Asymptotic behaviour of travelling waves for the delayed Fisher-KPP equation.

Arnaud Ducrot[a] and Grégoire Nadin [b]

[a] Univ. Bordeaux, IMB, UMR 5251, F-33076 Bordeaux, France
CNRS, IMB, UMR 5251, F-33400 Talence, France

[b] Laboratoire Jacques-Louis Lions, UPMC Univ. Paris 6 and CNRS UMR 7598, F-75005, Paris

October 15, 2013

Abstract

In this work we study the behaviour of travelling wave solutions for the diffusive Hutchinson equation with time delay. Using a phase plane analysis we prove the existence of travelling wave solution for each wave speed $c \geq 2$. We show that for each given and admissible wave speed, such travelling wave solutions converge to a unique maximal wavetrain. As a consequence the existence of a nontrivial maximal wavetrain is equivalent to the existence of travelling wave solution non-converging to the stationary state $u = 1$.

Key words: Travelling wave solutions, time delay, maximal wavetrain, oscillations.
2010 Mathematical Subject Classification: 35C07, 35J15, 35B09, 92D25.

The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP/2007-2013) / ERC Grant Agreement n.321186 - ReaDi -Reaction-Diffusion Equations, Propagation and Modelling.

1 Introduction

The aim of this article is to study the entire bounded and positive orbits of the following second order delay differential equation:

$$-u''(z) + cu'(z) = u(z)\big(1 - u(z-h)\big) \quad \text{for } z \in \mathbb{R}, \tag{1.1}$$

with $c > 0$ and $h > 0$, together with the conditions at infinity

$$\lim_{z \to -\infty} u(z) = 0 \quad \text{and} \quad \liminf_{z \to \infty} u(z) > 0. \tag{1.2}$$

The above problem arises when looking at travelling wave solutions with speed c of the so-called Hutchinson equation, also refereed as the diffusive delayed logistic equation, which reads:

$$(\partial_t - \Delta)\, U(t,x) = U(t,x)\,[1 - U(t-\tau, x)]\,,\ t > 0,\ x \in \mathbb{R}^N. \tag{1.3}$$

A travelling wave solution with speed c in the unit direction $e \in \mathbb{S}^{N-1}$ for the above equation is an entire solution of the form

$$U(t,x) \equiv u(z) \quad \text{ with } z = xe + ct,$$

so that the profile u satisfies (1.1) with $h = c\tau$.

When $h = 0$, that is $\tau = 0$, we recover the classical Fisher-KPP equation. It is known since the pionnering works of Kolmogorov, Petrovsky and Piskunov [16] and Fisher [9] in the 30's that for all $c \geq 2$, equations (1.1)-(1.2) with $h = 0$ admits a travelling wave solution u, which is increasing and converges to 1 at $+\infty$. The travelling wave with minimal speed $c = 2$ attracts, in a sense, the solutions of initial value problems associated with compactly supported initial data. Hence, such solutions model population invasion processes.

The introduction of delayed or nonlocal effects in reaction-diffusion equations is known to give rise to nontrivial periodic steady states since the pioneering paper of Turing [24]. The equation

$$-u'' + cu' = u\big(1 - \phi \star u\big), \tag{1.4}$$

where ϕ is an even probability distribution, has been introduced in [5, 12, 10] in an evolutionary dynamics framework. Nontrivial periodic steady states could then be interpretated as the emergence of new species. The existence of waves for such equations has been investigated by Berestycki, Perthame, Ryzhik and the second author in [3]. The convergence of such waves to 1 at $+\infty$ is unclear, which lead these authors to introduce a generalized notion of travelling waves, that we now adapt to equation (1.1).

Definition 1.1 *[3] We say that a positive solution $u \in \mathcal{C}^2(\mathbb{R})$ of (1.1) is a* travelling wave *(of speed $c > 0$) if it is bounded, it converges to 0 at $-\infty$ and $\liminf_{z\to+\infty} u(z) > 0$. In other words, a travelling wave is a solution of:*

$$\begin{cases} -u''(z) + cu'(z) = u(z)\big(1 - u(z-h)\big) & \text{in } \mathbb{R}, \\ u \text{ is positive and bounded over } \mathbb{R}, \\ u(-\infty) = 0, \quad \liminf_{z\to+\infty} u(z) > 0. \end{cases} \tag{1.5}$$

The existence of such waves for the nonlocal equation (1.4) was proved in [3] for all $c \geq 2$. The monotonicity of such waves was completely characterized in [7]. Numerics indicate that such waves might always converge to 1 [21], but this conjecture is only proved for large speeds [1] or when the Fourier transform of ϕ is positive [3]. The investigation of the initial value problem started in [13].

The delayed equation (1.1)-(1.2) was investigated at the same time. On one hand, the derivation of bounds for the solutions of the delayed equation is much more difficult than for the nonlocal equation (1.4). On the other hand, the theory of delayed differential equations (DDE) applies to (1.1) and provides many useful tools.

Numerics show three different types of behaviours for travelling waves:

- monotone travelling waves, connecting 0 to 1,
- non-monotone travelling waves converging to 1 at $+\infty$,
- non-monotone and non-converging travelling waves oscillating around 1 at $+\infty$.

Gomez and Trofimchuk [11] and Kwong and Ou [17] identified in parallel a necessary and sufficient condition for the existence of increasing travelling waves, namely, such solutions exists if and only if $\tau \leq \tau^*(c) := \sup_{\gamma>0} \frac{\ln \gamma(\gamma+c)}{\gamma c}$. Moreover, monotone travelling waves are unique. The inside dynamics of these waves was investigated in [4]. If $\tau(= h/c)$ is large, then even the existence of travelling waves in a general meaning was unclear until recently. Hasik and Trofimchuk [14] proved the existence of such travelling waves using Mallet-Paret-Sell's Lyapounov functional for delayed differential equations [20]. This functional enabled them to show that such a travelling wave is necessarily slowly oscillating, that is, two consecutive local maximum are separated by an interval of lenght $h = c\tau$ at least. Then they showed that if $\tau \leq 3/2$, any travelling wave converges to 1 [15], which is a Wright's type result [25]. On the other hand, if τ is larger than an explicit threshold depending on c, then travelling waves cannot converge to 1 [14]. We also would like to mention that the question of the behaviour at infinity of travelling waves arises in many models of population dynamics involving time delay and spatial dispersal. One may refer to the so-called diffusive Nicholson's blowflies equation. In such case, travelling wave solutions correspond to solutions of the following second order delay differential equation:

$$\begin{cases} u''(z) - cu'(z) - u(z) + g\left(u(z - c\tau)\right) = 0, \; z \in \mathbb{R}, \\ u(-\infty) = 0 \text{ and } \liminf_{z\to\infty} u(z) > 0. \end{cases}$$

We refer to Faria and Trofimchuk [8] for a study of this problem for large speed and to Trofimchuk et al [23] for a study of the oscillations at $z = +\infty$ for unimodal nonlinearity. We also refer to Ducrot [6] for a study of the oscillations for a similar kind of problem with distributed time delay.

In the present paper, instead of invoking the Mallet-Paret and Sell's theory for DDE, we recovered the related results through direct arguments. This enabled us to show that the solutions of (1.1) do not intersect in the phase-plane under appropriate conditions (this could also be derived from the Mallet-Paret-Sell theory, see the Remark after the proof of Proposition 3.1 below). This is a very powerful result which enables us to provide an alternative proof to the existence of travelling waves and to characterize the convergence of travelling waves when 1 is not attracting, which is a new result. Indeed, such waves necessarily converge to an object that we called the *maximal wavetrain* (see Definition 2.2 below) since it is the larger Jordan curve in the phase-plane. This illustrates how well-fitted our phase-plane approach is.

2 Statement of the results

2.1 Existence of generalized transition waves

We start by showing that travelling waves always exist when $c \geq 2$.

Theorem 2.1 (Existence) *For all $c \geq 2$ and $h > 0$, there exists a travelling wave solution of (1.1)-(1.2) of speed c.*

This result has recently been proved through a different method by Hasik and Trofimchuk [14]. The main difficulty is to show that the travelling waves are globally bounded. This was proved through an involved derivation of a priori bounds in [14]. In the present paper, we first construct exponentially bounded travelling waves through a sub and supersolution approach, and then we show that such a solution is necessarily bounded using a phase-plane analysis.

2.2 Convergence to the maximal wavetrain

Let now characterize the convergence of travelling waves at $+\infty$.

Definition 2.2 (Wavetrain) *Let $c > 0$ and $h > 0$ be given. Let $w \in \mathcal{C}^2(\mathbb{R})$ be a positive solution of (1.1).*

1. *Function u is said to be a* **wavetrain** *(of speed c) if it is periodic. In other words, a wavetrain is a solution of:*

$$\begin{cases} -w''(z) + cw'(z) = w(z)\big(1 - w(z-h)\big) \quad \text{in } \mathbb{R}, \\ w(z) > 0 \quad \text{over } \mathbb{R}, \\ \exists L > 0 |\ \forall z \in \mathbb{R}, \quad w(z+L) \equiv w(z). \end{cases} \tag{2.6}$$

   If the period satisfies $L \geq h$ then the wavetrain w is said to be **slowly oscillating**.

2. *We say that a wavetrain w is said to be a* **maximal wavetrain** *if the curve*

$$\mathcal{S} = \{\big(w(z), w'(z)\big),\ z \in \mathbb{R}\}$$

   is a Jordan curve and if for all wavetrain $\widetilde{w}$, one has

$$\{\big(\widetilde{w}(z), \widetilde{w}'(z)\big),\ z \in \mathbb{R}\} \ \subset\ \mathcal{S} \cup \mathcal{S}^{int}$$

   where $\mathcal{S}^{int}$ is the interior of the Jordan curve $\mathcal{S}$.

Note that the existence and the uniqueness of a maximal wavetrain is far from being clear. Indeed, we will prove these as a side product of our results. Also, note that the constant function $w \equiv 1$ is a trivial wavetrain but it may happen that it is the maximal wavetrain, meaning that no other wavetrain exists.

We are now in position to state our main result.

Theorem 2.3 (Convergence to the maximal wavetrain) *Let $c \geq 2$ be given. Then Equation (1.1) has a unique (up to translation) maximal wavetrain $w \equiv w(z)$. If $u \equiv u(z)$ is a travelling wave solution of (1.1)-(1.2) then there exists a translation $\tau \in \mathbb{R}$ such that*

$$\lim_{z \to +\infty} \big(u(z) - w(z+\tau)\big) = 0.$$

When nontrivial, this unique maximal wavetrain is slowly oscillating around the stationary state 1*.*

This result implies in particular that if there exists a travelling wave converging to 1, then all the travelling waves converge to 1 since $w \equiv 1$ is the maximal wavetrain in this case. Moreover from the above result one obtains that, for each given $c \geq 2$, (1.1) has a nontrivial maximal and slowly oscillating wavetrain if and only if there exists a travelling solution of (1.1)-(1.2) that does not converge to 1 as $z \to \infty$. As a corollary of Theorem 2.3 coupled with the uniqueness theorem of Hasik and Trofimchuk in [14] one can split the parameter space $(c, h) \in [2, \infty) \times [0, \infty)$ into three disjoint regions:

- there exists $u \equiv u(z)$ monotone solution of (1.1)-(1.2), and thus $u(\infty) = 1$;
- there exists $u \equiv u(z)$ non-monotone solution of (1.1)-(1.2) such that $u(\infty) = 1$;
- there exists $u \equiv u(z)$ non-monotone and non-converging travelling waves oscillating around 1 at $+\infty$.

Remark 2.4 *Let $c \geq 2$ be given. Assume that (1.1) has a nontrivial maximal wave train w, then this wavetrain is strictly maximal in the sense that if $\widetilde{w}$ is a different wavetrain (up to translation) then*

$$\big\{\big(\widetilde{w}(z), \widetilde{w}'(z)\big),\ z \in \mathbb{R}\big\} \subset \mathcal{S}^{int}$$

where $\mathcal{S}^{int}$ denotes the interior of the Jordan curve $\mathcal{S} := \big\{\big(w(z), w'(z)\big),\ z \in \mathbb{R}\big\}$. In particular one gets:

$$\min_{\mathbb{R}} w < \min_{\mathbb{R}} \widetilde{w} \leq \max_{\mathbb{R}} \widetilde{w} < \max_{\mathbb{R}} w.$$

A reader who is familiar with the Mallet-Paret and Sell's theory might believe that Theorem 2.3 is indeed a trivial corollary of their Poincaré-Bendixon theorem for DDE [19]. This is not the case. First, as already underlined, we provide here a direct, and thus simpler, proof of the convergence to a periodic solution (that is, a wavetrain). Second, the Poincaré-Bendixon theorem for DDE [19] states that the solution of the DDE either converges to a periodic solution or contains an homoclinic orbit in its $\omega-$limit set. At some point in the coming proof, we will indeed need to exclude the existence of solutions of (1.1) converging to 1 (see the proof of Proposition 5.6 below), which will be a highly nontrivial step.

Travelling waves for the Fisher-KPP equation with advanced saturation effect, that is, solutions of (1.1)-(1.2) with $h < 0$, was investigated in [22] through a toy model. The results are completely different, the limit of travelling waves at $+\infty$ is not unique and all types of travelling waves might coexist: non-montonone and monotone travelling waves converging to 1, and travelling waves converging to various wavetrains. This emphasizes how important is the class of nonlocality we add in the classical Fisher-KPP equation. In particular, for even nonlocalities such as in (1.4), we expect a compromise between the two extreme situations presented in [22] and in the present paper.

Theorem 2.3 might remind the reader of similar results for the Wright's equation. This equation corresponds to the underlying DDE associated with (1.3), which reads:

$$\dot{u}(t) = u(t)\big(1 - u(t-\tau)\big),\ t > 0, \tag{2.7}$$

supplemented together with some initial data $u_0(.) \in C\left([-\tau, 0]\right) \setminus \{0\}$. The question about the global stability of the zero solution is challenging issue in delay differential equation. Wright conjectured in 1955 [25] that u necessarily converges to 1 if $\tau \leq \pi/2$, and proved this conjecture for $\tau \leq 3/2$. Recently Bánhelyi et al [2] proved that such a global stability of 1 for (2.7) is equivalent to the non-existence of slowly oscillating periodic solution.

In the same spirit, in this work, we show that a travelling wave solution converges to 1 at $+\infty$ if and only if equation (1.1) does not have any slowly oscillating maximal wavetrain. Hasik and Trofimchuk proved the convergence to 1 when $\tau \leq 3/2$ and the non-convergence to 1 when $\tau \geq \pi/2$ and $c \geq c^\star(h)$, with an explicit $c^\star(h)$. It now remains to determine some necessary and sufficient conditions ensuring the convergence to 1, but of course we expect this open question to be at least as difficult as Wright's conjecture for (2.7).

Open problem 1: Find a necessary and sufficient condition guaranteeing the convergence to 1 of travelling waves, that is, the non-existence of a nontrivial wavetrain.

Another question that we do not solve in the present paper is that of the uniqueness of travelling waves. It was proved in [14] that if there exists a monotone travelling wave, then any travelling wave converging to 1 is indeed a translation of this monotone travelling wave. Indeed, Theorem 2.3 shows that in this framework, any travelling wave converges to 1 and thus there exists a unique travelling wave up to translation, which is indeed monotone.

When monotone travelling waves do not exist, it is not clear whether travelling waves are unique or not. Theorem 2.3 ensures that travelling waves converge to the same limit at $+\infty$ up to translation and Proposition 3.1 yields that their graphs do not intersect in the phase-plane and Proposition 4.10 ensures that such waves cannot be monotone near $+\infty$, but this is not sufficient to guarantee the uniqueness.

Open problem 2: Prove that travelling waves are unique up to translation.

3 The key tool: non-intersection of the curves in the phase-plane

Proposition 3.1 *Consider two solutions $u, v \in \mathcal{C}^2(\mathbb{R})$ of (1.1). Assume that*

$$\liminf_{z\to+\infty} \frac{u(z)}{v(z)} \leq 1 \leq \limsup_{z\to+\infty} \frac{u(z)}{v(z)}$$

and that there exists $A < 0$ such that $u(z) > v(z)$ for all $z < A$. Then there does not exist any $X \in \mathbb{R}$ such that $u(X) = v(X)$ and $u'(X) = v'(X)$.

Proof. If $u > v$ over $\mathbb{R}$, the result clearly holds. Assume that u and v crosses and let x_0 the smallest solution of $u(x_0) = v(x_0)$ (which is well-defined since $u(z) > v(z)$ for all $z > A$).

Assume that $u'(x_0) = v'(x_0)$. Let $w := u/v$. This function satisfies

$$-w'' + \left(c - 2\frac{v'}{v}\right) w' = w(z)\left[v(z-h) - u(z-h)\right] \quad \text{in } \mathbb{R}, \tag{3.8}$$

with $w(x_0) = 1$ and $w'(x_0) = 0$. As $u(x_0 - h) > v(x_0 - h)$, one has $w''(x_0) > 0$. Hence, x_0 is a local minimizer of w.

Assume that w is not nondecreasing over (x_0, ∞) and let $\overline{z}$ the first positive local maximizer of w. As $w(z) = u(z)/v(z) > 1$ for all $z < x_0$, $w''(x_0) > 0$ and w is nondecreasing in $(x_0, \overline{z})$, it follows from (3.8) that

$$-w'' + \big(c - 2\frac{v'}{v}\big)w' < 0 \quad \text{in } (-\infty, \overline{z} + h).$$

As $w''(\overline{z}) \leq 0$ and $w'(\overline{z}) = 0$, w cannot satisfies this equation and we have thus proved by contradiction that w is nondecreasing over (x_0, ∞). On the other hand, we know that

$$\liminf_{z\to+\infty} w(z) = \liminf_{z\to+\infty} \frac{u(z)}{v(z)} \leq 1$$

by hypothesis. This contradicts the fact that w is nondecreasing, with $w(x_0) = 1$ and $w''(x_0) > 0$. We have thus proved by contradiction that $u'(x_0) \neq v'(x_0)$, which implies $u'(x_0) < v'(x_0)$ since $u > v$ in $(-\infty, x_0)$.

Next, assume that $u-v$ crosses 0 at n points $x_0 < ... < x_{n-1}$, with $u'(x_k) \neq v'(x_k)$ for all $k = 0, ..., n-1$ and $x_k - x_{k-2} \geq h$ for all $k = 2, ..., n-1$, $u(z) \neq v(z)$ if $z < x_{n-1}$ with $z \notin \{x_0, ..., x_{n-1}\}$. Assume that $u - v$ admits an additional zero $x_n > x_{n-1}$ and let prove that $x_n - x_{n-2} \geq h$ and $u'(x_n) \neq v'(x_n)$ in order to conclude by iteration.

We can assume that $u(z) < v(z)$ for $z \in (x_{n-1}, x_n)$, the case where the other inequality is satisfied being treated similarly. Define w as above. Take $\underline{z}_n \in (x_{n-1}, x_n)$ such that $\min_{(x_{n-1},x_n)} w = w(\underline{z}_n)$. Equation (3.8) yields $w(\underline{z}_n - h) \geq 1$. As $w(z) < 1$ for all $z \in (x_{n-3}, x_{n-2})$ and $x_{n-1} - x_{n-3} \geq h$ for all $k = 2, ..., n-1$, one has $\underline{z}_n - h \in (x_{n-2}, x_{n-1})$. Hence, $x_n \geq \underline{z}_n \geq x_{n-2} + h$.

Next, as $w(z) < 1 = w(x_n)$ for all $z \in (x_{n-1}, x_n)$ and $w'(x_n) = 0$ since we have assumed $u'(x_n) = v'(x_n)$, one has $w''(x_n) \leq 0$. Equation (3.8) gives $w(x_n - h) \leq 1$. Then as $x_n - x_{n-2} \geq h$, one has $x_n - h \in (x_{n-1}, x_n)$.

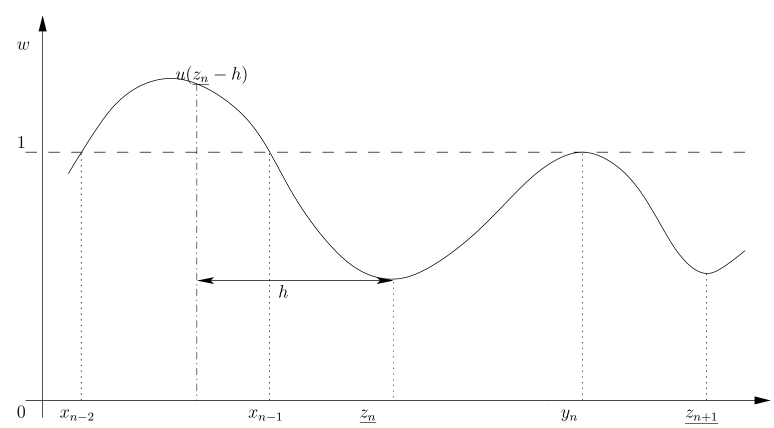
Consider first the case where $w(x_n - h) = 1$, that is, $x_n - h = x_{n-1}$. One has $w'(x_{n-1}) < 0$ since $u'(x_{n-1}) \neq v'(x_{n-1})$ and $w(z) < 1$ for all $z \in (x_{n-1}, x_n)$, and thus $w(z) > 1$ for $x_{n-2} < z < x_{n-1} = x_n - h$. It follows from (3.8) that

$$-w'' + \big(c - 2\frac{v'}{v}\big)w' < 0 \quad \text{on the left neighbourhood of } x_n.$$

Thus, the Hopf Lemma applies: as $w(z) < 1 = w(x_n)$ for all $z < x_n$ close to x_n, $w'(x_n) = 0$ is a contradiction.

Let handle the second case where $w(x_n - h) < 1$. Then $x_{n-1} < x_n - h$ and (3.8) gives $w''(x_n) < 0$ and thus x_n is a local maximizer. Assume that there exists a first local minimizer $\underline{z} > x_n$. Then $w(z) \leq 1$ for all $z \in (x_{n-1}, \underline{z})$ and thus $u(z-h) \leq v(z-h)$ over $(x_{n-1} + h, \underline{z} + h)$, leading to

$$-w'' + \big(c - 2\frac{v'}{v}\big)w' \geq 0 \quad \text{in } (x_{n-1} + h, \underline{z} + h).$$



We already know that $x_n - h \in (x_{n-1}, x_n)$, and thus $\underline{z}$ is an interior local minimizer of w over $(x_{n-1} + h, \underline{z} + h)$. The strong maximum principle gives $w \equiv w(\underline{z})$ over $(x_{n-1} + h, \underline{z} + h)$. It follows from (3.8) that $u \equiv v$, that is, $w \equiv 1$, over $(x_{n-1}, \underline{z})$, which is a contradiction since $w'(x_{n-1}) < 0$. We have thus proved that w does not admit any local minimizer on (x_n, ∞), meaning that w is nonincreasing on this set. We can then conclude with the same arguments as above, using $\limsup_{z\to+\infty} \frac{u(z)}{v(z)} \geq 1$. We have thus reached a contradiction in all cases, meaning that $w'(x_n) \neq 0$, that is, $u'(x_n) \neq v'(x_n)$. The result follows by iteration. $\square$

Remark 3.2 *This result could be derived from the results of Mallet-Paret and Sell in [20] on the number of zeros of solutions of delayed differential equations. Indeed, define $w := 1 - u/v$. Then the function pair $\left(x^0, x^1\right)$ defined by $x^0 = 1 - u/v$ and $x^1 = \left(x^0\right)'$ satisfies the following system of equations on $\mathbb{R}$:*

$$\begin{cases} \frac{dx^0}{dz}(z) = x^1(z), \\ \frac{dx^1}{dz}(z) = \left[c - 2\frac{v'(z)}{v(z)}\right] x^1(z) + \frac{u(z)}{v(z)} v(z-h) x^0(z-h). \end{cases} \tag{3.9}$$

Note that the above system is a non-autonomous unidirectional cyclic delay differential system with positive feedback. Next consider the set $\mathbb{K} = [-h, 0] \cup \{1\}$ and let us define for each $\varphi \in C(\mathbb{K}, \mathbb{R}) \setminus \{0\}$ the sign change as

$$\mathrm{sc}\big(\varphi\big) := \sup\{n \in \mathbb{N},\ \exists z_1, ..., z_N \in \mathbb{K} \mid z_i < z_{i+1},\ \varphi(z_i)\varphi(z_{i+1}) < 0\}. \tag{3.10}$$

Next recall (see [20]) that if we set for each $z \in \mathbb{R}$ $x_z \in C(\mathbb{K}, \mathbb{R})$ defined by

$$x_z(\theta) = \begin{cases} x^0(z+\theta) \textit{if } \theta \in [-h, 0], \\ x^1(z) \textit{ if } \theta = 1 \end{cases},$$

then the map $z \mapsto V(x_z)$ is nonincreasing on $\mathbb{R}$ and that $V\big(w(z-3h)\big) > V\big(w(z)\big)$ if $x^0(z) = 0$ and $x^1(z) = 0$. Here we have set $V : C(\mathbb{K}, \mathbb{R}) \setminus \{0\} \to \{0, 2, ...\}$ defined by:

$$V(\varphi) = \begin{cases} \mathrm{sc}(\varphi) \textit{ if } \mathrm{sc}(\varphi) \textit{ is even} \\ \mathrm{sc}(\varphi) + 1 \textit{ else.} \end{cases}$$

As a consequence since $u(z) < v(z)$ for all $z < A$ (namely $w(z) > 0$ for all $z < A$) then for each $z < A - h$ one has $\mathrm{sc}(x_z) \in \{0, 1\}$. Hence for each $z \in \mathbb{R}$ one has $V(x_z) \in \{0, 2\}$. We claim that under the condition of Proposition 3.1 one has $V(x_z) \equiv 2$ for all $z \in \mathbb{R}$, that prevents from phase plane intersections. Note that if $V(x_z) = 0$ for $z \to -\infty$ then $V(x_z) = 0$ for all $z \in \mathbb{R}$ and the map w is positive and increasing. This contradicts the condition $\limsup_{z\to\infty} \frac{u(z)}{v(z)} \geq 1$. Hence $V(x_z) = 2$ as $z \to -\infty$. Next note that if phase plane intersection occurs then $V(x_z) = 0$ for z large enough. This means that for z large enough, either w is positive and increasing or negative and decreasing. Both cases contradict the assumed behaviour at infinity.

4 Construction of the travelling waves

This section is devoted to the proof of Theorem 2.1. We will start as in [3] by constructing appropriate sub and supersolutions when $c > 2$. The difference with [3] is that we have no global a priori bound on the solutions. Such bounds will come from a phase-plane analysis and Proposition 3.1.

4.1 Sub and supersolutions when $c > 2$

Assume that $c > 2$. We can define $\lambda < \Lambda$ the two positive solutions of

$$P_c(\lambda) := \lambda^2 - \lambda c + 1 = 0. \tag{4.11}$$

We will use the following super and subsolutions:

$$\overline{u}(z) := e^{\lambda z}, \quad \underline{u}(z) := \max\{0, e^{\lambda z} - Ae^{(\lambda+\gamma)z}\} \quad \text{for all } z \in \mathbb{R}, \tag{4.12}$$

for some well-chosen $A > 0$ and $\gamma \in (0, \lambda)$ such that $\lambda + \gamma < \Lambda$. Indeed, this choice of γ yields that $P_c(\lambda + \gamma) < 0$, and thus we can choose $A > 0$ large enough so that

$$1 \leq -A^{\lambda/\gamma} P_c(\lambda + \gamma). \tag{4.13}$$

We can also assume, taking A sufficiently large, that $\sup_{\mathbb{R}} \underline{u} < 1$.

Lemma 4.1 *One has $\overline{u} \geq \underline{u}$ in $\mathbb{R}$ and*

$$-\overline{u}'' + c\overline{u}' - \overline{u} = 0 \quad \text{in } \mathbb{R}, \tag{4.14}$$

$$-\underline{u}'' + c\underline{u}' - \underline{u} < -\overline{u}^2 \quad \text{in } \mathbb{R}. \tag{4.15}$$

Proof. The inequality $\underline{u} \leq \overline{u}$ is obvious. Equation (4.14) follows from $P_c(\lambda) = 0$. Take $z \in \mathbb{R}$ such that $\underline{u}(z) > 0$, that is, $Ae^{\gamma z} < 1$. Then:

$$\begin{aligned} -\underline{u}''(z) + c\underline{u}'(z) - \underline{u}(z) + \overline{u}^2(z) &= AP_c(\lambda + \gamma)e^{(\lambda+\gamma)z} + e^{2\lambda z} \\ &\leq e^{2\lambda z}\Big(1 - A^{1-\frac{\lambda}{\gamma}} e^{(\gamma-\lambda)z}\Big) \\ &< e^{2\lambda z}\Big(1 - A^{1-\frac{\lambda}{\gamma}} A^{\frac{\lambda}{\gamma}-1}\Big) = 0. \end{aligned}$$

□

4.2 Construction of an exponentially bounded solution when $c > 2$

For all $a > 0$, define

$$\Gamma_a := \{u \in \mathcal{C}^2((-a, a)) \cap \mathcal{C}^0([-a, a]),\ \underline{u} \leq u \leq \overline{u} \text{ in } (-a, a)\},$$

and

$$F_a: \begin{array}{rcl} \mathcal{C}^2((-a,a))\cap\mathcal{C}^0([-a,a]) & \to & \mathcal{C}^2((-a,a))\cap\mathcal{C}^0([-a,a]) \\ u_0 & \mapsto & u_1 \end{array} \tag{4.16}$$

where u_1 satisfies $u_1(\pm a) = \underline{u}(\pm a)$ and

$$-u_1''(z) + cu_1(z) = u_0(z)\big(1 - u_0(z-h)\big) \quad \text{in } (-a,a), \tag{4.17}$$

where we extend u_0 by $\underline{u}(-a)$ in $(-\infty,-a)$.

Lemma 4.2 *One has $F_a(\Gamma_a) \subset \Gamma_a$.*

Proof. Take $u_0 \in \Gamma_a$ and assume that $\max_{[-a,a]}\big(u_1 - \overline{u}\big) > 0$ by contradiction, with $u_1 = F_a(u_0)$. Take $z \in [-a,a]$ such that this maximum is reached at z. As $u_1(\pm a) = \underline{u}(\pm a) \le \overline{u}(\pm a)$, one has $z \in (-a,a)$. As $u_1''(z) \le \overline{u}''(z)$, $u_1'(z) = \overline{u}'(z)$ and $u_0 \ge 0$, equation (4.17) yields

$$u_0(z) \ge u_0(z)\big(1 - u_0(z-h)\big) = -u_1''(z) + cu_1'(z) \ge -\overline{u}''(z) + c\overline{u}'(z) = \overline{u}(z),$$

which contradicts $u_0 \in \Gamma_a$. Hence $\max_{\mathbb{R}}(u_1 - \overline{u}) \le 0$, that is, $u_1 \le \overline{u}$.

Next, assume by contradiction that $\min_{[-a,a]}\big(u_1 - \underline{u}\big) = \big(u_1(z) - \underline{u}(z)\big) < 0$. Then $z \in (-a,a)$ and one has:

$$u_0(z)\big(1 - u_0(z-h)\big) = -u_1''(z) + cu_1'(z) \le -\underline{u}''(z) + c\underline{u}'(z) < \underline{u}(z) - \overline{u}^2(z),$$

which is a contradiction since $u_0 \ge \underline{u}$ and $u_0(z)u_0(z-h) \le \overline{u}(z)\overline{u}(z-h) \le \overline{u}^2(z)$. Hence, $u_1 \ge \underline{u}$. $\square$

Lemma 4.3 *Assume that $c > 2$. Then equation (1.1) admits a solution u such that $\underline{u} \le u \le \overline{u}$ in $\mathbb{R}$.*

Proof. Clearly F_a is compact, continuous and Γ_a is convex and bounded. Hence, the Schauder fixed point theorem gives the a solution $u_a \in \Gamma_a$ of $F_a(u_a) = u_a$, that is:

$$-u_a''(z) + cu_a'(z) = u_a(z)\big(1 - u_a(z-h)\big) \quad \text{over } (-a,a)$$

with $\underline{u} \le u_a \le \overline{u}$, where u_a is extended by $\underline{u}(-a)$ in $(-\infty,-a)$ and by $\underline{u}(a)$ in (a,∞). As u_a is locally bounded since $u_a \le \overline{u}$, the Schauder elliptic estimates apply and the Ascoli Theorem gives a sequence $(a_n)_n$, with $\lim_{n\to+\infty} a_n = +\infty$ such that $(u_{a_n})_n$ converges in $\mathcal{C}^2_{loc}$. We thus get a solution u of

$$-u'' + cu' = u\big(1 - u(z-h)\big) \quad \text{in } \mathbb{R} \tag{4.18}$$

such that $\underline{u} \le u \le \overline{u}$. $\square$

4.3 Boundedness and construction of travelling waves when $c > 2$

This is where we start to provide real new arguments compared with [3], in order to derive the boundedness of the solution.

Lemma 4.4 *Let u as in Lemma 4.3 and assume that u does not converge to* 1 *at* $+\infty$*. Then there does not exist any $z_+ > 0$ such that $u(z) \geq 1$ for all $z \geq z_+$.*

Proof. In order to prove this claim, assume that such a z_+ exists. Taking $z_+ + h$ instead of z_+, we can assume that $u(z) \geq 1$ for all $z \geq z_+ - h$. We can also assume that $u'(z_+) > 0$, if not, u would be nonincreasing near $+\infty$ and thus it would converge since $u(z) \geq 1$ for all $z \geq z_+$. Passing to the limit in (4.18), one would get $u(+\infty) = 1$, which is excluded by hypothesis.

Next, as $-u'' + cu' \leq 0$ over (z_+, ∞), integrating between z_+ and $z > z_+$, one gets

$$-u'(z) + cu(z) \leq -u'(z_+) + cu(z_+) =: \alpha < cu(z_+) \quad \text{for all } z \geq z_+.$$

We could rewrite this inequality $-\big(ue^{-cz}\big)' \leq \alpha e^{-cz}$ and thus one gets

$$-u(z)e^{-cz} + u(z_+)e^{-cz_+} \leq \frac{\alpha}{c}e^{-cz_+} - \frac{\alpha}{c}e^{-cz} \quad \text{over } (z_+, \infty),$$

that is

$$u(z) \geq \big(u(z_+) - \frac{\alpha}{c}\big)e^{c(z-z_+)} + \frac{\alpha}{c} \quad \text{for all } z \geq z_+.$$

On the other hand, we have constructed u in such a way that $u(z) \leq \overline{u}(z) = e^{\lambda z}$, where λ is the smallest solution of (4.11). Comparing the two exponential growth rates as $z \to +\infty$, we thus get $\lambda \geq c$. But as $P_c(c) = c^2 - c \times c + 1 > 0$ and $P_c'(c) = c > 0$, one necessarily has $\lambda < c$, which gives the final contradiction and proves the claim. □

Lemma 4.5 *Let u as in Lemma 4.3 and assume that u does not converge to* 1 *at* $+\infty$*. Then there does not exist any $z_- > 0$ such that $u(z) \leq 1$ for all $z \geq z_-$.*

Proof. As above, assume that such a z_- exists, with $u(z) \leq 1$ for all $z \geq z_- - h$ and $u'(z_-) < 0$. As $-u'' + cu' \geq 0$ over (z_-, ∞), the same types of computations as in the proof of Lemma 4.4 lead to

$$u(z) \leq \big(u(z_-) - \frac{\alpha}{c}\big)e^{c(z-z_-)} + \frac{\alpha}{c} \quad \text{for all } z \geq z_-,$$

with $\alpha = -u'(c_-) + cu(z_-) > cu(z_+)$. Hence, $u(z)$ becomes negative when z is large enough, a contradiction. □

Lemma 4.6 *Assume that $u \in \mathcal{C}^2(\mathbb{R})$ satisfies (1.5). Then there exists $A < 0$ such that u is increasing over $(-\infty, A)$.*

Proof. As $u(-\infty) = 0$, there exists $A < 0$ such that $u(z-h) < 1$ for all $z < A$. Hence $-u'' + cu' > 0$ over $(-\infty, A)$ and u does not admit any local minimum over $(-\infty, A)$. As u is positive and $u(-\infty) = 0$, u is necessarily increasing over $(-\infty, A)$. □

Lemma 4.7 *There do not exist any $z_1 < z_2$ in $\mathbb{R}$ such that $u(z_1) = u(z_2)$ and $u'(z_1) = u'(z_2)$, where u is the solution constructed in Lemma 4.3.*

Proof. We just apply Proposition 3.1 to u and $v = u(\cdot + z_1 - z_2)$. Lemma 4.6 ensures that $v(z) = u(z + z_1 - z_2) < u(z)$ for all $z < B$. Moreover, if $\liminf_{z\to+\infty} u(z)/v(z) > 1$, then there exist $\kappa > 1$ and $A > 0$ such that $u(z) \geq \kappa u(z - \sigma)$ for all $z \geq A$, with $\sigma = z_2 - z_1 > 0$. But as u is positive and continuous, one has $\varepsilon = \inf_{[A-\sigma,A]} u > 0$. It follows by iteration that $\inf_{[A+n\sigma,A+(n+1)\sigma]} u \geq \varepsilon\kappa^{n+1}$ for all $n \in \mathbb{N}$. In particular, one has $u(z) \geq 1$ when z is large enough, a contradiction. Hence, $\liminf_{z\to+\infty} u(z)/v(z) \leq 1$. Similarly, if $\limsup_{z\to+\infty} u(z)/v(z) < 1$, then the same types of arguments as above give $u(z) \leq 1$ when z is large enough and thus Lemma 4.5 gives a contradiction. The hypotheses of Proposition 3.1 are thus satisfied and the conclusion follows. □

Lemma 4.8 *For all $c > 2$, the solution u constructed in Lemma 4.3 satisfies $u(z) \leq e^{ch}$.*

Proof. The result is clear if $u \leq 1$. If not, Lemma 4.4 ensures that u cannot stay above 1 at large z. Hence, there necessarily exists a first local maximizer $\overline{z}_0 \in \mathbb{R}$, such that u is nondecreasing over $(-\infty, \overline{z}_0)$. Moreover, Lemma 4.5 yields that u admits a first local minimizer $\underline{z}_0$ on the right of $\overline{z}_0$, such that u is nonincreasing on $(\overline{z}_0, \underline{z}_0)$. Consider the curve:

$$\mathcal{S}_0 := \{\big(u(z), u'(z)\big),\ z \in (-\infty, \underline{z}_0)\} \cup \big([0, u(\underline{z}_0)] \times \{0\}\big).$$

Remembering that $u(-\infty) = u'(-\infty) = 0$ and that $u'(\underline{z}_0) = 0$, it is clear that $\mathcal{S}_0$ does not self-intersect and that it is a closed curve. That is, $\mathcal{S}_0$ is a Jordan curve.

Next, assume by contradiction that the curve associated with u in the phase plane leaves the interior of the Jordan curve $\mathcal{S}_0$ and let

$$z_* := \inf\{z > \underline{z}_0,\ \big(u(z), u'(z)\big) \notin \mathcal{S}_0 \cup \mathcal{S}_0^{int}\}.$$

As $\mathcal{S}_0^{int}$ is an open set, one has $\big(u(z_*), u'(z_*)\big) \in \mathcal{S}_0$. Lemma 4.7 ensures that $\big(u(z_*), u'(z_*)\big) \notin \{\big(u(z), u'(z)\big),\ z \in (-\infty, \underline{z}_0]\}$, hence, $\big(u(z_*), u'(z_*)\big) \in \big[0, u(\underline{z}_0)\big) \times \{0\}$, that is, $u'(z_*) = 0$ and $u(z_*) < u(\underline{z}_0)$. As $\big(u(z), u'(z)\big) \in \mathcal{S}_0 \cup \mathcal{S}_0^{int}$ for all $z < z_*$, as u is $\mathcal{C}^1$ and as $u(z_*) < u(\underline{z}_0)$, necessarily $u'(z) > 0$ in a left neighbourhood of z_*. We can thus define $\xi := \max\{z < z_*,\ u'(z) < 0\}$. Clearly $u'(\xi) = 0$ by continuity. As $u'(z) \geq 0$ for all $z \in (\xi, z_*)$, one has $0 < u(\xi) < u(z_*)$. On the other hand, the definition of ξ yields that there

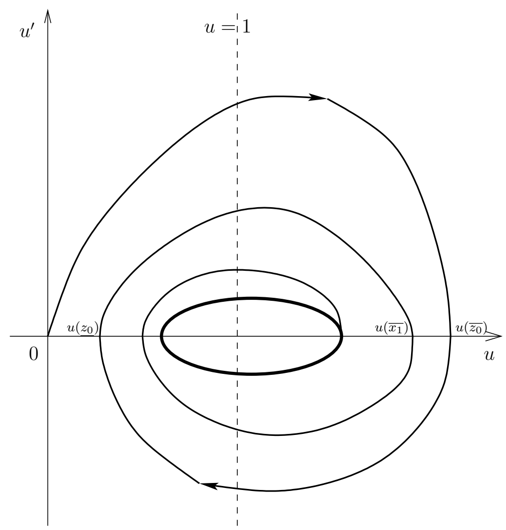


Figure 1: The curve $\mathcal{C}$ in the phase plane (u,u').

exists $\varepsilon > 0$ arbitrarily small so that $u'(\xi-\varepsilon) < 0$. Taking ε small, one can get $u(\xi-\varepsilon) < u(\underline{z}_0)$ and thus $\big(u(\xi-\varepsilon), u'(\xi-\varepsilon)\big) \in \mathcal{S}_0^{ext}$, which contradicts the definition of z_*. We have thus proved that $\big(u(z), u'(z)\big) \in \mathcal{S}_0 \cup \mathcal{S}_0^{int}$ for all $z \in \mathbb{R}$, which implies in particular that $u \le u(\overline{z}_0)$ over $\mathbb{R}$.

Lastly, as u reaches a local maximum at $\overline{z}$, equation (1.1) gives $u(\overline{z}_0 - h) \le 1$. Indeed, as u is nondecreasing over $(-\infty, \overline{z}_0)$, one gets $u \le 1$ over $(-\infty, \overline{z}_0 - h)$. Thus, $-u'' + cu' \ge 0$ in $(-\infty, \overline{z}_0)$. Integrating this inequality from $-\infty$ to z, one gets $u' \le cu$ in $(-\infty, \overline{z}_0)$. Applying the Gronwall inequality between $\overline{z}_0 - h$ and $\overline{z}_0$, we eventually get $u(\overline{z}_0) \le u(\overline{z}_0 - h)e^{ch} \le e^{ch}$, which concludes the proof since $u \le u(\overline{z}_0)$. □

Proof of Theorem 2.1 when $c > 2$. We are now in position to conclude the proof. Right now we have constructed a positive solution u of equation (1.1) such that $u(-\infty) = 0$ (Lemma 4.3) which is bounded (Lemma 4.8). It now remains to prove that $\liminf_{z\to+\infty} u(z) > 0$. Assume by contradiction that $\liminf_{z\to+\infty} u(z) = 0$.

Let first prove that u is necessarily nonincreasing over (A, ∞) for A large enough. If this was not true, one would be able to construct a sequence $(z_n)_n$ of local minimizers of u such that $z_n \to +\infty$ and $u(z_n) \to 0$ as $n \to +\infty$. Considering the equation satisfied by u at z_n, one would get $u(z_n - h) \ge 1$. On the other hand, as u is bounded, the Harnack inequality applies to $-u'' + cu' = \big(1 - u(z-h)\big)u$ (just consider $1 - u(z-h)$ as a fixed bounded zero order term) and gives a constant $C > 0$ such that $u(z_n - h) \le Cu(z_n)$, which is a contradiction. Thus u is nonincreasing over (A, ∞) for A large enough.

Next, define the sequence $u_n(z) := u(z+z_n)/u(z_n)$. The Harnack inequality yields that $(u_n)_n$ is a locally bounded sequence. Moreover, it satisfies

$$-u_n'' + cu_n' = u_n\big(1 - u(z_n)u_n(z-h)\big) \quad \text{over } \mathbb{R}.$$

Hence, Schauder elliptic estimates yield that one can assume, up to extraction, that the sequence $(u_n)_n$ converges to a limit $u_\infty \in \mathcal{C}^2(\mathbb{R})$ satisfying

$$-u_\infty'' + cu_\infty' = u_\infty \quad \text{over } \mathbb{R}, \quad u_\infty(0) = 1, \quad u_\infty \geq 0.$$

One can thus write $u_\infty(z) = \alpha e^{\lambda z} + \beta e^{\Lambda z}$, with $\alpha, \beta \in \mathbb{R}$. The nonnegativity of u_∞ yields $\alpha \geq 0$ and $\beta \geq 0$. But as u is nonincreasing near $+\infty$ and $\Lambda > \lambda > 0$, one gets $\alpha = \beta = 0$. This is a contradiction since $u_\infty(0) = 1$. Hence, we have proved by contradiction that $\liminf_{z\to+\infty} u(z) > 0$, which concludes the proof. □

4.4 Construction of travelling waves when $c = 2$

Lemma 4.9 *There exists $\varepsilon > 0$ such that if $c \in [2,3]$ and w is a solution of (1.1) over $\mathbb{R}$ such that $0 < w \leq e^{ch}$, and $\inf_{\mathbb{R}} w > 0$, then $\inf_{\mathbb{R}} w \geq \varepsilon$.*

Proof. This could be proved with exactly the same arguments as in Lemma 3.4 in [3]. □

Proof of Theorem 2.1 when $c = 2$. Let $\varepsilon > 0$ as in Lemma 4.9. We know that for all $n \geq 1$, there exists a travelling wave u_n of speed $c_n = 2 + 1/n$, with $u_n \leq e^{c_n h}$ over $\mathbb{R}$. As $\liminf_{z\to+\infty} u_n(z) > 0$, it easily follows from Lemma 4.9 that $\liminf_{z\to+\infty} u_n(z) \geq \varepsilon$, otherwise, one would be able to construct an entire solution w of (1.1) with $\inf_{\mathbb{R}} w = \liminf_{+\infty} u_n < \varepsilon$. As $u_n(-\infty) = 0$, one could thus translate u_n so that $u_n(0) = \varepsilon/2$.

Next, the Schauder elliptic estimates yield that some extraction of the sequence $(u_n)_n$ converges to a limit u in $\mathcal{C}^2_{loc}(\mathbb{R})$. As $\lim_{n\to+\infty} c_n = 2$, the function u satisfies (1.1) with $c = 2$. Moreover, one has $0 \leq u \leq e^{2h}$ over $\mathbb{R}$ and $u(0) = \varepsilon/2$. The strong maximum principle ensures the positiveness of u. Moreover, if $\inf_{\mathbb{R}} u > 0$, then Lemma 4.9 would give $u \geq \varepsilon$, which is impossible since $u(0) = \varepsilon/2$. Hence $\inf_{\mathbb{R}} u = 0$. Thus, there exists a sequence $(z_k)_n$, with $\lim_{k\to+\infty} |z_k| = +\infty$, such that $\lim_{k\to+\infty} u(z_k) = 0$.

Assume by contradiction that $\limsup_{k\to+\infty} z_k = +\infty$, meaning that $\liminf_{z\to+\infty} u(z) = 0$. Then the same arguments as in the case $c > 2$ would give that u is nonincreasing over (A, ∞) for some $A > 0$ and that the sequence $u_k(z) := u(z+z_k)/u(z_k)$ converges, up to extraction, to a limit u_∞ satisfying $-u_\infty'' + 2u_\infty' = u_\infty$ over $\mathbb{R}$, with $u_\infty \geq 0$ and $u_\infty(0) = 1$. Hence one could write $u_\infty(z) = (\alpha z + \beta)e^z$, with $\alpha, \beta \in \mathbb{R}$. As $u_\infty \geq 0$ over $\mathbb{R}$, one gets $\alpha = 0$ and $\beta \geq 0$. On the other hand, as u is nonincreasing over (A, ∞), one has $\beta \leq 0$. Hence $\alpha = \beta = 0$, which is a contradiction since $u_\infty(0) = 1$. We have thus prove by contradiction that $\liminf_{z\to+\infty} u(z) > 0$ and that $\lim_{k\to+\infty} z_k = -\infty$.

Lastly, one can easily prove using the same arguments as in the case $c > 2$ that u is nondecreasing over $(-\infty, A)$ since $\lim_{k\to+\infty} u(z_k) = 0$ with $\lim_{k\to+\infty} z_k = -\infty$. It easily follows that $u(-\infty) = 0$, which concludes the proof. □

4.5 Monotonicity at $+\infty$

We end this section by a result of independent interest analysing wave-like solutions that are monotone near $+\infty$, which will be useful in the coming section.

Proposition 4.10 *Assume that there exists a positive solution $w \in \mathcal{C}^2\big((x_0 - h, \infty)\big)$, with $x_0 \in \mathbb{R}$, of*

$$-w''(z) + cw'(z) = w(z)\big(1 - w(z-h)\big) \qquad \text{over } (x_0, \infty)$$

such that w is monotone over (x_0, ∞) and $w(+\infty) = 1$. Then (1.1)-(1.2) admits an increasing travelling wave.

Lemma 4.11 *Take w as in Proposition 4.10. Then there exists $C > 0$ such that*

$$\forall z \in (x_0, \infty), \quad |w(z-h) - 1| \le C|w(z) - 1|.$$

Proof. This might be derived from the proof of Lemma 3.1.1 of [18], but we directly apply the arguments of [18] to our problem in order to get a self-contained proof.

We only consider the case were w is nonincreasing, the nondecreasing case being treated similarly. Let $v := w - 1$, as $w(+\infty) = 1$, the function v is nonnegative. Moreover, it satisfies

$$-v''(z) + cv'(z) = -\big(1 + v(z)\big)v(z-h) \le -v(z) \qquad \text{for all } z > x_0 + h.$$

Let $\alpha_- < 0 < \alpha_+$ the two roots of equation $-\alpha^2 + c\alpha = -1$. It is easy to check that $v(z') \ge Ce^{\alpha_-(z'-z)} + De^{\alpha_+(z'-z)}$ for all $z' > z \ge x_0 + h$, where

$$C = \frac{\alpha_+ v(z) - v'(z)}{\alpha_+ - \alpha_-} \quad \text{and} \quad D = \frac{-\alpha_- v(z) + v'(z)}{\alpha_+ - \alpha_-}.$$

As v is bounded, one has $D \le 0$, meaning that

$$\forall z \ge x_0 + h, \quad v'(z) \le \alpha_- v(z). \tag{4.19}$$

This implies $v(z-h) \le e^{\alpha_-(z-x_1)}v(x_1 - h)$ for all $z > x_1 > x_0 + h$.

Next, let $\xi(z) := v(z)e^{\mu z}$, where $\mu < 0$ is such that $\mu^2 + \mu c \ge 0$. Clearly ξ is decreasing. Take $x_1 > x_0 + h$. For all $z \in (x_1, x_1 + h)$, one has:

$$-\xi''(z) + (c+2\mu)\xi'(z) - (\mu^2 + c\mu)\xi(z) = -e^{\mu h}\xi(z-h)\big(1+v(z)\big) \ge -e^{\mu h}\xi(x_1 - h)\big(1 + v(x_1)\big).$$

This gives:

$$\big(\xi'(z)e^{-(c+2\mu)z}\big)' \le e^{\mu h}\xi(x_1 - h)\big(1 + v(x_1)\big)e^{-(c+2\mu)z}.$$

Integrating over (x_1, z) for any $z \in (x_1, x_1 + h)$, one gets:

$$\xi'(z)e^{-(c+2\mu)z} - \xi'(x_1)e^{-(c+2\mu)x_1} \le \frac{-1}{c+2\mu}e^{\mu h}\xi(x_1 - h)\big(1 + v(x_1)\big)\big(e^{-(c+2\mu)z} - e^{-(c+2\mu)x_1}\big).$$

As ξ is decreasing, this yields

$$\xi'(z) \leq \frac{-1}{c+2\mu} e^{\mu h} \xi(x_1 - h)\big(1 + v(x_1)\big)(1 - e^{(c+2\mu)(z-x_1)}).$$

Integrating over $(x_1, x_1 + h)$, as $v > 0$, we eventually obtain

$$-\xi(x_1) \leq \xi(x_1 + h) - \xi(x_1) \leq e^{\mu h} \xi(x_1 - h)\big(1 + v(x_1)\big)\Big(\frac{e^{(c+2\mu)h} - 1}{(c+2\mu)^2} - \frac{h}{c+2\mu}\Big).$$

Taking $\mu < 0$ small enough so that $\frac{e^{(c+2\mu)h}-1}{c+2\mu} > h$, as $c + 2\mu < 0$, we could rewrite this inequality:

$$\xi(x_1) \geq e^{\mu h} \xi(x_1 - h) \times C^{-1} \quad \text{ with } C^{-1} = \big(1 + \|v\|_\infty\big)\Big(\frac{h}{c+2\mu} - \frac{e^{(c+2\mu)h} - 1}{(c+2\mu)^2}\Big) > 0.$$

As $\xi(z) = v(z) e^{\mu z}$, this concludes the proof. □

Proof of Proposition 4.10. Lemma 4.11 will enable us to linearize the equation near $w = 1$, as in the proof of Proposition 5.1 in [3].

Take a sequence $(z_n)_n$ such that $\lim_{n\to+\infty} z_n = +\infty$ and define $v_n(z) := \frac{w(z+z_n)-1}{w(z_n)-1}$ for all $z \in \mathbb{R}$. Lemma 4.11 yields that $|v_n(z-h)| \leq C|v_n(z)|$ for all $n \in \mathbb{N}$ and $z \in \mathbb{R}$. It easily follows from the equation satisfied by v_n, Schauder elliptic estimates and the Ascoli theorem that one can extract a subsequence converging to a limit v_∞ in $\mathcal{C}^2_{loc}(\mathbb{R})$ which satisfies:

$$\forall z \in \mathbb{R}, \quad v''_\infty(z) - cv'_\infty(z) - v_\infty(z-h) = 0, \quad v_\infty(0) = 1,$$

and v_∞ is either nonincreasing and nonnegative or nondecreasing and nonpositive. Taking $-v_\infty$ instead of v_∞ if necessary, one can assume that v_∞ is nonnegative and nonincreasing. It is easy to check that $v_\infty(+\infty) = v'_\infty(+\infty) = 0$. Lemma 4.11 yields $v_\infty(z-h) \leq C v_\infty(z)$ for all $z \in \mathbb{R}$. As v_∞ is nonincreasing, we could thus find $A > 0$ and $\gamma > 0$ such that $v_\infty(z) \geq Ae^{-\gamma z}$ for all $z \in \mathbb{R}$. Define

$$\overline{A} := \sup\{A > 0, \quad \forall z \in \mathbb{R},\ v_\infty(z) \geq Ae^{-\gamma z}\}.$$

For all $z \in \mathbb{R}$, one has:

$$v''_\infty(z) - cv'_\infty(z) \geq \overline{A} e^{-\gamma(z-h)}.$$

Integrating on (z, ∞), one gets

$$-v'_\infty(z) + cv_\infty(z) = -\big(v_\infty(z)e^{-cz}\big)' e^{cz} \geq \frac{\overline{A}}{\gamma} e^{-\gamma(z-h)}.$$

Integrating one more time gives

$$v_\infty(z) \geq \frac{\overline{A} e^{\gamma h}}{\gamma(\gamma + c)} e^{-\gamma z}.$$

The definition of $\overline{A}$ thus yields

$$\frac{e^{\gamma h}}{\gamma(\gamma+c)} \leq 1.$$

It now follows from Gomez and Trofimchuk's result [11] that, as this inequality implies the existence of a solution $\gamma > 0$ of $\gamma^2 + \gamma c = e^{\gamma h}$, there exists an increasing travelling wave solution to (1.1)-(1.2) . $\square$

5 Convergence to a wave-train

The aim of this section is to prove Theorem 2.3. Through all this section we will consider a given

travelling wave u of speed c which does not converges to 1 at $+\infty$

(otherwise Theorem 2.3 is obvious).

5.1 Estimation of the width of the oscillations

Lemma 5.1 *There exist two infinite sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{R}}$ such that for all $n \in \mathbb{N}$:*

$$u < 1 \text{ in } (-\infty, x_0) \text{ and in } (y_n, x_{n+1}) \quad \text{and} \quad u > 1 \text{ in } (x_n, y_n).$$

Proof. First, if $u \leq 1$ over $\mathbb{R}$, then $u'' - cu' \leq 0$ over $\mathbb{R}$ and the maximum principle yields that u cannot reach any local minimum over $\mathbb{R}$. Hence it would be a nondecreasing function which admits a limit ℓ as $z \to +\infty$, and (1.5) would yield that $\ell(1-\ell) = 0$. As u is nondecreasing and positive, $\ell = 1$, which would contradict the fact that u does not converge to 1 as $z \to +\infty$.

Hence u crosses 1 at least one time. We have thus constructed x_0. Let prove that $u'(x_0) > 0$ so that x_0 is isolated. Clearly $u'(x_0) \geq 0$ since $u(z) < 1 = u(x_0)$ for all $z < x_0$. If $u'(x_0) = 0$, then as $u(x_0 - h) < 1$,

$$-u''(x_0) = -u''(x_0) + cu'(x_0) = u(x_0)\big(1 - u(x_0 - h)\big) > 0$$

and thus x_0 is a local maximizer. Moreover, $u \leq 1$ over $\mathbb{R}$ is impossible and thus u admits a first local minimizer $\underline{z} > x_0$, with $u'(\underline{z}) = 0$. But as $u(z) \leq 1$ or all $z \leq \underline{z}$, one has $-u'' + cu' \geq 0$ over $(-\infty, \underline{z} + h)$. Hence, the strong maximum principle applies and gives $u \equiv u(\underline{z})$ in $(-\infty, \underline{z} + h)$, a contradiction since $u(-\infty) = 0$ and $u(x_0) = 1$. We have thus proved by contradiction that $u'(x_0) > 0$, which implies in particular that $u > 1$ in the right neighbourhood of x_0.

Now if $u(z) > 1$ for all $z > x_0$, then $u''(z) - cu'(z) \geq 0$ for all $z > x_0 + h$ and thus u could not admit a local maximum over $(x_0 + h, \infty)$ and would be monotone. As u is bounded, it would converge to a limit, which is necessarily 1 with the same arguments as above. This contradicts our hypothesis. We could thus construct y_0.

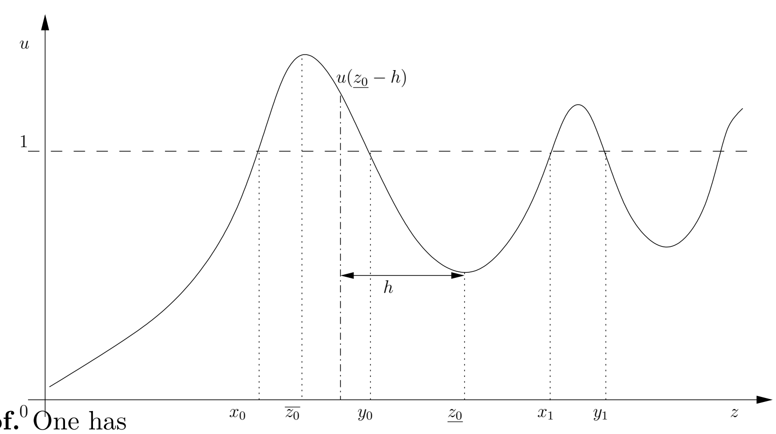
Assume that $u'(y_0) = 0$ by contradiction. Then as $u(z) > 1 = u(y_0)$ for all $z \in (x_0, y_0)$, one has $u''(y_0) \geq 0$ and thus equation (1.5) gives $u(y_0 - h) \geq 1$, that is, $y_0 - h \geq x_0$. First, if $u(y_0 - h) = 1$, then $u(z-h) \leq 1$ for all $z \leq y_0$ ans thus $-u'' + cu' = u\big(1 - u(z-h)\big) \geq 0$ over $(-\infty, y_0 + h)$. As $u(z) > 1 = u(y_0)$ for all $z \in (x_0, y_0)$, the Hopf lemma applies and gives $u'(y_0) < 0$, a contradiction. Second, if $u(y_0 - h) > 1$, that is, $y_0 - h > x_0$, then $u''(y_0) > 0$ and thus y_0 is a local minimizer. If u does not admit any local maximizer $\overline{z} > y_0$, then u would be nondecreasing and, as it is bounded, it would converge to a limit which would be strictly larger than 1, a contradiction. Hence such a $\overline{z}$ exists. Moreover, as $u(z) \geq 1$ for all $z \in (x_0, \overline{z})$, one has $-u'' + cu' \leq 0$ over $(x_0 + h, \overline{z} + h))$. As $\overline{z} > y_0 > x_0 + h$, $\overline{z}$ is an interior maximizer and the strong maximum principle gives $u \equiv u(\overline{z})$ over $(x_0 + h, \overline{z} + h))$, which is a contradiction since $u''(y_0) > 0$. Hence $u'(y_0) = 0$ is impossible and thus $u'(y_0) < 0$, which implies in particular that y_0 is isolated.

We could then continue the construction of the sequences by iteration with similar monotonicity arguments. □

Proposition 5.2 *One has*

$$\forall n \in \mathbb{R}, \quad x_{n+1} - x_n > h, \quad y_{n+1} - y_n > h.$$

Moreover, letting $\overline{z}_n \in (x_n, y_n)$ *and* $\underline{z}_n \in (y_n, x_{n+1})$ *such that* $u(\overline{z}_n) = \max_{(x_n, y_n)} u$ *and* $u(\underline{z}_n) = \min_{(y_n, x_{n+1})} u$*, then* u *is nondecreasing in* $[\underline{z}_n, \overline{z}_n]$ *and nonincreasing in* $[\overline{z}_n, \underline{z}_{n+1}]$ *for all* $n \in \mathbb{N}$*.*



Proof. One has

$$0 \geq -u''(\underline{z}_0) + cu'(\underline{z}_0) = u(\underline{z}_0)\big(1 - u(\underline{z}_0 - h)\big).$$

Thus $u(\underline{z}_0 - h) \geq 1$, which necessarily implies $\underline{z}_0 - h \in [x_0, y_0]$ by definition of x_0, y_0. Hence, $x_1 - x_0 > \underline{z}_0 - \big(\underline{z}_0 - h\big) = h$.

Next, as u reaches a local maximum at $\overline{z}_1$, one has $u(\overline{z}_1 - h) \leq 1$. Thus $\overline{z}_1 - h \leq x_0$ or $\overline{z}_1 - h \in [y_0, x_1]$. But as $x_1 - x_0 > h$, one has $\underline{z}_1 - h >$

$x_1 - h > x_0 + h - h = x_0$. We thus conclude that $\overline{z}_1 - h \in [y_0, x_1]$, and thus $y_1 - y_0 > \overline{z}_1 - (\overline{z}_1 - h) = h$.

The conclusion of the first part of the Proposition follows from similar arguments by iteration.

Assume that u is not nonincreasing in $[\overline{z}_n, \underline{z}_{n+1}]$. Then u admits a local minimizer z_* and a local maximizer z^*, with $\overline{z}_n < z_* < z^* < \underline{z}_{n+1}$. One gets $u(z_* - h) \geq 1$ and $u(z^* - h) \leq 1$ from (1.5). As $\overline{z}_n \geq y_{n-1} + h$ for all n from the first part of the Lemma, $u < 1$ in (y_{n-1}, x_n) and $u > 1$ in (x_n, y_n), it necessarily follows that $z_* - h \geq x_n$ and $z^* - h \leq x_n$. As $z_* < z^*$, this gives a contradiction. Hence u is nonincreasing in $[\overline{z}_n, \underline{z}_{n+1}]$. The monotonicity on $[\underline{z}_n, \overline{z}_n]$ follows from similar arguments. □

Lemma 5.3 *Define $(x_n)_n$ as in Lemma 5.1 and let $(\overline{z}_n)_n$ and $(\underline{z}_n)_n$ such that $u(\overline{z}_n) = \max_{(x_n, y_n)} u$ and $u(\underline{z}_n) = \min_{(y_n, x_{n+1})} u$. Then $\big(u(\overline{z}_n)\big)_{n\in\mathbb{N}}$ is nonincreasing and $\big(u(\underline{z}_n)\big)_{n\in\mathbb{N}}$ is nondecreasing.*

Proof. We have already observed in Lemma 4.7 that the curve $\mathcal{C} := \{\big(u(z), u'(z)\big) \in \mathbb{R}^2,\ z \in \mathbb{R}\}$ does not self-intersect. It is thus clear from Figure 4.3 that $\big(u(\overline{z}_n)\big)_n$ is nonincreasing and $\big(u(\underline{z}_n)\big)_{n\in\mathbb{N}}$ is nondecreasing. This could be proved rigorously iterating the arguments used in the proof of Lemma 4.8. □

Let

$$M := \lim_{n\to+\infty} u(\overline{z}_n) \quad \text{and} \quad m := \lim_{n\to+\infty} u(\underline{z}_n). \tag{5.20}$$

Lemma 5.4 *Assume that u does not converge to 1 at $+\infty$. Then $m < 1 < M$.*

Proof. Lemma 5.3 ensures that $u(\overline{z}_n) = \min_{(x_n, x_{n+1})} u = \min_{(x_n, +\infty)} u$. In other words, $u(z) \leq u(\overline{z}_n)$ for all $z \geq x_n$. Similarly, $u(z) \geq u(\underline{z}_n)$ for all $z \geq x_n$. Note that as $x_{n+1} - x_n \geq h$ for all n by Proposition 5.2, one has $\lim_{n\to+\infty} x_n = +\infty$. Hence, one has $\liminf_{z\to+\infty} u(z) = m$ and $\limsup_{z\to+\infty} u(z) = M$ and thus as u does not converge to 1, one has $m < 1$ or $M > 1$. Assume first that $m < 1$ and assume by contradiction that $M = 1$.

The Schauder elliptic estimates yields that one can assume the convergence, up to extraction, of the sequence $\big(u(\cdot + \underline{z}_n)\big)_n$. Let w its limit. One has $w(0) = m < 1$ and

$$-w''(z) + cw'(z) = w(z)\big(1 - w(z-h)\big) \quad \text{in } \mathbb{R}.$$

Take $z \in \mathbb{R}$ and let $k_n = \sup\{k \in \mathbb{N},\ x_k \leq z + \underline{z}_n\}$. The sequence $(k_n)_n$ is strictly increasing and thus $\lim_{n\to+\infty} k_n = +\infty$. As $u(z + \underline{z}_n) \leq u(\overline{z}_{k_n})$ since $z + \underline{z}_n \geq x_{k_n}$, one eventually gets $w(z) \leq M = 1$ by letting $n \to +\infty$. As this is true for any $z \in \mathbb{R}$, one gets $-w'' + cw' \leq 0$ over $\mathbb{R}$. But one can prove through similar arguments that $w \geq m$ over $\mathbb{R}$ and as $w(0) = m$, w reaches a minimum at 0 and equation (1.1) yields $w(-h) \geq 1$, that is, $w(-h) = 1$. Thus w reaches a global maximum at $-h$ and the strong maximum principle would give $w \equiv 1$, a contradiction since $w(0) = m < 1$. Hence $M > 1$ if $m < 1$. Similarly, one can prove that $m < 1$ implies $M > 1$. As we have already proved that either $m < 1$ or $M > 1$, we conclude that $m < 1 < M$. □

Lemma 5.5 *Assume that u does not converge to 1 at $+\infty$. Let $\sigma \in (m, M)$ and consider an increasing sequence $(z_k)_k$ such that $u(z_k) = \sigma$ and $u'(z_k) \leq 0$ (resp. > 0) for all k and $\lim_{k\to+\infty} z_k = +\infty$. Then the sequence $\big(u'(z_k)\big)_k$ is increasing (resp. decreasing) and converges.*

Proof. This could be proved through phase-plane analysis. As the arguments are very similar to the proof of Lemma 5.3, we leave the details to the reader. □

Proposition 5.6 *Assume that u does not converge to 1 at $+\infty$. Consider a function $w \in \mathcal{C}^2(\mathbb{R})$ associated with an extraction $(\overline{z}_{\varphi(n)})_n$ such that $u(z + \overline{z}_{\varphi(n)}) \to w(z)$ as $n \to +\infty$ in $\mathcal{C}^2_{loc}(\mathbb{R})$. Then there exist four increasing families $(X_n)_{n\in\mathbb{Z}}$, $(Y_n)_{n\in\mathbb{Z}}$, $(\underline{Z}_n)_{n\in\mathbb{Z}}$ and $(\overline{Z}_n)_{n\in\mathbb{Z}}$ such that $\overline{Z}_0 = 0$, $\overline{Z}_n \geq \underline{Z}_n + h$ for all $n \in \mathbb{Z}$ and*

$$\begin{array}{l} X_n < \overline{Z}_n < Y_n < \underline{Z}_n, \\ w' < 0 \quad \text{in } (\overline{Z}_n, \underline{Z}_n), \\ w' > 0 \quad \text{in } (\underline{Z}_n, \overline{Z}_{n+1}), \\ w(\overline{Z}_n) = M, \quad w(\underline{Z}_n) = m. \end{array} \tag{5.21}$$

Note that the Schauder elliptic estimates ensures that such an extraction $(\overline{z}_{\varphi(n)})_n$ and such a limit w do exist.

Proof. The function w satisfies:

$$-w''(z) + cw'(z) = w(z)\big(1 - w(z-h)\big) \quad \text{over } \mathbb{R} \tag{5.22}$$

and $w(0) = M = \max_{\mathbb{R}} w > 1$. Moreover, as $\liminf_{z\to+\infty} u(z) > 0$, one has $\inf_{\mathbb{R}} w > 0$.

Assume by contradiction that $w \geq 1$ over $(0, \infty)$. Then as $-w''+cw' \leq 0$ over (h, ∞), w could not attains a local maximum and thus w would be monotone in (h, ∞). It would then follow from Proposition 4.10 that there exists an increasing travelling wave u. Proposition 3.1 applies since $\lim_{z\to-\infty} u(z) = 0 < \inf_{\mathbb{R}} w$,

$$\limsup_{z\to+\infty} w(z)/u(z) = \limsup_{z\to+\infty} w(z) \geq 1 \geq \liminf_{z\to+\infty} w(z) = \liminf_{z\to+\infty} w(z)/u(z).$$

Hence, the graphs of u and w do not intersect in the phase-plane. On the other hand, if $w \geq 1$ over $\mathbb{R}$, then w would be nonincreasing over $\mathbb{R}$ and thus as it is bounded (since u is bounded), it would admit a limit at $-\infty$, which would necessarily be 1 due to equation (5.22), a contradiction to $w(0) = M > 1$. Thus there exists $z_- < 0$ such that $w(z_-) < 1$, and one can assume that $w'(z_-) = 0$. But then the curve $\{\big(w(z), w'(z)\big), z \in [z_-, \infty)\}$ necessarily crosses $\{\big(u(z), u'(z)\big), z \in \mathbb{R}\}$, a contradiction. Hence there exists $X > 0$ such that $w(X) < 1$.

It follows that $u(\overline{z}_{\varphi(n)} + X) < 1$ when n is large enough. The properties of u (see Lemma 5.1 and Proposition 5.2) yield that $X + \overline{z}_{\varphi(n)} > y_{\varphi(n)}$. Hence $(y_{\varphi(n)} - \overline{z}_{\varphi(n)})_n$ is a bounded and positive sequence and one can extract a subsequence converging to a limit Y_0. As $u(y_{\varphi(n)}) = 1$ and u is nonincreasing on $(\overline{z}_{\varphi(n)}, y_{\varphi(n)})$, one has

$$w(Y_0) = 1 \quad \text{and} \quad w \text{ is nonincreasing over } (0, Y_0).$$

Similarly, one can prove that there exists $X_0 < 0$ such that

$$w(X_0) = 1 \quad \text{and} \quad w \text{ is nondecreasing over } (X_0, 0).$$

Now if w stays below 1 over (Y_0, ∞), as $-w'' + cw' \geq 0$ over $(Y_0 + h, \infty)$ and thus w would be monotone and its only possible limit would be 1, a contradiction. Hence there exists $Z > Y_0$ such that $W(Z) > 1$ and thus $u(\overline{z}_{\varphi(n)} + Z) > 1$ when n is large enough. As Y_0 is an accumulation point of $(y_{\varphi(n)} - \overline{z_{\varphi(n)}})_n$ and $Z > Y_0$, one has $\overline{z}_n + Z > x_{n+1}$. Thus the sequences $(\underline{z}_{\varphi(n)+1} - \overline{z}_n)_n$ and $(x_{n+1} - \overline{z}_n)_n$ converge to some accumulation points $\underline{Z}_0$ and X_1. As $m = \lim_{n\to+\infty} u(\underline{z}_n) = m$, u is nonincreasing over $(y_n, \underline{z}_n)$ and nondecreasing over $(\underline{z}_n, x_{n+1})$, it follows that

$$w \text{ is nonincreasing over } (Y_0, \underline{Z}_0), \quad w \text{ is nondecreasing over } (\underline{Z}_0, X_1) \text{ and } w(\underline{Z}_0) = m.$$

Moreover, as $\underline{z}_n - h \in (x_n, y_n)$ for all n by Proposition 5.2, one has

$$X_0 \leq \underline{Z}_0 - h \leq Y_0. \tag{5.23}$$

Going on the construction on $(0, \infty)$ and similarly on $(-\infty, 0)$, one gets 4 increasing families $(X_n)_{n\in\mathbb{Z}}$, $(Y_n)_{n\in\mathbb{Z}}$, $(\underline{Z}_n)_{n\in\mathbb{Z}}$ and $(\overline{Z}_n)_{n\in\mathbb{Z}}$, with $\overline{Z}_0 = 0$, such that

$$\begin{array}{l} X_n < \overline{Z}_n < Y_n < \underline{Z}_n, \\ w \text{ is nonincreasing over } (\overline{Z_n}, \underline{Z}_n), \\ w \text{ is nondecreasing over } (\underline{Z}_n, \overline{Z}_{n+1}), \\ w(\overline{Z}_n) = M, \quad w(\underline{Z}_n) = m. \end{array}$$

Let now prove that $w' < 0$ in $(\overline{Z_n}, \underline{Z}_n)$. Assume by contradiction that there exists $\zeta \in (\overline{Z_n}, \underline{Z}_n)$ such that $w'(z_n) = 0$. As w is nonincreasing in $(\overline{Z_n}, \underline{Z}_n)$, one has $w' \leq 0$ and thus ζ is a local maximizer for w', which implies in particular $w''(\zeta) = 0$. It follows from (5.22) that $w(\zeta - h) = 1$. Thus Proposition 5.2 and the monotonicity of w give $w(z) \geq 1$ for all $z \in (\zeta, Y_n)$. We thus derive from (5.22):

$$-w'' + cw' \leq 0 \quad \text{in } (\zeta + h, Y_n + h).$$

The Gronwall inequality yields $w'(z) \geq w'(\zeta)e^{c(z-\zeta)} = 0$ for all $z \in (\zeta + h, Y_n + h)$. But as w is nonincreasing in $(\overline{Z}_n, \underline{Z}_n)$ and $\underline{Z}_n \leq Y_n + h$, we get $w' \equiv 0$ on $(\zeta, \underline{Z}_n)$. Thus $w(\zeta) = w(\underline{Z}_n) = m$, in particular, $w'(\zeta) = 0$. On the other hand, as $w(\zeta - h) = 1$, one has $w \leq 1$ in $(Y_{n-1}, \zeta - h)$ and thus

$$-w'' + cw' \geq 0 \quad \text{in } (Y_{n-1} + h, \zeta).$$

As $w(\zeta) = m = \min_{\mathbb{R}} w$, it follows from the Hopf Lemma that $w'(\zeta) < 0$, a contradiction. Hence, $w' < 0$ in $(\overline{Z_n}, \underline{Z}_n)$ and similarly one can prove that $w' > 0$ in $(\underline{Z}_n, \overline{Z}_{n+1})$. □

As w is invertible over $(0, \underline{Z}_0)$, one can define

$$\forall \eta \in (m, M), \quad F_-(\eta) := w'\big(w^{[-1]}(\eta)\big), \tag{5.24}$$

where $w^{[-1]}$ is the inverse of w restricted to $(0, \underline{Z}_0)$. The local inversion theorem yields that F_- is of class $\mathcal{C}^1$ over (m, M). Similarly, define $F_+(\eta) := w'\big(w^{\langle -1\rangle}(\eta)\big)$ where $w^{\langle -1\rangle}$ is the inverse of w restricted to $(\underline{Z}_{-1}, 0)$.

Proposition 5.7 *Assume that u does not converge to 1 at $+\infty$. Consider a function $w^* \in \mathcal{C}^2(\mathbb{R})$ associated with an extraction $(\overline{z}_{\psi(k)})_k$ such that $u(z + \overline{z}_{\psi(k)}) \to w^*(z)$ as $k \to +\infty$ in $\mathcal{C}^2_{loc}(\mathbb{R})$. Associate with w^* four increasing families $(X^*_n)_{n\in\mathbb{Z}}$, $(Y^*_n)_{n\in\mathbb{Z}}$, $(\underline{Z}^*_n)_{n\in\mathbb{Z}}$ and $(\overline{Z}^*_n)_{n\in\mathbb{Z}}$ through Proposition 5.7. Then for all n, w^* is a solution of*

$$\begin{array}{lll} \frac{d}{dz}w^* = F_-(w^*) & \text{in } (\overline{Z}^*_n, \underline{Z}^*_n), & w^*(Y^*_n) = 1, \\ \frac{d}{dz}w^* = F_+(w^*) & \text{in } (\underline{Z}^*_n, \overline{Z}^*_{n+1}), & w^*(X^*_n) = 1. \end{array} \tag{5.25}$$

Proof. Assume that $n = 0$, the proof on the other intervals being similar. We remind to the reader that $\overline{Z}^*_0 = 0$ by definition. Let $\xi \in (\overline{Z}^*_0, \underline{Z}^*_0)$ and $\sigma := w^*(\xi) \in (m, M)$. As $\sigma \in (m, M)$, Lemma 5.3 yields that for all k, there exists $z_k \in (\overline{z}_k, \underline{z}_k)$ such that $u(z_k) = \sigma$. As $(w^*)'(\xi) < 0$ since $\xi \in (\overline{Z}^*_0, \underline{Z}^*_0)$, Lemma 5.5 applies and gives that $\big(u'(z_k)\big)_k$ is increasing and converges. Let

$$D = \lim_{k\to+\infty} u'(z_k).$$

Next, as $\lim_{k\to+\infty} u(\xi + \delta + \overline{z}_{\psi(k)}) = w(\xi + \delta)) < \sigma$ for δ small enough since $w'(\xi) < 0$ and as u is nonincreasing over $(\underline{z}_k, \overline{z}_k)$, one gets $\xi + \delta + \overline{z}_{\psi(k)} > z_{\psi(k)}$ when k is large enough. In other words $\limsup_{k\to+\infty} \big(z_{\psi(k)} - \overline{z}_{\psi(k)}\big) \leq \xi$. Similarly, one can prove that $\liminf_{k\to+\infty} \big(z_{\psi(k)} - \overline{z}_{\psi(k)}\big) \geq \xi$ and thus

$$\lim_{k\to+\infty} \big(z_{\psi(k)} - \overline{z}_{\psi(k)}\big) = \xi.$$

It follows that

$$D = \lim_{k\to+\infty} u'\big(z_{\psi(k)} - \overline{z}_{\psi(k)} + \overline{z}_{\psi(k)}\big) = \lim_{k\to+\infty} u'\big(\xi + \overline{z}_{\psi(k)}\big) = \big(w^*\big)'(\xi).$$

As this does not depend on the extraction ψ since the full sequence $\big(u'(z_k)\big)_k$ converges to D, we also have $D = w'(\zeta)$, where η is the unique element of $(0, \underline{Z}_0)$ such that $w(\eta) = \sigma$. In other words, $\eta = w^{[-1]}(\sigma)$ and thus $D = w'\big(w^{[-1]}(\sigma)\big) = F_-(\sigma)$. As $D = \big(w^*\big)'(\xi)$ and $\sigma = w^*(\xi)$, we eventually get

$$\big(w^*\big)'(\xi) = F_-\big(w^*(\xi)\big) \quad \text{ for all } \xi \in (0, \underline{Z}^*_0).$$

The proof on $(\underline{Z}^*_0, \overline{Z}^*_1)$ is similar. $\square$

Corollary 5.8 *The function w is $L-$periodic, with $L = X_1 - X_0$.*

Proof. The functions w and $\widetilde{w} := w(\cdot + L)$ are both solutions of $\frac{d}{dz}w = F_+(w)$ on I, $w(X_0) = 1$, with $I = \big(\max\{\underline{Z}_1 - L, \underline{Z}_0\}\ ,\ \min\{\overline{Z}_1 - L, \overline{Z}_0\}\big)$, and as F_+ is of class $\mathcal{C}^1$, the Cauchy-Lipschitz theorem holds and give $\widetilde{w} \equiv w$ over I. It easily follows that $\underline{Z}_1 = \underline{Z}_0 + L$ and $\overline{Z_1} = \overline{Z}_0 + L$, hence $I = (\underline{Z}_0, \overline{Z}_0)$. In particular, $w(\cdot + L) \equiv w$ over I.

Similarly, one can prove using F_- that $w(\cdot + L') \equiv w$ over $\big(\overline{Z}_0, \underline{Z}_0\big)$, where $L' = Y_1 - Y_0$, and that $\underline{Z}_1 = \underline{Z}_0 + L'$ and $\overline{Z}_1 = \overline{Z}_0 + L'$, which implies $L' = L$.

Hence, $w \equiv w(\cdot + L)$ in $(\overline{Z}_0, \overline{Z}_1)$. As $\overline{Z}_1 - \overline{Z}_0 \geq h$, we conclude from the well-posedness of delayed differential equations that $w \equiv w(\cdot + L)$ in $\mathbb{R}$. $\square$

Corollary 5.9 *One has* $w^* \equiv w$.

Proof. This immediately follows from the Cauchy-Lipschitz theorem together with Proposition 5.7. $\square$

Proposition 5.10 *Assume that u is a travelling wave of speed c. Then there exists a wavetrain w such that*

$$\lim_{z\to+\infty} \big(u(z) - w(z)\big) = 0.$$

Proof. If u converges to 1 at $+\infty$, then the result is obvious. Otherwise, the results of this section holds and thus we conclude that the full sequence $\big(u(\cdot + \overline{z}_n\big)_n$ converges to w in $\mathcal{C}^2_{loc}(\mathbb{R})$, where w is a periodic solution of (1.1), that is, w is a wavetrain. As this convergence is uniform and as $\big(\overline{z}_{n+1} - \overline{z}_n\big)_n$ is a bounded sequence (since it clearly converges to $\overline{Z}_1$), we indeed get the convergence of the full function $u = u(z)$ to $w(z)$ as $z \to +\infty$. $\square$

6 Uniqueness of the attracting wave-train

Proof of Theorem 2.3. We know from Proposition 5.10 that $\lim_{z\to+\infty} \big(u(z) - w(z)\big) = 0$, where w is a wavetrain. Consider any wavetrain $\widetilde{w}$. In order to prove that w is a maximal wavetrain, we need to prove that $\widetilde{\mathcal{S}} \subset \mathcal{S} \cup \mathcal{S}^{int}$, where $\mathcal{S}$ is the Jordan curve associated with w in the phase plane, $\mathcal{S}^{int}$ is its interior, and $\widetilde{\mathcal{S}}$ is the curve associated with $\widetilde{w}$ in the phase plane. The case $\widetilde{w} \equiv 1$ is trivial and we thus assume $\widetilde{w} \not\equiv 1$.

As a first step we will prove that Proposition 3.1 applies to u and some translation of $\widetilde{w}$. The hypothesis $u < \min \widetilde{w}$ over $(-\infty, A)$ is clearly satisfied since $u(-\infty) = 0$ while $\widetilde{w}$ is periodic and positive. Take $\underline{Z} \in \mathbb{R}$ so that $\min_{\mathbb{R}} w = w(\underline{Z})$ and translate $\widetilde{w}$ by $a \in \mathbb{R}$ so that $\widetilde{w}(a + \underline{Z}) = \max \widetilde{w}$. We already know that $w(\underline{Z}) \leq 1 \leq \widetilde{w}(\underline{Z} + a)$, from which we get

$$\liminf_{z\to+\infty} u(z)/\widetilde{w}(z + a) = \inf_{\mathbb{R}} \frac{w}{\widetilde{w}(\cdot + a)} = \frac{w(\underline{Z})}{\widetilde{w}(\underline{Z} + a)} \leq 1.$$

Moreover, equation (1.1) yields $w(\underline{Z} - h) \geq 1 \geq \widetilde{w}(\underline{Z} + a - h)$, which gives

$$\limsup_{z\to+\infty} \frac{u(z)}{\widetilde{w}(z + a)} = \sup_{\mathbb{R}} \frac{w}{\widetilde{w}(\cdot + a)} \geq \frac{w(\underline{Z} - h)}{\widetilde{w}(\underline{Z} + a - h)} \geq 1.$$

Hence, the hypotheses of Proposition 3.1 are satisfied and thus the curves associated with u and $\widetilde{w}(\cdot + a)$ (and thus $\widetilde{w}$) do not intersect in the phase plane.

As a second step, assume that there exists $z_e \in \mathbb{R}$ such that $X_e := \big(\widetilde{w}(z_e), \widetilde{w}'(z_e)\big) \in \mathcal{S}^{ext}$. For all $n \in \mathbb{N}$, let us define

$$J_n := \{\big(u(z), u'(z)\big), \underline{z}_{n-1} < z < \underline{z}_n\} \cup \Big([u(\underline{z}_{n-1}), u(\underline{z}_n)] \times \{0\}\Big)$$

wherein we have set by convention $\underline{z}_{-1} = -\infty$. Note that for each $n \in \mathbb{N}$, J_n defines a Jordan curve by Proposition 5.2 and the fact that the curve associated with u does not self-intersect. Moreover, Lemma 5.3 yields the inclusion between the interiors: $J_n^{int} \subset J_{n-1}^{int}$ for all n.

Assume first that $X_e = \big(\widetilde{w}(z_e), \widetilde{w}'(z_e)\big) \in J_0^{ext}$. Let $z_* > z_e$ the smallest number such that w reaches a local minimum at z_*, with $\widetilde{w}(z_*) < 1$. This quantity is well-defined since $\min_{\mathbb{R}} \widetilde{w} < 1$, otherwise, the same arguments as above would show that $\widetilde{w}$ is nonincreasing, and thus at it is periodic it would be constant equal to 1, a contradiction. As $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ do not self-intersect and $\widetilde{w}'(z_*) = 0$, one has $\widetilde{w}(z_*) \in \big[0, u(\underline{z}_0)\big)$. As $\widetilde{w}$ is periodic (with period $\widetilde{T}$), one has $\big(\widetilde{w}(z_e + \widetilde{T}), \widetilde{w}'(z_e + \widetilde{T})\big) = X_e$, which is impossible: one cannot draw a curve in the phase-plane going from the interval $\big[0, u(\underline{z}_0)\big) \times \{0\}$ to X_e without crossing $\{\big(u(z), u'(z)\big), -\infty < z < \underline{z}_0\}$.

Next, assume that $X_e = \big(\widetilde{w}(z_e), \widetilde{w}'(z_e)\big) \in J_0 \cup J_0^{int}$. If $X_e \in J_0$, as $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ do not intersect the only possibility is that $X_e \in [0, u(\underline{z}_0)) \times \{0\}$, which has been ruled out in the previous case. Thus $X_e \in J_0^{int}$. As u converges to w and as $X_e \notin \mathcal{S} \cup \mathcal{S}^{int}$, there exists $n \in \mathbb{N} \setminus \{0\}$ such that $X_e \in J_n^{ext} \setminus J_{n-1}^{ext}$. Defining z_* the first local minimizer of $\widetilde{w}$ larger than z_e and such that $\widetilde{w}(z_*) < 1$, one gets as above: $\widetilde{w}(z_*) \in \big(u(\underline{z}_{n-1}), u(\underline{z}_n)\big)$. This gives a contradiction since one cannot draw a curve in the phase-plane going from the interval $\big(u(\underline{z}_{n-1}), u(\underline{z}_n)\big) \times \{0\}$ to X_e without crossing $\{\big(u(z), u'(z)\big), \underline{z}_{n-1} < z < \underline{z}_n\}$. This gives the final contradiction and concludes the proof. $\square$

We end this section by proving the strict maximality of nontrivial maximal wavetrain stated in Remark 2.4. To do so we shall make use of the discrete Lyapunov functional in order to give a short justification of this statement. Let $u \equiv u(z)$ be a travelling wave non-converging to one and $w \equiv w(z)$ be its limit maximal wave train. Let $\widetilde{w}$ be an other wavetrain with period $\widetilde{L}$. Assume that $\mathcal{S} \cap \widetilde{\mathcal{S}} \neq \emptyset$ and let us prove that there exists $\tau \in \mathbb{R}$ such that $w(z + \tau) \equiv \widetilde{w}(z)$.

Using the same arguments and notations as in Remark 3.2 one has:

$$V\left[x_z^n\right] = 2, \;\; \forall n \geq 0\, z \in \mathbb{R},$$

wherein we have set

$$x_z^n(\theta) = \begin{cases} W^n(z+\theta) \text{ for } \theta \in [-h, 0] \\ \left(W^n\right)'(z) \text{ if } \theta = 1 \end{cases} \qquad \text{and } W^n(z) = 1 - \frac{u\left(z + n\widetilde{L}\right)}{\widetilde{w}(z)}.$$

Next due to Theorem 2.3, possibly along a subsequence, there exists $\tau \in \mathbb{R}$ such that

$$\lim_{n\to\infty} W^n(z) = W(z) := 1 - \frac{w\,(z+\tau)}{\widetilde{w}(z)}, \text{ locally uniformly for } z \in \mathbb{R},$$

Now let us set for each $z \in \mathbb{R}$: $x_z(\theta) = \begin{cases} W(z+\theta) \text{ for } \theta \in [-h,0] \\ (W)'(z) \text{ if } \theta = 1 \end{cases}$. We aim to show that $x_z \equiv 0$, that is $w\,(z+\tau) \equiv \widetilde{w}(z)$. Assume by contradiction that $x_z \neq 0$ for some $z \in \mathbb{R}$, hence for all $z \in \mathbb{R}$ (due to periodicity). Now since $\mathcal{S} \cap \widetilde{\mathcal{S}} \neq \emptyset$, one obtains (see Remark 3.2) that $V\,[x_z] = 0$, $\forall z \in \mathbb{R}$. Thus W is monotone on $\mathbb{R}$ and satisfies:

$$W''(z) - \left(c - \frac{2\widetilde{w}'(z)}{\widetilde{w}}\right) W'(z) = \frac{w(z+\tau)\widetilde{w}(z-h)}{\widetilde{w}(z)} W(z-h).$$

Therefore $W(z) \to 0$ as $z \to \pm\infty$ and it follows that $W(z) \equiv 0$, a contradiction. This completes the proof of the statement.

References

[1] M. Alfaro; J. Coville. Rapid traveling waves in the nonlocal Fisher equation connect two unstable states. *Appl. Math. Lett.*, 25:2095–2099, 2012.

[2] B. Bánhelyi; T. Csendes; T. Krisztin; A. Neumaier. Global attractivity of the zero solution for Wright's equation. *preprint*, 2013.

[3] H. Berestycki; G. Nadin; B. Perthame; L. Ryzhik. The non-local Fisher-KPP equation: traveling waves and steady states. *Nonlinearity*, 22(12):2813–2844, 2009.

[4] O. Bonnefon; J. Garnier; F. Hamel; L. Roques. Inside dynamics of delayed travelling waves. *Math. Mod. Nat. Phen.*, 2013.

[5] N. Britton. Spatial structures and periodic traveling waves in an integro-differential reaction-diffusion population model. *SIAM J. Appl. Math.*, 50(6):1663–1688, 1990.

[6] A. Ducrot. Travelling waves for a size and space structured model in population dynamics: Point to sustained oscillating solution connections. *J. Differential Equations*, 250(1): 410–449, 2011.

[7] J. Fang; X.-Q. Zhao. Monotone wavefronts of the nonlocal Fisher-KPP equation. *Nonlinearity*, 24(11):3043–3054, 2011.

[8] T. Faria; S. Trofimchuk. Nonmonotone travelling waves in a single species reactionâĂŞdiffusion equation with delay *J. Differential Equations*, 228(1): 357–376, 2006.

[9] R. A. Fisher. The advance of advantageous genes. *Ann. Eugenics*, 7:335–369, 1937.

[10] S. Genieys; V. Volpert; P. Auger. Pattern and waves for a model in population dynamics with nonlocal consumption of resources. *Math. Modelling Nat. Phenom.*, 1:65–82, 2006.

[11] A. Gomez; S. Trofimchuk. Monotone traveling wavefronts of the KPP-Fisher delayed equation. *J. Differential Equations*, 250(4):1767–1787, 2011.

[12] S. Gourley. Traveling front solutions of a nonlocal Fisher equation. *J. Math. Biol.*, 41(3):272–284, 2000.

[13] F. Hamel; L. Ryzhik. On the nonlocal Fisher-KPP equation: steady states, spreading speed and global bounds. *preprint*, 2013.

[14] K. Hasik; S. Trofimchuk. Slowly oscillating wavefronts of the KPP-Fisher delayed equation. *preprint*, 2013.

[15] K. Hasik; S. Trofimchuk. An extension of the Wright's 3/2-theorem for the KPP-Fisher delayed equation. *preprint*, 2013.

[16] A.N. Kolmogorov; I.G. Petrovsky; N.S. Piskunov. Etude de l équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique. *Bulletin Université d'Etat à Moscou (Bjul. Moskowskogo Gos. Univ.)*, 1–26, 1937.

[17] M.K. Kwong; C. Ou. Existence and nonexistence of monotone traveling waves for the delayed Fisher equation. *J. Differential Equations*, 249(3):728–745, 2010.

[18] H.J. Hupkes; S.M. Verduyn Lunel. Analysis of Newton's method to compute travelling wave solutions to lattice differential equations. *Technical Report, Mathematical Institute Leiden*, 2003.

[19] J. Mallet-Paret; G. Sell. The Poincaré-Bendixson theorem for monotone cyclic feedback systems with delay. *J. Differential Equations*, 125(2):441–489, 1996.

[20] J. Mallet-Paret; G. Sell. Systems of differential delay equations: Floquet multipliers and discrete Lyapunov functions. *J. Differential Equations,* 125(2):385–440, 1996.

[21] G. Nadin; B. Perthame; M. Tang. Can a traveling wave connect two unstable states? The case of the nonlocal Fisher equation. *C. R. Math. Acad. Sci. Paris*, 349(9-10):553–557, 2011.

[22] G. Nadin; L. Rossi; L. Ryzhik; B. Perthame. Wave-like Solutions for Nonlocal Reaction-diffusion Equations: a Toy Model. *Math. Mod. Nat. Phen.*, 8(03):33–41, 2013.

[23] E. Trofimchuk; V. Tkachenko; S. Trofimchuk. Slowly oscillating wave solutions of a single species reaction-diffusion equation with delay. *J. Differential Equations*, 245(08):2307–2332, 2008.

[24] A. Turing. The chemical basis of morphogenesis. *Philosophical Transactions of the Royal Society of London B*, 327:37–72, 1952.

[25] E. M. Wright. A nonlinear difference-differential equation. *J. Reine Angew. Math.*, 194:66-87, 1955.