STABLE MINIMAL HYPERSURFACES IN $\mathbb{R}^6$

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Abstract. We prove that, in $\mathbb{R}^6$, a complete, two-sided, stable minimal hypersurfaces is flat. The proof follows the strategy developed by Chodosh, Li, Minter and Stryker, and use the spectral volume estimate of Antonelli and Xu.

1. introduction

A minimal hypersurface M^n of $\mathbb{R}^{n+1}$ is a critical point of the n-volume functional. It is characterized by its vanishing mean curvature. If a unit normal vectorfield ν is defined along M and φ is a function with compact support on M, one can consider a deformation of M with initial speed $\varphi\nu$. The computation of the second derivative of the n-volume along this deformation at initial time gives

$$\int_M |\nabla\varphi|^2 - |A_M|^2\varphi^2$$

where A_M is the second fundamental form of M. So asking that this quantity is non negative for any φ means that M is a minimum at order 2 of the n-volume. Such a minimal hypersurface is called stable.

The stable Bernstein problem asks wether a complete stable minimal hypersurface is a flat affine hyperplane. We give a positive answer in the case $n = 5$.

Theorem 1.1. *Let $M^5 \looparrowright \mathbb{R}^6$ be an immersed, complete, connected, two-sided, stable minimal hypersurface. Then M is a Euclidean hyperplane.*

A particular class of stable minimal hypersurface is given by minimal graphs over $\mathbb{R}^n$. In [4], Bernstein proved that a minimal graph over $\mathbb{R}^2$ has to be a plane. In the sixties, the same question for higher dimensions was studied in a series of paper by Fleming [17], De Giorgi [14], Almgren [1] and Simons [26]. They proved that minimal graphs over $\mathbb{R}^n$ are planes if $n \leq 7$. For $n \geq 8$, Bombieri, De Giorgi and Giusti [5] were able to construct counter-examples and gave also in $\mathbb{R}^8$ an example of a stable minimal hypersurfaces that is not a hyperplane.

Concerning the stable Bernstein problem, the question was solved positively in $\mathbb{R}^3$ by Do Carmo and Peng [15], Fischer-Colbrie and Schoen [16] and Porogelov [22] in the early eighties. In higher dimension, Schoen, Simon and Yau [23, 24] were able to settle the stable Bernstein in $\mathbb{R}^{n+1}$, $n \leq 6$, under a Euclidean volume growth assumption (see also the recent work of Bellettini [3]).

Recently Chodosh and Li [9] were able to answer positively the stable Bernstein problem in $\mathbb{R}^4$. Later two alternative proofs came out: one by Catino, Mastrolia and Roncoroni [8] and one by Chodosh and Li [10]. Actually in [10], Chodosh and Li develop a second strategy to prove the result. Then, in a joint work Minter and Stryker [12], they were able to apply this strategy in the case of $\mathbb{R}^5$ to solve the stable Bernstein problem in this dimension as well.

As in [13, 12], it is well known that the solution to the stable Bernstein problem (Theorem 1.1) implies corollaries like curvature estimates for stable minimal immersions in 6-dimensional manifolds and characterization of finite Morse index minimal hypersurfaces in $\mathbb{R}^6$. For example, we have

Corollary 1.1. *Let (X^6, g) be a complete Riemannian manifold whose sectional curvature satisfies $|sec_g| \leq K$. Then any compact, two-sided, stable minimal immersion $M^5 \looparrowright X$ satisfies*

$$|A_M|(q) \min(1, d_M(q, \partial M)) \leq C(K)$$

for $q \in M$.

The proof of [13, Corollary 2.5] extends to dimension 6 to prove the above result.

The basic idea to prove Theorem 1.1 is to obtain a Euclidean growth estimate for the volume of M and then apply the work of Schoen, Simon and Yau. The strategy of Chodosh and Li is a way towards this estimate. We refer to [10, 12] for a good presentation of their ideas. Let us give some elements. Let M be a stable minimal hypersurface in $\mathbb{R}^{n+1}$ with induced metric g. Inspired by the work of Gulliver and Lawson [18], they consider the conformal metric $\tilde{g} = r^{-2} g$ where r is the Euclidean distance to 0 in $\mathbb{R}^{n+1}$. If M was a hyperplane passing through the origin $(M \setminus \{0\}, \tilde{g})$ would be isometric to the Euclidean product $\mathbb{S}^{n-1} \times \mathbb{R}$. In the general case, the idea of Chodosh and Li is that the stability assumption implies that the geometry of $(M \setminus \{0\}, \tilde{g})$ should look like $\mathbb{S}^{n-1} \times \mathbb{R}$. In [12], the authors consider the bi-Ricci curvature which is a certain combination of sectional curvatures. The bi-Ricci curvature was introduced by Shen and Ye in [25], already to study minimal surfaces (see precise definition in Section 2). Notice that on $\mathbb{S}^{n-1} \times \mathbb{R}$, the bi-Ricci curvature is lower bounded by $n - 2$. In [12], the authors prove that the stability of M implies a positive spectral lower bound for the bi-Ricci curvature of $(M \setminus \{0\}, \tilde{g})$. More precisely they prove that, on $(M \setminus \{0\}, \tilde{g})$, the operator $-\widetilde{\Delta} + (\widetilde{\mathrm{BRic}}_- - 1)$ is non-negative where $\widetilde{\mathrm{BRic}}_-$ is the punctual minimum of the bi-Ricci curvature of $\tilde{g}$. This should be understood as a weak formulation of the inequality $\widetilde{\mathrm{BRic}} \geq 1$.

The second step of the strategy consists in the construction of a μ-bubble in $(M \setminus \{0\}, \tilde{g})$ with a spectral lower bound for its Ricci curvature. In some sense, they identify in any sufficiently large part of $(M \setminus \{0\}, \tilde{g})$ a hypersurface that play the role of $\mathbb{S}^{n-1} \times \{t\}$ in $\mathbb{S}^{n-1} \times \mathbb{R}$.

The last step is to obtain an upper-bound for the volume of the μ-bubble. In [12], the authors obtain a Bishop-Gromov volume estimate under the spectral lower bound on the Ricci curvature. In their paper, the proof of this volume estimate was specific to dimension 3. Recently, Antonelli and Xu [2] have proved such a Bishop-Gromov estimate in any dimension.

Theorem 1.2 (Antonelli and Xu [2])**.** *Let (M^k, g) be a compact Riemannian k-manifold with $k \geq 3$ and let $0 \leq \gamma \leq \frac{k-1}{k-2}$ and $\lambda > 0$. Assume that there is a positive function $u \in C^\infty(M)$ such that, for any $(p, v) \in UM$,*

$$\gamma \Delta u(p) \leq (\mathrm{Ric}(v, v) - (k-1)\lambda) u(p)$$

Then $\mathrm{Vol}(M) \leq \lambda^{-k/2} \mathrm{Vol}(\mathbb{S}^k)$.

Once the $\tilde{g}$-volume of the μ-bubble is controlled, this gives an estimate of its volume in the original metric g and then control the growth of the volume of M tanks to an isoperimetric inequality due to Michael and Simon [20] and Brendle [6].

In the present paper, we also follow the above strategy of [12]. Let us first notice that it is possible to obtain a spectral lower bound for the bi-Ricci curvature also when $n = 5$, however this lower bound is far from being sufficient to perform the μ-bubble construction. In order to solve this difficulty, we consider a weighted bi-Ricci curvature BRic_α where the parameter α does not give the same weight to all sectional curvatures in the combination (a similar idea appears in the recent article by Hong and Yan [19]). We prove a spectral lower bound for the weighted bi-Ricci curvature: the operator $-a\widetilde{\Delta} + (\widetilde{\mathrm{BRic}}_{\alpha-} - \delta)$ is non-negative where $a, \delta \in \mathbb{R}$. At that step, a, α are two parameters that should be chosen such that $\delta > 0$.

By imposing some new constraints on a and α, we are then able to construct the μ-bubble with a spectral lower bound on the Ricci curvature. At the last step, we apply the Bishop-Gromov estimate of Antonelli and Xu [2]. In order to do so, this imposes some new constraints on the parameters a and α. Nevertheless, the choice $a = \frac{11}{10}$ and $\alpha = \frac{40}{43}$ meets all the constraints. Let us notice that, for $n = 6$, no choice of a and α satisfies all the constraints. Moreover, the computations have to be done the most precisely possible in order to allow such a choice when $n = 5$. The end of the proof then follows the line of [12].

Organization. In Section 2, we fix some notations that we use all along the paper. Section 3 is devoted to the proof the spectral lower bound for $\widetilde{\mathrm{BRic}}_\alpha$ for the Gulliver-Lawson metric on a stable minimal hypersurface. In Section 4, we construct the μ-bubble with a spectral lower Ricci bound. We end the proof of Theorem 1.1 in Section 5. Along the paper, we specify the value of n, a and α only when it is necessary, we hope this allows to understand where the constraints come from.

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2. Preliminaries

Let (M^n, g) be a Riemannian manifold and $(e_i)_{1\le i\le n}$ be an orthonormal basis of T_pM. For $\alpha \in \mathbb{R}$, we recall or define

- the Ricci curvature $\mathrm{Ric}(e_1, e_1) = \sum_{i=2}^n R(e_1, e_i, e_i, e_1)$,
- the punctual minimum of the Ricci curvature $\lambda(p) = \min_{v\in T_pM, |e|=1} \mathrm{Ric}(e, e)$,
- the weighted bi-Ricci or α-bi-Ricci curvature

$$\mathrm{BRic}_\alpha(e_1, e_2) = \sum_{i=2}^n R(e_1, e_i, e_i, e_1) + \alpha \sum_{j=3}^n R(e_2, e_j, e_j, e_2)$$

- the minimum of the α-bi-Ricci curvature $\Lambda_\alpha(p) = \min_{(e,f) \text{ orthonormal in } T_pM} \mathrm{BRic}_\alpha(e, f)$

Notice that for $\alpha = 1$, BRic_1 is the classical bi-Ricci curvature as defined in [25].

If $\Sigma \looparrowright M$ is a hypersurface with unit normal ν. We use the following conventions:

- the second fundamental form of Σ is $A_\Sigma(X, Y) = (\nabla_X \nu, Y) = -(\nabla_X Y, \nu)$ and
- the mean curvature of Σ is $H = \operatorname{tr} A_\Sigma$.

If Ω is a subset of M, we denote by $\mathcal{N}_\rho(\Omega)$ the ρ-tubular neighborhood of Ω: the set of points at distance less than ρ from Ω.

We finish by a simple remark that we use in Section 3.

Remark 1. Let $A \in \mathcal{M}_n(\mathbb{R})$ be a positive definite symmetric matrix and $B \in \mathbb{R}^n$. Then the function $f : X \in \mathbb{R}^n \mapsto X^\top AX + B^\top X \in \mathbb{R}$ is lower bounded and its minimum is given by $-\frac{1}{4}B^\top A^{-1}B$.

3. Spectral lower bound for the weighted bi-Ricci curvature

Let $F : M^n \looparrowright \mathbb{R}^{n+1}$ be a complete two-sided minimal hypersurface and g its induced metric. We consider the Gulliver-Lawson conformal metric $\tilde{g} = r^{-2}g$ where r is the Euclidean distance function to 0. Notice that if $F(p) = 0$, $\tilde{g}$ is not defined. So we consider $N = M \setminus F^{-1}(0)$. As it was observed by Gulliver and Lawson [18], the metric $(N, \tilde{g})$ is complete.

The first step of the proof of Theorem 1.1 consists in proving that the stability assumption can be translated in a spectral lower bound for the α-bi-Ricci curvature of the metric $\tilde{g}$. Actually we have the following result.

Theorem 3.1. *Let $M^n \looparrowright \mathbb{R}^{n+1}$ be a two-sided stable minimal hypersurface. Suppose $n = 5$, then, for $a = \frac{11}{10}$, $\alpha = \frac{40}{43}$ and $\delta = \frac{3}{10}$, there is a smooth function V such that*

$$V \geq \delta - \widetilde{\Lambda}_\alpha$$

and

$$\int_N |\nabla\varphi|^2_{\tilde{g}} dv_{\tilde{g}} \geq \int_N \frac{1}{a} V\varphi^2 dv_{\tilde{g}} \tag{1}$$

for any $\varphi \in C^1_c(N)$.

3.1. Recalling some computations.

We first recall some computations and results of [12].

We denote by ν the unit normal to M and by $|dr|$ the norm of the differential of r along M with respect to the metric g.

Let $(e_i)_{1\leq i\leq n}$ be an orthonormal basis for the metric g then, for the conformal metric $\tilde{g}$, an orthonormal basis is given by $\tilde{e}_i = re_i$. The sectional curvatures of g and $\tilde{g}$ are related by

$$\tilde{R}_{ijji} = r^2 R_{ijji} + 2 - |dr|^2 - dr(e_i)^2 - dr(e_j)^2 - (p, \nu)(A_{ii} + A_{jj}) \tag{2}$$

(see [12, Proposition 3.5]).

The second result that we want to recall is the writing of the stability inequality in the conformal metric $\tilde{g}$. We have

$$\int_N |\nabla\varphi|^2 dv_{\tilde{g}} \geq \int_N \left(r^2|A|^2 - \frac{n(n-2)}{2} + \frac{n^2-4}{4}|dr|^2 \right) \varphi^2 dv_{\tilde{g}} \tag{3}$$

for any $\varphi \in C^1_c(N)$ (see [12, Proposition 3.10]).

3.2. Estimating the curvature terms.

In this subsection, we want to relate the curvature term in the stability inequality (3) to the α-bi-Ricci curvature. We use the notations of the preceding subsection.

Proposition 3.1.

$$\begin{aligned}\widetilde{\mathrm{BRic}}_\alpha(\tilde{e}_1, \tilde{e}_2) =& r^2 \,\mathrm{BRic}_\alpha(e_1, e_2) + 2(n - 1 + \alpha(n-2)) - (n + \alpha(n-1))|dr|^2 \\ &- ((n-2-\alpha)dr(e_1)^2 + \alpha(n-3)dr(e_2)^2) \\ &- (p, \nu)((n-2-\alpha)A_{11} + \alpha(n-3)A_{22})\end{aligned}$$

Proof. Summing (2) and using $\operatorname{tr} A = 0$, we have

$$\begin{aligned}\widetilde{\mathrm{BRic}}_\alpha(\tilde e_1,\tilde e_2) =&\sum_{i=2}^n \tilde R_{1ii1} + \alpha\sum_{j=3}^n \tilde R_{2jj2}\\ =&r^2\,\mathrm{BRic}_\alpha(e_1,e_2) + 2(n-1) - (n-1)|dr|^2 - (n-1)dr(e_1)^2\\ &-(|dr|^2 - dr(e_1)^2) - (p,\nu)((n-1)A_{11} - A_{11}) + 2\alpha(n-2)\\ &-\alpha(n-2)|dr|^2 - \alpha(n-2)dr(e_2)^2 - \alpha(|dr|^2 - dr(e_1)^2 - dr(e_2)^2)\\ &-\alpha(p,\nu)((n-2)A_{22} - A_{11} - A_{22})\\ =&r^2\,\mathrm{BRic}_\alpha(e_1,e_2) + 2(n-1+\alpha(n-2)) - (n+\alpha(n-1))|dr|^2\\ &-((n-2-\alpha)dr(e_1)^2 + \alpha(n-3)dr(e_2)^2)\\ &-(p,\nu)((n-2-\alpha)A_{11} + \alpha(n-3)A_{22})\end{aligned}$$

□

Proposition 3.2.

$$\mathrm{BRic}_\alpha(e_1,e_2) = -\sum_{i=1}^n A_{1i}^2 - \alpha\sum_{j=2}^n A_{2j}^2 - \alpha A_{11}A_{22}$$

Proof. Applying Gauss formula and $\operatorname{tr} A = 0$, we have

$$\begin{aligned}\mathrm{BRic}_\alpha(e_1,e_2) =&\sum_{i=2}^n (A_{11}A_{ii} - A_{1i}^2) + \alpha\sum_{j=3}^n (A_{22}A_{jj} - A_{2j}^2)\\ =&-\sum_{i=1}^n A_{1i}^2 + \alpha(-A_{22}(A_{11}+A_{22}) - \sum_{j=3}^n A_{2j}^2)\\ =&-\sum_{i=1}^n A_{1i}^2 - \alpha\sum_{j=2}^n A_{2j}^2 - \alpha A_{11}A_{22}\end{aligned}$$

□

Using the above computation, we obtain the following estimate of the curvature term. This estimate introduces some constraints on α and a second parameter a.

Proposition 3.3. *Let $a, \alpha > 0$ such that $a > \frac{1}{2}$, $2a \geq \alpha$ and*

$$W = (a - \frac{1}{2})\big(a - \frac{n-2}{2n}(1+2\alpha)\big) - \frac{n-2}{4n}(1-\alpha)^2 > 0$$

Let us define

$$f = \frac{(n-2)^2}{8W}\Big((a-\frac{1}{2})\frac{n-2}{n}(1+\alpha\frac{n-4}{n-2})^2 + (1-\alpha)^2(a + \frac{n-2}{2n} - \frac{2}{n}\alpha)\Big)$$

Then

$$ar^2|A|^2 + f(1-|dr|^2) \geq -r^2\,\mathrm{BRic}_\alpha(e_1,e_2) + (p,\nu)((n-2-\alpha)A_{11} + \alpha(n-3)A_{22})$$

Proof. By Proposition 3.2, the right-hand side of the expected inequality satisfies to

$$\begin{aligned}-r^2\operatorname{BRic}_\alpha(e_1,e_2)+&(p,\nu)((n-2-\alpha)A_{11}+\alpha(n-3)A_{22})\\ =&r^2\Big(\sum_{i=1}^n A_{1i}^2+\alpha\sum_{j=2}^n A_{2j}^2+\alpha A_{11}A_{22}\\ &\quad+(\frac{p}{r^2},\nu)((n-2-\alpha)A_{11}+\alpha(n-3)A_{22})\Big)\\ =&r^2\Big(A_{11}^2+\alpha A_{22}^2+\alpha A_{11}A_{22}+\sum_{i=2}^n A_{1i}^2+\alpha\sum_{j=3}^n A_{2j}^2\\ &\quad+(\frac{p}{r^2},\nu)((n-2-\alpha)A_{11}+\alpha(n-3)A_{22})\Big)\end{aligned}\tag{4}$$

The vector $A_\Delta=(A_{11},\cdots,A_{nn})$ belongs to the sub-space $F_n=\{X\in\mathbb{R}^n\mid x_1+\cdots+x_n=0\}$. We write a decomposition in an orthonormal basis of F_n as

$$\begin{pmatrix}A_{11}\\ \vdots\\ A_{nn}\end{pmatrix}=\sum_{i=1}^{n-3}\begin{pmatrix}0\\0\\E_i\end{pmatrix}x_i+\frac{1}{\sqrt{2n(n-2)}}\begin{pmatrix}n-2\\n-2\\-2\\ \vdots\\ -2\end{pmatrix}z_1+\frac{1}{\sqrt2}\begin{pmatrix}1\\-1\\0\\ \vdots\\0\end{pmatrix}z_2$$

where $(E_i)_{1\le i\le n-3}$ is an orthonormal basis of F_{n-2}. So we have

$$\begin{aligned}A_{11}^2+&\alpha A_{22}^2+\alpha A_{11}A_{22}+(\frac{p}{r^2},\nu)((n-2-\alpha)A_{11}+\alpha(n-3)A_{22})\\ =&(\frac{\sqrt{n-2}}{\sqrt{2n}}z_1+\frac{z_2}{\sqrt2})^2+\alpha(\frac{\sqrt{n-2}}{\sqrt{2n}}z_1-\frac{z_2}{\sqrt2})^2+\alpha(\frac{n-2}{2n}z_1^2-\frac12 z_2^2)\\ &+(\frac{p}{r^2},\nu)\big(\frac{\sqrt{n-2}}{\sqrt{2n}}(n-2+\alpha(n-4))z_1+\frac{n-2}{\sqrt2}(1-\alpha)z_2\big)\\ =&\frac{n-2}{2n}(1+2\alpha)z_1^2+\sqrt{\frac{n-2}{n}}(1-\alpha)z_1z_2+\frac12 z_2^2\\ &+(\frac{p}{r^2},\nu)\frac{n-2}{\sqrt2}\big(\sqrt{\frac{n-2}{n}}(1+\alpha\frac{n-4}{n-2})z_1+(1-\alpha)z_2\big)\end{aligned}\tag{5}$$

For $a>0$, we are interested in the minimum (if it exists) of

$$\begin{aligned}a(z_1^2+z_2^2)-&\frac{n-2}{2n}(1+2\alpha)z_1^2-\sqrt{\frac{n-2}{n}}(1-\alpha)z_1z_2-\frac12 z_2^2\\ &-(\frac{p}{r^2},\nu)\frac{n-2}{\sqrt2}(\sqrt{\frac{n-2}{n}}(1+\alpha\frac{n-4}{n-2})z_1+(1-\alpha)z_2)\end{aligned}\tag{6}$$

The matrix of the quadratic part of the above expression is

$$\begin{pmatrix}a-\frac{n-2}{2n}(1+2\alpha) & -\sqrt{\frac{n-2}{4n}}(1-\alpha)\\ -\sqrt{\frac{n-2}{4n}}(1-\alpha) & a-\frac12\end{pmatrix}$$

This matrix is positive definite if $a > \frac{1}{2}$ and its determinant is positive:

$$W = (a - \frac{1}{2})\big(a - \frac{n-2}{2n}(1+2\alpha)\big) - \frac{n-2}{4n}(1-\alpha)^2 > 0$$

If it's the case, by Remark 1 with vector $B = -(\frac{p}{r^2},\nu)\frac{n-2}{\sqrt{2}}\big(\sqrt{\frac{n-2}{n}}(1+\alpha\frac{n-4}{n-2}),(1-\alpha)\big)$, the quantity in (6) is lower bounded by

$$\begin{aligned}
&-(\frac{p}{r^2},\nu)^2\frac{(n-2)^2}{8W}\Big((a-\frac{1}{2})\frac{n-2}{n}(1+\alpha\frac{n-4}{n-2})^2 + \frac{n-2}{n}(1-\alpha)^2(1+\alpha\frac{n-4}{n-2}) \\
&\qquad + (a - \frac{n-2}{2n}(1+2\alpha))(1-\alpha)^2\Big) \\
&= -(\frac{p}{r^2},\nu)^2\frac{(n-2)^2}{8W}\Big((a-\frac{1}{2})\frac{n-2}{n}(1+\alpha\frac{n-4}{n-2})^2 + (1-\alpha)^2(a+\frac{n-2}{2n}-\frac{2}{n}\alpha)\Big) \\
&= -(\frac{p}{r^2},\nu)^2 f
\end{aligned}$$

Since $(\frac{p}{r},\nu)^2 = (1-|dr|^2)$, we have then proved that

$$\begin{aligned}
a(z_1^2+z_2^2) + \frac{f}{r^2}(1-|dr|^2) \geq & \frac{n-2}{2n}(1+2\alpha)z_1^2 + \sqrt{\frac{n-2}{n}}(1-\alpha)z_1z_2 + \frac{1}{2}z_2^2 \\
& + (\frac{p}{r^2},\nu)\frac{n-2}{\sqrt{2}}(\sqrt{\frac{n-2}{n}}(1+\alpha\frac{n-4}{n-2})z_1 + (1-\alpha)z_2)
\end{aligned}$$

Combining this with (4) and (5), if $2a \geq \alpha$, we have

$$\begin{aligned}
a|A|^2 + \frac{f}{r^2}(1-|dr|^2) \geq & a(|A_\Delta|^2 + \sum_{i\neq j} A_{ij}^2) + \frac{f}{r^2}(1-|dr|^2) \\
\geq & A_{11}^2 + \alpha A_{22}^2 + \alpha A_{11}A_{22} + (\frac{p}{r^2},\nu)((n-2-\alpha)A_{11} + \alpha(n-3)A_{22}) \\
& + a\sum_{i\neq j} A_{ij}^2 \\
\geq & A_{11}^2 + \alpha A_{22}^2 + \alpha A_{11}A_{22} + \sum_{i=2}^n A_{1i}^2 + \alpha\sum_{j=3}^n A_{2j}^2 \\
& + (\frac{p}{r^2},\nu)((n-2-\alpha)A_{11} + \alpha(n-3)A_{22}) \\
\geq & -\mathrm{BRic}_\alpha(e_1,e_2) + \frac{(p,\nu)}{r^2}((n-2-\alpha)A_{11} + \alpha(n-3)A_{22})
\end{aligned}$$

This is the expected estimate. □

3.3. **Proof of Theorem 3.1.** Let us assume that the basis is chosen such that $\widetilde{\Lambda}_\alpha = \widetilde{\mathrm{BRic}}_\alpha(\tilde{e}_1,\tilde{e}_2)$. From (3), we are looking for a lower bound for $r^2|A|^2 - \frac{n(n-2)}{2} + \frac{n^2-4}{4}|dr|^2$. Under the assumptions of Proposition 3.3, $\alpha \leq 1$ (such that $n-2-\alpha \geq \alpha(n-3)$) and using

Proposition 3.1, we have

$$\begin{aligned}
a\Big(r^2|A|^2-\frac{n(n-2)}{2}+&\frac{n^2-4}{4}|dr|^2\Big)\\
&\ge -r^2\operatorname{BRic}_\alpha(e_1,e_2)+(p,\nu)((n-2-\alpha)A_{11}+\alpha(n-3)A_{22})\\
&\quad -f(1-|dr|^2)-a\frac{n(n-2)}{2}+a\frac{n^2-4}{4}|dr|^2\\
&\ge -\widetilde{\operatorname{BRic}}_\alpha(\tilde e_1,\tilde e_2)+2(n-1+\alpha(n-2))-(n+\alpha(n-1))|dr|^2\\
&\quad -\big((n-2-\alpha)dr(e_1)^2+\alpha(n-3)dr(e_2)^2\big)\\
&\quad -f(1-|dr|^2)-a\frac{n(n-2)}{2}+a\frac{n^2-4}{4}|dr|^2\\
&\ge C(|dr|^2)-\widetilde\Lambda_\alpha
\end{aligned}$$

where

$$C(t)=2(n-1+\alpha(n-2))-(2n-2+\alpha(n-2))t-f(1-t)-a\frac{n(n-2)}{2}+a\frac{n^2-4}{4}t$$

C is an affine function and $0\le|dr|^2\le 1$, so $C(|dr|^2)\ge\min(C(0),C(1))$. We have

$$\begin{aligned}
C(1)=&2(n-1+\alpha(n-2))-(2n-2+\alpha(n-2))-a\frac{n(n-2)}{2}+a\frac{n^2-4}{4}\\
=&\,\alpha(n-2)-a\frac{(n-2)^2}{4}=(n-2)(\alpha-a\frac{n-2}{4})
\end{aligned}$$

and

$$C(0)=2(n-1+\alpha(n-2))-f-a\frac{n(n-2)}{2}$$

If we consider $a=\frac{11}{10}$ and $\alpha=\frac{40}{43}$, we have $a>\frac12$, $2a\ge\alpha$, $\alpha\le 1$ and $W=\frac{26697}{184900}>0$. So the above computations apply. We have

$$C(0)=\frac{731975}{1530628}\simeq 0.47\quad\text{and}\quad C(1)=\frac{543}{1720}\simeq 0.31$$

So for these values of a and α, and with $\delta=\frac{3}{10}\le\min(C(0),C(1))$, we have

$$V=a\left(r^2|A|^2-\frac{n(n-2)}{2}+\frac{n^2-4}{4}|dr|^2\right)\ge\delta-\widetilde\Lambda_\alpha$$

By (3), the spectral estimate (1) is true. Theorem 3.1 is proved.

4. The μ-bubble construction

In this section, we produce a warped μ-bubble with a spectral Ricci curvature lower bound. So we start with a connected complete non-compact Riemannian manifold $(N^n,\bar g)$ with a spectral lower bound on the α-bi-Ricci curvature: there is a smooth function $\overline V$ on N such that

$$\overline V\ge\delta-\overline\Lambda_\alpha$$

and

$$\int_N|\overline\nabla\varphi|^2_{\bar g}dv_{\bar g}\ge\int_N\frac1a\overline V\varphi^2dv_{\bar g}\tag{7}$$

for any $\varphi \in C^1_c(N)$

Theorem 4.1. *Assume $(N, \bar g)$ as above with $n = 5$, $a = \frac{11}{10}$, $\alpha = \frac{40}{43}$ and $\delta = \frac{3}{10}$. Let Ω_+ be a domain in N (i.e. an open subset with compact smooth boundary) such that $N \setminus \overline{\mathcal{N}}_{100\pi}(\Omega_+) \neq \emptyset$. Then there is a domain Ω_* with*

- *$\Omega_+ \subset \Omega_* \subset \overline{\mathcal{N}}_{100\pi}(\Omega_+)$ and*
- *there is a smooth function V on $\Sigma = \partial\Omega_*$ such that*

$$V \geq \frac{\delta}{2} - \alpha\lambda^\Sigma$$

and

$$\frac{4}{4-a}\int_\Sigma |\nabla\varphi|^2 dv_g \geq \int_\Sigma V\varphi^2 dv_g \tag{8}$$

for any $\varphi \in C^1(\Sigma)$ where g is the induced metric on Σ.

4.1. **Construction of the μ-bubble.** Because of the spectral control (7) on N, we know (see [16]) that there is a positive function w on N such that

$$-a\overline{\Delta}w = \overline{V}w \geq (\delta - \overline{\Lambda}_\alpha)w \tag{9}$$

Let us recall quickly the construction of the μ-bubble. Let Ω_- be a domain in N such that $\Omega_+ \subset\subset \Omega_- \subset \overline{\mathcal{N}}_{100\pi}(\Omega_+)$. Let $h : \Omega_- \setminus \Omega_+ \to \mathbb{R}$ be a smooth function such that $\lim_{p\to\partial\Omega_+} h(p) = +\infty$ and $\lim_{p\to\partial\Omega_-} h(p) = -\infty$. Let $\underline{\Omega}$ be a domain with $\Omega_+ \subset\subset \underline{\Omega} \subset\subset \Omega_-$.

For any sets of finite perimeter Ω with $\Omega_+ \subset\subset \Omega \subset\subset \Omega_-$, we consider the quantity

$$\mathcal{A}(\Omega) = \int_{\partial^*\Omega} w^a - \int_U (\chi_\Omega - \chi_{\underline{\Omega}})hw^a$$

where $\partial^*\Omega$ is the reduced boundary of Ω. By similar argument to the ones in [11, 28], there there is a set of finite perimeter Ω_* ($\Omega_+ \subset\subset \Omega_* \subset\subset \Omega_-$) which minimize the functional $\mathcal{A}$. Moreover its reduced boundary $\partial^*\Omega_* = \Sigma$ is non empty ($N \setminus \overline{\mathcal{N}}_{100\pi}(\Omega_+) \neq \emptyset$) and smooth (see for example [21, 27]).

4.2. **Spectral Ricci-curvature bound of the μ-bubble.** We denote by $k = n-1$ the dimension of Σ and by η the outgoing unit normal to Σ.

As in [12, Proposition 4.2], if φ is a function on Σ, writing the first variation of $\mathcal{A}$ for a variation $\{\Omega_t\}$ of Ω_* generated by $\varphi\eta$ gives

$$0 = \frac{d}{dt}\mathcal{A}(\Omega_t)_{|t=0} = \int_\Sigma (Hw^a + aw^{a-1}dw(\eta) - hw^a)\varphi = \int_\Sigma (H + aw^{-1}dw(\eta) - h)w^a\varphi$$

Since this is true for any φ,

$$H = h - ad\ln w(\eta) \tag{10}$$

Computing the second derivative of $\mathcal{A}(\Omega_t)$, we obtain

$$0 \leq \frac{d^2}{dt^2}\mathcal{A}(\Omega_t)_{|t=0} = \int_\Sigma w^a\Big(-\varphi\Delta\varphi - (|B|^2 + \overline{\mathrm{Ric}}(\eta,\eta))\varphi^2 - aw^{-2}dw(\eta)^2\varphi^2 \\ + aw^{-1}\overline{\nabla}^2 w(\eta,\eta)\varphi^2 - aw^{-1}(\overline{\nabla}w, \nabla\varphi)\varphi - dh(\eta)\varphi^2\Big)$$

where B is the second fundamental form of Σ. So

$$0 \le \int_\Sigma -\operatorname{div}(w^a\varphi\nabla\varphi) + w^a\Big(|\nabla\varphi|^2 - (|B|^2 + \overline{\mathrm{Ric}}(\eta,\eta))\varphi^2 - aw^{-2}dw(\eta)^2\varphi^2 \\ + aw^{-1}\overline{\nabla}^2 w(\eta,\eta)\varphi^2 - dh(\eta)\varphi^2\Big)$$

Using $\overline{\nabla}^2 w(\eta,\eta) = \overline{\Delta} w - \Delta w - H dw(\eta)$, we obtain

$$0 \le \int_\Sigma w^a\Big(|\nabla\varphi|^2 - (|B|^2 + \overline{\mathrm{Ric}}(\eta,\eta))\varphi^2 - aw^{-2}dw(\eta)^2\varphi^2 \\ + aw^{-1}(\overline{\Delta} w - \Delta w - H dw(\eta))\varphi^2 - dh(\eta)\varphi^2\Big) \tag{11}$$

For $\varphi = w^{-a/2}\psi$, we have $\nabla\varphi = w^{-a/2}\nabla\psi - \frac{a}{2}w^{-a/2-1}\psi\nabla w$. So we can write

$$\begin{aligned}\int_\Sigma w^a(|\nabla\varphi|^2 - aw^{-1}\Delta w\varphi^2) &= \int_\Sigma |\nabla\psi|^2 - aw^{-1}\psi(\nabla w,\nabla\psi) + \frac{a^2}{4}\psi^2 w^{-2}|\nabla w|^2 - a\psi^2 w^{-1}\Delta w \\ &= \int_\Sigma |\nabla\psi|^2 - a\operatorname{div}(\psi^2 w^{-1}\nabla w) + aw^{-1}\psi(\nabla w,\nabla\psi) \\ &\qquad - (a - \frac{a^2}{4})\psi^2 w^{-2}|\nabla w|^2 \\ &= \int_\Sigma |\nabla\psi|^2 + aw^{-1}\psi(\nabla w,\nabla\psi) - (a - \frac{a^2}{4})\psi^2 w^{-2}|\nabla w|^2\end{aligned}$$

Using that $w^{-1}\psi(\nabla w,\nabla\psi) \le \varepsilon|\nabla\psi|^2 + \frac{1}{4\varepsilon}\psi^2 w^{-2}|\nabla w|^2$ with $\varepsilon = \frac{1}{4-a}$, we get

$$\int_\Sigma w^a(|\nabla\varphi|^2 - aw^{-1}\Delta w\varphi^2) \le \frac{4}{4-a}\int_\Sigma |\nabla\psi|^2$$

From (11) and (9), we then obtain

$$\begin{aligned}\frac{4}{4-a}\int_\Sigma |\nabla\psi|^2 &\ge \int_\Sigma \Big(|B|^2 + \overline{\mathrm{Ric}}(\eta,\eta) + aw^{-2}dw(\eta)^2 - aw^{-1}\overline{\Delta} w + aHd\ln(\eta)\Big)\psi^2 \\ &\qquad + adh(\eta)\psi^2 \\ &\ge \int_\Sigma \Big(|B|^2 + \overline{\mathrm{Ric}}(\eta,\eta) + \delta - \overline{\Lambda}_\alpha + aw^{-2}dw(\eta)^2 + aHd\ln(\eta)\Big)\psi^2 \\ &\qquad + adh(\eta)\psi^2\end{aligned} \tag{12}$$

Let $(e_1,\dots,e_k)$ be an orthonormal basis of Σ. Using Gauss equation, we have

$$\begin{aligned}\alpha\,\mathrm{Ric}^\Sigma(e_1,e_1) = \alpha\sum_{j=2}^k R^\Sigma_{1jj1} &= \alpha\sum_{j=2}^k(\overline{R}_{1jj1} + B_{11}B_{jj} - B_{1j}^2) \\ &= \overline{\mathrm{BRic}}_\alpha(\eta,e_1) - \overline{\mathrm{Ric}}(\eta,\eta) + \alpha\sum_{j=2}^k(B_{11}B_{jj} - B_{1j}^2)\end{aligned}$$

So assuming that e_1 is such that $\mathrm{Ric}^\Sigma(e_1,e_1)=\lambda^\Sigma$, we have

$$\overline{\mathrm{Ric}}(\eta,\eta)-\overline{\Lambda}_\alpha \geq \overline{\mathrm{Ric}}(\eta,\eta)-\overline{\mathrm{BRic}}_\alpha(\eta,e_1)=-\alpha\lambda^\Sigma+\alpha\sum_{j=2}^k(B_{11}B_{jj}-B_{1j}^2)$$

So using the above inequality and $\operatorname{tr} B=H$ in (12), we then get the inequality

$$\begin{aligned}\frac{4}{4-a}\int_\Sigma|\nabla\psi|^2 \geq \int_\Sigma\psi^2\Big(&\delta-\alpha\lambda^\Sigma+|B|^2+\alpha\sum_{j=2}^k(B_{11}B_{jj}-B_{1j}^2)+a(d\ln w(\eta))^2\\ &+aHd\ln w(\eta)+adh(\eta)\Big)\\ \geq \int_\Sigma\psi^2\Big(&\delta-\alpha\lambda^\Sigma+|B|^2+\alpha HB_{11}-\alpha\sum_{j=1}^k B_{1j}^2+a(d\ln w(\eta))^2\\ &+aHd\ln w(\eta)+adh(\eta)\Big)\end{aligned}$$

Using (10), we have

$$\begin{aligned}K:=|B|^2+\alpha HB_{11}-\alpha\sum_{j=1}^k B_{1j}^2+a(d\ln w(\eta))^2+aHd\ln w(\eta)=\\ |B|^2+\alpha HB_{11}-\alpha\sum_{j=1}^k B_{1j}^2+\frac{1}{a}(H-h)^2+H(h-H)\end{aligned}$$

Let us denote by Φ the traceless part of B and let Φ_Δ denote the vector $(\Phi_{11},\dots,\Phi_{kk})\in F_k$. Thus, for $\alpha\leq 2$, we have

$$K\geq\frac{1}{k}H^2+|\Phi_\Delta|^2+\frac{\alpha}{k}H^2+\alpha H\Phi_{11}-\alpha(\frac{1}{k}H+\Phi_{11})^2+\frac{1}{a}(H-h)^2+H(h-H)$$

We can write a decomposition of Φ_Δ in an orthonormal basis of F_k

$$\Phi_\Delta=\sum_{i=1}^{k-2}\begin{pmatrix}0\\E_i\end{pmatrix}x_i+\frac{1}{\sqrt{k(k-1)}}\begin{pmatrix}k-1\\-1\\\vdots\\-1\end{pmatrix}z$$

where $(E_i)_{1\leq i\leq k-2}$ is an orthonormal basis of F_{k-1}. We then have

$$\begin{aligned}K\geq&\frac{1}{k}H^2+z^2+\frac{\alpha}{k}H^2+\alpha H\sqrt{\frac{k-1}{k}}z-\alpha(\frac{1}{k}H+\sqrt{\frac{k-1}{k}}z)^2+\frac{1}{a}(H-h)^2+H(h-H)\\ \geq&(\frac{1}{k}+\frac{\alpha}{k}-\frac{\alpha}{k^2}+\frac{1}{a}-1)H^2+(1-\alpha\frac{k-1}{k})z^2+\frac{1}{a}h^2+\alpha\sqrt{\frac{k-1}{k}}(1-\frac{2}{k})Hz+(1-\frac{2}{a})Hh\end{aligned}$$

The above expression is a quadratic form in (H,z,h) associated to the matrix

$$G=\begin{pmatrix}\frac{1}{k}+\frac{\alpha}{k}-\frac{\alpha}{k^2}+\frac{1}{a}-1 & \frac{\alpha}{2}\sqrt{\frac{k-1}{k}}(1-\frac{2}{k}) & \frac{1}{2}-\frac{1}{a}\\ \frac{\alpha}{2}\sqrt{\frac{k-1}{k}}(1-\frac{2}{k}) & 1-\alpha\frac{k-1}{k} & 0\\ \frac{1}{2}-\frac{1}{a} & 0 & \frac{1}{a}\end{pmatrix}$$

Notice that this matrix is positive definite if $1-\alpha\frac{k-1}{k}>0$ and $\det(G)>0$. Actually for $k=4$, $a=\frac{11}{10}$, $\alpha=\frac{40}{43}$, we have $1-\alpha\frac{k-1}{k}=\frac{13}{43}>0$ and

$$\det\Big(G-\begin{pmatrix}0&0&0\\0&0&0\\0&0&\frac{1}{22}\end{pmatrix}\Big)=\frac{2599}{1789832}>0$$

So $K\ge\frac{1}{22}h^2$. Finally, for our values of the parameters, we have

$$\frac{4}{4-a}\int_\Sigma|\nabla\psi|^2\ge\int_\Sigma\psi^2\big(\frac{\delta}{2}-\alpha\lambda^\Sigma\big)+\psi^2\big(\frac{\delta}{2}+\frac{1}{22}h^2+adh(\eta)\big) \tag{13}$$

4.3. **End of the proof.** We need to choose the domain Ω_- and the function h. Let $\Psi: N\backslash\Omega_+\to\mathbb{R}_+$ be a smoothing of the distance function $d_{\bar g}(\cdot,\partial\Omega_+)$ such that $\frac12 d_{\bar g}(\cdot,\partial\Omega_+)\le\Psi\le 2d_{\bar g}(\cdot,\partial\Omega_+)$ and $|\overline{\nabla}\Psi|_{\bar g}\le 2$. Let $\varepsilon>0$ small be such that $(1+\varepsilon)11\pi\sqrt{\frac{88}{15}}$ is a regular value of Ψ. Let us define $\Omega_-=\Omega_+\cup\{\Phi\le(1+\varepsilon)11\pi\sqrt{\frac{88}{15}}\}$. On Ω_-, $d_{\bar g}(\cdot,\partial\Omega_+)\le 2(1+\varepsilon)11\pi\sqrt{\frac{88}{15}}\le 100\pi$, so $\Omega_-\subset\overline{\mathcal{N}}_{100\pi}(\Omega_+)$.

On $\{0<\Psi<(1+\varepsilon)11\pi\sqrt{\frac{88}{15}}\}$, we consider the function h defined by $h=k\circ\frac{\Psi}{1+\varepsilon}$ where

$$k(t)=-\sqrt{\frac{33}{10}}\tan\big(\frac{1}{11}\sqrt{\frac{15}{88}}t-\frac{\pi}{2}\big)$$

for $t\in(0,11\pi\sqrt{\frac{88}{15}})$. We have $\lim_{p\to\partial\Omega_\pm}h(p)=\pm\infty$. Notice that k solves $-k'=\frac{3}{44}+\frac{5}{242}k^2$ so

$$|adh(\eta)|=a|k'(\frac{\Psi(p)}{1+\varepsilon})|\frac{|\Psi'(p)|}{1+\varepsilon}\le\frac{2a}{1+\varepsilon}(\frac{3}{44}+\frac{5}{242}h^2)\le\frac{3}{20}+\frac{1}{22}h^2=\frac{\delta}{2}+\frac{1}{22}h^2$$

Hence, the above construction applies and, for our choices of parameters, (13) becomes

$$\frac{4}{4-a}\int_\Sigma|\nabla\psi|^2\ge\int_\Sigma\psi^2\big(\frac{\delta}{2}-\alpha\lambda^\Sigma\big)$$

This ends the proof of Theorem 4.1.

5. Stable Bernstein problem

In this section we prove Theorem 1.1. This a consequence of the following volume growth estimate.

Proposition 5.1. *Let $F: M^5\looparrowright\mathbb{R}^6$ be a complete, immersed, two-sided, simply-connected stable minimal hypersurface. Let $\mathcal{B}_\rho$ denote the geodesic ball of radius $\rho>0$ centered at some point p_0 in M (for the induced metric g). Then*

$$\mathrm{Vol}(\mathcal{B}_\rho)\le\mathrm{Vol}(\mathbb{B}^5)(\frac{800}{43})^{5/2}\big(2\exp(100\pi)\big)^5\rho^5$$

Proof. First, up to a translation, we may assume that $F(p_0)=0$. Let Ω_+ be a smooth compact domain in M such that $\mathcal{B}_\rho\subset\Omega_+\subset\mathcal{B}_{2\rho}$ and such that $0\notin F(\partial\Omega_+)$. We consider the Gulliver-Lawson conformal metric $\tilde g=r^{-2}g$. By Theorem 3.1 and Theorem 4.1, there is Ω_* a domain in M such that $\Omega_+\subset\Omega_*\subset\widetilde{\mathcal{N}}_{100\pi}(\Omega_+)$ and $\partial\Omega_*$ satisfies the spectral Ricci lower bound (8) for the metric induced by $\tilde g$.

By [7, Theorem 1], M has one end. We consider Ω_{**} the connected component of Ω_* that contains $\mathcal{B}_\rho$. We assume M is simply connected so the unbounded component of $M \setminus \Omega_{**}$ has only boundary component Σ_0. Let Ω' be the bounded component of $M \setminus \Sigma_0$. We have $\mathcal{B}_\rho \subset \Omega'$ and $\partial\Omega' \subset \widetilde{\mathcal{N}}_{100\pi}(\mathcal{B}_{2\rho})$.

On $\partial\mathcal{B}_{2\rho}$, the Euclidean distance function r is bounded by 2ρ. So, by [10, Lemma 6.2], on $\widetilde{\mathcal{N}}_{100\pi}(\mathcal{B}_{2\rho})$, the Euclidean distance function r is bounded by $2\rho\exp(100\pi)$.

Now, because of the spectral Ricci lower bound (8) and since $\frac{4}{(4-a)\alpha} = \frac{43}{29} < \frac{3}{2} = \frac{k-1}{k-2}$, we can apply the volume estimate of Antonelli and Xu [2, Theorem 1] for the metric $\tilde{g}$ and obtain

$$\mathrm{Vol}_{\tilde{g}}(\Sigma_0) \le (\frac{\delta}{6\alpha})^{-2}\,\mathrm{Vol}(\mathbb{S}^4) = (\frac{800}{43})^2\,\mathrm{Vol}(\mathbb{S}^4)$$

So scaling back to the Euclidean metric

$$\mathrm{Vol}(\Sigma_0) \le (\frac{800}{43})^2\,\mathrm{Vol}(\mathbb{S}^4)\big(2\exp(100\pi)\big)^4\rho^4$$

Finally we can apply the isoperimetric inequality for minimal hypersurfaces in $\mathbb{R}^{n+1}$ [6, 20] to obtain

$$\mathrm{Vol}_g(\mathcal{B}_\rho) \le \mathrm{Vol}_g(\Omega') \le \mathrm{Vol}(\mathbb{B}^5)(\frac{800}{43})^{5/2}\big(2\exp(100\pi)\big)^5\rho^5$$

□

Proof of Theorem 1.1. Let $M \looparrowright \mathbb{R}^6$ be an immersed, connected,complete, two-sided, stable minimal hypersurface. The stability assumption lifts to the universal cover, so we can assume M to be simply connected. By Proposition 5.1, M has Euclidean volume growth. So by [23] (see also [3]), we obtain that M is a flat hyperplane. □

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