( \psi \) are given by the equation
\[ \sin(s_c) = \pm |x|(s_c + k\pi), \]
hence those of \( \psi(2^{-l}) \) by \( r = 2^l s_c \). If \( r \) is supposed to lie in our domain of integration, then \( |x| \sim 2^{-l}k^{-1} \). For such \( x \), there is then in fact exactly one critical point \( s_c = s_{c,k}(|x|) \sim 2^{-l} \) sufficiently close to our domain of integration, given by
\begin{equation}
\sin(s_c) = |x|(s_c + k\pi).
\end{equation}
For later use, let us also put
\[ \gamma_k(|x|) := \psi(s_{c,k}(|x|)) = \frac{1}{s_{c,k}(|x|) + k\pi} - |x|^2 \cot(s_{c,k}(|x|)). \]
Observe that, by (4.4), this can be re-written as
\begin{equation}
\gamma_k(|x|) = \frac{s_k(|x|) \sin s_k(|x|) \cos s_k(|x|)}{s_k(|x|)^2},
\end{equation}
if we put \( s_k(|x|) := s_{c,k}(|x|) + k\pi \).

Assume that \( \lambda \geq 2^l k \).

In this case, one gains from the \( r \)-integration in \( F_l \), which we perform first. To this end, recall that \( \psi(s) \) has now a unique critical point \( s_c = s_{c,k}(|x|) \sim 2^{-l} \) if \( |x| \sim 2^{-l}k^{-1} \).

We may thus apply the method of stationary phase to the integration in \( r \). Noticing that the critical point of the phase \( \psi_l \) is given by \( 2^l s_{c,k}(|x|) \), one eventually finds that
\begin{align}
F_l(x, v) &= \chi^{m+n+1/2} k^{m+n-1/2} (m-1) \int \chi(t)a(\lambda kt 4|v|) \\
&\quad b(k^{2^l 2^{2l}|x|^2} \lambda k^{-1} 2^{-l} t)e^{i\lambda kt \left( \psi(s_{c,k}(|x|) + 4v) \right)} dr dt,
\end{align}
where $\chi(t)$ is again a smooth cut-off function supported where $t \sim 1$, and where $b \in S^{-1/2}$ is a symbol of order $-1/2$ on $T^*\mathbb{R}$, which may in fact depend also on $k$ and $l$, but which lie in a bounded set of $S^{-1/2}$ with respect to the natural Fréchet topology on $S^{-1/2}$.

Consequently,

$$A_{\lambda,k,l,+a}(x, \pm |u|) = \lambda^{m+n+1/2}k^{m+n-1/2}2^{(m-1)/l} \int \chi(t) a(\lambda kt |u|) b(k^22^l|x|^2, \lambda k^{-1/2}t) e^{i\lambda kt \left( \gamma_k(|x|) \pm 4|u| \right)} dt,$$

where $\gamma_k(|x|) := \psi(s_c, k(|x|))$ is explicitly given by (4.5).

Now, integrations by parts in (4.6) easily yield that, for every $N \in \mathbb{N}$,

$$|A_{\lambda,k,l,+a}(x, \pm |u|)| \leq C N \lambda^{m+n}k^{m+n-1/2}2^{(m-1)/l} \cdot (1 + \lambda k |u|)^{-\frac{n-1}{2}} \left( 1 + \lambda k |\gamma_k(|x|) \pm 4|u| \right)^{-N}.$$

(4.7)

The actual details of these arguments are more involved, and a complete estimate of $A_{\lambda,k,l,+a}$ requires a careful case analysis and domain decompositions.

**Remarks 4.1.** (a) Observe that (4.7) shows that the singularities of $A_{\lambda,k,l,+}(x, u)$ are located where $|u| = \gamma_k(|x|)$. In view of (4.4) and (4.5), these are thus located where the pair of norms $(|x|, 4|u|)$ lies on the piece of the parametric curve

$$\gamma : s \mapsto \left( \frac{\sin s}{s}, \left| \frac{s - \sin s \cos s}{s^2} \right| \right),$$

corresponding to the range of parameters where $s - k\pi \sim 2^{-l}$.

This corresponds exactly to the set $\Sigma_1$, in view of (2.4).

(b) Analogous estimates and results hold true for $A_{\lambda,k,l,-}(x, u)$. Here, we have $s_c(|x|) < 0$, and the singularities correspond to the range of parameters where $s - k\pi \sim -2^{-l}$.
Let us put
\[ \alpha(p) := (d - 1) \left| \frac{1}{p} - \frac{1}{2} \right|, \quad 1 \leq p < \infty. \]

We need to estimate \( (1 + L)^{-\alpha(p)/2} e^{i\sqrt{L}\delta} \) on \( L^p(G) \) in Theorem 1.1. By means of a dyadic decomposition, this can easily be reduced to estimating the operator
\[
\sum_{j=0}^{\infty} 2^{-\alpha(p)j} \tilde{\chi} \left( \frac{\sqrt{L}}{2^j} \right) e^{i\sqrt{L}},
\]
for a suitable cut-off function \( \tilde{\chi} \) supported in \([1/16, 4]\). In view of (3.10), this means that we have to estimate the convolution operators on \( L^p(G) \) given by the integral kernels
\[
E_{k,J}^{p,\pm}(x, u) := \sum_{j=0}^{J} \sum_{l=0}^{\infty} 2^{-\alpha(p)j} A_{2j,k,l,\pm}(x, u)
\]
and the corresponding kernels, with \( A_{\lambda,k,l,\pm} \) replaced by \( B_{\lambda,k,l,\pm} \). These kernels will converge, in the sense of distributions, to distributions \( E_k^{p,\pm} \) as \( J \to \infty \), and we shall have to show that the estimates sum in \( k \).
Our estimates will follow from some rather elementary estimate for the case $p = 2$, and a deeper estimate for $p = 1$ on a suitable local Hardy space $h_k$, whose definition will depend on $k$, by means of complex interpolation. The corresponding interpolation argument is standard (cf. [35],[21]).

Let’s here concentrate on the case $p = 1$, and the kernels

$$E^+_{k,J} := E^{1,+}_{k,J} = \sum_{j=0}^{J} \sum_{l=0}^{\infty} 2^{\frac{(m-1)}{2} j} A_{2j,k,l}.$$

What can be proved in the end for these kernels is a result, which, somewhat simplified, reads as follows:

**Proposition 5.1.** Let $k \geq 1$. There exist kernel functions $K^\pm_{\lambda,k,l} \in S(G)$, for $\lambda \geq 2^l k$, and $R_{k,J} \in L^1(G)$ so that

$$E^+_{k,J} = \sum_{l=0}^{\infty} \sum_{j=l+\log_2 k}^{J} (K^+_{2j,k,l} + K^-_{2j,k,l}) + R_{k,J},$$

and such that the following hold true:

(a) The kernel $K^\pm_{\lambda,k,l}$ is supported in the set

$$\{(x,u) \in G : |x| \sim 2^{-l} k^{-1} \text{ and } \lambda k|u| \geq 1\},$$

and can be estimated, for every $\alpha \in \mathbb{N}^{2m}, \beta \in \mathbb{N}^n$, by

$$|\partial^\alpha_x \partial^\beta_u K^\pm_{\lambda,k,l}(x,u)| \leq C_{N,\alpha,\beta} \lambda^{\alpha} (\lambda k)^{|\beta|} \lambda^{\frac{m+n-1}{2}} k^{m+n-\frac{1}{2}} 2^{(m-\frac{1}{2})l} (1 + \lambda k|u|)^{-N},$$

for every $N \in \mathbb{N}$, with constants $C_{N,\alpha,\beta}$ which independent of $\lambda, k$ and $l$, and where $\gamma_k(|x|)$ is given by (4.5). In particular,

$$\|\partial^\alpha_x \partial^\beta_u K^\pm_{\lambda,k,l}\|_1 \lesssim \lambda^{\alpha} (\lambda k)^{|\beta|} (2^l k)^{-m-\frac{1}{2}}.$$

(b) The sequence $\{R_{k,J}\}_J$ converges for $J \to \infty$ in $L^1(G)$ towards a function $R_k \in L^1(G)$, and

$$\|R_k\|_1 \leq C k^{-m-\frac{1}{2}},$$

with a constant $C$ not depending on $k$. 
5.1. Anisotropic re-scaling for $k$ fixed. Let $k \geq 1$, and denote by

$$\tilde{F}(x,u) := k^{-D}F(k^{-1}x,k^{-2}u)$$

the $L^1$-norm preserving scaling of a function $F$ on $G$ by $\delta_{k^{-1}}$. Let us also put

$$\hat{\Gamma}_k(v) := k^2\gamma_k(k^{-1}v) - \frac{k}{\pi}.$$ 

We can then make the interesting observation that, by (5.3), (5.4), the non-isotropically re-scaled kernels $\hat{K}_{\lambda,k,l}^{\pm}$ are supported in the set

$$\{(x,u) \in G : |x| \sim 2^{-l} \text{ and } \frac{\lambda}{k}|u| \gtrsim 1\},$$

and satisfy the isotropic estimates

$$|\partial^\alpha_x \partial^\beta_u \hat{K}_{\lambda,k,l}^{\pm}(x,u)| \leq C_{N,\alpha,\beta} \left( \frac{\lambda}{k} \right)^{|\alpha|+|\beta|} k^{\frac{m+n+1}{2}} 2^{(m-\frac{1}{2})l} (1 + \frac{\lambda}{k}|u| - \hat{\Gamma}_k(|x| \pm 4|u|))^{-N}$$

and

$$\|\partial^\alpha_x \partial^\beta_u \hat{K}_{\lambda,k,l}^{\pm}\|_1 \lesssim \left( \frac{\lambda}{k} \right)^{|\alpha|+|\beta|} (2^l k)^{-m-\frac{1}{2}}.$$ 

Moreover, one easily verifies that

$$\hat{\Gamma}_k(v) = \pi^{-2}\arcsin(\pi v) + \pi^{-1}v\sqrt{1 - (\pi v)^2} + O\left(\frac{1}{k}\right).$$

6. $L^2$-estimates for components of the wave propagator

Denote by $A_{\lambda,k,l,\pm}$ the convolution operator

$$A_{\lambda,k,l,\pm}f := f \ast A_{\lambda,k,l,\pm}$$

on $L^2(G)$. It follows from Proposition 3.1 that

$$A_{\lambda,k,l,\pm}^{\nu} = \lambda^{1/2} \left( \int \chi_0(t)\chi_1(\pm 2^l(2\pi t|\mu|/\lambda - k\pi)) \eta_0(2\pi t|\mu|/\lambda - k\pi) e^{it\lambda L^\nu} dt \right) \delta.$$ 

Comparing with (9.13) and (9.12) in the Appendix, which obtained by means of Plancherel’s formula on $G$, we then find that the operator norm of $A_{\lambda,k,l,\pm}$ on $L^2(G)$ is given by

$$\|A_{\lambda,k,l,\pm}\| = \lambda^{1/2} \sup\{|\alpha_{\lambda,k,l,\pm}(\nu,q)| : \nu \geq 0, q \in \mathbb{N}\},$$
where
\[ \alpha_{\lambda,k,l,\pm}(\nu, q) := \int e^{i(\delta + t_{\pm}^{m+2q})} dt. \]

The method of stationary phase then implies that
\[ \|A_{\lambda,k,l,\pm}\| \leq C, \]
uniformly in \( \lambda, k \) and \( l \). By the same method, we also get, for any \( \alpha \geq 0 \),
\[ \| \sum_{l=0}^{\infty} \sum_{j \geq j_0} 2^{-\alpha j} A_{2^j,k,l,\pm} \| \leq C 2^{-\alpha j_0}, \]
uniformly in \( l \) and \( j_0 \).

7. Estimation for \( p = 1 \)

We introduce a non-standard atomic local Hardy space \( h^1 \) on \( G \) as follows:

An atom centered at the origin is an \( L^2 \)-function \( a \) supported in a Euclidean ball \( B_r(0) \) of radius \( r \leq 1 \) satisfying the normalizing condition
\[ \|a\|_2 \leq r^{-d/2} \]
and, if \( r < 1 \), in addition the moment condition
\[ \int_G a(x) \, dx = 0. \]
Notice that in particular \( \|a\|_1 \lesssim 1 \). An atom centered at \( z \in G \) is the left-translate by \( z \) of an atom centered at the origin. The local Hardy space \( h^1(G) \) consists of all \( L^1 \)-functions which admit an atomic decomposition
\[ f = \sum_j \lambda_j a_j, \]
with atoms \( a_j \) and coefficients \( \lambda_j \) such that \( \sum_j |\lambda_j| < \infty \). We define the norm \( \|f\|_{h^1} \) as the infimum of the sums \( \sum_j |\lambda_j| < \infty \) over all possible atomic decompositions (7.3) of \( f \).

In comparison, let us denote by \( h^1_E := h^1(\mathbb{R}^{2m} \times \mathbb{R}^n) \) the classical Euclidean local Hardy space introduced by Goldberg [10]. Since atoms
in this space which are supported in a ball of radius \( r \geq 1 \) need not satisfy (7.2), such atoms can be decomposed into atoms supported in balls of radius 1, so that one can give the same atomic definition for \( h^1_E \) as for \( h^1 \), only with the non-commutative group translation in \( G \) replaced by Euclidean translation.

Consider the cylindrical sets
\[
Z(y) := \{(x, u) \in G : |x - y| \leq 10\},
\]
for \( y \in \mathbb{R}^{2m} \). Note the following easy, but important observations:

**Lemma 7.1.** (a) Any atom \( a \) supported in \( Z(0) \) is indeed also a Euclidean atom in \( h^1_E \) (possibly up to a fixed factor depending only on \( G \)), and vice versa.

(b) If \( H^1_E := H^1(\mathbb{R}^{2m} \times \mathbb{R}^n) \) denote the classical Euclidean Hardy space, and if \( \eta \in C_0^\infty(\mathbb{R}^{2m}) \), then multiplication with \( \eta \otimes 1 \) maps \( H^1_E \) continuously into \( h^1_E \).

(b) is known for compactly supported smooth multipliers, and the proof carries over easily.

Since \( G \) can be covered by sets \( Z(y_j) \) with bounded overlap, if we choose a suitable \( \varepsilon \)-separated set of points \( \{y_j\}_j \) for a suitable constant \( \varepsilon > 0 \), one can use a) and b) in combination with classical arguments [6] to show that our non-standard Hardy space \( h^1(G) \) interpolates with \( L^2(G) \) by the complex method.

We shall not do this here, but indicate the main idea in the interpolation argument explained later.

**Proposition 7.2.** There is a constant \( C > 0 \), such that
\[
\|f \ast \overline{E^1_k} \|_1 \leq Ck^{-m}\|f\|_{h^1}
\]
for every \( f \in h^1(G) \).

**Proof.** We again consider only \( E^{1,+}_k = E^+_k \). In view of the atomic decomposition of \( f \) and because of left-invariance, one can show that it suffices to assume that \( f = a \) is an atom centered at the origin and supported in a ball \( B_r(0) \). We shall also assume that \( r < 1 \), so that \( a \) has vanishing integral, since the case \( r = 1 \) is somewhat easier. In view of Proposition 5.1, we have to understand in particular the functions
\[
a * \overline{K^{\pm}_{2^j,k,l}}.
\]
To simplify the argument a bit, assume that \( n = 1 \), so that \( |u| = \pm u \).

Let’s also assume that we consider the part of \( \widetilde{K}_{\lambda, k, l} \) where \( u > 0 \), and that \( \widetilde{K}_{\lambda, k, l} \) were not only essentially, but actually supported in the set
\[
\left\{ \left( x, u \right) \in G : |x| \sim 2^{-l} \ \text{and} \ |u - \frac{1}{2} \Gamma_k(|x|)| \leq \frac{k}{\lambda} \right\},
\]
and that
\[
\left| \frac{k}{\pi} - \Gamma_k(|x|) + 4u \right| \leq \frac{k}{\lambda}.
\]
Translating these kernels along the center by \( k \frac{\pi}{4} \), in view of (5.6) we can then make the (slightly oversimplified) assumption that

\[
\text{(7.4) } \text{supp} \widetilde{K}_{\lambda, k, l} \subset \left\{ \left( x, u \right) \in G : |x| \sim 2^{-l} \ \text{and} \ |u - \frac{1}{4} \Gamma_k(|x|)| \leq \frac{k}{\lambda} \},
\]
and that

\[
\text{(7.5) } |\partial^\alpha x^\beta u \widetilde{K}_{\lambda, k, l}(x, u)| \leq C_{N, \alpha, \beta} \left( \frac{\lambda}{k} \right)^{\alpha + |\beta|} \lambda k^{-(m+n+\frac{1}{2})2(m-n+\frac{1}{2})}.
\]

Define for \( \rho > 0 \) the set
\[
\Omega_\rho := \left\{ \left( x, u \right) \in G : |x| \lesssim 1 \ \text{and} \ |u - \frac{1}{4} \Gamma_k(|x|)| \leq C \rho \right\}
\]
where \( C \) is a sufficiently large constant, and choose \( j_0 \) so that

\[
k^{2-j_0} = r.
\]

Observe that with this choice of \( j_0 \)

\[
\text{(7.6) } \text{supp} a * (\sum_l \sum_{j \geq j_0} \widetilde{K}_{\lambda, k, l} \subset \Omega_r),
\]
since \( \text{supp} a \subset B_r(0) \). We therefore split
\[
E_{k, r}^{1, \pm} = E_{k, r} + F_{k, r},
\]
where
\[
E_{k, r} := \sum_{l=0}^{\infty} \sum_{j \geq j_0} 2^{-\alpha(1)j} A_{2^l k, k, l, \pm},
\]
\[
F_{k, r} := \sum_{l=0}^{\infty} \sum_{j < j_0} 2^{-\alpha(1)j} A_{2^l k, k, l, \pm}.
\]

We estimate \( a * \widetilde{E}_{k, r} \) separately on the set \( \Omega_r \) and its complement. By Hölder’s inequality and (6.2), we get

\[
\|a * \widetilde{E}_{k, r}\|_{L^1(\Omega_r)} \lesssim |\Omega_r|^{1/2} \|a\|_2 2^{-\alpha(1)j_0} \lesssim r^{1/2} r^{-(m+1/2)2} r^{-(d-1)} j_0 \leq k^{-m}.
\]

Moreover, since by Proposition 5.1 \( E_{k, r} = \sum_l \sum_{j \geq j_0} \widetilde{K}_{\lambda, k, l} + \widetilde{R}_{k, r} \), where \( \|R_{k, r}\|_1 \lesssim k^{m-\frac{1}{2}} \), we obtain from (7.6) that

\[
\|a * \widetilde{R}_{k, r}\|_{L^1(G \setminus \Omega_r)} \lesssim \|a * \widetilde{R}_{k, r}\|_{L^1(G)} \lesssim k^{m-\frac{1}{2}},
\]
so that
\[(7.7) \quad \|a * E_{k,r}\|_1 \lesssim k^{-m}.
\]

Consider next \(a * F_{k,r}\). Again, by Proposition 5.1, we can write \(\widehat{F_{k,r}} = \sum_l \sum_{j < j_0} \widehat{K_{2^j,k,l}}^{\pm} + \widehat{R'_{k,r}}\), where \(\|R'_{k,r}\|_1 \lesssim k^{-m-\frac{1}{2}}\). And, for \(j < j_0\), we have, by (7.4),
\[
\text{supp } a * \left( \sum_l \widehat{K_{2^j,k,l}}^{\pm} \right) \subset \Omega_{k2^{-j}}.
\]
Moreover, making use of the cancellation \(\int a dx = 0\) and (7.5) in estimating the convolution product \(a * K_{2^j,k,l}^{\pm}\), we gain a factor of order \(O(r^{k2^{-j}})\) compared to (5.7) and obtain
\[
\|a * \left( \sum_l \widehat{K_{2^j,k,l}}^{\pm} \right)\|_1 \lesssim \sum_{l=0}^{\infty} \frac{r}{k2^{-j}} (2^j k)^{m-\frac{1}{2}} \lesssim 2^{j} \frac{r}{k} k^{-m-\frac{1}{2}}
\]
Summing in \(j\), this gives
\[(7.8) \quad \|a * F_{k,r}\|_1 \lesssim k^{-m-\frac{1}{2}}.
\]
The estimates (7.7) and (7.8) imply the proposition. \(\square\)

**Remark 7.3.** Let us denote by \(h^1_k(G)\) the re-scaled local Hardy space consisting of all function
\[
f(x,u) := k^{D} \tilde{f}(kx,k^2u), \quad \tilde{f} \in h^1(G).
\]
Then Proposition 7.1 can be re-stated as
\[
\|f * E^{1,\pm}_k\|_1 \leq C k^{-m} \|f\|_{h^1_k}
\]
for every \(f \in h^1_k(G)\). Thus, \(h^1_k(G)\) is a natural space associated with the k-th piece \(E^{1,\pm}_k\) of the wave propagator whose definition depends on the parameter \(k\).

### 8. Proof of Theorem 1.1

Observe that by (6.2) convolution with the kernel
\[
E^{2,\pm}_k := \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} A_{2^j,k,l,\pm}
\]
is bounded on \(L^2(G)\), and
\[(8.1) \quad \|f * E^{2,\pm}_k\|_2 \leq C \|f\|_2,
\]
where $C$ is independent of $k$. Consider now the analytic family of operators

$$T_\alpha : f \mapsto f \ast \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} 2^{-\alpha j} \widetilde{A_{2j,k,l,\pm}}, \quad 0 \leq \text{Re } \alpha \leq \frac{(d-1)}{2}.$$ 

By ignoring some error terms, we may assume that all $\widetilde{A_{2j,k,l,\pm}}$ are supported in $Z(0)$ (even a smaller cylindrical set). This implies that if $f$ is supported in a cylindrical set $Z(y)$, then $T_\alpha f$ is supported in the “cylindrical double” $2Z(y) := \{(x,u) \in G : |x-y| \leq 20\}$, for every $y \in \mathbb{R}^{2m}$, which allows to reduce to functions $f$ supported in a set $Z(y)$, and we may even assume $y = 0$, by translation invariance.

Choosing a suitable cut-off function $\eta \in C^\infty_0(\mathbb{R}^{2m})$ so that $\eta \otimes 1$ localizes to $Z(0)$, we therefore consider the localized operators

$$T^0_\alpha f := T_\alpha((\eta \otimes 1)f).$$

By Lemma 7.1 and Proposition 7.3, $T^0_{\frac{d-1}{2}} : H^1_E \to L^1$ continuously, and since the proof of Proposition 7.3 remains valid for $T^0_{\frac{d-1}{2} + i\beta}$, we have

$$\|T^0_{\frac{d-1}{2} + i\beta} f\|_1 \leq C k^{-m} \|f\|_{H^1_E} \quad \forall \beta \in \mathbb{R}.$$ 

Similarly, by (8.1), we have

$$\|T^0_{i\beta} f\|_2 \leq C \|f\|_2 \quad \forall \beta \in \mathbb{R}.$$ 

We can thus apply the complex interpolation result by C. Fefferman and Stein in [6] to obtain

$$\|T^0_{\alpha(p)} f\|_p \leq C k^{-\frac{2-1}{p}m} \|f\|_p, \quad 1 < p \leq 2.$$ 

In view of the afore-mentioned support property of the convolution kernels involved, this immediately implies

$$\|f \ast E^\alpha(p,\pm)\|_p \leq C k^{-\frac{2-1}{p}m} \|f\|_p, \quad 1 < p \leq 2.$$ 

If $m \geq 2$, these estimates sum in $k$ for $p$ sufficiently close to 1, so that

$$(8.2) \quad \|(1 + L)^{-\alpha(p)/2} e^{i\sqrt{L}} f\|_p \leq C \|f\|_p$$

for these $p$. Interpolating these estimates with the corresponding trivial estimate for $p = 2$ we then obtain Theorem 1.1 for $1 < p \leq 2$, and the case $p > 2$ follows by duality.

For $m = 1$, the result can be obtained by means of a minor refinement of the arguments.
The $L^1$-estimate in Theorem 1.1 follows a lot more easily from Proposition 5.1, since for $\alpha > \frac{d-1}{2}$ we have an additional factor $\lambda^{-(\alpha-\frac{d-1}{2})}$ in the corresponding estimate (5.4), so that the corresponding norms $\|K_{2^j,k,l}^\pm\|_1$ simply sum in $j$, $k$ and $l$. 
9. Appendix: The Fourier transform on a group of Heisenberg type

Let us first briefly recall some facts about the unitary representation theory of a Heisenberg type group $G$ (compare [36],[5]). For the convenience of the reader, we shall show how this representation theory can easily be reduced to the well-known case of the Heisenberg group $\mathbb{H}_m = \mathbb{R}^{2m} \times \mathbb{R}$. In slightly modified notation, the product in $\mathbb{H}_m$ is given by

$$(z, t) \cdot (z', t') = (z + z', t + t' + \frac{1}{2} \omega(z, z')),$$

where $\omega$ denotes the canonical symplectic form

$$(9.1) \quad \omega(z, w) := \langle Jz, w \rangle, \quad J := \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix},$$

on $\mathbb{R}^{2m}$. Let us split coordinates $z = (x, y) \in \mathbb{R}^m \times \mathbb{R}^m$ in $\mathbb{R}^{2m}$, and consider the associated natural basis of left-invariant vector fields

$$\tilde{X}_j := \frac{\partial}{\partial x_j} - \frac{1}{2} y_j \frac{\partial}{\partial t}, \quad \tilde{Y}_j := \frac{\partial}{\partial y_j} + \frac{1}{2} x_j \frac{\partial}{\partial t}, \quad j = 1, \ldots, m,$$

and $T := \frac{\partial}{\partial t}$, of the Lie algebra of $\mathbb{H}_m$.

For $\tau \in \mathbb{R}^\times := \mathbb{R} \setminus \{0\}$, the Schrödinger representation $\rho_\tau$ of $\mathbb{H}_m$ acts on the Hilbert space $L^2(\mathbb{R}^m)$ as follows:

$$[\rho_\tau(x, y, t) h](\xi) := e^{2\pi i \tau(t + y \xi + \frac{1}{2} y x)} h(\xi + x), \quad h \in L^2(\mathbb{R}^m).$$

This is an irreducible, unitary representation, and every irreducible unitary representation of $\mathbb{H}_m$ which acts non-trivially on the center is in fact unitarily equivalent to exactly one of these, by the Stone-von Neumann theorem (a good reference to these and related results is for instance [8]; see also [26]).

Next, if $\pi$ is any unitary representation, say, of a Heisenberg type group $G$, we denote by

$$\pi(f) := \int_G f(g) \pi(g) \, dg, \quad f \in L^1(G),$$

the associated representation of the group algebra $L^1(G)$. Going back to the Heisenberg group, if $f \in S(\mathbb{H}_m)$, then it is well-known and easily seen that $\rho_\tau(f) = \int_{\mathbb{R}^{2m}} f^{-\tau}(z) \rho_\tau(z, 0) \, dz$ is a trace class operator on $L^2(\mathbb{R}^m)$, and its trace is given by

$$(9.2) \quad \text{tr}(\rho_\tau(f)) = |\tau|^{-m} \int_{\mathbb{R}} f(0, 0, t) e^{2\pi i \tau t} \, dt = |\tau|^{-m} f^{-\tau}(0, 0),$$
for every \( \tau \in \mathbb{R}^\times \).

From these facts, it is easy to derive the Plancherel formula for our Heisenberg type group \( G \). Given \( \mu \in \mathfrak{g}^*_2 = \mathbb{R}^n \), \( \mu \neq 0 \), consider the matrix \( J_\mu \) introduced in Section 3. If \( |\mu| = 1 \), then \( J^2_\mu = -I \), because of (1.1). In particular, \( J_\mu \) has only eigenvalues \( \pm i \), and since it is orthogonal, it is easy to see that there exists an orthonormal basis

\[
X_{\mu,1}, \ldots, X_{\mu,m}, Y_{\mu,1}, \ldots, Y_{\mu,m}
\]

of \( \mathfrak{g}_1 = \mathbb{R}^{2m} \) which is symplectic with respect to the form \( \omega_\mu \), i.e., \( \omega_\mu \) is represented by the standard symplectic matrix \( J \) in (9.1).

This means that, for every \( \mu \in \mathbb{R}^n \setminus \{0\} \), there is an orthogonal matrix \( R_\mu = R_\mu |\mu|^{-1} \in O(2m, \mathbb{R}) \) such that

\[
J_\mu = |\mu| R_\mu J R_\mu.
\]

Notice also that \( R_\mu = R_\mu^{-1} \), and that condition (9.3) is in fact equivalent to \( G \) being of Heisenberg type.

We remark that (9.3) easily implies that the subalgebra \( L^1_r(G) \) of \( L^1(G) \), consisting of all radial functions \( f(x,u) \) in the sense that they depend only on \( |x| \) and \( u \), is commutative. This fact is well-known and easy to prove for Heisenberg groups ([8],[26]). And, by means of (9.3), one easily checks that

\[
(\varphi *_\mu \psi) \circ R_\mu = (\varphi \circ R_\mu) \times |\mu| (\psi \circ R_\mu)
\]

for all functions \( \varphi, \psi \in L^1(\mathbb{R}^{2m}) \). Therefore, if \( f, g \in L^1_r(G) \), then \( f^\mu \circ R_\mu, g^\mu \circ R_\mu \) are radial on \( \mathbb{R}^{2m} \), and thus \( f^\mu \circ R_\mu \times |\mu| g^\mu \circ R_\mu = g^\mu \circ R_\mu \times |\mu| f^\mu \circ R_\mu \), which is again radial. Consequently, \( f^\mu *_\mu g^\mu = g^\mu *_\mu f^\mu \), for every \( \mu \), and thus

\[
f \ast g = g \ast f \quad \text{for every } f, g \in L^1_r(G).
\]

The following lemma is easy to check and establishes a useful link between representations of \( G \) and those of \( \mathbb{H}_m \).

**Lemma 9.1.** The mapping \( \alpha_\mu : G \to \mathbb{H}_m \), given by

\[
\alpha_\mu(z,u) := (R_\mu z, \frac{u}{|\mu|}), \quad (z,u) \in \mathbb{R}^{2m} \times \mathbb{R}^n,
\]

is an epimorphism of Lie groups. In particular, \( G/\ker \alpha_\mu \) is isomorphic to \( \mathbb{H}_m \), where \( \ker \alpha_\mu = \mu^\perp \) is just the orthogonal complement of \( \mu \) in the center \( \mathbb{R}^n \) of \( G \).
Given \( \mu \in \mathbb{R}^n \setminus \{0\} \), we can now define an irreducible unitary representation \( \pi_\mu \) of \( G \) on \( L^2(\mathbb{R}^m) \) by simply putting
\[
\pi_\mu := \rho_{|\mu|} \circ \alpha_\mu .
\]
Observe that then \( \pi_\mu(0, u) = e^{2\pi i \mu \cdot u} I \). In fact, any irreducible representation of \( G \) with central character \( e^{2\pi i \mu \cdot u} \) factors through the kernel of \( \alpha_\mu \) and hence, by the Stone-von Neumann theorem, must be equivalent to \( \pi_\mu \).

One then computes that, for \( f \in \mathcal{S}(G) \),
\[
\pi_\mu(f) = \int_{\mathbb{R}^{2m}} f^{-\mu}(R_\mu z) \rho_{|\mu|}(z, 0) \, dz,
\]
so that the trace formula (9.2) yields the analogous trace formula
\[
\text{tr} \pi_\mu(f) = |\mu|^{-m} f^{-\mu}(0)
\]
on \( G \). The Fourier inversion formula in \( \mathbb{R}^n \) then leads to \( f(0, 0) = \int_{\mu \in \mathbb{R}^n \setminus \{0\}} \text{tr} \pi_\mu(f) |\mu|^m \, d\mu \). When applied to \( \delta_{g^{-1}} * f \), we arrive at the Fourier inversion formula
\[
(9.5) \quad f(g) = \int_{\mu \in \mathbb{R}^n \setminus \{0\}} \text{tr} (\pi_\mu(g)^* \pi_\mu(f)) |\mu|^m \, d\mu, \quad g \in G.
\]
Applying this to \( f^* * f \) at \( g = 0 \), where \( f^*(g) := \overline{f(g^{-1})} \), we obtain the Plancherel formula
\[
(9.6) \quad \|f\|_2^2 = \int_{\mu \in \mathbb{R}^n \setminus \{0\}} \|\pi_\mu(f)\|_{HS}^2 |\mu|^m \, d\mu,
\]
where \( \|T\|_{HS}^2 = \text{tr} (T^* T) \) denotes the Hilbert-Schmidt norm.

Let us next consider the group Fourier transform of our sub-Laplacian \( L \) on \( G \).

We first observe that \( d\alpha_\mu(X) = \iota R_\mu X \) for every \( X \in g_1 = \mathbb{R}^{2m} \), if we view, for the time being, elements of the Lie algebra as tangential vectors at the identity element. Moreover, by (9.3), we see that
\[
\iota R_\mu X_{\mu,1}, \ldots, \iota R_\mu X_{\mu,m}, \iota R_\mu Y_{\mu,1}, \ldots, \iota R_\mu Y_{\mu,m}
\]
forms a symplectic basis with respect to the canonical symplectic form \( \omega \) on \( \mathbb{R}^{2m} \). We may thus assume without loss of generality that this basis agrees with our basis \( \tilde{X}_1, \ldots, \tilde{X}_m, \tilde{Y}_1, \ldots, \tilde{Y}_m \) of \( \mathbb{R}^{2m} \), so that
\[
d\alpha_\mu(X_{\mu,j}) = \tilde{X}_j, \quad d\alpha_\mu(Y_{\mu,j}) = \tilde{Y}_j, \quad j = 1, \ldots, m.
\]
By our construction of the representation $\pi_\mu$, we thus obtain for the derived representation $d\pi_\mu$ of $\mathfrak{g}$ that

\begin{equation}
(9.7) \quad d\pi_\mu(X_{\mu,j}) = d\rho_{|\mu|}(\tilde{X}_j), \quad d\pi_\mu(Y_{\mu,j}) = d\rho_{|\mu|}(\tilde{Y}_j), \quad j = 1, \ldots, m.
\end{equation}

Let us define the sub-Laplacians $L_\mu$ on $G$ and $\tilde{L}$ on $H_m$ by

\begin{equation}
L_\mu := -\sum_{j=1}^{m}(X_{\mu,j}^2 + Y_{\mu,j}^2), \quad \tilde{L} := -\sum_{j=1}^{m}(\tilde{X}_j^2 + \tilde{Y}_j^2),
\end{equation}

where from now on we consider elements of the Lie algebra again as left-invariant differential operators. Then, by (9.6),

\begin{equation}
(9.8) \quad d\pi_\mu(L_\mu) = d\rho_{|\mu|}(\tilde{L}).
\end{equation}

Moreover, since the basis $X_{\mu,1}, \ldots, X_{\mu,m}, Y_{\mu,1}, \ldots, Y_{\mu,m}$ and our original bases $X_1, \ldots, X_{2m}$ of $\mathfrak{g}_1$ are both orthonormal bases, it is easy to verify that the distributions $L\delta_0$ and $L_{\mu}\delta_0$ do agree. Since $Af = f*(A\delta_0)$ for every left-invariant differential operator $A$, we thus have $L = L_\mu$, hence

\begin{equation}
(9.9) \quad d\pi_\mu(L) = d\rho_{|\mu|}(\tilde{L}).
\end{equation}

But, it is well-known that $d\rho_{|\mu|}(\tilde{L}) = \Delta_\xi - (2\pi|\mu|)^2|\xi|^2$ is just a re-scaled Hermite operator, and an orthonormal basis of $L^2(\mathbb{R}^m)$ is given by the tensor products

\begin{equation}
(9.10) \quad d\pi_\mu(U_j) = 2\pi i\mu_j I, \quad j = 1, \ldots, n.
\end{equation}

Now, the operators $L, -iU_1, \ldots, -iU_n$ form a commuting set of self-adjoint operators, with joint core $\mathcal{S}(G)$, so that they admit a joint spectral resolution, and we can thus give meaning to expressions like $\varphi(L, -iU_1, \ldots, -iU_n)$ for each continuous function $\varphi$ defined on the corresponding joint spectrum. For simplicity of notation we write

\begin{equation}
U := (-iU_1, \ldots, -iU_n).
\end{equation}
If $\varphi$ is bounded, then $\varphi(L, U)$ is a bounded, left invariant operator on $L^2(G)$, so that it is a convolution operator

$$\varphi(L, U)f = f \ast K_\varphi, \quad f \in S(G),$$

with a convolution kernel $K_\varphi \in S'(G)$ which will also be denoted by $\varphi(L, U)\delta$. Moreover, if $\varphi \in S(\mathbb{R} \times \mathbb{R}^n)$, then $\varphi(L, U)\delta \in S(G)$ (compare [29], [30]). Since functional calculus is compatible with unitary representation theory, we obtain in this case from (9.9), (9.10) that

$$\pi_{\mu}(\varphi(L, U)\delta) h_{\alpha}^{\mu} = \varphi(2\pi|\mu|(m+2|\alpha|), 2\pi \mu) h_{\alpha}^{\mu}$$

(this identity in combination with the Fourier inversion formula could in fact be taken as the definition of $\varphi(L, U)\delta$). In particular, the Plancherel theorem implies then that the operator norm on $L^2(G)$ is given by

$$\|\varphi(L, U)\| = \sup\{|\varphi(|\mu|(m+2q), \mu)| : \mu \in \mathbb{R}^n, q \in \mathbb{N}\}.$$  

(9.12)

Finally, observe that

$$K_{\phi}^{\mu} = \varphi(L^{\mu}, 2\pi \mu)\delta;$$

this follows for instance by applying the unitary representation induced from the character $e^{2\pi \mu \cdot u}$ on the center of $G$ to $K_{\phi}$.  

\textbf{References}


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