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A Conditioned Local Limit Theorem for Nonnegative Random Matrices

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Abstract
For any fixed real $a > 0$ and $x \in \mathbb{R}^d, d \geq 1$, we consider the real-valued random process $(S_n)_{n\geq 0}$ defined by $S_0 = a, S_n = a + \ln|g_n \cdots g_1 x|, n \geq 1$, where the $g_k, k \geq 1$, are i.i.d. nonnegative random matrices. By using the strategy initiated by Denisov and Wachtel to control fluctuations in cones of d-dimensional random walks, we obtain an asymptotic estimate and bounds on the probability that the process $(S_n)_{n\geq 0}$ remains nonnegative up to time n and simultaneously belongs to some compact set $[b, b + \ell] \subset \mathbb{R}_*^+$ at time n.

Keywords Local limit theorem · Random walk · Product of random matrices · Markov chains · First exit time

Mathematics Subject Classification (2020) 60B15 · 60F15

1 Introduction and Main Results

1.1 Motivation

Random walks conditioned to staying positive is a popular topic in probability. In addition to their own interest, such as information about the maxima and the minima, the ladder variables and the ladder epoch of random walks on $\mathbb{R}$, they are also important in view of their applications, for instance in queuing theory, in coding the genealogy

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of Galton–Watson trees or else as models for polymers and interfaces; we refer to [1] and references therein.

The first interesting question is to determine the asymptotic behavior of the exit time from the half line $[0, +\infty[$ and then to prove limit theorems for the process restricted to this half line or conditioned to remain there. More precisely, let $(S_n)_{n\geq 1}$ be a random walk in $\mathbb{R}$ whose increments are independent with common distribution. Assume that $(S_n)_{n\geq 1}$ is centered and let τ be its exit time from $[0, +\infty[$. Then, for any $a, b, \ell > 0$, as $n \to +\infty$,

$$\mathbb{P}_a(\tau > n, S_n \in [b, b+\ell[) \sim c\, \frac{h^+(a)h^-(b)}{n^{3/2}}\, \ell,$$

where c is a positive constant and h^+ and h^- are the renewal functions associated with $(S_n)_{n\geq 1}$, based on ascending and descending ladder heights (in particular these functions are positive). The increments being independent and identically distributed, the classical approach relies on the Wiener–Hopf factorization and related identities associated with the names of Baxter, Pollaczek and Spitzer; important references in the field are given by Feller and Spitzer in their books [2, 3].

Important conceptual difficulties arise both when the random walk $(S_n)_{n\geq 1}$ is $\mathbb{R}^d$-valued with $d \geq 2$ (the half line being replace by a general cone of the Euclidean space), or when the increments of the random walk are no longer independent. As far as we know, equivalent theory based on factorizations for these processes does not exist. In dimension $d \geq 2$, the Wiener–Hopf factorization method works when the cone is a half space but breaks down for more general cones. Any attempt to develop a theory of fluctuations for higher-dimensional random walks deals with the question: what would play the role of ladder epochs and ladder variables? [4]; Kingman showed in particular the impossibility of extending Baxter and Spitzer approach to random walks in higher dimension [5].

In 2015, Denisov and Wachtel developed a new approach to study the exit time from a cone of a random walk and several consequent limit theorems [6]. Their strategy, based on the approximation of these walks suitably normalized by a Brownian motion, with a strict control of the speed of convergence, is promising, powerful and flexible. It allows in particular to approach the random walks whose jumps are not i.i.d.

This flexible approach could be adapted to the quantity $S_n(x) := \ln |g_n \cdots g_2 g_1 x|$, where $(g_k)_{k\geq 1}$ is a sequence of i.i.d. random matrices, x is a non-nul vector in $\mathbb{R}^d$ and $|\cdot|$ is the ℓ_1 norm in $\mathbb{R}^d$; this process falls within the general framework of Markov walks on $\mathbb{R}$ satisfying some spectral gap assumption. The behavior of the tail of the distribution of $\tau_{x,a} := \inf\{n \geq 1 : a + \ln |g_n \cdots g_2 g_1 x| \leq 0\}$ is known for a few years when the random matrices are invertible or nonnegative [7, 8]. This is extended by Grama et al. [9] to the case of Markov walks, under a spectral gap assumption. Nevertheless, the question of a local limit theorem for $\ln |g_n \cdots g_2 g_1 x|$ confined in a half line still resists. In [10], such a statement holds for conditioned Markov walks over a finite state space, in which case the dual driving Markov chain also satisfies nice spectral gap properties; unfortunately, such a property does not hold for product of random matrices since it is not realistic to assume that the random matrices M_n act projectively on a finite set.

1.2 Notations and Assumptions

We endow $\mathbb{R}^d$ with the L_1 norm $|\cdot|$ defined by $|x| := \sum_{i=1}^d |x_i|$ for any column vector $x = (x_i)_{1\le i\le d}$. Let $\mathbf{1}$ (resp., $\mathbf{0}$) be the column vector of $\mathbb{R}^d$ whose all coordinates equal 1 (resp., 0).

Let $\mathcal{S}$ be the set of $d \times d$ matrices with positive entries. We endow $\mathcal{S}$ with the standard multiplication of matrices, and then, the set $\mathcal{S}$ is a semigroup. For any $g = (g(i,j))_{1\le i,j\le d} \in \mathcal{S}$, we define v, endow $|\cdot|$ a norm on $\mathcal{S}$ and define N as follows,

$$v(g) := \min_{1\le j\le d}\Big(\sum_{i=1}^d g(i,j)\Big); \quad |g| := \sum_{i,j=1}^d g(i,j) \quad \text{and} \quad N(g) := \max\left(\frac{1}{v(g)}, |g|\right).$$

Notice that $N(g) \ge 1$ for any $g \in \mathcal{S}$.

Let $\mathcal{C}$ be the cone of column vectors defined by $\mathcal{C} := \{x \in \mathbb{R}^d \mid \forall 1 \le i \le d, x_i \ge 0\}$ and $\mathbb{X}$ be the limited standard simplex defined by $\mathbb{X} := \{x \in \mathcal{C} \mid |x| = 1\}$. For any $x \in \mathcal{C}$, we denote by $\tilde{x}$ the corresponding row vector and set $\tilde{\mathcal{C}} = \{\tilde{x} \mid x \in \mathcal{C}\}$ and $\tilde{\mathbb{X}} = \{\tilde{x} \mid x \in \mathbb{X}\}$.

We consider the following actions:

- the linear action of $\mathcal{S}$ on $\mathcal{C}$ (resp. $\tilde{\mathcal{C}}$) defined by $(g, x) \mapsto gx$ (resp. $(g, \tilde{x}) \mapsto \tilde{x}g$) for any $g \in \mathcal{S}$ and $x \in \mathcal{C}$,
- the projective action of $\mathcal{S}$ on $\mathbb{X}$ (resp. $\tilde{\mathbb{X}}$) defined by $(g, x) \mapsto g \cdot x := \frac{gx}{|gx|}$ (resp. $(g, \tilde{x}) \mapsto \tilde{x} \cdot g = \frac{\tilde{x}g}{|\tilde{x}g|}$) for any $g \in \mathcal{S}$ and $x \in \mathbb{X}$.

It is noticeable that $0 < v(g)\,|x| \le |gx| \le |g|\,|x|$ for any $x \in \mathcal{C}$.

From now on, we consider a sequence $(g_k)_{k\ge 1}$ of i.i.d. random variables with values in $\mathcal{S}$, and with common distribution μ and set $L_{n,1} := g_n \cdots g_2 g_1$ for $n \ge 1$. For any fixed $x \in \mathbb{X}$ and $a > 0$, we denote by $\tau_{x,a}$ the first time the random process $(a + \ln |L_{n,1}x|)_n$ becomes negative, i.e.

$$\tau_{x,a} := \min\{n \ge 1 : a + \ln |L_{n,1}x| \le 0\}.$$

We impose the following assumptions on μ.

Hypotheses

P1 _Moment assumption:_ There exists $\delta_1 > 0$ such that $\int_{\mathcal{S}} N(g)^{\delta_1} \mu(\mathrm{d}g) < +\infty$.

P2 _Irreducibility assumption:_ There exists no affine subspaces A of $\mathbb{R}^d$ such that $A \cap \mathcal{C}$ is non-empty and bounded and invariant under the action of all elements of the support of μ.

This assumption is classical in the context of product of positive random matrices; it ensures in particular that the central limit theorem satisfied by these products is meaningful since the variance is positive (see Corollary 3 in [11]).

P3 There exists $B > 0$ such that for μ-almost all g in $\mathcal{S}$ and any $1 \le i, j, k, l \le d$

$$\frac{g(i,j)}{B} \le g(k,l) \le B\, g(i,j). \tag{1}$$

This is a classical assumption for product of random matrices with positive entries; it was first introduced by Furstenberg and Kesten [12].

P4 *Centering:* The upper Lyapunov exponent γ_μ is equal to 0.

P5 There exists $\delta_5 > 0$ such that $\mu\{g \in \mathcal{S} : \forall x \in \mathbb{X}, \ln|gx| \ge \delta_5\} > 0$.

Condition **P5** ensures that uniformly in $x \in \mathbb{X}$, the probability that the process $(a + \ln|L_{n,1}x|)_{n\ge1}$ remains in the half line $]0, +\infty[$ is positive. It is satisfied for instance when $\mu\{g \mid v(g) > 1\} > 0$.

As it is usual in studying local probabilities, one has to distinguish between "lattice" and "non-lattice" cases. The "non-lattice" assumption ensures that the $\mathbb{R}$-component of the trajectories of the Markov walk $(X_n, S_n)_{n\ge0}$ does not live in the translation of a proper subgroup of $\mathbb{R}$; in the contrary case, when μ is lattice, a phenomenon of cyclic classes appears which involves some complications which are not interesting in our context. We refer to equality (4) in Sect. 2 for a precise definition in the context of products of random matrices.

P6 *Non-lattice assumption:* The measure μ is non-lattice.

The tail of the distribution of $\tau_{x,a}$ has been the subject of an extensive study in [8]: under hypotheses **P1–P5**, there exists a positive Borel function $V : \mathbb{X} \times \mathbb{R}^+ \to \mathbb{R}^+$ such that as $n \to +\infty$,

$$\mathbb{P}(\tau_{x,a} > n) \sim \frac{2}{\sigma\sqrt{2\pi n}} V(x,a).$$

In the sequel, we also need to consider the process $(b - \ln|\tilde{x} R_{1,n}|)_n$, $\tilde{x} \in \tilde{\mathbb{X}}$, $b \in \mathbb{R}^+$, where $R_{1,n}$ denotes the "right random walk" $R_{1,n} := g_1 g_2 \cdots g_n$ for $n \ge 1$. We thus also consider the stopping time

$$\tilde{\tau}_{\tilde{x},b} := \min\{n \ge 1 : \ b - \ln|\tilde{x} R_{1,n}| \le 0\}.$$

As above, there exists a positive Borel function $\tilde{V} : \mathbb{X} \times \mathbb{R}^+ \to \mathbb{R}^+$ such that as $n \to +\infty$,

$$\mathbb{P}(\tilde{\tau}_{\tilde{x},b} > n) \sim \frac{2}{\sigma\sqrt{2\pi n}} \tilde{V}(\tilde{x}, b).$$

At last, as $n \to +\infty$, the sequence $\left(\dfrac{a + \ln|L_{n,1}x|}{\sigma\sqrt{n}}\right)_n$ conditioned to the event $(\tau_{x,a} > n)$ converges weakly toward the Rayleigh distribution on $\mathbb{R}^+$ whose density equals $y\, e^{-y^2/2}\mathbf{1}_{\mathbb{R}^+}(y)$. Properties of the function V are precisely stated in Sect. 2.

The natural question is to get a local limit theorem for the process $(a+\ln|L_{n,1}x|)_{n\ge1}$ forced to stay positive up to time n, in other words to describe the behavior of the quantity $\mathbb{P}(\tau_{x,a} > n, a + \ln |L_{n,1}x| \in [b, b+\ell])$ as $n \to +\infty$, where $a, b > 0$ and $\ell > 0$.

1.3 Main Statements

We first state a version of the Gnedenko local limit theorem.

Theorem 1 *Assume hypotheses* **P1**–**P6**. *Then, as $n \to +\infty$, for any $x \in \mathbb{X}$, $a, b > 0$ and $\ell > 0$,*

$$\lim_{n\to+\infty} \left| n\mathbb{P}(\tau_{x,a} > n, a + \ln |L_{n,1}x| \in [b, b+\ell]) - \frac{2\sqrt{2\pi}}{\sigma^2\sqrt{n}} V(x,a)\, b\, e^{-b^2/2n}\, \ell \right| = 0,$$

the convergence being uniform in $x \in \mathbb{X}$ and $b \geq 0$.

Notice that Theorem 1 says only that the probability

$$\mathbb{P}(\tau_{x,a} > n, a + \ln |L_{n,1}x| \in [b, b+\ell])$$

is $o(n^{-1})$. The following theorem describes an asymptotic behavior of this quantity; the constant Δ that appears in this statement is $\Delta := \ln \delta$ where δ is defined in Lemma 4.

Theorem 2 *Assume hypotheses* **P1**–**P6**. *There exist strictly positive constants c and C such that for any $x \in \mathbb{X}$, $a, b \geq 0$ and $\ell > 0$,*

$$n^{3/2}\mathbb{P}(\tau_{x,a} > n, a + \ln |L_{n,1}x| \in [b, b+\ell]) \leq C\; V(x,a)\; \tilde{V}(x,b)\, \ell. \tag{2}$$

Furthermore, there exist strictly positive constants ℓ_0 and Δ such that, for $\ell > \ell_0$ and $b \geq \Delta$,

$$\liminf_{n\to+\infty} n^{3/2}\mathbb{P}(\tau_{x,a} > n, a + \ln |L_{n,1}x| \in [b, b+\ell]) \geq c\; V(x,a)\; \tilde{V}(x,b)\, \ell. \tag{3}$$

As for random walks with i.i.d. increments, it is expected that this probability is in fact equivalent to $n^{3/2}$ up to a positive constant. The argument relies on a combination of what is sometimes called "reverse time" and "duality" in the classical theory of random walks; roughly speaking, it relies on the fact that, for a classical random walk $(S_n)_{n\geq 1}$ with i.i.d. increments, the vectors $(S_1, S_2, \ldots, S_n)$ and $(S_n - S_{n-1}, S_n - S_{n-2}, \ldots, S_n)$ have the same distribution. In [9], this idea is developed in the context of Markov walks over a Markov chain with finite state space, it is technically much more difficult and so far, it escapes from the framework of random matrix products (see the paragraph before Lemma 5 for more detailed explanations). In the case of nonnegative random matrices, the difference between $\ln |L_{n,1}x|$ and $\ln |L_{n,1}|$ is uniformly bounded (see Lemma 4), one can thus avoid the precise study of the associated dual chain[1] to

[1] This study would require restrictive conditions on μ, for example the existence of a density.

obtain the above result, a bit less precise but still worth of interest. There are in particular interesting and deep applications in the theory of branching processes (see for instance [13]).

Notation. Let c be a strictly positive constant and ϕ, ψ be two functions of some variable x; we denote by $\phi \overset{c}{\preceq} \psi$ (or simply $\phi \preceq \psi$) when $\phi(x) \leq c\, \psi(x)$ for any value of x. The notation $\phi \overset{c}{\asymp} \psi$ (or simply $\phi \asymp \psi$) means $\phi \overset{c}{\preceq} \psi \overset{c}{\preceq} \phi$.

2 Preliminaries

2.1 The Killed Markov Walk on $\mathbb{X} \times \mathbb{R}$ and its Harmonic Function

We consider a sequence of i.i.d. $\mathcal{S}$-valued matrices $(g_k)_{k\geq 1}$ with common distribution μ and denote the left and right product of matrices $L_{n,k} := g_n \cdots g_k$ and $R_{k,n} = g_k \cdots g_n$ for any $n \geq k \geq 1$.

We fix a $\mathbb{X}$-valued random variable X_0 and consider the Markov chain $(X_n)_{n\geq 0}$ defined by $X_n^{X_0} := L_{n,1} \cdot X_0$ for any $n \geq 1$; when $X_0 = x$, we set for simplicity $X_n = X_n^x$. Similarly, the $\tilde{\mathbb{X}}$-valued Markov chain $(\tilde{X}_n)_{n\geq 0}$ is defined by $\tilde{X}_n := \tilde{X}_0 \cdot R_{1,n}$ for any $n \geq 1$, where $\tilde{X}_0$ is a fixed $\tilde{\mathbb{X}}$-valued random variable.
Notice that the sequence $(g_{n+1}, X_n^x)_{n\geq 0}$ (resp. $(g_{n+1}, \tilde{X}_n^{\tilde{x}})_{n\geq 0}$) is a $\mathcal{S} \times \mathbb{X}$ valued (resp. $\mathcal{S} \times \tilde{\mathbb{X}}$ valued) Markov chain with initial distribution $\mu \otimes \delta_x$ (resp. $\mu \otimes \delta_{\tilde{x}}$), where δ_x is the Dirac distribution at x. Their respective transition probability P and Q are defined by: for any $(g, x) \in \mathcal{S} \times \mathbb{X}$ and any bounded Borel function $\varphi : \mathcal{S} \times \mathbb{X} \to \mathbb{C}, \phi : \mathcal{S} \times \tilde{\mathbb{X}} \to \mathbb{C}$,

$$P\varphi(g, x) := \int_{\mathcal{S}} \varphi(h, g \cdot x)\mu(\mathrm{d}h) \quad \text{and} \quad Q\phi(g, \tilde{x}) := \int_{\mathcal{S}} \phi(h, \tilde{x} \cdot g)\mu(\mathrm{d}h).$$

We denote by $(\Omega = (\mathcal{S} \times \mathbb{X})^{\otimes \mathbb{N}}, \mathcal{F} = \mathcal{B}(\mathcal{S} \times \mathbb{X})^{\otimes \mathbb{N}}, (g_{n+1}, X_n^x)_{n\geq 0}, \theta, \mathbb{P}_x)$ the canonical probability space associated with $(g_{n+1}, X_n^x)_{n\geq 0}$, where θ is the classical "shift operator" on $(\mathcal{S} \times \mathbb{X})^{\otimes \mathbb{N}}$. Similarly, $(\tilde{\Omega}, \tilde{\mathcal{F}}, (g_{n+1}, \tilde{X}_n^{\tilde{x}})_{n\geq 0}, \tilde{\theta}, \mathbb{P}_{\tilde{x}})$ denotes the canonical probability space associated with $(g_{n+1}, \tilde{X}_n^{\tilde{x}})_{n\geq 0}$.

We introduce next the functions ρ ant $\tilde{\rho}$ defined for any $g \in \mathcal{S}$ and $x \in \mathbb{X}$ by

$$\rho(g, x) := \ln |gx| \quad \text{and} \quad \tilde{\rho}(g, \tilde{x}) := \ln |\tilde{x}g|.$$

Notice that $gx = e^{\rho(g,x)} g \cdot x$ and that ρ satisfies the "cocycle property":

$$\rho(gh, x) = \rho(g, h \cdot x) + \rho(h, x), \quad \forall g, h \in \mathcal{S} \text{ and } x \in \mathbb{X}.$$

This yields to the following basic decomposition

$$\ln |L_{n,1}x| = \sum_{k=0}^{n-1} \rho(g_{k+1}, X_k^x) \quad \text{and} \quad \ln |\tilde{x} R_{1,n}| = \sum_{k=0}^{n-1} \rho(g_{k+1}, \tilde{X}_k^{\tilde{x}}).$$

Thus, it is natural to introduce the following Markov walks on $\mathbb{X} \times \mathbb{R}$ and $\tilde{\mathbb{X}} \times \mathbb{R}$ defined by $S_n = S_0 + \sum_{k=0}^{n-1} \rho(g_{k+1}, X_k^x)$ and $\tilde{S}_n = \tilde{S}_0 - \sum_{k=0}^{n-1} \tilde{\rho}(g_{k+1}, \tilde{X}_k^{\tilde{x}})$ where S_0 and $\tilde{S}_0$ are real-valued random variables.

Notice that the sequences $(X_n, S_n)_{n\geq 0}$ and $(\tilde{X}_n, \tilde{S}_n)_{n\geq 0}$ are Markov chains on $\mathbb{X}\times\mathbb{R}$ and $\tilde{\mathbb{X}} \times \mathbb{R}$, respectively, with transition probability $\widehat{P}$ and $\widehat{Q}$ defined by: for any $(x, a) \in \mathbb{X}\times\mathbb{R}$ and any bounded Borel functions $\Phi : \mathbb{X}\times\mathbb{R} \to \mathbb{C}$, $\Psi : \tilde{\mathbb{X}}\times\mathbb{R} \to \mathbb{C}$,

$$\widehat{P}\Phi(x, a) = \int_{\mathcal{S}} \Phi(g \cdot x, a + \rho(g, x))\mu(\mathrm{d}g) \quad \text{and}$$
$$\widehat{Q}\Psi(\tilde{x}, a) = \int_{\mathcal{S}} \Psi(\tilde{x} \cdot g, a - \tilde{\rho}(g, \tilde{x}))\mu(\mathrm{d}g).$$

For any $(x, a) \in \mathbb{X} \times \mathbb{R}$, we denote by $\mathbb{P}_{x,a}$ the probability measure on $(\Omega, \mathcal{F})$ conditioned to the event $(X_0 = x, S_0 = a)$ and by $\mathbb{E}_{x,a}$ the corresponding expectation; for simplicity, we set $\mathbb{P}_{x,0} = \mathbb{P}_x$ and $\mathbb{E}_{x,0} = \mathbb{E}_x$.
Hence for any $n \geq 1$,

$$\widehat{P}^n\Phi(x, a) = \mathbb{E}[\Phi(L_{n,1} \cdot x, a + \ln |L_{n,1}x|)] = \mathbb{E}_{x,a}[\Phi(X_n, S_n)].$$

Next we consider the restriction $\widehat{P}_+$ to $\mathbb{X} \times \mathbb{R}^+$ of $\widehat{P}$ defined for any bounded Borel functions $\Phi : \mathbb{X} \times \mathbb{R} \to \mathbb{C}$ and any $(x, a) \in \mathbb{X} \times \mathbb{R}$ by:

$$\widehat{P}_+\Phi(x, a) = \widehat{P}(\Phi\mathbf{1}_{\mathbb{X}\times\mathbb{R}^+})(x, a).$$

Let us emphasize that $\widehat{P}_+$ may not be a Markov kernel on $\mathbb{X}\times\mathbb{R}^+$. Furthermore, if $\tau := \min\{n \geq 1 : S_n \leq 0\}$ is the first time the random process $(S_n)_{n\geq 1}$ becomes negative, it holds for any $(x, a) \in \mathbb{X} \times \mathbb{R}^+$ and any bounded Borel function $\Phi : \mathbb{X} \times \mathbb{R} \to \mathbb{C}$,

$$\widehat{P}_+\Phi(x, a) = \mathbb{E}_{x,a}[\Phi(X_1, S_1); \tau > 1] = \mathbb{E}[\Phi(g_1 \cdot x, a + \ln |g_1x|); a + \ln |g_1x| > 0].$$

A positive $\widehat{P}_+$-harmonic function V is any function from $\mathbb{X} \times \mathbb{R}^+$ to $\mathbb{R}^+$ satisfying $\widehat{P}_+V = V$. We extend V by setting $V(x, a) = 0$ for $(x, a) \in \mathbb{X} \times \mathbb{R}_*^-$. In other words, the function V is $\widehat{P}_+$-harmonic if and only if for any $x \in \mathbb{X}$ and $a \geq 0$,

$$V(x, a) = \mathbb{E}_{x,a}[V(X_1, S_1); \tau > 1].$$

Similarly, if $\tilde{\tau} := \min\{n \geq 1 : \tilde{S}_n \leq 0\}$ is the first time the random process $(\tilde{S}_n)_{n\geq 1}$ becomes negative, then for any $(x, b) \in \tilde{\mathbb{X}} \times \mathbb{R}^+$ and any bounded Borel function $\Psi : \tilde{\mathbb{X}} \times \mathbb{R} \to \mathbb{C}$,

$$\widehat{Q}_+\Phi(x, a) = \mathbb{E}_{\tilde{x},b}[\Psi(\tilde{X}_1, \tilde{S}_1); \tilde{\tau} > 1] = \mathbb{E}[\Psi(\tilde{x} \cdot g_1, b - \ln |\tilde{x}g_1|); b - \ln |\tilde{x}g_1| > 0].$$

From Theorem II.1 in [14], when the support of μ contains matrices with positive entries (in particular when condition **P3** holds), there exists a unique probability measure ν on $\mathbb{X}$ such that for any bounded Borel function φ from $\mathbb{X}$ to $\mathbb{R}$,

$$(\mu * \nu)(\varphi) = \int_{\mathcal{S}} \int_{\mathbb{X}} \varphi(g \cdot x)\nu(\mathrm{d}x)\mu(\mathrm{d}g) = \int_{\mathbb{X}} \varphi(x)\nu(\mathrm{d}x) = \nu(\varphi).$$

Such a measure is said to be μ-invariant. When $\int_{\mathcal{S}} |\ln |g||\mu(\mathrm{d}g) < +\infty$, the upper Lyapunov exponent associated with μ is finite and is expressed by

$$\gamma_\mu = \int_{\mathcal{S}} \int_{\mathbb{X}} \rho(g, x)\nu(\mathrm{d}x)\mu(\mathrm{d}g).$$

We are now able to give a precise definition of a lattice distribution μ. We say that the measure μ is *lattice* if there exist $t > 0, \epsilon \in [0, 2\pi[$ and a function $\psi : \mathbb{X} \to \mathbb{R}$ such that

$$\forall g \in T_\mu, \forall x \in \mathbb{X}, \quad \exp\{it\rho(g, x) - i\epsilon + i(\psi(g \cdot x) - \psi(x))\} = 1, \tag{4}$$

where T_μ is the closed sub-semigroup generated by the support of μ.

It is also noticeable that the process $(X_n, S_n)_n$ is a semi-Markovian random walk on $\mathbb{X} \times \mathbb{R}$ with the strictly positive variance $\sigma^2 := \lim_{n\to+\infty} \frac{1}{n}\mathbb{E}_x[S_n^2]$, for any $x \in \mathbb{X}$. Condition **P2** implies that $\sigma^2 > 0$; we refer to Theorem 5 in [11].

In [8], C. Pham establishes the asymptotic behavior of $\mathbb{P}(\tau_{x,a} > n)$ by studying the $\widehat{P}_+$-harmonic function V. Firstly, she proves the existence of a $\widehat{P}_+$-harmonic function and describes some of its properties (Proposition 1.1), then obtains the behavior of the tail distribution of $\tau_{x,a}$ and a conditional central limit theorem (Theorems 1.2 and 1.3). We summarize all these results in the following statement.

Proposition 3 *Assume hypotheses* **P1–P5**. *Then, there exists a $\widehat{P}_+$-harmonic Borel function $V : \mathbb{X} \times \mathbb{R}^+ \to \mathbb{R}^+$ such that the function $a \mapsto V(x, a)$ is increasing on $\mathbb{R}^+$ for any $x \in \mathbb{X}$ and satisfies the following properties: there exist strictly positive constants c, C and A such that for any $x \in \mathbb{X}$ and $a \geq 0$,*

$$c \vee (a - A) \leq V(x, a) \leq C\ (1 + a) \quad \text{and} \quad \lim_{a\to+\infty} \frac{V(x, a)}{a} = 1.$$

Furthermore, for any $x \in \mathbb{X}$, $a \geq 0$ and $n \geq 1$,

$$\sqrt{n}\mathbb{P}(\tau_{x,a} > n) \leq C\ V(x, a)$$

and as $n \to +\infty$,

$$\mathbb{P}(\tau_{x,a} > n) \sim \frac{2}{\sigma\sqrt{2\pi n}} V(x, a).$$

At last, as $n \to +\infty$, the sequence $\left(\frac{a + \ln|L_{n,1}x|}{\sigma\sqrt{n}}\right)_n$ conditioned to $(\tau_{x,a} > n)$ converges weakly toward the Rayleigh distribution on $\mathbb{R}^+$ whose density equals $y\, e^{-y^2/2} 1_{\mathbb{R}^+}(y)$, relatively to $\mathbb{P}_{x,a}$ for any $x \in \mathbb{X}$ and $a \geq 0$.

2.2 Product of Positive Random Matrices

Products of random matrices are first studied by Furstenberg and Kesten [12] for matrices satisfying condition (1) and then being extended to elements of $\mathcal{S}$ by several authors (see [11] and references therein). The restrictive condition of Furstenberg and Kesten considerably simplifies the study. The following statement (see [12] Lemma 2) is a key argument in the sequel to control the asymptotic behavior of the norm of products of matrices satisfying condition (1).

Lemma 4 *Let g be a product of positive matrices satisfying condition (1).*
Then, for any $1 \leq i, j, k, l \leq d$,

$$g(i, j) \overset{B^2}{\asymp} g(k, l).$$

In particular, there exists $\delta > 1$ such that for any products g, h of matrices satisfying condition (1) and any $x \in \mathbb{X}$, $\tilde{y} \in \tilde{\mathbb{X}}$,

1. $|gx| \overset{\delta}{\asymp} |g|$ *and* $|\tilde{y}g| \overset{\delta}{\asymp} |g|$,
2. $|\widetilde{y}gx| \overset{\delta}{\asymp} |g|$,
3. $|g||h| \overset{\delta}{\preceq} |gh| \leq |g||h|$.

As a direct consequence, properties 1., 2. and 3. above are satisfied for any elements g, h of the closed sub-semigroup T_μ generated by the support of μ; hence, $\mathbb{P}$-almost surely, the sequence $(\ln|L_{n,1}x| - \ln|L_{n,1}|)_{n\geq 0}$ is bounded uniformly in $x \in \mathbb{X}$. This property is crucial in the sequel in order to apply the “reverse time" trick, an essential argument in the proofs of our main results.

When studying fluctuations of random walks $(S_n)_{n\geq 1}$ with i.i.d. increments Y_k on $\mathbb{R}^d, d \geq 1$, it is useful to “reverse time” as follows. For any $1 \leq k \leq n$, the random variables $S_n - S_k = Y_{k+1} + \cdots + Y_n$ and $S_{n-k} = Y_1 + \cdots + Y_{n-k}$ have the same distribution. In the case of products of random matrices, the cocycle property $S_n(x) = \ln|L_{n,1}(x)| = S_k(x) + S_{n-k}(X_k)$ is more subtle and the same argument cannot be applied directly. The fact that the g_k do satisfy condition (1) comes to our rescue here, but the price to pay is the appearance of the constant $\Delta = \ln\delta$ (where δ is the constant which appears in Lemma 4) that disturbs the estimates as follows. Up to this constant Δ, we can compare the distribution of $S_n(x) - S_k(x) =: S_{n-k}(X_k)$ to the one of $\ln|g_{k+1}\cdots g_n|$, then to the one of $\ln|g_1\cdots g_{n-k}|$ and at last to the one of $\ln|\tilde{y}g_1\cdots g_{n-k}| =: -\tilde{S}_{n-k}(y)$, for any $x, y \in \mathbb{X}$ (notice here that for this last quantity, the non-commutativity of the product of matrices forces us to consider the right linear action of the matrices $R_{1,n-k}$). It is the strategy that we apply repeatedly to obtain the following result.

Notation When the starting couple (x, a) or $(\tilde{x}, b)$ is well indicated, we may shorten the notation by writing τ or $\tilde{\tau}$.

Lemma 5 *For any $x, y \in \mathbb{X}, a, b, \ell > 0$ and $n \geq 1$*

$$\mathbb{P}_{x,a}(\tau > n, S_n \in [b, b+\ell]) \leq \mathbb{P}_{\tilde{y},\ b+\ell+\Delta}(\tilde{\tau} > n, \tilde{S}_n \in [a, a+\ell+2\Delta]). \tag{5}$$

Similarly, for $a \geq \ell > 2\Delta > 0$ and $b \geq \Delta$,

$$\mathbb{P}_{x,a}(\tau > n, S_n \in [b, b+\ell]) \geq \mathbb{P}_{\tilde{y},\ b-\Delta}(\tilde{\tau} > n, \tilde{S}_n \in [a-\ell, a-2\Delta]). \tag{6}$$

Proof We begin with the demonstration of (5). For any $n \geq 1$ and $a, b > 0, \ell > 0$, it holds

$$\begin{aligned}
&\mathbb{P}_{x,a}(\tau > n,\ S_n \in [b, b+\ell]) \\
&\quad = \mathbb{P}_x(a + S_1 > 0, \ldots, a + S_{n-1} > 0, a + S_n \in [b, b+\ell]) \\
&\quad = \mathbb{P}_x(a + S_n - S_{n-1} \circ \theta > 0, \ldots, a + S_n - S_1 \circ \theta^{n-1} > 0, a + S_n \in [b, b+\ell]) \\
&\quad \leq \mathbb{P}_x(b + \ell - S_{n-1} \circ \theta > 0, \ldots, b + \ell - S_1 \circ \theta^{n-1} > 0, b + \ell - S_n \in [a, a+\ell]),
\end{aligned}$$

where θ is the shift operator and $S_{n-k} \circ \theta^k = \ln |L_{n,k+1} X_k^x|$ $\mathbb{P}_x$-a.s. for $0 \leq k < n$. By Lemma 4, for any $\tilde{y} \in \tilde{\mathbb{X}}$ and $0 \leq k \leq n-1$, the quantities $\ln |L_{n,k+1} X_k^x|$ and $\ln |\tilde{y} L_{n,k+1}|$ both belong to the interval $[\ln |L_{n,k+1}| - \Delta, \ln |L_{n,k+1}|]$. Therefore, $S_{n-k} \circ \theta^k \in [\ln |\tilde{y} L_{n,k+1}| - \Delta, \ln |\tilde{y} L_{n,k+1}| + \Delta]$ and as a result

$$\begin{aligned}
&\mathbb{P}_{x,a}(\tau > n,\ S_n \in [b, b+\ell]) \\
&\quad \leq\ \mathbb{P}(b + \ell + \Delta - \ln |\tilde{y} L_{n,2}| > 0, \ldots, b + \ell + \Delta - \ln |\tilde{y} L_{n,n}| > 0, \\
&\quad b + \ell + \Delta - \ln |\tilde{y} L_{n,1}| \in [a, a+\ell+2\Delta]) \\
&\quad =\ \mathbb{P}(b + \ell + \Delta - \ln |\tilde{y} R_{1,n-1}| > 0, \ldots, b + \ell + \Delta - \ln |\tilde{y} R_{1,1}| > 0, \\
&\quad b + \ell + \Delta - \ln |\tilde{y} R_{1,n}| \in [a, a+\ell+2\Delta]) \\
&\quad \text{by using the fact that } (g_1, \ldots, g_n) \text{ and } (g_n, \ldots, g_1) \text{ have the same distribution} \\
&\quad =\ \mathbb{P}_{\tilde{y},\ b+\ell+\Delta}(\tilde{\tau} > n, \tilde{S}_n \in [a, a+\ell+2\Delta]).
\end{aligned}$$

Similarly, for $a > \ell > 2\Delta > 0$ and $b > 0$, we obtain the proof of (6) as follows.

$$\begin{aligned}
&\mathbb{P}_{x,a}(\tau > n,\ S_n \in [b, b+\ell]) \\
&\quad = \mathbb{P}_x(a + S_1 > 0, \ldots, a + S_{n-1} > 0, a + S_n \in [b, b+\ell]) \\
&\quad = \mathbb{P}_x(a + S_n - S_{n-1} \circ \theta > 0, \ldots, a + S_n - S_1 \circ \theta^{n-1} > 0, b \leq a + S_n \leq b + \ell) \\
&\quad \geq \mathbb{P}_x(b - S_{n-1} \circ \theta > 0, \ldots, b - S_1 \circ \theta^{n-1} > 0, a - \ell \leq b - S_n \leq a) \\
&\quad \geq \mathbb{P}(b - \Delta - \ln |\tilde{y} L_{n,2}| > 0, \ldots, b - \Delta - \ln |\tilde{y} L_{n,n}| > 0, \\
&\quad a - \ell \leq b - \Delta - \ln |\tilde{y} L_{n,1}| \leq a - 2\Delta) \\
&\quad = \mathbb{P}(b - \Delta - \ln |\tilde{y} R_{1,n-1}| > 0, \ldots, b - \Delta - \ln |\tilde{y} R_{1,1}| > 0,
\end{aligned}$$

$$b - \Delta - \ln|\tilde{y}R_{1,n}| \in [a-\ell, a-2\Delta])$$
$$= \mathbb{P}_{\tilde{y},\, b-\Delta}(\tilde{S}_1 > 0, \ldots, \tilde{S}_{n-1} > 0, \tilde{S}_n \in [a-\ell, a-2\Delta])$$
$$= \mathbb{P}_{\tilde{y},\, b-\Delta}(\tilde{\tau} > n, \tilde{S}_n \in [a-\ell, a-2\Delta]).$$

Since $a > \ell > 2\Delta > 0$, the interval $[a-\ell, a-2\Delta]$ is not empty. □

2.3 Limit Theorem for Product of Positive Random Matrices

In this section, we state some classical results and preparatory lemmas, useful for the demonstration of Theorems 1 and 2. The following result plays a crucial role in this article.

Theorem 6 ([15], Theorem 3.2.2) *Assume hypotheses* **P1–P6** *hold. Then for any continuous function* $u : \mathbb{X} \to \mathbb{R}$ *and any continuous function with compact support* $\varphi : \mathbb{R} \to \mathbb{R}$*, it holds*

$$\lim_{n\to+\infty}\left|\sqrt{n}\mathbb{E}_{x,a}\left[u(X_n)\varphi(S_n)\right] - \frac{\nu(u)}{\sigma\sqrt{2\pi}}\int_{\mathbb{R}}\varphi(y)e^{-(y-a)^2/2\sigma^2 n}\mathrm{d}y\right| = 0,$$

where the convergence is uniform in $(x, a) \in \mathbb{X} \times \mathbb{R}^{*+}$.

Other necessary elementary estimations are proved below.

Lemma 7 *There exist constants* $c, C > 0$ *such that for every* $x \in \mathbb{X}$, a, b *in* $\mathbb{R}$, $\ell > 0, n \geq 1$,

$$\mathbb{P}_{x,a}(S_n \in [b, b+\ell]) \leq \frac{c}{\sqrt{n}}\,\ell \tag{7}$$

and, for any $t > 0$,

$$\limsup_{n\to+\infty}\ \sup_{|a-b|>t\sqrt{n}} \sqrt{n}\,\mathbb{P}_{x,a}(S_n \in [b, b+\ell]) \leq C\,\ell\, e^{-ct^2}. \tag{8}$$

Proof Assertion (7) is a consequence of Theorem 6. Assertion (8) is more precise than (7) for large values of the starting point a, namely when $a \succeq \sqrt{n}$, as proved below. We fix $\ell, t > 0$ and let $m := \lfloor n/2 \rfloor$ be the lower round of $n/2$. We decompose $\mathbb{P}_{x,a}(S_n \in [b, b+\ell])$ as follows.

$$\begin{aligned}
\mathbb{P}_{x,a}(S_n \in [b, b+\ell]) &= \mathbb{P}_x(a + S_n \in [b, b+\ell]) \\
&= \underbrace{\mathbb{P}_x(a + S_n \in [b, b+\ell], |S_m| > t\sqrt{n}/2)}_{P_1(n,x,a,b,\ell)} \\
&\quad + \underbrace{\mathbb{P}_x(a + S_n \in [b, b+\ell], |S_m| \leq t\sqrt{n}/2)}_{P_2(n,x,a,b,\ell)}.
\end{aligned}$$

On the one hand, from the Markov property and inequality (7), there exists a strictly positive constant c_1 such that, uniformly in x, a and b,

$$\begin{aligned} P_1(n, x, a, b, \ell) &= \int_{\mathbb{X}\times[-t\sqrt{n}/2, t\sqrt{n}/2]^c} \mathbb{P}_{x'}(a + a' + S_{n-m} \\ &\in [b, b+\ell]) \quad \mathbb{P}_x(X_m \in \mathrm{d}x', S_m \in \mathrm{d}a') \\ &\le c_1 \frac{\ell}{\sqrt{n-m}} \mathbb{P}_x(|S_m| > t\sqrt{n}/2). \end{aligned} \tag{9}$$

On the other hand, when $|a - b| > t\sqrt{n}$, conditions $|S_m| \le t\sqrt{n}/2$ and $a + S_n \in [b, b+\ell]$ yield $|S_n - S_m| \ge t\sqrt{n}/2 - \ell$. Hence,

$$\begin{aligned} &P_2(n, x, a, b, \ell) \le \mathbb{P}_x(a + S_n \in [b, b+\ell], |S_n - S_m| > t\sqrt{n}/4) \\ &= \mathbb{P}_x(a + S_m + S_{n-m} \circ \theta^m \in [b, b+\ell], |S_{n-m} \circ \theta^m| > t\sqrt{n}/4) \\ &= \mathbb{P}_x(a + \ln|L_{m,1}X_0| + \ln|L_{n,m+1}X_m| \in [b, b+\ell], \ln|L_{n,m+1}X_m| > t\sqrt{n}/4) \\ &\le \mathbb{P}(a + \ln|L_{m,1}x| + \ln|L_{n,m+1}x| \in [b-\Delta, b+\ell+\Delta], \\ &\qquad\qquad |\ln|L_{n,m+1}x|| > t\sqrt{n}/4 - \Delta) \qquad \text{by Lemma 4} \\ &\le \int_{\{|\alpha| > t\sqrt{n}/4-\Delta\}} \underbrace{\mathbb{P}(a + \ln|L_{m,1}x| + \alpha \in [b-\Delta, b+\ell+\Delta])}_{\le c\,\frac{\ell+2\Delta}{\sqrt{n}} \text{ by(7)}} \mathbb{P}(\ln|L_{n,m+1}x| \in \mathrm{d}\alpha) \\ &\le\ c\ \frac{\ell + 2\Delta}{\sqrt{n}}\ \mathbb{P}_x(|S_{n-m}| > t\sqrt{n}/4 - \Delta). \end{aligned} \tag{10}$$

We conclude the proof by combining (9), (10) and the central limit theorem for products of random matrices [11]. □

The next statement is analogous to the previous lemma when the random walk $(a + S_n)_n$ is forced to remain positive up to time n. For this reason, we assume here $a, b > 0$.

Lemma 8 *There exists a constant $C > 0$ such that for all $x \in \mathbb{X}, a, b, \ell > 0$ and $n \ge 1$,*

$$\mathbb{P}_{x,a}(\tau > n, S_n \in [b, b+\ell]) \ \le\ C\ \frac{1+a}{n}\ \ell. \tag{11}$$

Furthermore, there exists a constant $C > 0$ such that for any $\ell, t > 0$,

$$\limsup_{n\to+\infty} \sup_{x\in\mathbb{X}} \sup_{\substack{a > t\sqrt{n}+\ell+2\Delta \\ b > t\sqrt{n}+\Delta}} \sqrt{n}\, \mathbb{P}_{x,a}(\tau \le n, S_n \in [b, b+\ell]) \le C\ \ell\ e^{-ct^2}. \tag{12}$$

Proof For any $1 \le m \le n$,

$$\mathbb{P}_{x,a}(\tau > n,\ S_n \in [b, b+\ell]) \le \mathbb{P}_{x,a}(\tau > m, S_n \in [b, b+\ell])$$

$$= \int_{\mathbb{X}\times\mathbb{R}_*^+} \mathbb{P}_{x',a'}(S_{n-m} \in [b, b+\ell])\, \mathbb{P}_{x,a}(\tau > m, (X_m, S_m) \in \mathrm{d}x'\mathrm{d}a')$$
$$\preceq \frac{\mathbb{P}_{x,a}(\tau > m)}{\sqrt{n-m}}\ell \quad \text{by (7)}$$
$$\leq c\,\frac{1+a}{n}\,\ell \text{ by Proposition 3.}$$

The proof of assertion (12) is decomposed in two steps. Let $m = \lfloor n/2 \rfloor$.

Step 1. When $b > t\sqrt{n}$, by using the Markov property, we get

$$\mathbb{P}_{x,a}(\tau \leq m, S_n \in [b, b+\ell])$$
$$= \sum_{k=1}^{m} \mathbb{P}_{x,a}(\tau = k, S_n \in [b, b+\ell])$$
$$= \sum_{k=1}^{m} \int_{\mathbb{X}\times\mathbb{R}^-} \mathbb{P}_{x',a'}(S_{n-k} \in [b, b+\ell]) \quad \mathbb{P}_{x,a}(\tau = k, (X_k, S_k) \in \mathrm{d}x'\mathrm{d}a')$$
$$\leq \max_{n-m\leq k'\leq n} \left(\sup_{\substack{x'\in\mathbb{X} \\ |a'-b|>t\sqrt{n}}} \mathbb{P}_{x',a'}(S_{k'} \in [b, b+\ell]) \right)$$
$$\underbrace{\sum_{k=1}^{m} \int_{\mathbb{X}\times\mathbb{R}^-} P_{x,a}(\tau = k, (X_k, S_k) \in \mathrm{d}x'\mathrm{d}a')}_{=\mathbb{P}_{x,a}(\tau\leq m)}$$
$$\leq \max_{n-m\leq k'\leq n} \left(\sup_{\substack{x'\in\mathbb{X} \\ |a'-b|>t\sqrt{k'}}} \mathbb{P}_{x',a'}(S_{k'} \in [b, b+\ell]) \right).$$

Hence, by (8), there exists $c, C > 0$ such that, for any $x \in \mathbb{X}$ and $a > 0$

$$\limsup_{n\to+\infty} \sqrt{n}\, \mathbb{P}_{x,a}(\tau \leq m, S_n \in [b, b+\ell]) \leq C\, \ell\, e^{-ct^2}.$$

Step 2. We control the term $\mathbb{P}_{x,a}(m < \tau \leq n, S_n \in [b, b+\ell])$. By using the same argument to prove (5), it follows that

$$\mathbb{P}_{x,a}(m < \tau \leq n,\ S_n \in [b, b+\ell])$$
$$= \mathbb{P}_x(\exists k \in \{m+1, \ldots, n-1\} : a + S_k \leq 0, a + S_n \in [b, b+\ell])$$
$$= \mathbb{P}(\exists k \in \{m+1, \ldots, n-1\} : a + \ln|L_{n,1}x| - \ln|L_{n,k+1}X_k^x|$$
$$\leq 0, a + \ln|L_{n,1}x| \in [b, b+\ell])$$
$$\leq \mathbb{P}(\exists k \in \{m+1, \ldots, n-1\} : b - \Delta - \ln|\tilde{y}L_{n,k+1}|$$
$$\leq 0, a + \ln|\tilde{y}L_{n,1}| \in [b-\Delta, b+\ell+\Delta])$$

$$= \mathbb{P}(\exists k \in \{m+1, \ldots, n-1\} : b - \Delta - \ln|\tilde{y}R_{1,n-k}| \\ \leq 0, a + \ln|\tilde{y}R_{1,n}| \in [b - \Delta, b + \ell + \Delta])$$

(by using again the fact that since again $(g_1, \ldots, g_n)$ and $(g_n, \ldots, g_1)$ have the same distribution)

$$\leq \mathbb{P}(\exists \ell \in \{1, \ldots, m\} : b - \Delta - \ln|\tilde{y}R_{1,\ell}| \\ \leq 0, b - \Delta - \ln|\tilde{y}R_{1,n}| \in [a - \ell - 2\Delta, a])$$

$$= \mathbb{P}_{\tilde{y}, b-\Delta}(\tilde{\tau} \leq m, \tilde{S}_n \in [a - \ell - 2\Delta, a]).$$

Now, when $a - \ell - 2\Delta > t\sqrt{n}$ and $b > \Delta$, we may apply Step 1 with the couple $(\tilde{\tau}, \tilde{S}_n)$ instead of (τ, S_n). This achieves the proof. □

3 Proof of Theorem 1

We adapt the proof of Theorem 5 in [6] and point out on the main differences. We fix two positive constants A and ϵ such that $A > 2\epsilon > 0$ and split $\mathbb{R}_*^+$ into three intervals: $[A\sqrt{n}; +\infty[$, $]0, 2\epsilon\sqrt{n}[$ and $I_{n,\epsilon,A} = [2\epsilon\sqrt{n}, A\sqrt{n}[$. The proof is decomposed into three steps.

Step 1.

$$\lim_{A \to +\infty} \limsup_{n \to +\infty} \left[n \sup_{\substack{x \in \mathbb{X} \\ b \geq A\sqrt{n}}} \mathbb{P}_{x,a}(\tau > n, S_n \in [b, b + \ell]) \right] = 0.$$

Step 2.

$$\lim_{\epsilon \to 0} \limsup_{n \to +\infty} \left[n \sup_{\substack{x \in \mathbb{X} \\ 0 < b \leq 2\epsilon\sqrt{n}}} \mathbb{P}_{x,a}(\tau > n, S_n \in [b, b + \ell]) \right] = 0.$$

Step 3. For any $A > 0$,

$$\lim_{\epsilon \to 0} \limsup_{n \to +\infty} \sup_{\substack{x \in \mathbb{X} \\ b \in I_{n,\epsilon,A}}} \left| n\mathbb{P}_{x,a}(\tau > n, S_n \in [b, b+\ell]) - \frac{2}{\sigma\sqrt{2\pi n}} V(x, a)\, b\, \ell\, e^{-b^2/2n} \right| = 0.$$

Theorem 1 follows by combining these three steps; the convergence is obviously uniform over x.

We set $m = \lfloor n/2 \rfloor$.

Proof of Step 1. Let $a > 0$ and $b \geq A\sqrt{n}$. We write

$$\mathbb{P}_{x,a}(\tau > n, S_n \in [b, b + \ell]) = P_n(x, a, b, A) + Q_n(x, a, b, A)$$

where

$$P_n(x, a, b, A) = \mathbb{P}_{x,a}(\tau > n, S_m \leq A\sqrt{m}, S_n \in [b, b + \ell])$$

and

$$Q_n(x, a, b, A) = \mathbb{P}_{x,a}(\tau > n, S_m > A\sqrt{m}, S_n \in [b, b+\ell]).$$

By the Markov property,

$$\begin{aligned}
P_n(x, a, b, A) &\leq \mathbb{P}_{x,a}(\tau > m, S_m \leq A\sqrt{m}, S_n \in [b, b+\ell]) \\
&\leq \int_{\mathbb{X}\times]0, A\sqrt{m}]} \mathbb{P}_{x,a}(\tau > m, X_m \in \mathrm{d}x', S_m \in \mathrm{d}a')\, \mathbb{P}_{x',a'}(S_{n-m} \in [b, b+\ell]) \\
&\leq \mathbb{P}_{x,a}(\tau > m, S_m \leq A\sqrt{m}) \sup_{\substack{x'\in\mathbb{X} \\ 0<a'\leq A\sqrt{m}}} \mathbb{P}_{x',a'}(S_{n-m} \in [b, b+\ell]) \\
&\leq \mathbb{P}_{x,a}(\tau > m) \sup_{\substack{x'\in\mathbb{X} \\ |b-a'|>A\sqrt{n}/4}} \mathbb{P}_{x',a'}(S_{n-m} \in [b, b+\ell]).
\end{aligned}$$

Consequently, by Proposition 3 and inequality (8), there exist constants $c, C > 0$ such that, for any $x \in \mathbb{X}$ and $a > 0$,

$$\limsup_{n\to+\infty} n \sup_{b\geq A\sqrt{n}} P_n(x, a, b, A) \leq C(1+a)\, \ell\, e^{-cA^2}. \tag{13}$$

Similarly,

$$\begin{aligned}
Q_n(x, a, b, A) &\leq \mathbb{P}_{x,a}(\tau > m, S_m > A\sqrt{m}, S_n \in [b, b+\ell]) \\
&\leq \mathbb{P}_{x,a}\left(S_m > A\sqrt{m} \middle| \tau > m\right) \mathbb{P}_{x,a}(\tau > m) \\
&\quad \sup_{(x',a')\in\mathbb{X}\times\mathbb{R}_*^+} \mathbb{P}_{x',a'}\left(S_{n-m} \in [b, b+\ell]\right) \\
&\preceq \mathbb{P}_{x,a}\left(\frac{S_m}{\sigma\sqrt{m}} > \frac{A}{\sigma} \middle| \tau > m\right) \frac{1+a}{\sqrt{m}} \frac{\ell}{\sqrt{n-m}}
\end{aligned}$$

with $\lim_{m\to+\infty} \mathbb{P}_{x,a}\left(\frac{S_m}{\sigma\sqrt{m}} > \frac{A}{\sigma} \middle| \tau > m\right) = \int_{A/\sigma}^{+\infty} t e^{-t^2/2}\mathrm{d}t = e^{-A^2/2\sigma^2}$. Thus, there exists $C > 0$ such that, for any $x \in \mathbb{X}$ and $a, b > 0$,

$$\limsup_{n\to+\infty} n\, Q_n(x, a, b, A) \leq C(1+a)\, \ell\, e^{-A^2/2\sigma^2}. \tag{14}$$

We conclude, by combining (13) and (14).

Proof of Step 2. Assume now $0 < b < 2\epsilon\sqrt{n}$. The Markov property and Proposition 3 yield

$$\begin{aligned}
&\mathbb{P}_{x,a}(\tau > n,\ S_n \in [b, b+\ell]) \\
&\leq \sum_{i\in\mathbb{N}} \mathbb{P}_{x,a}(\tau > n, S_m \in [i, i+1[, S_n \in [b, b+\ell])
\end{aligned}$$

$$\leq \sum_{i\in\mathbb{N}} \mathbb{P}_{x,a}(\tau > m, S_m \in [i, i+1[) \sup_{\substack{x'\in\mathbb{X} \\ a'\in[i,i+1[}} \mathbb{P}_{x',a'}(\tau > n-m, S_{n-m} \in [b, b+\ell])$$

$$\overset{\text{by (5)}}{\leq} \sum_{i\in\mathbb{N}} \mathbb{P}_{x,a}(\tau > m, S_m \in [i, i+1[) \, \mathbb{P}_{\tilde{x},b+\ell+\Delta}(\tilde{\tau} > n-m, \tilde{S}_{n-m} \in [i, i+\ell+2\Delta+1])$$

$$\overset{\text{by (11)}}{\leq} C\frac{1+a}{m} \sum_{i\in\mathbb{N}} \mathbb{P}_{\tilde{x},b+\ell+\Delta}(\tilde{\tau} > n-m, \tilde{S}_{n-m} \in [i, i+\ell+2\Delta+1])$$

$$\preceq \frac{1+a}{n}\mathbb{P}_{\tilde{x},b+\ell+\Delta}(\tilde{\tau} > n-m)$$

$$\preceq \frac{1+a}{n} \times \frac{1+b+\ell+\Delta}{\sqrt{n-m}} \quad \preceq \quad \frac{(1+a)(1+2\epsilon\sqrt{n})}{n^{3/2}}.$$

We conclude the proof of Step 2 letting $n \to +\infty$, then $\epsilon \to 0$.

Proof of Step 3. We assume $b \in I_{n,\epsilon,A}$ and set $m_\epsilon = \lfloor \epsilon^3 n \rfloor$. We rewrite $\mathbb{P}_{x,a}(\tau > n, S_n \in [b, b+\ell])$ as follows.

$$\begin{aligned}\mathbb{P}_{x,a}(\tau > n, S_n \in [b, b+\ell]) &= \underbrace{\mathbb{P}_{x,a}(\tau > n, |S_{n-m_\epsilon} - b| > \epsilon\sqrt{n}, S_n \in [b, b+\ell])}_{\Sigma_1(n,\epsilon)} \\ &\quad + \underbrace{\mathbb{P}_{x,a}(\tau > n, |S_{n-m_\epsilon} - b| \leq \epsilon\sqrt{n}, S_n \in [b, b+\ell])}_{\Sigma_2(n,\epsilon)}\end{aligned}$$

On the one hand, by the Markov property,

$$\begin{aligned}\Sigma_1(n,\epsilon) &= \int_{\mathbb{X}\times[b-\epsilon\sqrt{n}, b+\epsilon\sqrt{n}]^c} \mathbb{P}_{x',a'}(\tau > m_\epsilon, S_{m_\epsilon} \in [b, b+\ell]) \\ &\quad \mathbb{P}_{x,a}\Big(\tau > n-m_\epsilon, (X_{n-m_\epsilon}, S_{n-m_\epsilon}) \in \mathrm{d}x'\mathrm{d}a'\Big) \\ &\leq \sup_{\substack{x'\in\mathbb{X} \\ |a'-b|>\epsilon\sqrt{n}}} \mathbb{P}_{x',a'}(\tau > m_\epsilon, S_{m_\epsilon} \in [b, b+\ell]) \, \mathbb{P}_{x,a}(\tau > n-m_\epsilon, S_{n-m_\epsilon} \in [b-\epsilon\sqrt{n}, b+\epsilon\sqrt{n}]^c) \\ &\leq \sup_{\substack{x'\in\mathbb{X} \\ |a'-b|>\frac{1}{\sqrt{\epsilon}}\sqrt{m_\epsilon}}} \mathbb{P}_{x',a'}(S_{m_\epsilon} \in [b, b+\ell]) \underbrace{\mathbb{P}_{x,a}(\tau > n-m_\epsilon)}_{\preceq \frac{1+a}{\sqrt{n-m_\epsilon}}}.\end{aligned}$$

By (8), there exist constants $c, C > 0$ such that

$$\limsup_{n\to+\infty} \sqrt{n} \sup_{\substack{x'\in\mathbb{X} \\ |a'-b|>\frac{1}{\sqrt{\epsilon}}\sqrt{m_\epsilon}}} \mathbb{P}_{x',a'}(S_{m_\epsilon} \in [b, b+\ell]) \leq C\,\ell\,\frac{e^{-c/\epsilon}}{\epsilon^{3/2}}$$

Therefore,

$$\lim_{\epsilon\to 0}\limsup_{n\to+\infty}\sup_{\substack{x\in\mathbb{X}\\ b\in I_{n,\epsilon,A}}}|n\Sigma_1(n,\varepsilon)| \preceq \lim_{\epsilon\to 0}(1+a)\,\ell\,\frac{e^{-c/\epsilon}}{\epsilon^{3/2}\sqrt{1-\epsilon^3}}=0. \tag{15}$$

On the other hand, we rewrite $\Sigma_2(n,\epsilon)$ as

$$\begin{aligned}
&\Sigma_2(n,\epsilon)\\
&\quad=\int_{\mathbb{X}\times[b-\epsilon\sqrt{n},b+\epsilon\sqrt{n}]}\mathbb{P}_{x',a'}(\tau>m_\epsilon, S_{m_\epsilon}\in[b,b+\ell])\\
&\quad\mathbb{P}_{x,a}\Big(\tau>n-m_\epsilon,(X_{n-m_\epsilon},S_{n-m_\epsilon})\in \mathrm{d}x'\mathrm{d}a'\Big)\\
&\quad=\Sigma_2'(n,\epsilon)-\Sigma_2''(n,\epsilon),
\end{aligned}\tag{16}$$

where

$$\begin{aligned}
\Sigma_2'(n,\epsilon):=&\int_{\mathbb{X}\times[b-\epsilon\sqrt{n},b+\epsilon\sqrt{n}]}\mathbb{P}_{x',a'}(S_{m_\epsilon}\in[b,b+\ell])\\
&\mathbb{P}_{x,a}\Big(\tau>n-m_\epsilon,(X_{n-m_\epsilon},S_{n-m_\epsilon})\in \mathrm{d}x'\mathrm{d}a'\Big)
\end{aligned}$$

and

$$\begin{aligned}
&\Sigma_2''(n,\epsilon)\\
&\quad:=\int_{\mathbb{X}\times[b-\epsilon\sqrt{n},b+\epsilon\sqrt{n}]}\mathbb{P}_{x',a'}(\tau\le m_\epsilon, S_{m_\epsilon}\in[b,b+\ell])\\
&\quad\mathbb{P}_{x,a}\Big(\tau>n-m_\epsilon,(X_{n-m_\epsilon},S_{n-m_\epsilon})\in \mathrm{d}x'\mathrm{d}a'\Big).
\end{aligned}$$

We first treat the term $\Sigma_2''(n,\epsilon)$. Since $b\ge 2\epsilon\sqrt{n}$, it holds $a'\ge\epsilon\sqrt{n}\ge\frac{\sqrt{m_\epsilon}}{\epsilon}$ for any $a'\in[b-\epsilon\sqrt{n},b+\epsilon\sqrt{n}]$. Hence, by (12), there exist constants $c,C>0$ such that

$$\limsup_{n\to+\infty}\sup_{x'\in\mathbb{X}}\sup_{\substack{a'\in[b-\epsilon\sqrt{n},b+\epsilon\sqrt{n}]\\ b\ge 2\epsilon\sqrt{n}}}\sqrt{n}\,\mathbb{P}_{x',a'}(\tau\le m_\epsilon,S_{m_\epsilon}\in[b,b+\ell])\le C\ell\,\frac{e^{-c/\epsilon}}{\epsilon^{3/2}}.$$

Consequently,

$$\begin{aligned}
n\Sigma_2''(n,\epsilon)\le&\left(\sup_{x'\in\mathbb{X}}\sup_{\substack{a'\in[b-\epsilon\sqrt{n},b+\epsilon\sqrt{n}]\\ b\ge 2\epsilon\sqrt{n}}}\sqrt{n}\,\mathbb{P}_{x',a'}(\tau\le m_\epsilon,S_{m_\epsilon}\in[b,b+\ell])\right)\\
&\int_{\mathbb{X}\times[b-\epsilon\sqrt{n},b+\epsilon\sqrt{n}]}\sqrt{n}\,\mathbb{P}_{x,a}(\tau>n-m_\epsilon,(X_{n-m_\epsilon},S_{n-m_\epsilon})\in\mathrm{d}x'\mathrm{d}a').
\end{aligned}$$

$$\leq \left(\sup_{x' \in \mathbb{X}} \sup_{\substack{a' \in [b-\epsilon\sqrt{n}, b+\epsilon\sqrt{n}] \\ b \geq 2\epsilon\sqrt{n}}} \sqrt{n}\, \mathbb{P}_{x',a'}(\tau \leq m_\epsilon, S_{m_\epsilon} \in [b, b+\ell]) \right)$$
$$\sqrt{n}\, \mathbb{P}_{x,a}(\tau > n - m_\epsilon)$$

which implies

$$\lim_{\epsilon \to 0} \limsup_{n \to +\infty} \sup_{\substack{x \in \mathbb{X} \\ b \in I_{n,\epsilon,A}}} |n \Sigma_2''(n, \varepsilon)| \preceq (1+a)\, \ell \lim_{\epsilon \to 0} \frac{e^{-c/\epsilon}}{\epsilon^{3/2}\sqrt{1-\epsilon^3}} = 0. \qquad (17)$$

It remains to control the term $\Sigma_2'(n, \epsilon)$. By Theorem 6, uniformly in $b \in I_{n,\epsilon,A}$,

$$\Sigma_2'(n, \epsilon) = \int_{\mathbb{X} \times [b-\epsilon\sqrt{n}, b+\epsilon\sqrt{n}]} \mathbb{P}_{x',a'}(S_{m_\epsilon} \in [b, b+\ell])$$
$$\mathbb{P}_{x,a}(\tau > n - m_\epsilon, (X_{n-m_\epsilon}, S_{n-m_\epsilon}) \in \mathrm{d}x'\mathrm{d}a')$$
$$= \int_{\mathbb{X} \times [b-\epsilon\sqrt{n}, b+\epsilon\sqrt{n}]} \frac{1}{\sigma\sqrt{2\pi m_\epsilon}} e^{-(b-a')^2/2\sigma^2 m_\epsilon}\, \ell\, (1 + o_n(1))$$
$$\mathbb{P}_{x,a}(\tau > n - m_\epsilon, (X_{n-m_\epsilon}, S_{n-m_\epsilon}) \in \mathrm{d}x'\mathrm{d}a')$$
$$\text{(with } o_n \text{ uniform in } b, a', \epsilon)$$
$$= \frac{\ell}{\sigma\sqrt{2\pi m_\epsilon}} (1 + o_n(1)) \mathbb{E}_{x,a} \Big[e^{-(b-S_{n-m_\epsilon})^2/2\sigma^2 m_\epsilon}; b - \epsilon\sqrt{n}$$
$$\leq S_{n-m_\epsilon} \leq b + \epsilon\sqrt{n}; \tau > n - m_\epsilon \Big]$$
$$= \frac{1}{\sigma^2\pi}\, V(x, a)\, \frac{\ell}{\sqrt{m_\epsilon(n - m_\epsilon)}}\, (1 + o_n(1))$$
$$\times\, \mathbb{E}_{x,a} \left[e^{-(b-S_{n-m_\epsilon})^2/2\sigma^2 m_\epsilon} 1_{[b-\epsilon\sqrt{n},\ b+\epsilon\sqrt{n}]}(S_{n-m_\epsilon}) / \tau > n - m_\epsilon \right]. \qquad (18)$$

The limit theorem for $(S_n)_n$ conditioned to stay in $\mathbb{R}^+$ (see Proposition 3) combined with the second Dini's theorem yields: for every fixed $\epsilon > 0$, as $n \to +\infty$,

$$\sup_{(x,b) \in \mathbb{X} \times I_{n,\epsilon,A}} \Bigg| \mathbb{E}_{x,a} \left[e^{-(b-S_{n-m_\epsilon})^2/2\sigma^2 m_\epsilon} 1_{[b-\epsilon\sqrt{n},\ b+\epsilon\sqrt{n}]}(S_{n-m_\epsilon}) / \tau > n - m_\epsilon \right]$$
$$- \int_{|\sqrt{1-\epsilon^3}t - \frac{b}{\sqrt{n}}| < \epsilon} t e^{-t^2/2} e^{-(b/\sqrt{n} - \sqrt{1-\epsilon^3}t)^2/2\epsilon^3} \mathrm{d}t \Bigg| \longrightarrow 0. \qquad (19)$$

Since this last integral equals $\frac{b}{\sqrt{n}} e^{-b^2/2n} (2\pi\epsilon)^{3/2} + o(\epsilon^{3/2})$ (see [6] for the details), by combining (18) and (19), we obtain

$$\lim_{\epsilon \to 0} \limsup_{n \to +\infty} \sup_{(x,b) \in \mathbb{X} \times I_{n,\epsilon,A}} \left| n \Sigma_2'(n, \epsilon) - \frac{2\sqrt{2\pi}}{\sigma^2}\, V(x, a)\, \frac{b\, \ell}{\sqrt{n}} e^{-b^2/2n} \right| = 0. \qquad (20)$$

The proof of Step 3 is complete by combining (15), (16), (17) and (20).

4 Proof of Theorem 2

Inequality (2) is proved in [16] Corollary 3.7. The proof of the lower bound (3) is based on Theorem 1 and is valid for $\ell > 2\Delta + 1$ and $b \geq \Delta$. As previously, we set $m = \lfloor n/2 \rfloor$. By the Markov property and (6),

$$\begin{aligned}
&\mathbb{P}_{x,a}(\tau > n, S_n \in [b, b+\ell]) \\
&\quad \geq \mathbb{P}_{x,a}(\tau > n, S_m \in [\sqrt{n}, \sqrt{2n}], S_n \in [b, b+\ell]) \\
&\quad \geq \sum_{\substack{k\in\mathbb{N} \\ \sqrt{n}\leq k\leq\sqrt{2n}-1}} \mathbb{P}_{x,a}\Big(\tau > n, k \leq S_m \leq k+1[,\ b \leq S_m + S_{n-m}\circ\theta^m \leq b+\ell\Big) \\
&\quad \geq \sum_{\substack{k\in\mathbb{N} \\ \sqrt{n}\leq k\leq\sqrt{2n}-1}} \int_{\mathbb{X}\times[k,k+1]} \mathbb{P}_{x,a}(\tau > m, (X_m, S_m) \in \mathrm{d}x'\mathrm{d}a') \\
&\quad \mathbb{P}_{x',a'}(\tau > n-m, b \leq S_{n-m} \leq b+\ell) \\
&\quad \geq \sum_{\substack{k\in\mathbb{N} \\ \sqrt{n}\leq k\leq\sqrt{2n}-1}} \int_{\mathbb{X}\times[k,k+1]} \mathbb{P}_{x,a}\Big(\tau > m, (X_m, S_m) \in \mathrm{d}x'\mathrm{d}a') \\
&\quad \mathbb{P}_{\tilde{x},b-\Delta}(\tilde{\tau} > n-m, a'-\ell \leq \tilde{S}_{n-m} \leq a'-2\Delta) \\
&\quad \geq \sum_{\substack{k\in\mathbb{N} \\ \sqrt{n}\leq k\leq\sqrt{2n}-1}} \mathbb{P}_{x,a}\Big(\tau > m, k \leq S_m \leq k+1\Big) \\
&\quad \mathbb{P}_{\tilde{x},b-\Delta}\Big(\tilde{\tau} > n-m, k+1-\ell \leq \tilde{S}_{n-m} \leq k-2\Delta\Big).
\end{aligned} \tag{21}$$

By Theorem 1, there exists a constant $C_0 > 0$ such that for any $k \in \mathbb{N}$ satisfying $\sqrt{n} \leq k \leq \sqrt{2n} - 1$,

$$\liminf_{n\to+\infty} n\mathbb{P}_{x,a}\Big(\tau > m, k \leq S_m \leq k+1\Big) \geq C_0$$

and

$$\liminf_{n\to+\infty} n\mathbb{P}_{\tilde{x},b-\Delta}\Big(\tilde{\tau} > n-m, k-1 \leq \tilde{S}_{n-m} \leq k-2\Delta\Big) \geq C_0(\ell - 2\Delta - 1).$$

Hence, inequality (21) yields, for n large enough,

$$n^2\mathbb{P}_{x,a}(\tau > n, S_n \in [b, b+\ell]) \geq \frac{C_0^2}{2}(\sqrt{2n} - \sqrt{n})(\ell - 2\Delta - 1),$$

which implies, for such n,

$$\mathbb{P}_{x,a}(\tau > n, S_n \in [b, b+\ell]) \succeq \frac{\ell - 2\Delta - 1}{n^{3/2}}.$$

This achieves the proof of inequality (3), taking for instance $\ell_0 = 4\Delta + 2$.

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