Inter Spike Intervals probability distribution and Double Integral Processes

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LIF models of neurons

- Membrane potential:

\[ \tau_m dV(t) = \left( - (V(t) - V_{rest}) + I_e(t) \right) dt + dl_s(t) \]
LIF models of neurons

- Membrane potential:

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- Synaptic currents:

\[ \tau_s dl_s(t) = -l_s(t) dt + \sigma dW_t \]
Integrate the linear SDE:

\[
V(t) = V_{\text{rest}}(1 - e^{-\frac{t}{\tau_m}}) + \frac{1}{\tau_m} \int_0^t e^{\frac{s-t}{\tau_m}} I_e(s) \, ds + \frac{I_s(0)}{1 - \frac{\tau_m}{\tau_s}} \left( e^{-\frac{t}{\tau_s}} - e^{-\frac{t}{\tau_m}} \right) + \frac{\sigma}{\tau_m \tau_s} e^{-\frac{t}{\tau_m}} \int_0^t e^{\frac{s}{\tau_s}} \left( \int_0^s e^{\frac{s'}{\tau_s}} dW_{s'} \right) \, ds
\]

with \( \alpha = \frac{1}{\tau_m} - \frac{1}{\tau_s} \).
Reaching the threshold

- A spike is emitted when $V(t)$ reaches the threshold $\theta(t)$
Reaching the threshold

- A spike is emitted when $V(t)$ reaches the threshold $\theta(t)$
- Same as first hitting time of

$$X_t = \int_0^t e^{s/\alpha} \left( \int_0^s e^{s'/\tau_s} dW_{s'} \right) \, ds$$
Reaching the threshold

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$$X_t = \int_0^t e^{\frac{s}{\alpha}} \left( \int_0^s e^{\frac{s'}{\tau_s}} dW_{s'} \right) ds$$

- to the deterministic boundary $a(t)$

$$\frac{\sigma}{\tau_m \tau_s} e^{-\frac{t}{\tau_m}} a(t) = \theta(t) - V_{\text{rest}} \left( 1 - e^{-\frac{t}{\tau_m}} \right) + \frac{1}{\tau_m} \int_0^t e^{-\frac{s-t}{\tau_m}} I_e(s) \, ds +$$

$$\int_0^t e^{-\frac{t}{\tau_s}} \left( e^{-\frac{s}{\tau_s}} - e^{-\frac{s}{\tau_m}} \right) ds$$
Stopping times

Definition
A positive real random variable is called a stopping time with respect to the filtration $\mathcal{F}_t$ provided that $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$. 
Stopping times and diffusion equations

- SDE:

\[ dX(t) = b(X, t)dt + B(X, t)dW_t \]
\[ X(0) = X_0 \]
Stopping times and diffusion equations

- SDE:

\[
dX(t) = b(X, t)dt + B(X, t)dW_t
\]

\[X(0) = X_0\]

- Let \(E\) be a non-empty open or closed set of \(\mathbb{R}^n\), then

\[\{\tau = \inf_{t \geq 0} | X(t) \in E\}\]

is a stopping time.
Stopping times and diffusion equations

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- Let \( E \) be a non-empty open or closed set of \( \mathbb{R}^n \), then

\[ \{ \tau = \inf_{t \geq 0} \mid X(t) \in E \} \]

is a stopping time.

- Connection between SDEs and PDEs through the Feynman-Kac formulae.
Stopping times and diffusion equations
A neural network
Countdown process and reset variable

For each neuron $i$, define $X^{(i)}(t) \geq 0$ to be the remaining time until the next emission of a spike by neuron $i$ if it does not receive any spike meanwhile.

This process has a very simple dynamics:

$$\frac{dX^{(i)}(t)}{dt} = -1$$
Countdown process and reset variable

- At time $t$, the next spike will occur in neuron $i = \text{Arg Min}_{j \in \{1...N\}} X^{(j)}(t)$ at time $t + X^{(i)}(t)$.
Countdown process and reset variable

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- The countdown value is also reset to a value $Y_i$ corresponding to the next spike time of this neuron if nothing occurs meanwhile.
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- This value is a random variable, the reset random variable.
## Countdown process and reset variable

- At time $t$, the next spike will occur in neuron $i = \text{Arg Min}_{j \in \{1 \ldots N\}} X^{(j)}(t)$ at time $t + X^{(i)}(t)$.
- At spike time, the membrane potential of the neuron that just spiked is reset.
- The countdown value is also reset to a value $Y_i$ corresponding to the next spike time of this neuron if nothing occurs meanwhile.
- This value is a random variable, the *reset random variable*.
- Depending upon the neurone model, its law is that of the first hitting time of a Brownian, an IWP or a DIP to a deterministic boundary.
Countdown process and reset variable

The interaction random variable $\eta_{ij}$ between neurons $i$ and $j$ is the modification of the time to the next spike of neuron $j$ caused by its receiving a spike from neuron $i$. 
Countdown process and reset variable
Markov description of the network

For a large variety of IF and LIF models, the state of the network can be described by a Markov chain (or process) (Touboul, Faugeras, in preparation), e.g. \((X(t), I_s(t), t)\).
Neural networks and queuing theory

- A lot can probably be gained in the study of neural networks by looking at the work in queuing theory.
- The countdown process is called an hourglass model (introduced by Marie Cottrell 1992).
- In order to apply this modeling we need to define in each case the reset and the interaction random variables.
Double Integrated Process

Definition (DIP)

Let \( f \in L^2(\mathbb{R}) \) and \( g \in L^1(\mathbb{R}) \). Let \( M_t \) be the martingale defined by \( M_t := \int_0^t f(s) dW_s \).

The double integral process (DIP) associated to the functions \( f \) and \( g \) is defined for all \( t \geq 0 \) by:

\[
X_t = \int_0^t g(s) M_s ds = \int_0^t g(s) \left( \int_0^s f(u) dW_u \right) ds
\]
Double Integrated Process

The LIF model:

\[ X_t = \int_0^t e^{\frac{s}{\alpha}} \left( \int_0^s e^{\frac{u}{\tau_s}} dW_u \right) ds \]
Double Integrated Process

Proposition

The two-dimensional process \((X_t, M_t)\) is a Gaussian Markov process.
A special case, the IWP

Definition (IWP)

The Integrated Wiener Process is a special case of the DIP where the functions $f$ and $g$ are identically equal to 1:

$$X_t = \int_0^t W_s \, ds \quad M_s = W_s$$
A special case, the IWP

Its transition measure reads:

\[
\mathbb{P} \left[ X_{t+s} \in du, W_{t+s} \in dv \mid X_s = x, W_s = y \right] \overset{\text{def}}{=} p_t(u, v; x, y) du \, dv = \frac{\sqrt{3}}{\pi t^2} \exp \left[ -\frac{6}{t^3} (u-x-ty)^2 + \frac{6}{t^2} (u-x-ty)(v-y) - \frac{2}{t} (v-y)^2 \right] du \, dv
\]
Describing the problem
First hitting time to a constant boundary

Consider \( U_t = (X_t + x + ty, W_t + y) \) where \( X_t \) is the standard IWP.
First hitting time to a constant boundary

Consider \( U_t = (X_t + x + ty, W_t + y) \) where \( X_t \) is the standard IWP.

Denote by

\[
\tau_a = \inf \{ t > 0 ; \ X_t + x + ty = a \}
\]

the first passage time at \( a \) of the first component of the two-dimensional Markov process \( U_t \).
A bit of history

McKean (1963) computes the joint law of \((\tau_a, W_{\tau_a})\) for \(x = a\):

\[
\mathbb{P}\left[\tau_a \in dt ; \ |W_{\tau_a}| \in dz \mid U_0 = (a, y)\right] \overset{\text{def}}{=} \mathbb{P}_{(a, y)}(\tau_a \in dt ; \ |W_{\tau_a}| \in dz) = \frac{3z}{\pi \sqrt{2} t^2} e^{-\left(2/t\right)(y^2 - |y|z + z^2)} \left(\int_{0}^{4|y|z/t} e^{-3\theta/2} \frac{d\theta}{\sqrt{\pi}\theta}\right) \mathbb{1}_{[0, +\infty)}(z) dz dt \overset{\text{def}}{=} m^a(t, y, z)
\]
A bit of history

Goldman (1971) computes the distribution of the random variable $\tau_a$ in the case where $x < a$ and $y \leq 0$:

$$
P[\tau_a \in dt \mid U_0 = (x, y)] = dt \left[ \sqrt{\frac{3}{8\pi t^3}} \left( \frac{3(a-x)}{t} - y \right) e^{-\frac{3(a-x-ty)^2}{2t^3}} \right]$$

$$+ \int_0^{+\infty} zdz \int_0^t \int_0^\infty P[\tau_0 \in ds \mid |W_{\tau_0}| \in d\mu \mid U_0 = (0, z)] q_{t-s}(x, y; a, z)$$

where $q_t(x, y; u, v) = p_t(x, y; u, v) - p_t(x, y; u, -v)$
A bit of history

Lachal (1991) extends these results and gives the joint distribution of the pair \((\tau_a, W_{\tau_a})\) in all cases:

\[
\mathbb{P}_{(x,y)} [\tau_a \in dt ; \ W_{\tau_a} \in dz] = |z| \left[ p_t(x, y; a, z) - \int_0^t \int_0^{+\infty} m^0(s, -|z|, \mu) p_{t-s}(x, y; a, -\varepsilon \mu) \, d\mu \, ds \right] \mathbb{1}_A(z) \, dz \, dt
\]

where \(A = [0, \infty)\) if \(x < a\), \(A = (-\infty, 0]\) if \(x > a\), \(\varepsilon = \text{sign}(a - x)\) and \(m^0(s, -|z|, \mu)\) is given by McKean’s formula. We denote this density by \(l_{x,y}^a(t, z)\).
The case of a cubic boundary

Lachal (1996) extends these results to the case of a cubic boundary. Idea of the proof

- Under a certain probability, the process $W_t + \frac{\beta}{2} t^2 + \alpha t + x$ is a Wiener process (Girsanov theorem)
The case of a cubic boundary

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- Under a certain probability, the process $W_t + \frac{\beta}{2}t^2 + \alpha t + x$ is a Wiener process (Girsanov theorem)

- Under this probability, the process $X_t + \frac{\beta}{6}t^3 + \frac{\alpha}{2}t^2 + tx + y$ has the law of an IWP.
The case of a cubic boundary

Lachal (1996) extends these results to the case of a cubic boundary. Idea of the proof

- Under a certain probability, the process $W_t + \frac{\beta}{2} t^2 + \alpha t + x$ is a Wiener process (Girsanov theorem).
- Under this probability, the process $X_t + \frac{\beta}{6} t^3 + \frac{\alpha}{2} t^2 + tx + y$ has the law of an IWP.
- The knowledge of the pdf of the first hitting time of the IWP to a constant yields that of the hitting time of the IWP to a cubic.
The case of a cubic boundary

Let $\tau_C$ be the first hitting time of the standard IWP to the cubic curve $C$ of equation

$$C(t-s) = a + b(t-s) + \frac{\alpha}{2}(t-s)^2 + \frac{\beta}{6}(t-s)^3. \quad t \geq s$$
The case of a cubic boundary

Theorem

*Under the reference probability $\mathbb{P}$, the law of the random variable $(\tau_C, W_{\tau_C})$ satisfies the equation:

$$
\mathbb{P}_{s,(x,y)}(\tau_C \in dt, W_{\tau_C} \in dz) = d^{-\alpha,-\beta}(s, x, y-b; t, a, z-b-\alpha(t-s)-\frac{\beta}{2}(t-s)^2) \times \mathbb{P}_{s,(x,y-b)}(\tau_a \in dt, W_{\tau_a} - b - \alpha(\tau_a-s) - \frac{\beta}{2}(\tau_a-s)^2 \in dz)
$$
The case of a cubic boundary

- The function $d^{\alpha,\beta}$ is given by the application of Girsanov’s theorem.
- The probability in the righthand side is that given by Lachal in 1991.
### Why a cubic?

- In the proof the IWP comes from the stochastic integration of the function $\alpha + \beta t$ with respect to the Brownian density.
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- Had we chosen a polynomial of degree greater than 1, the integration by parts would have produced higher-order integrals of the Brownian motion.
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- Had we chosen a polynomial of degree greater than 1, the integration by parts would have produced higher-order integrals of the Brownian motion.
- This method does not generalize to polynomial boundaries of degree larger than three.
Why a cubic?

- In the proof the IWP comes from the stochastic integration of the function $\alpha + \beta t$ with respect to the Brownian density.
- Had we chosen a polynomial of degree greater than 1, the integration by parts would have produced higher-order integrals of the Brownian motion.
- This method does not generalize to polynomial boundaries of degree larger than three.
- For general boundaries we perform a piecewise-cubic approximation.
Principle of the method

Compute the probability that the first hitting time of the IWP to a continuous piecewise function is greater than \( t \in [t_p, t_{p+1}] \).
Principles of the proof

Let $C(t)$ be a continuous piecewise cubic function defined on the interval $[0, T]$. 

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ISI and DIP

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Principles of the proof

- Let $C(t)$ be a continuous piecewise cubic function defined on the interval $[0, T]$.
- Let $(U_t)_{t \geq 0} = (X_t, W_t)_{t \geq 0}$ and $\tau^s_C = \inf \{ t > s \mid X_t = C(t) \}$
Principles of the proof

- Let \( C(t) \) be a continuous piecewise cubic function defined on the interval \([0, T]\).
- Let \((U_t)_{t \geq 0} = (X_t, W_t)_{t \geq 0}\) and \( \tau_C^s = \inf \{ t > s \mid X_t = C(t) \} \)
- Fix \( t \in [0, T]\), let \( p \) be the index of the bin \( t \) belongs to: \( t \in [t_p, t_{p+1}] \)
Principles of the proof

- Let $C(t)$ be a continuous piecewise cubic function defined on the interval $[0, T]$.
- Let $(U_t)_{t \geq 0} = (X_t, W_t)_{t \geq 0}$ and $\tau^s_C = \inf \{ t > s \mid X_t = C(t) \}$
- Fix $t \in [0, T]$, let $p$ be the index of the bin $t$ belongs to: $t \in [t_p, t_{p+1}]$
- We use the strong Markov property of $U_t$ to express $\tau^0_C$ recursively as an integral of a product of $p + 1$ terms
Principles of the proof

The event \( \{ U_{t_1} = u_1, \tau^0_C \geq t_1, U_0 \} \) is in \( \mathcal{F}^{U_{t_1}} \)

Therefore

\[
\mathbb{P} \left( \tau^0_C \geq t \mid U_{t_1} = u_1, \tau^0_C \geq t_1, U_0 \right) = \mathbb{P} \left( \tau^{t_1}_C \geq t \mid U_{t_1} = u_1 \right)
\]
Principles of the proof

\[ P\left( \tau^0_C \geq t \mid U_0 \right) = \]

\[ \int^{(2)} P\left( \tau^0_C \geq t \mid U_{t_1} = u_1, \tau^0_C \geq t_1, U_0 \right) P\left( U_{t_1} \in du_1, \tau^0_C \geq t_1 \mid U_0 \right) = \]

\[ \int^{(2)} P\left( \tau^{t_1}_C \geq t \mid U_{t_1} = u_1 \right) P\left( U_{t_1} \in du_1, \tau^0_C \geq t_1 \mid U_0 \right) \]

The first term in the integral is similar to the lefthand side of the equation: proceed recursively
Principles of the proof

\[
\mathbb{P}\left( \tau_C^0 \geq t \mid U_0 \right) = \int P\left( \tau_C^{t_2} \geq t \mid U_{t_2} = u_2 \right) \\
\times \mathbb{P}\left( U_{t_2} \in du_2, \tau_C^{t_1} \geq t_2 \mid U_{t_1} = u_1 \right) \\
\times \mathbb{P}\left( U_{t_1} \in du_1, \tau_C^0 \geq t_1 \mid U_0 \right) \\
\vdots
\]

Continuous piecewise cubic function
Principles of the proof

\[ P\left(\tau_C^0 \geq t \mid U_0\right) = \int^{(2p)} P\left(\tau_C^{t_p} \geq t \mid U_{t_p} = u_p\right) \]

\[ \times P\left(U_{t_p} \in du_p, \tau_C^{t_p-1} \geq t_p \mid U_{t_{p-1}} = u_{p-1}\right) \]

\[ \times P\left(U_{t_{p-1}} \in du_{p-1}, \tau_C^{t_{p-2}} \geq t_{p-1} \mid U_{t_{p-2}} = u_{p-2}\right) \]

\[ \times \ldots \]

\[ \times P\left(U_{t_1} \in du_1, \tau_C^0 \geq t_1 \mid U_0\right) \]
Principles of the proof

\[ \{ U_{t_k} \in du_k, \tau_C^{t_k-1} \geq t_k \} = \{ U_{t_k} \in du_k \} \setminus \{ U_{t_k} \in du_k, \tau_C^{t_k-1} < t_k \}, \]
Principles of the proof

\[
\mathbb{P}
\left(
U_{t_k} \in du_k, \ \tau_{C}^{t_k-1} \geq t_k \mid U_{t_{k-1}} = u_{k-1}
\right)
= \mathbb{P}
\left(U_{t_k} \in du_k \mid U_{t_{k-1}} = u_{k-1}
\right) - \mathbb{P}
\left(U_{t_k} \in du_k, \ \tau_{C}^{t_k-1} \leq t_k \mid U_{t_{k-1}} = u_{k-1}
\right)
\]
Principles of the proof

\[
P(U_{tk} \in du_k | U_{tk-1} = u_{k-1}) - \int_{t_{k-1}}^{t_k} P(U_{tk} \in du_k, \tau_{C}^{t_k-1} \in ds | U_{tk-1} = u_{k-1})
\]
Principles of the proof

\[
\begin{align*}
&= \mathbb{P}(U_{t_k} \in du_k \mid U_{t_k-1} = u_{k-1}) \\
&\quad - \int_{t_{k-1}}^{t_k} \int_{\mathbb{R}} \mathbb{P}(U_{t_k} \in du_k \mid \tau_C^{t_k} = s, W_s = y, U_{t_k-1} = u_{k-1}) \\
&\quad \times \mathbb{P}(\tau_C^{t_k} \in ds, W_s \in dy \mid U_{t_k-1} = u_{k-1})
\end{align*}
\]
Principles of the proof

\[
= \left( p_{t_k-t_{k-1}}(u_k; u_{k-1}) - \int_{t_{k-1}}^{t_k} \int_{\mathbb{R}} p_{t_k-s}(u_k; C(s), y) \mathbb{P} \left( \tau_C^{t_k-1} \in ds, W_s \in dy \ \bigg| \ U_{t_k-1} = u_{k-1} \right) du_k \right)
\]
Principles of the proof

Theorem

The law of the first hitting time of the IWP to a continuous piecewise cubic boundary is given by the formula:

\[
\mathbb{P}\left( \tau_0^C \geq t \middle| U_0 \right) = \int^{(2p)} \mathbb{P}\left( \tau_0^{t_p} \geq t \middle| U_{t_p} = u_p \right) \prod_{k=1}^{p} \left( p_{t_k-t_{k-1}}(u_k; u_{k-1}) \right)

- \int_{t_{k-1}}^{t_k} \int_{\mathbb{R}} p_{t_k-s}(u_k; C(s), y) \mathbb{P}\left( \tau_k^{t_k-1} \in ds, \ W_s \in dy \middle| U_{t_k-1} \right) \, du_k

Note that \( \mathbb{P}\left( \tau_k^{t_k-1} \in ds, \ W_s \in dy \middle| U_{t_k-1} \right) \) has been derived previously.
Principle of the method
Principles of the proof

Let $C : \mathbb{R} \mapsto \mathbb{R}$ be a continuously differentiable function
Let also $T > 0$ and

$$0 = t_0 < t_1 < \ldots < t_n = T$$

be a partition, noted $\pi$, of the interval $[0, T]$.
Denote by $\delta(\pi)$ the mesh step defined as:

$$\delta(\pi) = \max\{t_{i+1} - t_i, \ i = 0 \ldots n - 1\}$$
Principles of the proof

Let $C_\pi$ be a cubic spline approximation of $C$. It is a $C^2$ interpolation of $C$ which is an approximation of order four, i.e.

$$\|C - C_\pi\|_{\infty,T} = \sup_{t \in [0,T]} |C(t) - C_\pi(t)| \leq K(C)\delta(\pi)^4,$$

where $K(C)$ is a function of $C$ only.
Principles of the proof

Theorem

The first hitting time of the IWP to the curve $C_\pi$ before $T$ converges in law to the first hitting time of the IWP to the curve $C$ before $T$. 

Principles of the proof

- If $C$ is $C^2$ the convergence is of the same order as the approximation of $C$ by the cubic function $C_\pi$. 
Principles of the proof

- If $C$ is $C^2$ the convergence is of the same order as the approximation of $C$ by the cubic function $C_\pi$.
- Let $P(T, g) = \mathbb{P}\left(X_t \geq g(t) \text{ for some } t \in [0, T]\right)$. There exists a constant $\tilde{K}(C, T)$ such that:

$$|P(T, C) - P(T, C_\pi)| \leq \tilde{K}(C, T) \|C - C_\pi\|_{\infty, T}$$
Simplified DIP

Lemma

Let \((X_t)_{t \geq 0}\) be a DIP. Assume that \(f(s) \neq 0\) for all \(s \geq 0\). The study of the hitting times of the DIP \(X\) is equivalent to the study of the simpler process:

\[
\tilde{X}_t = \int_0^t h(s) W_s ds,
\]

where \(h\) is obtained from \(f\) and \(g\).
Simplified DIP

1. \( M_t = \int_0^t f(s) dW_s \)
Simplified DIP

1. \( M_t = \int_0^t f(s) dW_s \)

2. There exists a Brownian motion \((W_t)_t\) such that almost surely (Dubins-Schwarz)
   \[
   M_t = W_{\langle M \rangle_t}
   \]
Simplified DIP

1. $M_t = \int_0^t f(s) dW_s$

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3. Let $\Phi(t) = \langle M \rangle_t = \int_0^t f^2(s) ds$
Simplified DIP

1. \( M_t = \int_0^t f(s) dW_s \)
2. There exists a Brownian motion \((W_t)_t\) such that almost surely (Dubins-Schwarz)
   \[ M_t = W_{\langle M \rangle_t} \]
3. Let \( \Phi(t) = \langle M \rangle_t = \int_0^t f^2(s) ds \)
4. \( \Phi \) is one to one.
Simplified DIP

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   \[ M_t = W_{\langle M \rangle_t} \]

3. Let \( \Phi(t) = \langle M \rangle_t = \int_0^t f^2(s)ds \)

4. \( \Phi \) is one to one.

5. 
   \[
   X_t = \int_0^t g(s) M_s \, ds
   \]
   \[
   = \mathcal{L} \int_0^t g(s) W_{\Phi(s)} \, ds
   \]
   \[
   = \int_0^{\Phi^{-1}(t)} \frac{g(\Phi^{-1}(s))}{\Phi'(\Phi^{-1}(s))} W_s \, ds \equiv \int_0^t h(s) W_s \, ds
   \]
Principles of the method
Principles of the method

Let \( \pi \) be a partition of the interval \([0, T]\) with \( n \) intervals:

\[
0 = t_0 < t_1 < t_2 < \ldots < t_n = T
\]
Principles of the method

- Let $\pi$ be a partition of the interval $[0, T]$ with $n$ intervals:
  
  $$0 = t_0 < t_1 < t_2 < \ldots < t_n = T$$

- Denote by $h^{\pi}$ the piecewise constant approximation of $h$ defined by:
  
  $$h^{\pi}(t) = \sum_{i=0}^{n-1} h(t_i) 1_{[t_i, t_{i+1})}(t),$$
Principles of the method

- Let $\pi$ be a partition of the interval $[0, T]$ with $n$ intervals:

$$0 = t_0 < t_1 < t_2 < \ldots < t_n = T$$

- Denote by $h^\pi$ the piecewise constant approximation of $h$ defined by:

$$h^\pi(t) = \sum_{i=0}^{n-1} h(t_i) \mathbb{1}_{[t_i, t_{i+1})}(t),$$

- Denote by $X^\pi$ the associated DIP:

$$X^\pi_t = \int_0^t h^\pi(s) W_s \, ds.$$
Principles of the method

- Let $\pi$ be a partition of the interval $[0, T]$ with $n$ intervals:
  \[ 0 = t_0 < t_1 < t_2 < \ldots < t_n = T \]

- Denote by $h^\pi$ the piecewise constant approximation of $h$ defined by:
  \[ h^\pi(t) = \sum_{i=0}^{n-1} h(t_i) \mathbb{1}_{[t_i, t_{i+1})}(t), \]

- Denote by $X^\pi$ the associated DIP:
  \[ X^\pi_t = \int_0^t h^\pi(s) W_s \, ds. \]

- $C^\pi$ is the piecewise cubic approximation of the boundary.
Proposition

The process $X_t^\pi$ converges almost surely to the process $X_t$. 
Results

Theorem

The first hitting time $\tau^{\pi}$ of the process $X^{\pi}$ to the curve $C^{\pi}$ converges in law to the first hitting time $\tau_C$ of the process $X$ to the curve $C$. 
Theorem

Let $h$ be a Lipschitz continuous real function, $T > 0$ and $\pi$ a partition of the interval $[0, T]$.

$$0 = t_0 < t_1 < \ldots < t_n = T$$

Let $C$ be a continuously differentiable function. The first hitting time $\tau^\pi$ of the approximated process $X^\pi$ to a cubic spline approximation of $C$ on the partition $\pi$, denoted by $C^\pi$, satisfies the equation on the next slide.
Results

\[ P(\tau^{\pi} \geq T | U_0) = \int^{(2n)} \prod_{k=1}^{n} \left\{ p_{t_k - t_{k-1}} \left( \frac{x_k - x_{k-1}}{h(t_{k-1})}, y_k - y_{k-1}; 0, 0 \right) - \right. \\
\int_{t_{k-1}}^{t_k} \int_{\mathbb{R}} p_{t_k - s} \left( \frac{x_k - C^\pi(s)}{h(t_{k-1})}, y_k - y; 0, 0 \right) \\
\left. \int_{\sigma}^{s} \left( \tau(C-x_{k-1})/h(t_{k-1}) \in ds, W_s \in dy \right) \right\} dx_k dy_k \]

where \( P(\tau_C \in ds, W_s \in dy) \) is given by Lachal’s or McKean’s density.
Principle of the method

- The expressions we found involve an integral on $\mathbb{R}^{2n}$ when there are $n + 1$ points in the mesh.
- Another approximation is done besides the previous ones.
- We express the integral as an expectation and use a Monte-Carlo algorithm to compute it.
Implementation

Corollary

► Let $h$ be a Lipschitz continuous real function, $(X_t, W_t)_{t \geq 0}$ be a standard IWP-Brownian motion pair, $T > 0$ and $\pi$ a partition of the interval $[0, T]$.

► Let $C$ be a continuously differentiable function. The first hitting time $\tau^\pi$ of the approximated process $X^\pi$ to a cubic spline approximation $C^\pi$ of $C$ on the partition $\pi$ can be computed as the expectation:

$$\mathbb{P}\left(\tau^\pi \geq t \mid U_0\right) = \mathbb{E}\left[\theta_{p,\pi}^h(t, X_{t_1}, W_{t_1}, \ldots, X_t, W_t) \mid U_0\right]$$

► The function $\theta_{p,\pi}^h$ is defined for $t \in [t_{p-1}, t_p[$ on the next slide.
Implementation

\[
\theta_p^{h, \pi}(x_1, y_1 \ldots, x, y) := \prod_{k=1}^{p-1} \left\{ \frac{p_{t_k-t_{k-1}}(x_k-x_{k-1}) \cdot y_k - y_{k-1}; 0, 0}{p_{t_k-t_{k-1}}(x_k, y_k, x_{k-1}, y_{k-1})} - \int_{t_{k-1}}^{t_k} \int_{\mathbb{R}} p_{t_{k-1}-t_{k-1}}(x_k, y_k, x_{k-1}, y_{k-1}) \right\} \]

\[
p_{t_k-s} \left( \frac{x_k - C(\pi)(s)}{h(t_{k-1})}, y_k - z; 0, 0 \right) \quad \mathbb{P}_{s,(0,y_s)}(\tau(C-x_{k-1})/h(t_{k-1}) \in ds, \ W_s \in dz) \}
\]

\[
\times \left\{ \frac{p_{t-t_{p-1}}(x-x_{p-1}) \cdot y - y_{p-1}; 0, 0}{p_{t-t_{p-1}}(x, y, x_{p-1}, y_{p-1})} \right\} \]

\[
- \int_{t_{p-1}}^{t} \int_{\mathbb{R}} p_{t-t_{p-1}}(x, y, x_{p-1}, y_{p-1}) \quad \mathbb{P}_{s,(0,z)}(\tau(C-x_{p-1})/h(t_{p-1}) \in ds, \ W_s \in dz) \}
\]
Results

Probability density function of the first hitting time of the IWP to the cubic curve: \( t \mapsto 1 - 2t - 2t^2 - t^3 \) with the initial condition \( X_0 = 0, \ W_0 = 0 \). The total mass is 1 in this case.
Probability density function of the first hitting time of the IWP to the cubic curve: $t \mapsto 1 - \frac{1}{2}t + t^3$ with the initial condition $X_0 = 0$, $W_0 = 0$. The total mass is $\approx 0.2578$ in this case.
Conclusion

- Method of approximation of the probability distribution of the first hitting time of a Double Integral Process (DIP) to a curved boundary. This is the first result for this problem.
1) We obtain a closed-form expression of the probability distribution of the first hitting time of the Integrated Wiener Process (IWP) to a continuous piecewise cubic boundary.
1) We obtain a closed-form expression of the probability distribution of the first hitting time of the Integrated Wiener Process (IWP) to a continuous piecewise cubic boundary.

2) By approximating a general smooth boundary with a piecewise cubic function we compute an approximation of the probability distribution of the first hitting time of the IWP to any smooth curved boundary, and prove convergence.
Conclusion

3) By approximating the DIP with a piecewise IWP we compute an approximation of the probability distribution of the first hitting time of the DIP to any smooth curved boundary, and prove convergence.
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3) By approximating the DIP with a piecewise IWP we compute an approximation of the probability distribution of the first hitting time of the DIP to any smooth curved boundary, and prove convergence.

We sketch a numerical procedure based on Monte-Carlo simulation to compute the probability distribution efficiently.
Conclusion

3) By approximating the DIP with a piecewise IWP we compute an approximation of the probability distribution of the first hitting time of the DIP to any smooth curved boundary, and prove convergence.

We sketch a numerical procedure based on Monte-Carlo simulation to compute the probability distribution efficiently.

These results have potential applications in many fields of physics and biology.
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