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The principal eigenvalue of a space–time periodic parabolic operator

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Abstract This paper deals with the generalized principal eigenvalue of the parabolic operator $\mathcal{L}\phi = \partial_t\phi - \nabla \cdot (A(t,x)\nabla\phi) + q(t,x) \cdot \nabla\phi - \mu(t,x)\phi$, where the coefficients are periodic in t and x. We give the definition of this eigenvalue and we prove that it can be approximated by a sequence of principal eigenvalues associated to the same operator in a bounded domain, with periodicity in time and Dirichlet boundary conditions in space. Next, we define a family of periodic principal eigenvalues associated with the operator and use it to give a characterization of the generalized principal eigenvalue. Finally, we study the dependence of all these eigenvalues with respect to the coefficients.

Keywords Generalized principal eigenvalue · Parabolic periodic operator

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Contents

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1 Introduction

We are interested in the eigenvalues of parabolic operators of the form:

$$\mathcal{L}\phi = \partial_t \phi - \nabla \cdot (A(t,x)\nabla \phi) + q(t,x) \cdot \nabla \phi - \mu(t,x)\phi \tag{1}$$

The coefficients A, q, μ are supposed to be periodic in t and x, namely, there exist some positive constants $T, L_1, \ldots, L_N$ such that for all t, x, i:

$$\begin{array}{lll} A(t+T,x) = A(t,x), & q(t+T,x) = q(t,x), & \mu(t+T,x) = \mu(t,x) \\ A(t,x+L_i) = A(t,x), & q(t,x+L_i) = q(t,x), & \mu(t,x+L_i) = \mu(t,x) \end{array} \tag{2}$$

The periods $T, L_1, \ldots, L_N$ will be fixed in the sequel and when a function is said to be periodic in t or in x, this periodicity will always refer to these given periods.

This kind of operators appears in the context of reaction–diffusion in space-periodic media, that is to say equation of the form:

$$\partial_t u - \nabla \cdot (A(t,x)\nabla u) + q(t,x) \cdot \nabla u = f(t,x,u) \quad \text{in } \mathbb{R} \times \mathbb{R}^N \tag{3}$$

where f may satisfies the following hypotheses:

$$\forall x \in \mathbb{R}^N, \forall t \in \mathbb{R},\ s \to f(t,x,s)/s \text{ decreases on } \mathbb{R}^{+*} \tag{4}$$

$$\exists\, M > 0 \mid \forall x \in \mathbb{R}^N, \forall t \in \mathbb{R}, \forall s \geq M,\ f(t,x,s) \leq 0 \tag{5}$$

This kind of equation arises in population genetics, combustion and population dynamics models (see [3,11,22]). It is a generalization of the following homogeneous equation $\partial_t u - \Delta u = u(1-u)$. The case of the heterogeneous equation is of particular interest in population dynamics and we would like to investigate the influence of the environment on the species survival.

In [4], Berestycki et al. proved that if $q \equiv 0$, under these hypotheses, there exists a unique positive stationary state, which is periodic and attractive, if and only if the principal eigenvalue associated to the linearized problem around zero was negative. Setting $\mu(t,x) = f'_u(t,x,0)$, this linearized operator is of the form (1). In [7], Berestycki et al. extended these results to a more general class of operators.

In [21], Pinsky studied the principal eigenvalue of a space-periodic elliptic operator, their positive harmonic functions and their dependence to perturbations. In [5], using their preceding results, Berestycki et al. stated that these equations have solutions of a particular form: the *pulsating traveling fronts solutions*. These fronts admit a minimal speed, which can be characterized, under some additional hypotheses, using the principal eigenvalues of a family of space-periodic elliptic operators. This minimal speed is useful to compute the spreading speed of a solution with compactly supported initial data (see [6,12]).

Recently, in [20], Nolen et al. proved that the existence of *pulsating traveling fronts* and the characterization of their minimal speed can be extended to space–time periodic parabolic operator. They assumed that the reaction term is nonnegative and has only to zeros: 0 and 1. It is left to prove that these results can be extended to a more general class of reaction terms. The first part of this work is the study of the space–time periodic states of the reaction–diffusion equation. This is carried out in [19]. In this article, we prove that the sign of the two generalized eigenvalues plays an important role. During this work, many problems occurred and it was not always possible to extend the methods that were used in [4]. An interesting issue occurred: is it possible to approximate a space–time periodic principal eigenvalue with time-periodic principal eigenvalues in increasing bounded domains? We answer this question in this article.

In the other hand, many studies have been carried out on time-periodic parabolic operator in bounded domains (see [15–17]). These results cannot always be extended to unbounded domain, as we will see.

Lastly, we will study the effects of perturbations on this principal eigenvalue. Some of the results we prove are only extensions of well-known results in the case of a periodic elliptic operator or of a time-periodic parabolic operator in a bounded domain, but some of them are totally new, sometimes even in those simpler cases.

2 Approximation and characterization of the generalized principal eigenvalue

2.1 Definition of the generalized principal eigenvalue λ_1

The diffusion matrix A is supposed to be uniformly elliptic and continuous: there exist some positive constants γ and Γ such that for all $\xi \in \mathbb{R}^N$, $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ one has:

$$\gamma \|\xi\|^2 \leq \sum_{1 \leq i,j \leq N} a_{i,j}(x,t)\xi_i\xi_j \leq \Gamma \|\xi\|^2 \tag{6}$$

where $\|\xi\| = (|\xi_1|^2 + \cdots + |\xi_N|^2)^{1/2}$.

We need the following Holder-regularity for the coefficients: there exists $0 < \delta < 1$ such that $q \in \mathcal{C}^{\frac{\delta}{2},\delta}(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^N)$, $\mu \in \mathcal{C}^{\frac{\delta}{2},\delta}(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ and $A \in \mathcal{C}^{\frac{\delta}{2},1+\delta}(\mathbb{R} \times \mathbb{R}^N, \mathcal{S}_N(\mathbb{R}))$.

Under these hypotheses, we are able to define the generalized principal eigenvalue λ_1:

Definition 2.1 The generalized principal eigenvalue λ_1 is defined by:

$$\lambda_1 = \sup\{\lambda \in \mathbb{R} \mid \exists \phi \in \mathcal{C}^{1,2}(\mathbb{R} \times \mathbb{R}^N),\ \phi > 0,\ \phi \text{ is T-periodic and } \mathcal{L}\phi \geq \lambda\phi \text{ in } \mathbb{R} \times \mathbb{R}^N\} \tag{7}$$

This eigenvalue is related to uniqueness problems for the entire solutions of equation (3) (see [19]). We underline that we do not take the supremum over the functions that are periodic in x, but that we force the periodicity in t. It is not possible to define this eigenvalue if we do not assume that the functions ϕ are periodic in t. In fact, since $(\mathcal{L}-\lambda)(\phi e^{\alpha t}) = (\mathcal{L}+\alpha-\lambda)(\phi)e^{\alpha t}$ for all $\alpha \in \mathbb{R}$, if we do not force the periodicity in t, this would yield that $\lambda_1 = \lambda_1 + \alpha$ for all α.

We will also investigate the set of the generalized principal eigenfunctions associated to λ_1, which are defined by:

Definition 2.2 A function $\phi \in \mathcal{C}^{1,2}(\mathbb{R} \times \mathbb{R}^N)$ is called a generalized principal eigenfunction if it satisfies:

$$\begin{cases} \mathcal{L}\phi = \lambda_1\phi \text{ in } \mathbb{R} \times \mathbb{R}^N \\ \phi > 0 \text{ in } \mathbb{R} \times \mathbb{R}^N \\ \phi \text{ is T-periodic} \end{cases} \tag{8}$$

In the case of a bounded and smooth domain, one can consider the eigenvalue associated with the same operator but in a bounded domain with Dirichlet boundary conditions:

$$\begin{cases} \mathcal{L}\phi = \lambda_1(\Omega)\phi \\ \phi > 0 \text{ in } \mathbb{R}\times\Omega \\ \phi \text{ is T-periodic} \\ \phi = 0 \text{ in } \mathbb{R}\times\partial\Omega \end{cases} \tag{9}$$

It has been proved (see [15] for example) that the eigenvalue $\lambda_1(\Omega)$ is well-defined and unique and that ϕ is unique up to multiplication by a positive constant. Moreover, this eigenvalue can be characterized with a formula similar to (7):

$$\lambda_1(\Omega) = \sup\{\lambda \mid \exists\phi \in \mathcal{C}^{1,2}(\mathbb{R}\times\Omega)\cap\mathcal{C}^{1,1}(\mathbb{R}\times\overline{\Omega}),\ \phi > 0 \text{ and } \mathcal{L}\phi \geq \lambda\phi \text{ in } \mathbb{R}\times\Omega\} \tag{10}$$

The first issue we investigate is that of the approximation of the generalized principal eigenvalue λ_1 by a sequence of eigenvalues associated to increasing domains. We state the following result:

Proposition 2.3 *Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of nonempty bounded open sets such that:*

$$\Omega_n \subset \Omega_{n+1}, \quad \bigcup_{n\in\mathbb{N}} \Omega_n = \mathbb{R}^N$$

Then $\lambda_1(\Omega_n) \searrow \lambda_1$ as $n \to +\infty$.

The proof of this theorem includes the proof of the existence of a generalized principal eigenfunction:

Proposition 2.4 *There exists a generalized principal eigenfunction associated with λ_1.*

This means that one can replace the supremum in formula (7) by a maximum. One deduces from Definition 7 and Proposition 2.4 that:

Proposition 2.5 *One has:*

$$\lambda_1 = \max_{\phi>0,\quad \phi \text{ is } T-periodic,} \inf_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} \frac{\mathcal{L}\phi}{\phi}.$$

We are not able to be more precise about this eigenfunction. All the theorems stated in this section will be proved in a more general case (see Sect. 2.4).

2.2 Characterization of λ_1 with the help of the periodic principal eigenvalues k_α

In the case of a space periodic operator, there already exist periodic principal eigenvalues associated with the whole domain $\mathbb{R}^N$. We would like to compare the generalized principal eigenvalue with the periodic principal eigenvalues. First of all, we need to define these periodic principal eigenvalues.

Set L_α the following modified operator:

$$\begin{aligned} L_\alpha\phi &= e^{-\alpha\cdot x}\mathcal{L}(e^{\alpha\cdot x}\phi) \\ &= \partial_t\phi - \nabla\cdot(A\nabla\phi) - 2\alpha A\nabla\phi + q\cdot\nabla\phi - (\alpha A\alpha + \nabla\cdot(A\alpha) - q\cdot\alpha + \mu)\phi \end{aligned}$$

where $\beta A\alpha = \sum_{i,j=1}^N \beta_i a_{i,j}\alpha_j$.

Definition 2.6 A periodic principal eigenfunction of the operator L_α is a function $\phi \in \mathcal{C}^{1,2}(\mathbb{R} \times \mathbb{R}^N)$ such that it exists a constant k so that:

$$\begin{cases} L_\alpha \phi = k\phi \\ \phi > 0 \\ \phi(.,. + T) = \phi \\ \phi(. + L_i e_i, .) = \phi \quad \text{for } i = 1, \dots, N \end{cases} \tag{11}$$

Such a k is called a periodic principal eigenvalue.

We first prove the existence and the uniqueness of these eigenelements:

Theorem 2.7 *There exists a couple (k, ϕ) that satisfies (11). Furthermore, k is unique and ϕ is unique up to multiplication by a positive constant.*

We define $k_\alpha = k$ and $\phi_\alpha = \phi$ the eigenelements associated with L_α and normalized by $\|\phi_\alpha\|_\infty = 1$.

The proof of this theorem includes the proof of the following proposition, which is of independent interest:

Proposition 2.8 *For all $\alpha \in \mathbb{R}^N$, there exists some $\beta_0 \in \mathbb{R}$ such that for all $\beta > \beta_0$, $g \in \mathcal{C}^0_{\text{per}}(\mathbb{R} \times \mathbb{R}^N)$, there exists a unique function $u \in \mathcal{C}^{1,2}_{\text{per}}(\mathbb{R} \times \mathbb{R}^N)$ that satisfies:*

$$L_\alpha u + \beta u = g$$

We now state a variational characterization for the periodic principal eigenvalues and an important concavity result that we will need several times in the sequel:

Proposition 2.9 *One has the following characterization for the periodic principal eigenvalues k_α:*

$$k_\alpha = \max_{\phi>0 \; in \; (t,x), \;\; \phi \; is \; T-periodic.} \min_{\mathbb{R}\times\mathbb{R}^N} \left(\frac{L_\alpha \phi}{\phi}\right) = \min_{\phi>0 \; in \; (t,x), \;\; \phi \; is \; T-periodic.} \max_{\mathbb{R}\times\mathbb{R}^N} \left(\frac{L_\alpha \phi}{\phi}\right).$$

Proposition 2.10 *For all A, q, set F the map:*

$$\begin{aligned} \mathbb{R}^N \times \mathcal{C}^0_{\text{per}}(\mathbb{R}^N \times \mathbb{R}) &\to \mathbb{R} \\ (\alpha, \mu) &\mapsto k_\alpha(\mu) \end{aligned}$$

Then F is concave and continuous.

This eigenelements family enables us to characterize the generalized principal eigenelements of (7):

Theorem 2.11 *If $\varphi \in \mathcal{C}^{1,2}(\mathbb{R} \times \mathbb{R}^N)$ is a generalized principal eigenfunction of (2.2), then it exists $\alpha \in \mathbb{R}^N$ such that $\phi_\alpha e^{\alpha \cdot x}$ is a generalized principal eigenvalue of (2.2).*

Theorem 2.12 *One has the following equality:*

$$\lambda_1 = \max_{\alpha \in \mathbb{R}^N} k_\alpha.$$

Furthermore, there exists a unique α that satisfies $\lambda_1 = k_\alpha$.

2.3 Comparison with another generalized principal eigenvalue

We now define another generalized principal eigenvalue with:

$$\lambda_1' = \inf\{\lambda \mid \exists \phi \in \mathcal{C}^{1,2}(\mathbb{R}\times\mathbb{R}^N)\cap W^{1,\infty}(\mathbb{R}\times\mathbb{R}^N), \phi \text{ is periodic in } t, \\ \phi > 0 \text{ and } \mathcal{L}\phi \le \lambda\phi \text{ in } \mathbb{R}\times\mathbb{R}^N\} \tag{12}$$

This second generalized principal eigenvalue is related to the existence problems for the periodic positive solutions of Eq. (3). Since the first one was associated with the uniqueness properties of these periodic solutions, this is interesting to investigate when the equality $\lambda_1 = \lambda_1'$ holds: this is the case where the periodic solution is unique when it exists. In the case of a time-homogeneous domain, if $q \equiv 0$, then one has $\lambda_1' = \lambda_1$ (see [7]). We now investigate if this assertion is still true in our context. First, we characterize the generalized principal eigenvalue λ_1' with the help of the periodic principal eigenvalues k_α.

Theorem 2.13 *One has the following equality:*

$$\lambda_1' = k_0.$$

With the help of this characterization, we are now able to try to find some cases where $\lambda_1' = \lambda_1$.

Proposition 2.14 *If A, q and μ have a common symmetry axis in t or in x, in other words if*:

$$\begin{aligned} &\exists x_0 \mid \forall t, x, \quad A(t,x_0+x) = A(t,x_0-x),\ q(t,x_0+x) = q(t,x_0-x) \\ &\qquad\qquad\qquad \textit{and}\quad \mu(t,x_0+x) = \mu(t,x_0-x), \\ &\textit{or if}\ \exists t_0 \mid \forall t, x, \quad A(t_0+t,x) = A(t_0-t,x),\ q(t_0+t,x) = q(t_0-t,x) \\ &\qquad\qquad\qquad \textit{and}\quad \mu(t_0+t,x) = \mu(t_0-t,x), \end{aligned} \tag{13}$$

and if q can be written $q = A\nabla Q$ where $Q \in \mathcal{C}^{0,1}(\mathbb{R}\times\mathbb{R}^N)$ with $\int_{(0,T)\times C} A^{-1}q = 0$, then $\lambda_1' = \lambda_1$.

This result is not true in general, even in the case of constant coefficients. It is easy to see that the inequality $\lambda_1' \le \lambda_1$ is always true using Theorems 2.12 and 2.13. But it is possible to get $\lambda_1' < \lambda_1$. For example, take:

$$\mathcal{L}\phi = -\phi'' + \phi'.$$

Taking $\phi \equiv 1$, one easily gets $\lambda_1' \le 0$ (in fact the equality holds). One can remark that:

$$e^{-\frac{x}{2}}\mathcal{L}(e^{\frac{x}{2}}\psi) = \mathcal{L}'\psi = -\psi'' + \frac{1}{4}\psi.$$

This modified operator is self-adjoint, thus one has $\lambda_1(\mathcal{L}') = \lambda_1(\mathcal{L}) = \frac{1}{4} > \lambda_1'$.

In the case of elliptic space-periodic operator, one has $\lambda_1 = \lambda_1'$ for general A, μ if q satisfies the same conditions as in proposition 2.14 (see [7]). It is not clear whether this assertion is still true in the case of space–time periodic parabolic operators and this would be interesting to prove it or to find a counterexample.

2.4 The case of general unbounded domains

We can wonder if the results that we have stated in the previous sections can be generalized to more general unbounded domains. Obviously, this will not be the case for the results that are related to the periodic principal eigenvalues k_α. Nevertheless, we are able to extend the other results to very general unbounded domains. The results of the first section are indeed particular cases of those which follow.

We define the principal eigenvalue of the operator $\mathcal{L}$ in the domain Ω as the quantity:

$$\lambda_1(\Omega) = \sup\{\lambda \mid \exists\phi \in \mathcal{C}^{1,2}(\mathbb{R}\times\Omega)\cap\mathcal{C}^{1,1}(\mathbb{R}\times\overline{\Omega}),\ \phi > 0, \\ \phi \text{ is T-periodic and } \mathcal{L}\phi \geq \lambda\phi \text{ in } \mathbb{R}\times\Omega\} \tag{14}$$

Obviously, this eigenvalue is nonincreasing with respect to Ω: if $\Omega \subset \Omega'$ then $\lambda_1(\Omega) \geq \lambda_1(\Omega')$. We also notice that if Ω is bounded and smooth, this definition is equivalent to the definition we gave in (9). First of all, we prove that this definition makes sense for general unbounded domains:

Proposition 2.15 *The principal eigenvalue is well-defined in any nonempty open domain Ω and $\lambda_1(\Omega) < +\infty$.*

Next, we prove that the approximation result still holds in the case of general domains. Furthermore, we can approximate such a domain with general, maybe unbounded or irregular, domains:

Theorem 2.16 *Let Ω be a general domain in $\mathbb{R}^N$ and $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of nonempty open sets such that*:

$$\Omega_n \subset \Omega_{n+1}, \quad \bigcup_{n\in\mathbb{N}} \Omega_n = \Omega$$

Then $\lambda_1(\Omega_n) \searrow \lambda_1(\Omega)$ as $n \to +\infty$.

This result is a generalization of the theorem for elliptic operators that was first proved for a bounded but not necessarily smooth domain in [8] and then for general domain in [7].

Lastly, we can define the generalized principal eigenfunctions in the same way as in definition 2.2. We are able to prove that such a function always exists:

Proposition 2.17 *For any general domain Ω, there exists a generalized principal eigenfunction associated with $\lambda_1(\Omega)$.*

This proposition gives the following $max - inf$ characterization for λ_1:

Proposition 2.18 *One has:*

$$\lambda_1 = \max_{\phi>0,\quad \phi \text{ is } T-periodic.} \ \inf_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} \frac{\mathcal{L}\phi}{\phi}.$$

3 Dependence with respect to the coefficients

Let us denote $\lambda_1'(A, q, \mu)$ and $\lambda_1(A, q, \mu)$ the two generalized eigenvalues associated with the diffusion matrix A, the advection term q and the intrinsic growth rate μ. We prove in [19] that the sign of $\lambda_1'(A, q, \mu)$ determines the existence of a positive periodic solution of Eq. (3) and that the sign of $\lambda_1(A, q, \mu)$ determines the convergence of the solution of the associated Cauchy problem to this periodic solution. Thus decreasing these eigenvalues enhances the possibility for the species to survive. Hence it is of particular interest to investigate how the diffusion matrix A, the advection term q and the growth rate μ affect these two eigenvalues. In this section, we will give results on the influence of the shape and the amplitude on these eigenvalues. We might omit A, q or μ in the notations if these quantities are not the main subject of our investigation.

Many of the following results can be extended to the periodic principal eigenvalues $k_\alpha(A, q, \mu)$. When this is possible, we state the results for the generalized principal eigenvalues in order to simplify the statements, but we prove the results for k_α and then use Theorem 2.12.

3.1 Particular cases

We first give the two particular cases of space-homogeneous environment and of the time-homogeneous environment without drift. These particular cases will be useful to get counter-examples in the sequel. The proof of these results can be found in [4, 15].

Proposition 3.1 *If μ, q and A do not depend on x, one has*:

$$\lambda_1'(A, q, \mu) = -\frac{1}{T}\int_0^T \mu.$$

Thus, the dependence between the environment and the generalized principal eigenvalues is very simple. In this case, the shape of the environment, the diffusion matrix A and the advection term q do not play any role.

In the case of a time-homogeneous environment, with no advection term, the operator is self-adjoint. This yields the following characterization for the principal eigenvalue:

Proposition 3.2 *If A and μ do not depend on t and $q \equiv 0$ then*:

$$\lambda_1 = \lambda_1' = \min_{\phi \in \mathcal{C}^2_{per}(\mathbb{R}^N),\ \phi > 0} \frac{\int_C \left(\nabla\phi A(x) \nabla\phi - \mu(x)\phi^2\right) \mathrm{d}x}{\int_C \phi^2 \mathrm{d}x} \tag{15}$$

In [4], Berestycki et al. analyzed the influence of the growth rate μ on the principal eigenvalue and obtained many results using this formula. The methods they used are not available now that there are two first-order terms. Nevertheless, we will now generalize these results using other methods.

3.2 Influence of the amplitude of the growth rate

Theorem 3.3 *Take ϕ_α an eigenfunction associated with L_α and $\widetilde{\phi}_\alpha$ the eigenfunction associated with the adjoint problem, normalized by $\int_{(0,T)\times C} \phi_\alpha \widetilde{\phi}_\alpha = 1$*:

$$\begin{cases} \partial_t \phi_\alpha - \nabla \cdot (A\nabla\phi_\alpha) - 2\alpha A \nabla\phi_\alpha + q \cdot \nabla\phi_\alpha - (\nabla \cdot (A\alpha) + \alpha A \alpha - q \cdot \alpha + \mu)\phi_\alpha \\ \quad = k_\alpha(\mu)\phi_\alpha, \ -\partial_t \widetilde{\phi}_\alpha - \nabla \cdot (A\nabla\widetilde{\phi}_\alpha) + 2\alpha A\nabla\widetilde{\phi}_\alpha - \nabla \cdot (q\widetilde{\phi}_\alpha) + (\nabla \cdot (A\alpha) - \alpha A\alpha + q \cdot \alpha - \mu)\widetilde{\phi}_\alpha \\ \quad = k_\alpha(\mu)\widetilde{\phi}_\alpha, \\ \phi_\alpha > 0, \ \widetilde{\phi_\alpha} > 0, \\ \phi_\alpha \text{ and } \widetilde{\phi_\alpha} \text{ are both periodic in } t \text{ and } x. \end{cases} \tag{16}$$

Take η a periodic continuous function. If $\int_{(0,T)\times C} \eta\phi_\alpha\widetilde{\phi}_\alpha > 0$, then the function $B \to k_\alpha(\mu + B\eta)$ is decreasing over $\mathbb{R}^+$. If $\int_{(0,T)\times C} \eta\phi_\alpha\widetilde{\phi}_\alpha = 0$, then $B \to k_\alpha(\mu + B\eta)$ is nonincreasing.

Remark 3.4 (1) In [4], this result was stated for a constant growth rate μ_0 and for $\alpha = 0$. In this case, $\phi_0 \equiv \widetilde{\phi}_0 \equiv 1$ and the condition for the monoticity is simpler: $\int_{(0,T)\times C} \eta \geq 0$.

(2) In the general case, the weight $\phi_\alpha\widetilde{\phi}_\alpha$ corresponds to the invariant measure of the stochastic process associated with the operator L_α.

This means that if the heterogeneity is favorable or neutral in average, then the more you increase the amplitude of the favorable *and* the unfavorable zone, the more the environment is globally favorable for the species.

In the case of a time-homogeneous environment, it has been proved in [4] that if the environment is unfavorable in average (i.e. $\int_C \mu < 0$), but if there exists a favorable zone (i.e. $\exists x_0|\ \mu(x_0) > 0$), then for a large enough amplitude, the principal eigenvalue was negative (i.e. $\exists B_0|\ \forall B > B_0,\ \lambda_1(B\mu) < 0$). This was a result with an interesting biological interpretation. This result does not hold true anymore in the case of a time-dependent system, because of proposition 3.1.

Nevertheless, the following proposition, which is the analogue of the result of [4], hold true:

Theorem 3.5 *If $\int_{(0,T)} \max_{x\in\mathbb{R}^N} \eta(t,x)dt > 0$ and (A, q, μ) are constant, then for all α, for B large enough, one has $k_\alpha(A, q, \mu + B\eta) < 0$.*

This result has been proved in the case of a bounded domain in [15]. We use this case to prove the theorem.

3.3 Influence of the diffusion

In the case of a time-homogeneous environment, the formula (15) yields that if $\gamma > \gamma'$, then $\lambda_1(\gamma A, 0, \mu) > \lambda_1(\gamma' A, 0, \mu)$. This result does not hold true in a time-dependent environment, as it was proved in [16]. It is only left to investigate the asymptotic behavior when $\gamma \to 0$ and $\gamma \to +\infty$.

Theorem 3.6 *For all A, q, μ, set*:

$$\overline{A}(t) = \frac{1}{|C|}\int_C A(t,x)\mathrm{d}x, \quad \overline{\mathrm{q}}(\mathrm{t}) = \frac{1}{|\mathrm{C}|}\int_\mathrm{C} \mathrm{q}(\mathrm{t},\mathrm{x})\mathrm{dx}, \quad \overline{\mu}(\mathrm{t}) = \frac{1}{|\mathrm{C}|}\int_\mathrm{C} \mu(\mathrm{t},\mathrm{x})\mathrm{dx}.$$

The following convergences holds as $\gamma \to +\infty$:

$$\lambda_1'(\gamma A, q, \mu) \to -\frac{1}{T|C|}\int_{(0,T)\times C} \mu = \lambda_1'(\overline{A}, \overline{q}, \overline{\mu})$$

$$\lambda_1(\gamma A, q, \mu) \to \lambda_1(\overline{A}, \overline{q}, \overline{\mu}).$$

For all A, q, μ, one has $\lambda_1'(\gamma A, \sqrt{\gamma} q, \mu) \to -\max_{x\in\mathbb{R}^N} \frac{1}{T}\int_{(0,T)} \mu(t,x)dt$ as $\gamma \to 0$.

This result means that a very large diffusion is favorable and a very small one is unfavorable, even if there is no monotonicity between those two limit cases. It was proved in the case of bounded environment in [15], in the particular case where A and q can be written as products $a(t)A_0(x)$ and $p(t)q_0(x)$, and in [10] in the case of time-independent coefficients.

3.4 Effect of the spatial variations

First of all, we discuss the influence of a heterogeneous function μ compared to the case where μ is constant in t, with the same average as μ.

Theorem 3.7 *For all (A, q, μ), if $\nabla \cdot q = 0$, the following comparisons holds*:

$$\lambda_1'(A, q, \mu) \leq \lambda_1'(\overline{A}, \overline{q}, \overline{\mu}) = \frac{-1}{|C|T} \int\limits_{(0,T)\times C} \mu.$$

This means that the spatial heterogeneity, in some sense, can enhance the possibility of a survival of the species. This result was proved in [4] in the case where the coefficients do not depend on t and in [16] in the case of a bounded domain, with $A = I_N$ and $q \equiv 0$. This generalization is new.

Next, we study the influence of the shape of the heterogeneity. To state our result, we first need to introduce the notion of Schwarz and Steiner periodic symmetrizations. For more details and properties about these notions, we refer the reader to [18].

Definition 3.8 [18] Assume that μ is a bounded measurable L-periodic function on the real line $\mathbb{R}$. Then there exists a unique bounded measurable L-periodic function μ^* such that:

(i) μ^* is symmetric with respect to $L/2$,
(ii) μ^* is nondecreasing on $(0, L/2)$,
(iii) μ^* has the same distribution function as μ, for all $\alpha \in \mathbb{R}$:

$$meas\{x, \mu(x) > \alpha\} = meas\{x, \mu^*(x) > \alpha\}.$$

The function μ^* is called the *Schwarz periodic symmetrization* of the function μ.

Consider now a function μ periodic on the set $\mathbb{R}^N$, with the period cell C. Fixing $(x_1, \ldots, x_{k-1}, x_k, \ldots, x_N)$, one can rearrange the function $x_k \mapsto \mu(x_1, \ldots, x_k, \ldots, x_N)$. This is called the Steiner rearrangement of the function μ in x_k. Performing successive rearrangement with respect to $x_1, \ldots, x_N$, one obtains a periodic function μ^* which is symmetric with respect to the planes $x_k = L_k/2$, nondecreasing in x_k on the set $\{x_k \in (0, L_k/2)\}$, with the same distribution function as μ. We underline that these conditions do not give a unique function μ^* and the way the symmetrization is carried out can lead to different rearranged functions. In the sequel, we will call μ^* the function that is obtained after succesive rearrangements in $x_1, \ldots, x_N$.

In [4], the authors proved that, if $A = I_N$, $q \equiv 0$ and μ does not depend on t, then $\lambda_1'(\mu) \geq \lambda_1'(\mu^*)$. The proof was based on the variational characterization of the principal eigenvalue as a Rayleigh quotient, which does not hold true anymore in the case of a time-dependent problem. Nevertheless, we give an alternate proof of this result in the case of a time-dependent problem, which is based on a strong result on Steiner periodic rearrangement that has been stated by Alvino et al. [2].

Theorem 3.9 *If $A = \gamma I_N$ and $q \equiv 0$, one has*:

$$\lambda_1(\mu^*) \leq \lambda_1(\mu)$$

where μ^* is the successive Steiner periodic symmetrizations in $x_1, \ldots, x_N$ of the function μ.

This is another way to compare two fragmented environments.

3.5 Effect of the temporal variations

We now study the influence of the amplitude of the first order term in t:

Theorem 3.10 *Set* $\lambda_1(\kappa)$ *the eigenvalue defined by*:

$$\lambda_1(\kappa) = \sup\{\lambda \in \mathbb{R},\ \exists\phi \in \mathcal{C}^{1,2}(\mathbb{R} \times \mathbb{R}^N),\ \phi > 0,\ \phi \text{ is } T\text{-periodic}, \kappa\partial_t\phi - \nabla \cdot (A\nabla\phi) + q \cdot \nabla - \mu\phi - \lambda\phi \geq 0 \text{ in } \mathbb{R} \times \mathbb{R}^N\} \tag{17}$$

Then $\lambda_1(\kappa) \to \lambda_1(\hat{A}, \hat{q}, \hat{\mu})$ *as* $\kappa \to +\infty$, *where*:

$$\hat{A}(x) = \frac{1}{T}\int_0^T A(t,x)\mathrm{d}t, \quad \hat{q}(x) = \frac{1}{T}\int_0^T q(t,x)\mathrm{d}t, \quad \hat{\mu}(x) = \frac{1}{T}\int_0^T \mu(t,x)\mathrm{d}t.$$

This result is new. In [16], some numerical computations had been carried out and suggested the existence of such a limit. It also suggested that the eigenvalue $\lambda_1(\kappa)$ was increasing in κ. This conjecture remains as an open problem. It is easy to prove that $\lambda_1(\kappa) = \kappa\lambda_1(\frac{A}{\kappa}, \frac{q}{\kappa}, \frac{\mu}{\kappa})$ and this formula and the theorem give another homogenization result.

Finally, we underline as another open problem the influence of the Steiner periodic rearrangement in t on the principal eigenvalue $\lambda_1'(\mu)$. This seems to be a difficult issue. The classical methods all use the symmetry of the solutions of the rearranged problems, which does not hold true in this case. This problem is linked to that of the influence of a drift on a rearrangement problem. As far as we know, there are only results for a time homogeneous problem with Dirichlet boundary conditions on this issue (see [1] and [13]).

3.6 An optimization result

Next, we state an optimization result for the generalized principal eigenvalues when the maximum, the minimum and the average of the function μ are fixed. This result was proved in [9] in the case of a time-homogeneous problem. We give here an alternate, simpler proof, which use classical optimization arguments.

Theorem 3.11 *Set*

$$F = \left\{\mu \in L^\infty([0;T] \times \overline{C}); \alpha \leq \mu(t,x) \leq \beta \text{ a.e. } t, x, \frac{1}{|C|T}\int_{(0,T)\times C} \mu = m\right\},$$

where α, β *and* m *are such that* F *is not empty.*

Then, the functions $\mu \mapsto \lambda_1(\mu)$ *and* $\mu \mapsto \lambda_1'(\mu)$ *reach their minima over* F *when* μ *is a function of the type* $\mu = \alpha 1_A + \beta 1_{(0,T)\times C\backslash A}$, *where* A *is a measurable subset of* $(0,T) \times C$ *such that* $\alpha\ |A| + \beta\ |(0,T) \times C\backslash A| = m\ |C|T$.

This means that it is better for the species survival to consider an environment with very favorable areas and very unfavorable areas, instead of a smooth environment, with slow evolution from an area to another.

This result leads to another issue. Namely, we would like to find the set A that minimizes the generalized principal eigenvalues $\lambda_1(\alpha 1_A + \beta 1_{(0,T)\times C-A})$ and $\lambda_1'(\alpha 1_A + \beta 1_{(0,T)\times C-A})$. In the case of a time-homogeneous environment with Dirichlet boundary conditions, it is well-known that this set is the ball when the shape of the domain is free and its measure is given. In the case of periodic boundary condition, even when the environment does not depend on t, there is no general result on this issue. We prove in the sequel that such a set must be Steiner-symmetric in x, but there are many Steiner-symmetric sets for a given area. In [14], Hamel and Roques give some numerical and theoretical results on this issue in the case of a time-homogeneous environment.

4 Proofs of the properties of the principal generalized eigenvalue

In this section, we prove Propositions 2.15, 2.16 and 2.17. The proofs of these propositions immediately give the proofs of propositions 2.3 and 2.4.

Proof of Proposition 2.15 As the function μ is bounded in $\mathbb{R}\times\Omega$, there exists a constant ν such that $\sup_{\mathbb{R}\times\Omega}(\mu+\nu)\le 0$. Considering the function $\phi\equiv 1$, one gets $\mathcal{L}\phi=-\mu\phi\ge\nu\phi$, which prove that $\nu\in\{\lambda|\exists\phi\in\mathcal{C}^{1,2}(\mathbb{R}\times\Omega)\cap\mathcal{C}^{1,1}(\mathbb{R}\times\overline{\Omega}),\ \phi>0 \text{ and } \mathcal{L}\phi\ge\lambda\phi \text{ in } \mathbb{R}\times\Omega\}$. Thus this set is not empty and $\lambda_1(\Omega)$ is defined in $\mathbb{R}\cup\{+\infty\}$.

On the other hand, observe that if $\Omega'\subset\Omega''$, then $\lambda_1(\Omega')\ge\lambda_1(\Omega'')$ by definition. As Ω is an open set, there exists an open ball $B\subset\Omega$. The principal eigenvalue $\lambda_1(B)$ corresponds with the classical eigenvalue defined by (9), thus $\lambda_1(\Omega)\le\lambda_1(B)<+\infty$. This ends the proof.

Proof of proposition 2.16 and Theorem 2.17 We reproduce the proof of proposition 4.2 of [7], which holds in the case of space-periodic elliptic operators. It is more convenient to deal with bounded and smooth domains. So consider a family $(\widetilde{\Omega}_n)_{n\in\mathbb{N}}$ of bounded, smooth domains such that:

$$\widetilde{\Omega}_n\subset\widetilde{\Omega}_{n+1},\quad \overline{\widetilde{\Omega}_n}\subset\Omega_n \quad\text{and}\quad \bigcup_{n\in\mathbb{N}}\widetilde{\Omega_n}=\Omega.$$

Call $\lambda_{1,n}=\lambda_1(\Omega_n),\widetilde{\lambda}_n=\lambda_1(\widetilde{\Omega}_n)$ and $\lambda_1=\lambda_1(\Omega)$. The sequences $(\lambda_{1,n})_{n\in\mathbb{N}}$ and $(\widetilde{\lambda}_n)_{n\in\mathbb{N}}$ are nonincreasing and bounded from below by λ_1, so that they converge and:

$$\lambda_1\le\lim_{n\to\infty}\lambda_{1,n}\le\lim_{n\to\infty}\widetilde{\lambda}_n,$$

thus we are back to the case of bounded, smooth domains. We define $\widetilde{\lambda}=\lim_{n\to\infty}\widetilde{\lambda}_n$.

Next, fix $x_0\in\widetilde{\Omega_0}$ and consider the sequence $(\widetilde{\phi}_n)_{n\in\mathbb{N}}$ of the time-periodic principal eigenfunctions of $-\mathcal{L}$ in $\widetilde{\Omega}_n$ with Dirichlet boundary conditions, normalized by $\widetilde{\phi}_n(0,x_0)=1$. Since the sequence $(\widetilde{\lambda}_n)_{n\in\mathbb{N}}$ is bounded, using the Krylov–Safonov Harnack inequality, for all m, we can find a positive constant $C(m)$ such that:

$$\forall n>m,\qquad \sup_{t\in[-T,0],x\in\widetilde{\Omega}_m}\widetilde{\phi}_n(t,x)\le C(m)\inf_{x\in\widetilde{\Omega}_m}\widetilde{\phi}_n(0,x)\le C(m).$$

Thus, the periodicity in t and the standard Schauder estimates yield that for all m there exists a subsequence of $(\widetilde{\phi}_n)_{n>m}$ that converges in $\mathcal{C}^{1+\frac{\widetilde{\delta}}{2},2+\widetilde{\delta}}(\mathbb{R}\times\overline{\widetilde{\Omega}_{m-1}})$, for any $\widetilde{\delta}\in[0,\delta)$, to a function ϕ_∞ satisfying:

$$\mathcal{L}\phi_\infty-\widetilde{\lambda}\phi_\infty=0 \text{ in } \mathbb{R}\times\widetilde{\Omega}_{m-1}.$$

Finally, using a diagonal extraction method, we can find a particular sub-sequence of $(\widetilde{\phi}_n)_{n\in\mathbb{N}}$ converging to ϕ_∞ in $\mathcal{C}_{\text{loc}}^{1+\frac{\widetilde{\delta}}{2},2+\widetilde{\delta}}(\mathbb{R}\times\Omega)$. Furthermore, $\phi_\infty(0,x_0)=1$, $\phi_\infty\ge 0$ and $\mathcal{L}\phi_\infty-\widetilde{\lambda}\phi_\infty=0$ in $\mathbb{R}\times\Omega$ and then the strong maximum principle yields $\phi_\infty>0$ in $\mathbb{R}\times\Omega$. Then ϕ_∞ is a generalized principal eigenfunction. Lastly, taking ϕ_∞ as a test super-solution in (7), one finds $\widetilde{\lambda}\le\lambda_1$.

Proof of Proposition 2.18 Set

$$\Lambda = \sup_{\phi>0,\phi \text{ is T-periodic}} \inf_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} \frac{\mathcal{L}\phi}{\phi}. \tag{18}$$

Taking ϕ_∞ a generalized principal eigenfunction as a test function in this formula, it is obvious that $\lambda_1 \leq \Lambda$.

Let us now assume that $\lambda_1 < \Lambda$ and set $\varepsilon = \Lambda - \lambda_1 > 0$. Then there exists a T-periodic function $\phi \in \mathcal{C}^{1,2}(\mathbb{R}\times\mathbb{R}^N)$ such that

$$\lambda_1 + \varepsilon \leq \frac{\mathcal{L}\phi}{\phi}.$$

This contradicts the definition of λ_1 and thus one has $\lambda_1 = \Lambda$. Furthermore, as $\mathcal{L}\phi_\infty = \lambda_1\phi_\infty$, the supremum in (18) is in fact a maximum.

5 Characterization of the principal generalized eigenvalue

First, we define the periodic principal eigenvalues associated with the parabolic operator L_α for all α.

Proof of Theorem 2.7 Take β a positive real number such that:

$$\eta = \beta - \|\alpha A\alpha\|_\infty - \|\mu\|_\infty - \|q\cdot\alpha\|_\infty - \frac{1}{2}\|\nabla\cdot q\|_\infty > 0.$$

We are looking for a T-periodic solution of:

$$\begin{cases} L_\alpha u + \beta u = g(t,x) \\ u(0,x) = u_0(x) \\ u(t,x+L_ie_i) = u(t,x) \text{ for all } i \end{cases} \tag{19}$$

where $u_0 \in L^2_{\text{per}}(\mathbb{R}^N)$ is a given initial data and $g \in \mathcal{C}^0_{\text{per}}(\mathbb{R}\times\mathbb{R}^N)$.

The classical parabolic theory yields the existence and uniqueness of a weak solution of (19) for all $t \geq 0$ in $\mathcal{C}^0_{\text{per}}(\mathbb{R}\times\mathbb{R}^N)$, which continuously depends on g. Then we can set:

$$G : L^2_{\text{per}}(\mathbb{R}^N) \mapsto L^2_{\text{per}}(\mathbb{R}^N)$$
$$u_0 \mapsto u(T,.)$$

Take $u_1, u_2 \in L^2_{\text{per}}(\mathbb{R}^N)$ and set $U(x,t) = (u_1(t,x) - u_2(t,x))e^{\eta t}$, then U satisfies:

$$\partial_t U - \nabla\cdot(A\nabla U) + q\cdot\nabla U - 2\alpha A\nabla U - (\alpha A\alpha + \nabla\cdot(A\alpha) - q\cdot\alpha + \mu + \eta - \beta)U = 0$$

Multiplying by U and integrating over $[0,T]\times C$, one gets:

$$\frac{1}{2}\left(\int_C U^2(T,.) - \int_C U^2(0,.)\right)$$
$$= \int_{[0,T]\times C} \left(-\nabla U A\nabla U + \left(\frac{1}{2}\nabla\cdot q + \alpha A\alpha - q\cdot\alpha + \mu + \eta - \beta\right)U^2\right)$$

Using the ellipticity property (6) and the definition of η, one gets:

$$\int_C U^2(0,.) \geq \int_C U^2(T,.)$$

This implies that:

$$\|u_1(T,.) - u_2(T,.)\|_{L^2(C)} \leq e^{-\eta T}\|u_1(0,.) - u_2(0,.)\|_{L^2(C)}$$

Then the map G is a contraction from $L^2_{\text{per}}(\mathbb{R}^N)$ into itself, and it admits a unique fixed point, which continuously depends on g. Furthermore, for all $u_0 \in L^2_{\text{per}}(\mathbb{R}^N)$, the Schauder regularity theorem and the classical Sobolev injections yield that $u(T,.) \in \mathcal{C}^2_{\text{per}}(\mathbb{R}^N)$. So, necessarily, the fixed point u_0 belongs to $\mathcal{C}^2_{\text{per}}(\mathbb{R}^N)$ and the associated T-periodic function belongs to $\mathcal{C}^{1,2}_{\text{per}}(\mathbb{R} \times \mathbb{R}^N)$.

We now set:

$$\begin{aligned} T : \mathcal{C}^0_{\text{per}}(\mathbb{R} \times \mathbb{R}^N) &\mapsto \mathcal{C}^0_{\text{per}}(\mathbb{R} \times \mathbb{R}^N) \\ g &\mapsto u \end{aligned}$$

where u is the unique fixed point associated with g. Obviously, T is a continuous compact linear map.

We want to apply the Krein–Rutman theorem to T in the cone K of the non-negative functions. It remains to prove that T is strongly positive on K. Set $g \in \mathcal{C}^0_{\text{per}}(\mathbb{R}^N \times \mathbb{R}) - \{0\}$ a non-negative function and u the associated function. Then multiplying the evolution equation by $u^- = \max(-u, 0)$ and integrating on $[0,T] \times C$ leads to:

$$-\int_{[0,T]\times C}\left(\nabla u^- A\nabla u^- - \left(\frac{1}{2}(\nabla\cdot q + \nabla\cdot(\alpha A)) + \alpha A\alpha - q\cdot\alpha + \mu - \beta\right)u^{-2}\right) = \int_{[0,T]\times C} g u^-$$

Since the left member is non-positive and the right member is non-negative, we have $u^- = 0$ and u is non-negative. The strong maximum principle and the T-periodicity yields that $u \in \text{Int}(K)$. This shows that T is strongly positive.

We then get a positive eigenfunction ϕ unique up to multiplication and a unique positive scalar r so that $T\phi = r\phi$. Set $k = \frac{1}{r} - \beta$, then k is a principal eigenvalue, which is unique.

Proof of Proposition 2.9 Since an eigenfunction ϕ_α belongs to $\mathcal{C}^{1,2}_{\text{per}}(\mathbb{R} \times \mathbb{R}^N)$, it is obvious that

$$k_\alpha \leq \max_{\phi>0,\phi\in\mathcal{C}^{1,2}_{\text{per}}(\mathbb{R}\times\mathbb{R}^N)} \inf_{\mathbb{R}\times\mathbb{R}^N}\left(\frac{L_\alpha\phi}{\phi}\right)$$

Let us now assume that there exists $\phi \in \mathcal{C}^{1,2}_{\text{per}}(\mathbb{R} \times \mathbb{R}^N)$ such that

$$k_\alpha < \inf_{\mathbb{R}\times\mathbb{R}^N}\left(\frac{L_\alpha\phi}{\phi}\right)$$

This implies that there exists a positive constant η such that:

$$L_\alpha\phi - k_\alpha\phi \geq \eta\phi \text{ in } \mathbb{R}\times\mathbb{R}^N$$

Let $w = \phi/\phi_\alpha$, then w is continuous and periodic with respect to t and x, so w reaches its minimum m on $\mathbb{R} \times \mathbb{R}^N$. A straightforward computation leads to:

$$\partial_t w - \nabla.(A\nabla w) + q\cdot\nabla w - 2\frac{\nabla\phi_\alpha}{\phi_\alpha}A\nabla w - 2\alpha A\nabla w \geq \eta w > 0$$

The parabolic strong maximum maximum principle and the periodicity in t yields that $w \equiv m$, i.e. $\phi \equiv m\phi_\alpha$. Putting this into the inequation $L_\alpha\phi - k_\alpha\phi \geq \eta\phi$ leads to $0 \geq \eta\phi$, which is impossible because both η and ψ are positive.

This proves that $k_\alpha \geq \inf_{\mathbb{R}\times\mathbb{R}^N}\left(\frac{L_\alpha\phi}{\phi}\right)$ for all $\phi \in \mathcal{C}^{1,2}_{\text{per}}(\mathbb{R}\times\mathbb{R}^N)$. Moreover, any eigenfunction reaches this infimum, so this infimum is a minimum and the property is proved.

The *min–max* characterization can be proved in a similar way.

Proof of Proposition 2.10 Let α_1, α_2 be two real numbers, $\mu_1, \mu_2 \in \mathcal{C}^0_{\text{per}}(\mathbb{R}^N \times \mathbb{R})$ and $r \in [0, 1]$. We want to show that:

$$F(r(\alpha_1, \mu_1) + (1-r)(\alpha_2, \mu_2)) \geq rF(\alpha_1, \mu_1) + (1-r)F(\alpha_2, \mu_2)$$

Set $\alpha = r\alpha_1 + (1-r)\alpha_2$ and $\mu = r\mu_1 + (1-r)\mu_2$. Set:

$$E_\alpha = \{\phi \in \mathcal{C}^{2,1}(\mathbb{R}^N \times \mathbb{R}), \phi > 0, \phi e^{\alpha\cdot x} \text{ is periodic}\}$$

Let ϕ_1, ϕ_2 be arbitrarily chosen in E_{α_1} and E_{α_2}, respectively. Define $z_1 = \ln(\phi_1)$, $z_2 = \ln(\phi_2)$, $z = rz_1 + (1-r)z_2$ and $\phi = e^z \in E_\alpha$. Therefore, it follows from the characterization of $k_\alpha(\mu)$ that:

$$k_\alpha(\mu) \geq \inf_{\mathbb{R}^N\times\mathbb{R}}\left(\frac{\partial_t\phi - \nabla\cdot(A\nabla\phi) + q\cdot\nabla\phi}{\phi} - \mu\right)$$

One the other hand, one can compute that:

$$\frac{\partial_t\phi - \nabla\cdot(A\nabla\phi) + q\cdot\nabla\phi}{\phi} = \partial_t z - \nabla\cdot(A\nabla z) - \nabla zA\nabla z + q\cdot\nabla z$$

and:

$$\begin{aligned}\nabla zA\nabla z &= r\nabla z_1A\nabla z_1 + (1-r)\nabla z_2A\nabla z_2 - r(1-r)(\nabla z_1 - \nabla z_2)A(\nabla z_1 - \nabla z_2)\\ &\leq r\nabla z_1A\nabla z_1 + (1-r)\nabla z_2A\nabla z_2\end{aligned}$$

Hence,

$$\begin{aligned}\frac{\partial_t\phi - \nabla\cdot(A\nabla\phi) + q\cdot\nabla\phi}{\phi} - \mu &\geq r(\partial_t z_1 - \nabla\cdot(A\nabla z_1) - \nabla z_1A\nabla z_1 + q\nabla z_1 - \mu_1)\\ &\quad +(1-r)(\partial_t z_2 - \nabla\cdot(A\nabla z_2) - \nabla z_2A\nabla z_2 + q\nabla z_2 - \mu_2)\\ &\geq r\left(\frac{\partial_t\phi_1 - \nabla\cdot(A\nabla\phi_1) + q\cdot\phi_1}{\phi_1} - \mu_1\right)\\ &\quad +(1-r)\left(\frac{\partial_t\phi_2 - \nabla.(A\nabla\phi_2) + q\cdot\phi_2}{\phi_2} - \mu_2\right)\end{aligned}$$

Then,

$$\begin{aligned}k_\alpha(\mu) &\geq \inf_{\mathbb{R}^N\times\mathbb{R}}\left(\tfrac{\partial_t\phi - \nabla.(A\nabla\phi)}{\phi} - \mu\right)\\ &\geq r\inf_{\mathbb{R}^N\times\mathbb{R}}\left(\frac{\partial_t\phi_1 - \nabla\cdot(A\nabla\phi_1) + q\cdot\nabla\phi_1}{\phi_1} - \mu_1\right)\\ &\quad +(1-r)\inf_{\mathbb{R}^N\times\mathbb{R}}\left(\frac{\partial_t\phi_2 - \nabla.(A\nabla\phi_2) + q\cdot\nabla\phi_2}{\phi_2} - \mu_2\right)\end{aligned}$$

Since ϕ_1 and ϕ_2 are arbitrarily chosen, this leads to

$$k_\alpha(\mu) \geq rk_{\alpha_1}(\mu_1) + (1-r)k_{\alpha_2}(\mu_2)$$

Then F is concave. This gives the continuity in α.

The first step of the proof of theorem 2.12 is the proof of theorem 2.11.

Proof of Theorem 2.11 Let φ a generalized principal eigenfunction associated with λ_1. Set $\psi(t,x) = \frac{\varphi(t,x+L_1e_1)}{\varphi(t,x)}$, then ψ satisfies:

$$\partial_t\psi - \nabla\cdot(A(t,x)\nabla\psi) + q(t,x)\cdot\nabla\psi - 2\frac{\nabla\varphi}{\varphi}A(t,x)\nabla\psi = 0$$

The Harnack inequality and the periodicity in t yield that ψ is bounded. Set $m = \sup_{\mathbb{R}\times\mathbb{R}^N}\psi > 0$ and $(x_n, t_n) \in [0,T]\times\mathbb{R}^N$ such that: $\psi(x_n,t_n) \to m$ as $n\to\infty$.

There exists $y_n \in \overline{C}$ so that for all n, $x_n - y_n \in L_1\mathbb{Z}\times\cdots\times L_N\mathbb{Z}$. We may assume that $y_n \to y_\infty \in \overline{C}$ and $t_n \to t_\infty \in [0,T]$.

Set $\psi_n(t,x) = \psi(t+t_n, x+x_n)$ and $\phi_n(t,x) = \frac{\varphi(t+t_n,x+x_n)}{\varphi(t_n,x_n)}$. The function ϕ_n satisfies:

$$\begin{aligned}&\partial_t\phi_n - \nabla\cdot(A(t+t_n,x+y_n)\nabla\phi_n)\\ &\quad+q(t+t_n,x+y_n)\cdot\nabla - \mu(t+t_n,x+y_n)\phi_n = \lambda_1\phi_n\end{aligned}$$

Using the classical parabolic estimates, we may suppose, up to extraction, that $\phi_n \to \phi_\infty$ in $C^{1,2}_{loc}(\mathbb{R}\times\mathbb{R}^N)$. The function ϕ_∞ satisfies:

$$\begin{cases}\partial_t\phi_\infty - \nabla\cdot(A(t+t_\infty,x+y_\infty)\nabla\phi_\infty) + q(t+t_\infty,x+y_\infty)\cdot\nabla\phi_\infty\\ -\mu(t+t_\infty,x+y_\infty)\phi_\infty = \lambda_1\phi_\infty\\ \phi_\infty \text{ periodic in t,}\\ \phi_\infty > 0,\ \phi_\infty(0,0) = 1\end{cases}$$

In the other hand, ψ_n satisfies:

$$\partial_t\psi_n - \nabla\cdot(A(t+t_n,x+y_n)\nabla\psi_n) + q(t+t_n,x+y_n)\cdot\nabla\psi_n - 2\frac{\nabla\phi_n}{\phi_n}A(x+y_n,t+t_n)\nabla\psi_n = 0$$

So, we may assume, up to extraction, that $\psi_n \to \psi_\infty$, where ψ_∞ satisfies:

$$\begin{aligned}&\partial_t\psi_\infty - \nabla\cdot(A(t+t_\infty,x+y_\infty)\nabla\psi_\infty) + q(t+t_\infty,x+y_\infty)\cdot\nabla\psi_\infty\\ &\quad-2\frac{\nabla\phi_\infty}{\phi_\infty}A(t+t_\infty,x+y_\infty)\nabla\psi_\infty = 0\end{aligned}$$

Furthermore, $\psi_\infty \le m$ and, as $\psi_n(0,0) = \psi(t_n,x_n) \to m$, $\psi_\infty(0,0) = m$. Using the strong parabolic maximum principle, we get $\psi_\infty \equiv m$.

As $m > 0$, we can define $\alpha_1 = \frac{1}{L_1}\ln(m)$. Then the function $\phi_\infty\exp(-\alpha_1 x_1)$ is L_1-periodic in x_1. Going on the construction, one can find a α_i for all i and then get a function θ satisfying:

$$\begin{cases}\partial_t\theta - \nabla\cdot(A(t+r_\infty,x+z_\infty)\nabla\theta) + q(t+r_\infty,x+z_\infty)\cdot\nabla\theta - \mu(t+r_\infty,x+z_\infty)\theta = \lambda_1\theta\\ \qquad\theta(t,x)\exp(-\alpha\cdot x) \text{ is periodic in } t, x_1,\ldots,x_N,\ \theta > 0,\ \theta(0,0) = 1\end{cases}$$

where $(r_\infty, z_\infty) \in \mathbb{R}\times\mathbb{R}^N$.

Therefore, since the periodic principal eigenvalue k_α is invariant under a translation in (t,x) of the coefficients, there exists a positive constant C such that the function $\theta e^{-\alpha\cdot x}$ is equal to $C\phi_\alpha$ and $\lambda_1 = k_\alpha$. This ends the proof.

Next, proposition 2.17 yields that there exists a generalized principal eigenfunction ϕ associated with the generalized principal eigenvalue λ_1. The preceding theorem yields that there exists $\alpha \in \mathbb{R}^N$ such that $\lambda_1 = k_\alpha$.

In the other hand, for all $\alpha \in \mathbb{R}^N$, the positive function $\psi_\alpha = \phi_\alpha e^{\alpha\cdot x}$ satisfies $\mathcal{L}\psi_\alpha = k_\alpha\psi_\alpha$. Taking ψ_α as a test-function in (7), one finds that $\lambda_1 \ge k_\alpha$ for all $\alpha \in \mathbb{R}^N$.

Next, assume that

$$k_\alpha + \min_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} (\alpha A\alpha + \nabla\cdot(A\alpha) - q\cdot\alpha + \mu) > 0,$$

then the zero order term of $L_\alpha - k_\alpha$ is negative. The weak maximum principle yields that there cannot exists a periodic function ϕ such that $(L_\alpha - k_\alpha)\phi = 0$, which gives a contradiction. Thus:

$$k_\alpha \le - \min_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} (\alpha A\alpha + \nabla\cdot(A\alpha) - q\cdot\alpha + \mu) \le -|\alpha|^2\gamma + \|\nabla\cdot(A\alpha) - q\cdot\alpha + \mu\|_\infty.$$

This finally gives the following inequality:

$$k_\alpha \le -\gamma\alpha^2 + \|\nabla\cdot(A\alpha) - q\cdot\alpha + \mu\|_\infty. \tag{20}$$

Using the classical perturbation theory, it is possible to prove (see [21]) that $\alpha \mapsto k_\alpha$ is analytic. As it is not constant, this function reaches its maximum for a unique α. This ends the proof of theorem 2.12.

6 Comparison between λ_1 and λ_1'

Proof of Proposition 2.13 Taking φ_0 a periodic principal eigenfunction associated with k_0 and using (12), one gets $\lambda_1' \le k_0$. Next, take $\lambda < k_0$ and assume that there exists a function $\phi \in \mathcal{C}^{1,2}(\mathbb{R}\times\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}\times\mathbb{R}^N)$ such that ϕ is periodic in t, positive and satisfies $\mathcal{L}\phi \le \lambda\phi$. We now search for a contradiction in order to prove that such a λ does not exist and that $\lambda_1' \ge k_0$.

Set $\gamma = \sup_{(0,T)\times C} \frac{\phi}{\varphi_0}$, then $0 < \gamma < \infty$ and one can define $z = \gamma\varphi_0 - \phi$. This function is nonnegative and $\inf z = 0$. Set $\varepsilon = (k_0 - \lambda)\min\varphi_0 > 0$. One has $(\mathcal{L} - \lambda)(z) \ge \gamma\varepsilon$.

Consider a nonnegative function $\theta \in \mathcal{C}^2(\mathbb{R}^N)$ that satisfies:

$$\theta(0) = 0, \quad \lim_{|x|\to+\infty} \theta(x) = 1, \quad \|\theta\|_{\mathcal{C}^2} < \infty.$$

There exists $\kappa > 0$ sufficiently large such that:

$$\forall y \in \mathbb{R}^N, \quad (\mathcal{L} - \lambda)(\tau_y\theta) > -\kappa\gamma\varepsilon/2,$$

where we denote $\tau_y\theta = \theta(. - y)$.

Since $\inf z = 0$, one can find some $(t_0, x_0) \in \mathbb{R}\times\mathbb{R}^N$ such that:

$$z(t_0, x_0) < \min\left\{\frac{1}{\kappa}, \frac{\gamma\varepsilon}{2\|\mu\|_\infty}\right\}$$

where $\|\mu\|_\infty = +\infty$ if $\mu \equiv 0$. Since $\lim_{|x|\to+\infty}\theta(x) = 1$, there exists a positive constant R such that $\tau_{x_0}\theta(x)/\kappa > z(t_0, x_0)$ if $|x - x_0| \ge R$. Consequently, setting $\widetilde{z} = z + \tau_{x_0}\theta(x)/\kappa$, one finds for all $|x - x_0| \ge R$, that:

$$\widetilde{z}(t, x) \ge \tau_{x_0}\theta(x)/\kappa > z(t_0, x_0) = \widetilde{z}(t_0, x_0).$$

Hence, if $\alpha = \min_{\mathbb{R}\times\mathbb{R}^N} \widetilde{z}$, this infimum is reached in $B_R(x_0)$. Moreover,

$$\alpha \le \widetilde{z}(t_0, x_0) = z(t_0, x_0) < \frac{\gamma\varepsilon}{2\|\mu\|_\infty}.$$

One can compute:

$$\begin{aligned}(\mathcal{L}-\lambda)(\widetilde{z}-\alpha) &= (\mathcal{L}-\lambda)(z) + \frac{1}{\kappa}(\mathcal{L}-\lambda)(\tau_{x_0}\theta(x)) - \mu(t,x)\alpha + \lambda\alpha \\ &> \gamma\varepsilon - \frac{\gamma\varepsilon}{2} - \|\mu-\lambda\|_\infty \alpha \\ &> 0\end{aligned}$$

for all $(t, x) \in \mathbb{R} \times B_R(x_0)$. Thus, the strong maximum principle and the periodicity yield that $\widetilde{z} \equiv \alpha$, which contradicts $(\mathcal{L}-\lambda)\ (\widetilde{z}-\alpha) > 0$.

Proof of Proposition 2.14 First case: symmetry in x, $q \equiv 0$

First of all, we assume that $q \equiv 0$ and that A and μ have a common symmetry axis in x. Up to translation, we can assume that A and μ are even in x.

Set ϕ_α the eigenfunction defined by:

$$\begin{cases} \partial_t\phi_\alpha - \nabla\cdot(A\nabla\phi_\alpha) - 2\alpha\cdot A\nabla\phi_\alpha - (\alpha A\alpha - \nabla\cdot(A\alpha)+\mu)\phi_\alpha = k_\alpha\phi_\alpha \\ \phi_\alpha > 0,\ \phi_\alpha \text{ is periodic in } t \text{ and } x,\ \|\phi_\alpha\|_\infty = 1 \end{cases} \tag{21}$$

Set: $\psi_\alpha(t, x) = \phi_{-\alpha}(t, -x)$, this function satisfies:

$$\begin{cases} (\partial_t\psi_\alpha - \nabla\cdot(A\nabla\psi_\alpha) - 2\alpha\cdot A\nabla\psi_\alpha - (\alpha A\alpha - \nabla\cdot(A\alpha)+\mu)\psi_\alpha)\,(t,-x) = k_\alpha\psi_\alpha(t,-x) \\ \psi_\alpha > 0,\ \psi_\alpha \text{ is periodic in } t \text{ and } x,\ \|\psi_\alpha\|_\infty = 1 \end{cases} \tag{22}$$

Since A and μ are even in x, the uniqueness of the principal eigenfunction yields that $\phi_\alpha = \psi_\alpha$ and then $k_\alpha = k_{-\alpha}$. As $\alpha \mapsto k_\alpha$ is concave, this gives $\lambda_1' = k_0 = \max_{\alpha\in\mathbb{R}^N} k_\alpha = \lambda_1$.

Second case: symmetry in t, $q \equiv 0$

We now assume that $q \equiv 0$ and that A and μ have a common symmetry axis in t. Up to translation, we can assume that A and μ are even in t.

We consider the adjoint operator:

$$P_\alpha^*\phi = -\partial_t\phi - \nabla\cdot(A(t,x)\nabla\phi) + 2\alpha\cdot A\nabla\phi - (\alpha A(t,x)\alpha - \nabla\cdot(A(t,x)\alpha) + \mu(t,x))\phi.$$

Set $\psi_\alpha(t, x) = \phi_{-\alpha}(-t, x)$, where $\phi_{-\alpha}$ is defined as in the first case. This new function is positive, periodic in t and x and using the symmetry in t, one can prove that it satisfies:

$$P_\alpha^*\psi_\alpha = k_{-\alpha}\psi_\alpha.$$

Thus, the uniqueness property of the principal eigenvalue yields that $k(-\alpha) = k(\alpha)^*$. But the principal eigenvalue associated with the adjoint operator is equal to the principal eigenvalue. This proves that k_α is even. We end the proof as in the first case.

The general case

We first prove that such a function Q is periodic. The periodicity in t is obvious since $\nabla Q = A^{-1}q$ is periodic in t. Set $Q^i(t, x) = Q(t, x + L_i e_i) - Q(t, x)$. Since q and A are periodic in x, the functions Q^i are constant. Next, one can compute:

$$\int\limits_{(0,T)\times C} Q^i(t,x)\mathrm{d}x\mathrm{d}t = \int\limits_{(0,T)\times C} \mathrm{d}x\mathrm{d}t \int\limits_0^{L_i} e_i.\nabla Q(t, x+se_i)\mathrm{d}s$$
$$= \int\limits_0^{L_i} \mathrm{d}s \int\limits_{(0,T)\times C} (A^{-1}q)_i(x+se_i)\mathrm{d}x\mathrm{d}t = 0.$$

Hence $Q^i \equiv 0$. This means that Q is periodic in x.

Next, set ϕ_α a positive eigenfunction associated with k_α and $\psi_\alpha(t,x) = \phi_\alpha(t,x)e^{-Q(x)/2}$. This new function satisfies:

$$\begin{cases} L_\alpha\psi_\alpha - \left(\frac{1}{2}\nabla\cdot(A\nabla Q) + \frac{1}{2}\partial_t Q - \frac{1}{4}\nabla Q A \nabla Q\right)\psi_\alpha = k_\alpha\psi_\alpha, \\ \psi_\alpha > 0, \\ \psi_\alpha \text{ is periodic in t and x.} \end{cases} \tag{23}$$

In other words, we wrote k_α as the periodic principal eigenvalue of an operator with $q \equiv 0$. We are then back to the first or the second case, which yields that $\max_{\alpha\in\mathbb{R}^N} k_\alpha = k_0$.

7 Proofs of the dependence results

7.1 Influence of the amplitude and the diffusion

Proof of Theorem 3.3 The concavity of the function $\mu \mapsto k_0(\mu)$ yields that the function $f : B \mapsto k_\alpha(\mu + B\eta)$ is concave. Take an arbitrary sequence $B_n \to 0$. Setting $\phi_\alpha(B)$ the eigenfunction associated with the zero order term $\mu + B\eta$, one has:

$$\int\limits_{(0,T)\times C} (\partial_t - \nabla\cdot A\nabla - 2\alpha A\nabla + q\cdot\nabla - (\nabla\cdot(A\alpha) + \alpha A\alpha - q\cdot\alpha + \mu))(\phi_\alpha(B) - \phi_\alpha(0))\widetilde{\phi}_\alpha(B)$$
$$= k_\alpha(\mu + B\eta) - k_\alpha(\mu) + B \int\limits_{(0,T)\times C} \eta\phi_\alpha(0)\widetilde{\phi}_\alpha(B).$$

The definition of $\widetilde{\phi}_\alpha(B)$ yields that the left member is null. Thus, f is of class $\mathcal{C}^1$ in 0 and using the continuity of $B \mapsto \widetilde{\phi}_\alpha(B)$, one can easily get:

$$f'(0) = -\int\limits_{(0,T)\times C} \eta\phi_\alpha(0)\widetilde{\phi}_\alpha(0)\mathrm{d}t\mathrm{d}x.$$

If $\int_{(0,T)\times C} \eta\phi_\alpha\widetilde{\phi}_\alpha dtdx > 0$, then $f'(0) < 0$ and the concavity of f gives its monoticity in $\mathbb{R}^+$. If $\int_{(0,T)\times C} \eta\phi_\alpha\widetilde{\phi}_\alpha dtdx = 0$, then $f'(0) = 0$ and f is nonincreasing in $\mathbb{R}^+$.

Proof of Theorem 3.5 Take $\mathcal{B} = B(0,1)$ the ball of center 0 and of radius 1. For all $\alpha \in \mathbb{R}^N$, one has $k_\alpha(A, q, \mu + B\eta) \le \lambda_1(A, q, \mu + B\eta, \mathcal{B})$. It has already been proved (see Lemma 15.4 in [15]) that the right member goes to $-\infty$ as $B \to +\infty$. Thus, the inequality yields that for all α:

$$k_\alpha(B\eta) \to -\infty \text{ as } B \to +\infty.$$

In particular, for B large enough, one has $k_\alpha(B\eta) < 0$.

Proof Theorem 3.6 (1) We first prove that for all $\alpha \in \mathbb{R}^N$, one has $k_\alpha(\gamma A, q, \mu) \to -\frac{1}{T|C|}\int_{(0,T)\times C}\mu$ when $\gamma \to +\infty$. We begin with the case $\alpha = 0$. We follow the proof of [16].

Take ϕ^γ a positive eigenfunction that satisfies:

$$\begin{cases} \partial_t\phi^\gamma - \gamma\nabla\cdot(A\nabla\phi^\gamma) + q\cdot\nabla\phi^\gamma - \mu\phi^\gamma = k_0(\gamma)\phi^\gamma \\ \phi^\gamma \text{ is periodic in } t \text{ and } x \\ \frac{1}{|C|}\int_{(0,T)\times C}\phi^{\gamma\,2} = 1 \end{cases} \tag{24}$$

where $k_0(\gamma) = k_0(\gamma A, q, \mu)$. Multiplying Eq. (24) by ϕ^γ and integrating, one gets:

$$\gamma \int_{(0,T)\times C} \nabla\phi^\gamma A\nabla\phi^\gamma = \int_{(0,T)\times C} (\mu + \nabla\cdot q + k_0(\gamma))\phi^{\gamma\,2}.$$

Thus there exists c_1 which does not depend on γ such that:

$$\int_{(0,T)\times C} \nabla\phi^\gamma A\nabla\phi^\gamma \leq \frac{c_1}{\gamma}.$$

Set $\psi^\gamma = \phi^\gamma - \overline{\phi^\gamma}$, where $\overline{\phi^\gamma}(t) = \frac{1}{|C|}\int_C \phi^\gamma(t,x)\mathrm{d}x$. Then $\nabla\psi^\gamma = \nabla\phi^\gamma$ and $\int_C \psi^\gamma = 0$, thus the Poincaré's inequality yields that there exists a constant $c > 0$ depending only on C such that $\int_C \nabla\psi^\gamma A\nabla\psi^\gamma \geq c\int_C \psi^{\gamma 2}$.

Now, integrating (24) over C, one gets:

$$\partial_t\overline{\phi^\gamma} - (\overline{\mu} + k_0(\gamma))\overline{\phi^\gamma} = \frac{1}{|C|}\int_C (\mu - \nabla\cdot q)\psi^\gamma$$

and the preceding computations yield that:

$$\int_{(0,T)\times C} \psi^{\gamma\,2} \leq \frac{c_1}{c\gamma}.$$

Thus, one can compute:

$$\overline{\phi^\gamma}(t) = \overline{\phi^\gamma}(0)e^{\int_0^t(\overline{\mu}+k_0(\gamma))} + O(1/\sqrt{\gamma}).$$

As $\overline{\phi^\gamma}(T) = \overline{\phi^\gamma}(0)$, we must have either $\int_0^T(\overline{\mu} + k_0(\gamma)) \to 0$ or $\overline{\phi^\gamma}(0) \to 0$ as $\gamma \to \infty$. If $\overline{\phi^\gamma}(0) \to 0$, then $\overline{\phi^\gamma}(t) \to 0$ uniformly in t and thus $\|\overline{\phi^\gamma}\|_{L^2} \to 0$. This finally gives that $\phi^\gamma = \psi^\gamma + \overline{\phi^\gamma}$ converges to 0 in L^2 as t goes to $+\infty$, which is impossible since $\|\phi^\gamma\|_2 = 1$ for all γ. This yields that $\int_0^T(\overline{\mu} + k_0(\gamma)) \to 0$. This gives $\lambda_1'(\gamma, A, q, \mu) \to -\frac{1}{T|C|}\int_{(0,T)\times C}\mu$ as $\gamma \to +\infty$.

As $k_\alpha(\gamma A, q, \mu) = \lambda_1'(\gamma A, q - 2\alpha\gamma A, \alpha\gamma A\alpha + \nabla\cdot(\gamma A\alpha) - q\cdot\alpha + \mu)$, we have:

$$k_\alpha(\gamma A, q, \mu) \to -\frac{1}{T|C|}\int_{(0,T)\times C}(-\alpha A\alpha - \nabla\cdot(A\alpha) + q\cdot\alpha + \mu) = k_\alpha(\overline{A}, \overline{q}, \overline{\mu})$$

as $\gamma \to +\infty$.

As $\alpha \mapsto k_\alpha(\gamma A, q, \mu)$ is concave for all γ, the Dini's theorem yields that this convergence is locally uniform in $\alpha \in \mathbb{R}^N$. For all γ, let α_γ be such that $k_{\alpha_\gamma}(A, q, \mu) = \lambda_1(A, q, \mu)$. Inequality (20) gives:

$$k_0(\gamma A, q, \mu) \leq k_{\alpha_\gamma}(\gamma A, q, \mu) \leq -\gamma|\alpha_\gamma|^2 + \|\gamma\nabla \cdot (A\alpha_\gamma) - q \cdot \alpha_\gamma + \mu\|_\infty.$$

Thus for all γ, one has:

$$\gamma(|\alpha_\gamma|^2 - \|\nabla \cdot (A\alpha_\gamma)\|_\infty) - \|q \cdot \alpha_\gamma\|_\infty \leq \mu\|_\infty - k_0(\gamma A, q, \mu).$$

For all $\gamma \geq 1$, if α_γ is large enough, this gives:

$$|\alpha_\gamma|^2 - \|\nabla \cdot (A\alpha_\gamma)\|_\infty - \|q \cdot \alpha_\gamma\|_\infty \leq \mu\|_\infty - k_0(\gamma A, q, \mu).$$

As the right-hand side converges when γ goes to $+\infty$, this inequality yields that the left-hand side is bounded and thus the family $(\alpha_\gamma)_{\gamma \geq 1}$ is bounded. We now take some subsequence $(\alpha_{\gamma_n})_n$ that converges to some α_∞ where $\gamma_n \to +\infty$ as $n \to +\infty$. As the convergence of $k_\alpha(\gamma A, q, \mu)$ is locally uniform in α, one has:

$$\lambda_1(\gamma_n A, q, \mu) = k_{\alpha_{\gamma_n}}(\gamma_n A, q, \mu) \to k_{\alpha_\infty}(\overline{A}, \overline{q}, \overline{\mu}).$$

Furthermore, for all α, one has $k_\alpha(\gamma_n A, q, \mu) \leq \lambda_1(\gamma_n A, q, \mu)$. Letting $n \to +\infty$, this gives:

$$k_\alpha(\overline{A}, \overline{q}, \overline{\mu}) \leq k_{\alpha_\infty}(\overline{A}, \overline{q}, \overline{\mu}).$$

Thus $k_{\alpha_\infty}(\overline{A}, \overline{q}, \overline{\mu}) = \lambda_1(\overline{A}, \overline{q}, \overline{\mu})$ and α_∞ is the unique α for which $k_\alpha(\overline{A}, \overline{q}, \overline{\mu}) = \lambda_1(\overline{A}, \overline{q}, \overline{\mu})$. As the limit of a converging subsequence $(\alpha_n)_n$ is necessarily equal to this unique α, one knows that the full family $(k_{\alpha_\gamma}(\gamma A, q, \mu))_{\gamma > 0}$ converges to $\lambda_1(\overline{A}, \overline{q}, \overline{\mu})$ as $\gamma \to +\infty$.

(2) Set $\lambda'_{1,n} = \lambda'_1(\gamma_n A, \sqrt{\gamma_n} q, \mu) \geq 0$ for all n, this sequence is bounded by $\|\mu\|_\infty$ so that one can assume, up to extraction, that it converges: $\lambda'_{1,n} \to \lambda'_{1,\infty}$. Take some $x_0 \in \mathbb{R}^N$. For all n, set ϕ_n the eigenfunction associated with $\lambda'_{1,n}$ and normalized by $\phi_n(0, x_0) = 1$. Set $\psi_n(t, x) = \phi_n(t, x_0 + \sqrt{\gamma_n} x)$ for all n, these functions satisfy:

$$\partial_t \psi_n - \nabla \cdot (A_n \nabla \psi_n) + q_n \nabla \psi_n - \mu_n \psi_n = \lambda'_{1,n} \psi_n$$

where $A_n(t, x) = A(t, x_0 + \sqrt{\gamma_n} x)$, $q_n(t, x) = q(t, x_0 + \sqrt{\gamma_n} x)$ and $\mu_n(t, x) = \mu(t, x_0 + \sqrt{\gamma_n} x)$. As $A_n(t, x) \to A(t, x_0)$, $q_n(t, x) \to q(t, x_0)$ and $\mu_n(t, x) \to \mu(t, x_0)$ uniformly on compact sets as $n \to \infty$, the Schauder classical estimates yields that one can assume, up to extraction, that $(\psi_n)_n$ converges to a function ψ_∞ in $\mathcal{C}^{1,2}_{loc}(\mathbb{R} \times \mathbb{R}^N)$. This function satisfies:

$$\partial_t \psi_\infty - \nabla \cdot (A(t, x_0) \nabla \psi_\infty) + q(t, x_0) \nabla \psi_\infty - \mu(t, x_0) \psi_\infty = \lambda'_{1,\infty} \psi_\infty$$

Furthermore, the function ψ_∞ is periodic in t, nonnegative and satisfies $\psi_\infty(0, 0) = 1$. The strong parabolic maximum principle yields that ψ_∞ is positive.

Thus, going back to the definition of the generalized principal eigenvalue λ_1, one gets:

$$\lambda'_{1,\infty} \leq \lambda_1(A(., x_0), q(., x_0), \mu(., x_0)).$$

But in this case, as the coefficients do not depend on t, this eigenvalue is equal to $-\frac{1}{T} \int_0^T \mu(t, x_0) \mathrm{d}t$, thus:

$$\lambda'_{1,\infty} \leq -\frac{1}{T} \int_0^T \mu(t, x_0) \mathrm{d}t$$

for all $x_0 \in \mathbb{R}^N$. One finally gets:

$$\limsup_{\gamma\to 0} \lambda_1'(\gamma A, \sqrt{\gamma} q, \mu) \leq -\min_{x\in\mathbb{R}^N} \frac{1}{T}\int_0^T \mu(t,x)\mathrm{d}t = -\max_{x\in\mathbb{R}^N} \hat{\mu}(x). \tag{25}$$

In the other hand, for all n, we can change the normalization and assume that $\|\phi_n\|_\infty = 1$. There exists some $(t_n, x_n) \in [0,T] \times \overline{C}$ such that $\phi_n(t_n, x_n) = 1$ for all n. Up to extraction, one may assume that $(t_n, x_n) \to (\bar{t}, \bar{x})$ as $n \to +\infty$. Set $\varphi_n(t,x) = \phi_n(t, \sqrt{\gamma_n}x + x_n)$. These functions satisfy:

$$\begin{aligned}\partial_t \varphi_n - \nabla \cdot (A(t, x_n + \sqrt{\gamma_n}x)\nabla\varphi_n) + q(t, x_n + \sqrt{\gamma_n}x)\nabla\varphi_n \\ -\mu(t, x_n + \sqrt{\gamma_n}x)\varphi_n = \lambda_{1,n}'\varphi_n.\end{aligned}$$

The Schauder parabolic estimates yield that one can assume that the sequence $(\varphi_n)_n$ converges, up to extraction, to some nonnegative function φ_∞ which satisfies:

$$\partial_t \varphi_\infty - \nabla \cdot (A(t,\bar{x})\nabla\varphi_\infty) + q(t,\bar{x})\nabla\varphi_\infty - \mu(t,\bar{x})\varphi_\infty = \lambda_{1,\infty}'\varphi_\infty,$$

where $\varphi_\infty(\bar{t}, 0) = 1 = \|\varphi_\infty\|_\infty$. The definition of the generalized principal eigenvalue $\lambda_1'(A(.,\bar{x}), q(.,\bar{x}), \mu(.,\bar{x}))$ yields that

$$-\frac{1}{T}\int_0^T \mu(t,\bar{x})\mathrm{d}t = \lambda_1'(A(.,\bar{x}), q(.,\bar{x}), \mu(.,\bar{x})) \leq \liminf_{\gamma\to 0} \lambda_1'(\gamma A, q, \mu). \tag{26}$$

Thus, the minimum that appears in the right hand-side of (25) is reached and (25) and (26) give the conclusion.

7.2 Distribution effects

This section is dedicated to the proof of theorem 3.9.

Proof of Theorem 3.9 Using the notations of part , we define the linear operator G_μ by:

$$\begin{aligned} G_\mu : L^\infty_{\mathrm{per}} &\to L^\infty_{\mathrm{per}} \\ u_0 &\mapsto u(T,.) \end{aligned} \tag{27}$$

where u is the solution of:

$$\begin{cases} \partial_t u - \gamma \Delta u - \mu(t,x)u = 0 \\ u(0,x) = u_0(x) \end{cases} \tag{28}$$

and we consider the eigenelements $(u_0, r_0(\mu)) \in \mathcal{C}^2_{\mathrm{per}}(\mathbb{R}^N) \times \mathbb{R}$ defined by :

$$\begin{cases} G_\mu u_0 = r_0(\mu) u_0 \\ u_0 > 0 \\ \|u_0\|_\infty = 1 \end{cases} \tag{29}$$

Finally, we define u as the solution of the Cauchy problem (28) associated with u_0. Next, consider the solution $V \in \mathcal{C}^{1,2}_{\mathrm{per}}(\mathbb{R} \times \mathbb{R}^N)$ of the equation:

$$\begin{cases} \partial_t V - \gamma \Delta V - \mu^*(t,x) V = 0 \\ V(0,x) = u_0^*(x) \end{cases}$$

Then, a result from [2] yields that

$$\forall t,\ \|u(t,.)\|_\infty \le \|V(t,.)\|_\infty.$$

Thus, one gets the following inequality:

$$\forall n \in \mathbb{N}^*, \quad r_0(\mu) = \|G_\mu^n(u_0)\|_\infty^{1/n} \le \|G_{\mu^*}^n(u_0^*)\|_\infty^{1/n} \le \|G_{\mu^*}^n\|_{\mathcal{L}(L^\infty_{\text{per}})}^{1/n}$$

When n goes to $+\infty$, the spectral radius formula yields that the right member converges to the principal eigenvalue of the operator G_{μ^*}, that is to say $r_0(\mu^*)$. Finally, we have obtained:

$$r_0(\mu) \le r_0(\mu^*).$$

It can easily be seen (see [15]) that:

$$\lambda_1(\mu) = -\frac{1}{T}\ln r_0(\mu).$$

This gives the conclusion.

7.3 Effect of the variations

Proof of Proposition 3.7 Consider some eigenfunction ϕ:

$$\partial_t\phi - \nabla\cdot(A\nabla\phi) + q\cdot\nabla\phi - \mu\phi = k_0\phi.$$

Dividing this equation by ϕ, integrating over $(0,T)\times C$ and using the periodicity of ϕ, one gets:

$$-\int_{(0,T)\times C} \frac{\nabla\phi A\nabla\phi}{\phi^2} - \int_{(0,T)\times C} (\nabla\cdot q) ln\phi - \int_0^T \overline{\mu} = T|C|k_0.$$

As A is elliptic this gives:

$$-\int_0^T \overline{\mu} \ge T|C|k_0.$$

Proposition 3.1 gives that $-\int_0^T \overline{\mu} = k_0(\overline{A},\overline{q},\overline{\mu})$.

Proof of Theorem 3.10 Take α so that $\lambda_1(\hat{A},\hat{q},\hat{\mu}) = k_\alpha(\hat{A},\hat{q},\hat{\mu})$ and $\kappa_n \to +\infty$ and (ϕ_n, k_n) the eigenelements defined by:

$$\begin{cases} \kappa_n\partial_t\phi_n - \nabla\cdot(A\nabla\phi_n) - 2\alpha A\nabla\phi_n + q\cdot\nabla\phi_n - (\alpha A\alpha + \nabla(A\alpha) - q\cdot\alpha + \mu)\phi_n = k_n\phi_n, \\ \phi_n \text{ is periodic in } t \text{ and } x, \\ \phi_n > 0, \\ \|\phi_n\|_{L^2((0,T)\times C} = 1. \end{cases} \tag{30}$$

First of all, $|k_n|$ is bounded by $\|\alpha A\alpha + \nabla(A\alpha) - q.\alpha + \mu\|_{L^\infty}$, thus one can assume, up to extraction, that $k_n \to k$ as $n \to +\infty$.

Multiplying Eq. (30) by ϕ_n and integrating over $(0,T)\times C$, one gets:

$$
\begin{aligned}
k_n &= -\int_{(0,T)\times C} \nabla(A.\nabla\phi_n)\phi_n + \int_{(0,T)\times C} (q-2\alpha A)\nabla\frac{\phi_n^2}{2} \\
&\quad - \int_{(0,T)\times C} (\mu+\alpha A\alpha+\nabla.(A\alpha)-q.\alpha)\phi_n^2 \\
&= \int_{(0,T)\times C} \nabla\phi_n A.\nabla\phi_n - \int_{(0,T)\times C} (\mu+\alpha A\alpha-\nabla.(A\alpha)-q.\alpha+\nabla.q)\phi_n^2
\end{aligned}
$$

by integrating by parts. Using the uniform ellipticity, one gets:

$$
\gamma\|\nabla\phi_n\|_{L^2}^2 \leq k_n + \|\mu+\alpha A\alpha-\nabla.(A\alpha)-q.\alpha+\nabla.q\|_{L^\infty}
$$

Multiplying Eq. (30) by $\partial_t\phi_n$ and integrating, one gets:

$$
\begin{aligned}
&\kappa_n \int_{(0,T)\times C} (\partial_t\phi_n)^2 - \int_{(0,T)\times C} \nabla(A.\nabla\phi_n)\partial_t\phi_n + \int_{(0,T)\times C} (q-2\alpha A)\nabla\phi_n\partial_t\phi_n \\
&\quad - \int_{(0,T)\times C} (\mu+\alpha A\alpha+\nabla.(A\alpha)-q.\alpha)\partial_t\frac{\phi_n^2}{2} \\
&= \kappa_n \int_{(0,T)\times C} (\partial_t\phi_n)^2 - \frac{1}{2}\int_{(0,T)\times C} \nabla\phi_n\partial_t A.\nabla\phi_n + \int_{(0,T)\times C} (q-2\alpha A)\nabla\phi_n\partial_t\phi_n \\
&\quad + \int_{(0,T)\times C} \partial_t(\mu+\alpha A\alpha-\nabla.(A\alpha)-q.\alpha+\nabla.q)\frac{\phi_n^2}{2} = 0
\end{aligned}
$$

This yields the following estimates:

$$
\begin{aligned}
\kappa_n \int_{(0,T)\times C} (\partial_t\phi_n)^2 &\leq \frac{1}{2}\|\partial_t A\|_{L^\infty}\|\nabla\phi_n\|_{L^2}^2 + \|q-2\alpha A\|_{L^\infty} \int_{(0,T)\times C} |\partial_t\phi_n||\nabla\phi_n| \\
&\quad + \frac{1}{2}\|\partial_t(\mu+\alpha A\alpha-\nabla.(A\alpha)-q.\alpha+\nabla.q)\|_{L^\infty} \\
&\leq \frac{1}{2}\|\partial_t A\|_{L^\infty}\|\nabla\phi_n\|_{L^2}^2 + \|q-2\alpha A\|_{L^\infty} \\
&\quad \times\Big(\frac{\kappa_n}{2\|q-2\alpha A\|_{L^\infty}} \int_{(0,T)\times C} (\partial_t\phi_n)^2 + \frac{\|q-2\alpha A\|_{L^\infty}}{2\kappa_n}\|\nabla\phi_n\|_{L^2}^2\Big) \\
&\quad + \frac{1}{2}\|\partial_t(\mu+\alpha A\alpha-\nabla.(A\alpha)-q.\alpha+\nabla.q)\|_{L^\infty}
\end{aligned}
$$

and finally:

$$
\begin{aligned}
\kappa_n\|\partial_t\phi_n\|_{L^2}^2 &\leq \left(\|\partial_t A\|_{L^\infty} + \frac{\|q-2\alpha A\|_{L^\infty}^2}{\kappa_n}\right)\|\nabla\phi_n\|_{L^2}^2 \\
&\quad + \|\partial_t(\mu+\alpha A\alpha-\nabla.(A\alpha)-q.\alpha+\nabla.q)\|_{L^\infty}
\end{aligned}
$$

These two estimates yield that $\|\nabla\phi_n\|_{L^2}$ is bounded and that $\|\partial_t\phi_n\|_{L^2} \to 0$ as $n \to +\infty$. Thus, up to extraction, one may assume that $\phi_n \to w$ in L^2, $\nabla\phi_n \rightharpoonup \nabla w$ and $\partial_t\phi_n \rightharpoonup \partial_t w$ in L^2. One has $\|\partial_t w\|_{L^2} \le \liminf \|\partial_t\phi_n\|_{L^2} = 0$ and thus w does not depend on t.

Passing to the limit $n \to +\infty$ in (30), one gets that w is a weak solution of:

$$-\nabla.(A\nabla w) - 2\alpha A\nabla w + q.\nabla w - (\alpha A\alpha + \nabla(A\alpha) - q.\alpha + \mu)w = kw$$

One can integrate over $(0, T)$, this yields:

$$\begin{aligned}&-\nabla.(\hat{A}\nabla w) - 2\alpha\hat{A}\nabla w + \hat{q}.\nabla w\\ &-\left(\alpha\hat{A}\alpha + \nabla(\hat{A}\alpha) - \hat{q}.\alpha + \hat{\mu}\right)w = kw\end{aligned}$$

The regularity Shauder estimates yield that $w \in \mathcal{C}^2(\mathbb{R}^N)$. Using the elliptic strong maximum principle, one gets $w > 0$. Thus w is the principal eigenfunction associated with $k_\alpha(\hat{A}, \hat{q}, \hat{\mu})$. The uniqueness of the principal eigenvalue leads to $k = k_\alpha(\hat{A}, \hat{q}, \hat{\mu}) = \lambda_1(\hat{A}, \hat{q}, \hat{\mu})$. This ends the proof.

7.4 The optimization result

Lemma 7.1 *Assume that (A_n, q_n, μ_n) is a bounded sequence in $(L^\infty)^3$, then one can extract a subsequence $(A_{n'}, q_{n'}, \mu_{n'})$ and find some coefficients $(A, q, \mu) \in (L^\infty)^3$ such that $k_\alpha(A_{n'}, q_{n'}, \mu_{n'}) \to k_\alpha(A, q, \mu)$.*

Proof of Lemma 7.1 First of all, we can find some coefficients $(A, q, \mu) \in (L^\infty)^3$ such that, up to extraction, $(A_n, q_n, \mu_n) \rightharpoonup^* (A, q, \mu)$ for the weak-* L^∞ topology. Set φ_n the eigenfunction associated with (A_n, q_n, μ_n) such that $\|\varphi_n\|_{H^1_{per}} = 1$. Up to extraction, we may assume that $\varphi_n \to \psi$ in L^2 and $\varphi_n \rightharpoonup \psi$ for the weak topology in H^1_{per}.

In the other hand, as (A_n, q_n, μ_n) is a bounded sequence in $(L^\infty)^3$, $k_\alpha(A_n, q_n, \mu_n)$ is also a bounded sequence. Up to extraction, we can assume that $k_\alpha(A_n, q_n, \mu_n) \to k$.

Choose $\theta \in \mathcal{D}((0, T) \times C)$ a test-function, then $(\theta\varphi_n)_n$ strongly converges to $\theta\psi$ in L^1, and:

$$k_\alpha(A_n, q_n, \mu_n) < \varphi_n, \quad \theta >_{\mathcal{D}'\times\mathcal{D}} \to k < \psi, \quad \theta >_{\mathcal{D}'\times\mathcal{D}},$$

$$< \mu_n\varphi_n, \theta >_{\mathcal{D}'\times\mathcal{D}} = < \mu_n, \varphi_n\theta >_{L^\infty\times L^1} \to < \mu, \psi\theta >_{L^\infty\times L^1} = < \mu\psi, \theta >_{\mathcal{D}'\times\mathcal{D}},$$

as $n \to +\infty$. This gives:

$$\begin{aligned}&< \partial_t\phi_n - \nabla\cdot(A_n\nabla\phi_n) - 2\alpha A_n\nabla\phi_n + q_n\cdot\nabla\phi_n - (\alpha A_n\alpha + \nabla\cdot(A_n\alpha)\\ &-q_n\cdot\alpha + \mu_n)\phi_n, \theta > \to < \partial_t\psi - \nabla\cdot(A\nabla\psi) - 2\alpha A\nabla\psi + q\cdot\nabla\psi\\ &-(\alpha A\alpha + \nabla\cdot(A\alpha) - q\cdot\alpha + \mu)\psi, \theta >_{\mathcal{D}'\times\mathcal{D}} .\end{aligned}$$

Then the uniqueness of the limit yields that

$$\partial_t\psi - \nabla\cdot(A\nabla\psi) - 2\alpha A\nabla\psi + q\cdot\nabla\psi - (\alpha A\alpha + \nabla\cdot(A\alpha) - q\cdot\alpha + \mu)\psi = k\psi$$

in $\mathcal{D}'((0, T) \times C)$. Furthermore, one knows that ψ is nonnegative.

In the other hand, a bootstrap method proves that, as $(k_\alpha(\mu_n))_n$ converges, up to extraction, one can assume that $\phi_n \to \psi$ in H^1_{per}. Thus $\|\psi\|_{H^1_{per}} = 1$ and ψ is not null. The strong maximum principle yields that $\psi > 0$. The uniqueness property for the principle eigenvalue gives $k = k_\alpha(A, q, \mu)$, which ends the proof.

Lemma 7.2 *Set ExF the set of the extremal points of F. We have the following equality*:

$$ExF = \{\gamma 1_A + \beta 1_{(0,T)\times C\backslash A};\ \gamma|A| + \beta|(0,T)\times C\backslash A| = m|C|T\}$$

Proof of Lemma 7.2 Set G the right hand-side set. Take $\mu \in ExF$, we want to prove that $\mu \in G$. It is sufficient to prove that, almost everywhere $\mu = \gamma$ or β. Set $Z = \{\mu \in]\gamma; \beta[\}$ and for all n, $Z_n = \{\mu \in]\gamma + \frac{1}{n}; \beta - \frac{1}{n}[\}$. Assume that the measure of Z_n is not equal to zero. In this case it exists a splitting of Z in two sets X and Y which have the same measure. Set $\mu_1 = \mu + \frac{1}{n}1_X - \frac{1}{n}1_Y$ and $\mu_2 = \mu - \frac{1}{n}1_X + \frac{1}{n}1_Y$. Then $\mu_1 \in F$, $\mu_2 \in F$ and $\mu = \frac{\mu_1+\mu_2}{2}$, this is in contradiction with the fact that μ is an extremal point. Thus $|Z| = \lim|Z_n| = 0$.

In the other hand, set $\mu \in G$, and assume that there exists μ_1 and μ_2 in F such that $\mu = \frac{\mu_1+\mu_2}{2}$. Then a.e x, $\mu_1(x) + \mu_2(x) \in \{2\gamma; 2\beta\}$, and necessarily, μ_1 and μ_2 take the same value γ or β, thus these functions are equal almost everywhere. This gives $\mu_1 = \mu_2 = \mu$ and μ is an extremal point.

Proof of Theorem 3.11 F is a closed, convex and bounded subset of $L^\infty([0;T]\times\overline{C})$, so it is compact for the weak-* topology. The Krein–Milman theorem implies that F is the closure of the convex hull of its extremal points.

The set ExF is obviously closed and bounded in $L^\infty([0;T]\times\overline{C})$. Thus it is a compact set for the weak-* topology. Lemma 7.1 yields that the function $\mu \to k_\alpha(\mu)$ is continuous on F for this topology, it raises its minimum over this set. Set μ_0 the associated extremal function.

Set $\mu = \sum_i t_i\mu_i$, where $\forall i$, $\mu_i \in Ex\ F$, $t_i \geq 0$ and $\sum_i t_i = 1$. Then:

$$k_\alpha(\mu) \geq \sum_i t_i k_\alpha(\mu_i) \geq \sum_i t_i k_\alpha(\mu_0) = k_\alpha(\mu_0)$$

As k_α is continuous, this inequality holds for the closure of the convex hull of the extremal points of F. This gives the conclusion.

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