Interacting Brownian particles with strong repulsion

Emmanuel Cépa* and Dominique Lépingle†

Abstract

We consider a finite system of interacting Brownian particles with strong repulsion on the line or on the circle. The involved equations may be interpreted as the multidimensional extension of some singular stochastic differential equations on a half-line or on a compact interval. As the number of particles increases without limit, we study the behaviour of the associated empirical measure process. This process converges to a deterministic measure-valued process which is the classical solution of a non-linear integro-partial differential equation.

1 Introduction

Our aim is to give an overview on recent results about the asymptotic behaviour of systems of N linear Brownian particles with electrostatic interaction when the number of particles grows to infinity. When N is fixed this system has a long story: it has been introduced by Dyson [7] as modeling the set of eigenvalues of random hermitian matrices with independent Brownian entries (except for the symmetry constraint). McKean [11] studied this system and proved the particles never collide. Later on, T.Chan [6] and L.C.G.Rogers-Z.Shi [14] investigated the asymptotic behaviour of the empirical measure process of the eigenvalues, after replacing Brownian motions with Ornstein-Uhlenbeck processes; they obtained a deterministic limiting measure-valued process which converges to the Wigner law as time goes to infinity. On the other hand, M.Métivier [12] considered several kinds of interactions between particles including electrostatic repulsion; this time the strength of the diffusion of each particle remains constant as N goes to infinity, whereas it was decreasing with $N^{-1/2}$ in the previous spectral model. He raised the question of asymptotic behaviour and propagation of chaos; we are precisely interested in this model and our main results deal with existence and smoothness of the limiting process.

*MAPMO, Université d'Orléans, F-45067 Orléans, cepa@labomath.univ-orleans.fr
†MAPMO, Université d'Orléans, F-45067 Orléans, lepingle@labomath.univ-orleans.fr

There is another way which leads to the same equations. When N independent Brownian particles are conditioned never to collide, the system of stochastic differential equations they satisfy is of the previous type [8]. If similarly we force the particles to stay in the so-called Weyl chamber $x_1 < \ldots < x_N < x_1 + 2\pi$, then the interaction term is now $1/2\cot(x/2)$ in place of $1/x$, as was proved by R.Pinsky [13]. This is the second model we are considering. A closely related third model is given by a $\coth(x)$ interaction on the line. The three settings are similar and in all cases the proof of the smoothness of the limiting process uses a holomorphic Burgers equation in a domain of the upper half-plane.

2 Three singular stochastic differential equations

Let B be a linear Brownian motion. For any $\gamma > 0$, we consider the equations

$$\begin{array}{llll} (B) & dX_t &=& dB_t + \gamma \dfrac{dt}{X_t} & 0 \le X_t \\ (L) & dX_t &=& dB_t + \dfrac{\gamma}{2}\cot\dfrac{X_t}{2}dt & 0 \le X_t \le 2\pi \\ (H) & dX_t &=& dB_t + \gamma\coth X_t dt & 0 \le X_t\,. \end{array}$$

Strong existence and uniqueness of solutions are easily proved. Equation (B) leads to a Bessel process with dimension $2\gamma + 1$ (if $\gamma = 1$ this is the equation of a Brownian motion conditioned never to hit 0). Equation (L) is similar to the Legendre equation

$$dX_t = dB_t + \frac{1}{2}\cot X_t\, dt$$

which represents the colatitude of a Brownian motion on the sphere S^2 [9]. By standard arguments using the scale function (see e.g. [10]), it can be proved that

(B) 0 is a.s. reached if and only if $\gamma < \frac{1}{2}$
(L) 0 and 2π are a.s. reached if and only if $\gamma < \frac{1}{2}$
(H) 0 is reached with positive probability if and only if $\gamma < \frac{1}{2}$.

3 Stochastic variational inequalities

We now consider a multidimensional setting. Let B be a N-dimensional Brownian motion, $\Phi : \mathbb{R}^N \to (-\infty, +\infty]$ a l.s.c. convex function such that Φ is C^1 in the interior of $D = \{\Phi < +\infty\}$, and $x_0 \in \bar{D}$. It has been proved in [2] and [3] there exists a unique strong solution (X, L) to the equation

$$dX_t = dB_t - \nabla\Phi(X_t)dt - n(X_t)dL_t \tag{1}$$

with $X_0 = x_0$, where X is a continuous adapted $\bar{D}$-valued process, L is a continuous adapted real non-decreasing process with $L_0 = 0$, $n(x)$ belongs to the set of unitary outwards normals to D at $x \in \partial D$, and

$$L_t = \int_0^t \mathbf{1}_{\{X_s \in \partial D\}} dL_s \ .$$

We now restrain the study to the following case. Let $0 < M \leq \infty$, $I = (0, M)$, $\phi : \mathbb{R} \to (-\infty, +\infty]$ a convex C^1-function on I such that $\phi = \infty$ on I^c, $\phi(0+) = \infty$, and $\phi(M-) = \infty$ if $M < \infty$. We set

$$D = \{x = (x^{(1)}, \dots, x^{(N)} : x^{(1)} < \dots < x^{(N)} < x^{(1)} + M\}$$

$$\Phi(x) = \begin{cases} \sum_{1 \leq i < j \leq N} \phi(x^{(j)} - x^{(i)}) & \text{on } D \\ \infty & \text{on } D^c \end{cases}$$

It has been proved in [5] that L in (1) identically vanishes in this case.

4 Interacting Brownian particles

We consider the three following systems of stochastic differential equations with boundary conditions

$$\begin{array}{lll} (B) & dX_t^i = dB_t^i + \gamma \sum_{1 \leq j \neq i \leq N} \frac{dt}{X_t^i - X_t^j} & X_t^1 \leq \dots \leq X_t^N \\ (L) & dX_t^i = dB_t^i + \frac{\gamma}{2} \sum_{1 \leq j \neq i \leq N} \cot \frac{X_t^i - X_t^j}{2} dt & X_t^1 \leq \dots \leq X_t^N \leq X_t^1 + 2\pi \\ (H) & dX_t^i = dB_t^i + \gamma \sum_{1 \leq j \neq i \leq N} \coth(X_t^i - X_t^j) dt & X_t^1 \leq \dots \leq X_t^N \end{array}$$

System (B) has been studied in [4] and [1], systems (L) and (H) in [5]. We apply the result in Section 3 to

$$\begin{array}{llll} (B) & \phi(x) = -\gamma \ln x & \text{on} & I = (0, \infty) \\ (L) & \phi(x) = -\gamma \ln \sin \frac{x}{2} & \text{on} & I = (0, 2\pi) \\ (H) & \phi(x) = -\gamma \ln \sinh x & \text{on} & I = (0, \infty) \ . \end{array}$$

This shows that each system has a unique strong solution with initial fixed condition x_0. As mentioned in case (L) by H.Spohn [15], it can be proved that there are a.s. no multiple collisions (more than two particles at the same position); there exist simple collisions if and only if $\gamma < 1/2$ in cases (B) and (L), and there exist simple collisions with positive probability if and only if $\gamma < 1/2$ in case (H).

5 Asymptotic behaviour of empirical measures

We first study system (B) with $\gamma = 2\lambda/N$, $\lambda > 0$, and observe the behaviour of the empirical measure

$$\mu_t^N = \frac{1}{N} \sum_{i=1}^N \delta_{X_t^i}$$

as the number N goes to infinity.

If μ_0^N converges in distribution to μ_0 in $\mathcal{P}(\mathbb{R})$, then μ^N converges in distribution on the metric space $\mathcal{C}([0,\infty);\mathcal{P}(\mathbb{R}))$ to the solution μ of the weak equation

$$\frac{\partial \mu_t}{\partial t} = \frac{1}{2}\frac{\partial^2 \mu_t}{\partial x^2} - 2\lambda \frac{\partial(\mu_t \mathcal{H}(\mu_t))}{\partial x}$$

where $\mathcal{H}$ is the Hilbert transform defined for any distribution ν by

$$\mathcal{H}(\nu) = p.v.(\frac{1}{x}) * \nu \ .$$

Theorem 1 *There is a unique solution μ and for any $t > 0$ μ_t has a density u_t which is the classical solution to the equation*

$$\begin{cases} u'_t = \frac{1}{2}u''_{xx} - 2\lambda(u\mathcal{H}(u))'_x \\ u_t(x)\,dx \to \mu_0 \ \textit{as}\ t \to 0\ ; \end{cases}$$

u and $\mathcal{H}(u)$ are real analytic on $\mathbb{R}_+^* \times \mathbb{R}$.

The proof is based on a complexification argument. For any $z = x + iy$ in the upper half-plane, we set

$$M_t(z) = \int \frac{\mu_t(du)}{z-u}\ .$$

Then M is the solution to the equation

$$M'_t = \frac{1}{2}M''_{zz} - 2\lambda M M'_z\ . \tag{2}$$

This is a holomorphic Burgers equation on the upper half-plane. Using the Hopf-Cole transformation, we are lead to a holomorphic heat equation which can be explicitely solved. Thus M_t can be extended to a holomorphic function in the neighbourhood of the closed upper half-plane; since $M_t(.+iy)$ converge to $\mathcal{H}(\mu_t) - i\pi\mu_t$ as $y \to 0$, we are done.

6 Self-similarity of the limiting function

Assume now that $\mu_0 = \delta_0$. Then the solution u in Section 5 and its Hilbert transform $\mathcal{H}(u)$ are self-similar: there exist analytic functions ϕ (even and positive) and ψ (odd) such that for any $t > 0$

$$\begin{aligned} u(t,x) &= \tfrac{1}{\sqrt{t}}\phi(\tfrac{x}{\sqrt{t}}) \\ \mathcal{H}u(t,x) &= \tfrac{1}{\sqrt{t}}\psi(\tfrac{x}{\sqrt{t}})\ . \end{aligned}$$

There is a closed expression for ϕ and ψ if $\lambda = \frac{1}{2}$:

$$\begin{aligned}\phi(x) &= \frac{1}{\sqrt{2\pi}} \times \frac{\exp\{x^2/2\}}{(\int_0^x \exp\{v^2/2\}dv)^2 + \pi/2} \\ \psi(x) &= x - \frac{\exp\{x^2/2\} \int_0^x \exp\{v^2/2\}dv}{(\int_0^x \exp\{v^2/2\}dv)^2 + \pi/2}\end{aligned}$$

Coming back to arbitrary $\lambda > 0$ we can show that $\psi(x) \sim 1/x$ as $x \to \pm\infty$. Therefore ψ is bounded and there exists a unique strong solution to the non-linear one-dimensional equation

$$\left\{\begin{array}{lcl} X_t &=& B_t + 2\lambda \int_0^t \mathcal{H}(u)(s, X_s)\, ds \\ \mathcal{L}(X_t) &=& u(t,x)\, dx \text{ for } t > 0 \ . \end{array}\right.$$

The non-linear equations are often used in proving propagation of chaos. In the present setting, we could not prove it.

7 Brownian particles on the circle

Let us consider system (L) in Section 4 with $\gamma = 2\lambda/N$. We set for $i = 1, \ldots, N$

$$Z_t^i = X_t^i \bmod 2\pi$$

with values in $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, and consider again

$$\mu_t^N = \frac{1}{N} \sum_{i=1}^N \delta_{Z_t^i} \ .$$

If μ_0^N converges to a probability measure μ_0 on $\mathbb{T}$, then μ^N converges to the solution of the weak equation

$$\frac{\partial \mu_t}{\partial t} = \frac{1}{2} \frac{\partial^2 \mu_t}{\partial x^2} - 2\lambda \frac{\partial(\mu_t\, \mathbb{H}(\mu_t))}{\partial x} \tag{3}$$

where

$$\mathbb{H}(\nu) = \frac{1}{2} p.v.(\cot \frac{x}{2}) * \nu \ .$$

Theorem 2 *There is a unique solution μ to (3) and for any $t > 0$ μ_t has a density u_t on $\mathbb{T}$ which is the classical solution to the equation*

$$\left\{\begin{array}{l} u_t' = \frac{1}{2} u_{xx}'' - 2\lambda (u \mathbb{H}(u))_x' \\ u_t(x)\, dx \to \mu_0 \text{ as } t \to 0 \ ; \end{array}\right.$$

u and $\mathbb{H}(u)$ are real analytic on $\mathbb{R}_+^* \times \mathbb{T}$.

To prove it we introduce for $z = x + iy$, $y > 0$,

$$M_t(z) = \frac{1}{2} \int_{\mathbb{T}} \cot \frac{z-u}{2} \mu_t(du) \ .$$

M is the solution of (2) with 2π-periodic initial condition M_0. Note that μ_t converges to the uniform measure on $\mathbb{T}$ as t goes to infinity.

8 Hyperbolic setting

The third situation (H) with $\gamma = 2\lambda/N$ is very similar. We replace $\mathcal{H}$ and $\mathbb{H}$ with

$$\tilde{H}(\nu) = p.v.(\coth x) * \nu \ .$$

In the proof of the smoothness theorem, note that we define

$$M_t(z) = \int \coth(z-u) \mu_t(du)$$

on the only strip $z = x + iy$, $0 < y < 2\pi$.

References

[1] A. Bonami, F. Bouchut, E. Cépa and D. Lépingle : *A nonlinear SDE involving Hilbert transform.* Journal of Functional Analysis, **2**, 390-406, 1999.

[2] E. Cépa : *Équations différentielles stochastiques multivoques.* Séminaire de Probabilités XXIX, 86-107, 1995.

[3] E. Cépa : *Problème de Skorokhod multivoque.* Annals of Probability, **26**, 500-532, 1998.

[4] E. Cépa and D. Lépingle : *Diffusing particles with electrostatic repulsion.* Prob. Th. Rel. Fields, **107**, 429-449, 1997.

[5] E. Cépa and D. Lépingle : *Brownian particles with electrostatic repulsion on the circle: Dyson's model for unitary random matrices revisited.* To appear.

[6] T. Chan : *The Wigner semi-circle law and eigenvalues of matrix-valued diffusions.* Prob. Th. Rel. Fields, **93**, 249-272, 1992.

[7] F.J. Dyson : *A Brownian model for the eigenvalues of a random matrix.* J. Math. Phys., **3**, 1191-1198, 1962.

[8] D.J. Grabiner : *Brownian motion in a Weyl chamber, non-colliding particles, and random matrices.* Ann. Inst. Henri Poincaré, **35**, 177-204, 1999.

[9] K. Ito and H.P. McKean : *Diffusion Processes and their Sample Paths.* Springer, Berlin Heidelberg New-York, 1974.

[10] I. Karatzas and S.E. Shreve : *Brownian motion and Stochastic Calculus.* Springer, Berlin Heidelberg New-York, 1988.

[11] H.P. McKean : *Stochastic Integrals.* Academic Press, New-York, 1969.

[12] M. Métivier : *Quelques problèmes liés aux systèmes infinis de particules et leurs limites.* Séminaire de Probabilités XX, 426-446, 1986.

[13] R.G. Pinsky : *On the convergence of diffusion processes conditioned to remain in a bounded region for large times to limiting positive recurrent diffusion processes.* Annals of Probability, **13**, 363-378, 1985.

[14] L.C.G. Rogers and Z. Shi : *Interacting Brownian particles and the Wigner law.* Prob. Th. Rel. Fields, **95**, 555-570, 1993.

[15] H. Spohn : *Dyson's model of interacting Brownian motions at arbitrary coupling strength.* Markov Proc. Rel. Fields, **4**, 649-661, 1998.