Oblique repulsion in the nonnegative quadrant

Dominique Lépingle
Université d'Orléans, FDP-MAPMO

January 24, 2014

Abstract

We consider the differential system $\dot{x} = \alpha/x + \beta/y$, $\dot{y} = \gamma/x + \delta/y$ in the nonnegative quadrant. Here α and δ are positive, β and γ are real constants. Under some condition on the constants there exists a unique global solution. The main difficulty is to prove uniqueness when starting at the corner of the quadrant.

1 Introduction.

We are interested in the question of existence and uniqueness of the solution $\mathbf{u}(.) = (x(.), y(.))$ to the following integral system

$$\begin{array}{rcl} x(t) & = & x + \alpha \int_0^t \frac{ds}{x(s)} + \beta \int_0^t \frac{ds}{y(s)} \\ y(t) & = & y + \gamma \int_0^t \frac{ds}{x(s)} + \delta \int_0^t \frac{ds}{y(s)} \end{array} \tag{1}$$

where $x(.)$ and $y(.)$ are continuous functions from $[0, \infty)$ to $[0, \infty)$ with the conditions

$$\begin{array}{ll} \int_0^t \mathbb{1}_{\{x(s)=0\}} ds = 0 & \int_0^t \mathbb{1}_{\{y(s)=0\}} ds = 0 \\ \int_0^t \mathbb{1}_{\{x(s)>0\}} \frac{ds}{x(s)} < \infty & \int_0^t \mathbb{1}_{\{y(s)>0\}} \frac{ds}{y(s)} < \infty \end{array} \tag{2}$$

for any $t \geq 0$. Here α, β, γ and δ are four real constants with $\alpha > 0$ and $\delta > 0$.

The system has a single singularity at each side of the nonnegative quadrant $S = \{(x, y) : x \geq 0, y \geq 0\}$ and a double singularity at the corner $\mathbf{0} = (0, 0)$. We write $S^\circ := S \setminus \{\mathbf{0}\}$ for the punctured quadrant.

We will note $\dot{x}(t)$ the derivative $dx(t)/dt$. So the integral system (1) may be written as an initial-value problem

$$\begin{array}{rcl} \dot{x} & = & \frac{\alpha}{x} + \frac{\beta}{y} \\ \dot{y} & = & \frac{\gamma}{x} + \frac{\delta}{y} \end{array} \tag{3}$$

with the inital condition $(x(0), y(0)) \in S$.

We first remark that if $\beta < 0$, $\gamma < 0$ and $\alpha\delta < \beta\gamma$, there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda\alpha + \mu\gamma < 0$ and $\lambda\beta + \mu\delta < 0$. Thus $z(t) := \lambda x(t) + \mu y(t)$ is decreasing, $\min(x(t), y(t)) \to 0$ and $\dot{z}(t) \to -\infty$ as $t \to t_f$ where $t_f < \infty$ and there is no solution. If $\beta < 0$, $\gamma < 0$ and $\alpha\delta = \beta\gamma$, then $v(t) := \alpha y(t) - \gamma x(t)$ remains equal to $\alpha y - \gamma x$ and there is a unique solution $(x(t), y(t))$ that converges to $(\frac{\gamma x - \alpha y}{\beta + \gamma}, \frac{\beta y - \delta x}{\beta + \gamma})$ except if $(x, y) = \mathbf{0}$ in which case there is no solution.

From now on we will make the following hypothesis:

$$(H) \qquad \max(\beta, \gamma) \geq 0 \;\text{ or }\; \beta\gamma < \alpha\delta.$$

This is equivalent to the existence of $\lambda \geq 0$ and $\mu \geq 0$ such that $\lambda\alpha+\mu\gamma > 0$ and $\lambda\beta+\mu\delta > 0$. This last formulation amounts to saying that the matrix

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

is an S-matrix in the terminology of [1]. In the sequel, we fix a pair (λ, μ) with $\lambda > 0$, $\mu > 0$, such that $\lambda\alpha + \mu\gamma > 0$ and $\lambda\beta + \mu\delta > 0$.

The aim of this note is to prove the following result.

Theorem 1 *Under condition (H), there exists a unique solution to* (1) *for any starting point* $(x, y) \in S$.

2 Some preliminary lemmata.

We begin with a comparison lemma.

Lemma 2 *Let x_1 and x_2 be nonnegative continuous functions on $[0, \infty)$ which are solutions to the system*

$$\begin{aligned} x_1(t) &= v_1(t) + \alpha \int_0^t \frac{ds}{x_1(s)} \\ x_2(t) &= v_2(t) + \alpha \int_0^t \frac{ds}{x_2(s)} \end{aligned}$$

where $\alpha > 0$, v_1 and v_2 are continuous functions such that $0 \leq v_1(0) \leq v_2(0)$, and $v_2 - v_1$ is nondecreasing. Then $x_1 \leq x_2$ on $[0, \infty)$.

Proof. Assume there exists $t > 0$ such that $x_1(t) > x_2(t)$. Set

$$\tau := \max\{s \leq t : x_1(s) \leq x_2(s)\}.$$

Then

$$\begin{aligned} x_2(t) - x_1(t) &= x_2(\tau) - x_1(\tau) + (v_2(t) - v_1(t)) - (v_2(\tau) - v_1(\tau)) + \alpha \int_\tau^t \left(\frac{1}{x_2(s)} - \frac{1}{x_1(s)}\right) ds \\ &\geq 0, \end{aligned}$$

a contradiction. ∎

Lemma 3 *Let the system*

$$\begin{aligned} \dot{x} &= \frac{\alpha}{x} + \phi(x, z) \\ \dot{z} &= \psi(x, z) \end{aligned} \tag{4}$$

with $x(0) = x_0 \geq 0$, $z(0) = z_0 \in \mathbb{R}$, $\alpha > 0$, ϕ and ψ two Lipschitz functions on $\mathbb{R}_+ \times \mathbb{R}$, and $|\phi| \leq c$ for some $c < \infty$. Then there exists a unique solution to (4)*. Moreover, for this solution, $x(t) > 0$ for any $t > 0$.*

Proof. Assume first $x_0 > 0$. Then the system (4) is Lipschitz on $[\min\{x_0, \frac{\alpha}{c}\}, \infty) \times \mathbb{R}$ and the solution does not step out of this domain, so there is a unique global solution. When $x_0 = 0$, we let $w_0(t) = 0$, $z_0(t) = z_0$ and for $n \geq 1$

$$\begin{aligned} w_n(t) &= 2\alpha t + 2\int_0^t \sqrt{w_{n-1}(s)}\, \phi(\sqrt{w_{n-1}(s)}, z_{n-1}(s)) ds \\ z_n(t) &= z_0 + \int_0^t \psi(\sqrt{w_{n-1}(s)}, z_{n-1}(s)) ds. \end{aligned}$$

Let $M > 0$ and assume $|w_{n-1}(t)| \leq M$ on some interval $[0, T]$. Then, for $0 \leq t \leq T$,

$$|w_n(t)| \leq T(2\alpha + 2c\sqrt{M})$$

and this is again $\leq M$ for T small enough. We also have $|z_n(t)| \leq M'$ for any $n \geq 0$ for T small enough. Equicontinuity of $(w_n, z_n)_{n\geq 0}$ is easily verified and from the Arzela-Ascoli theorem it follows there exists a subsequence (w_{n_k}, z_{n_k}) converging on $[0, T]$ to a solution (w, z) of the system

$$\begin{array}{rcl} \dot{w} & = & 2\alpha + 2\sqrt{w}\,\phi(\sqrt{w}, z) \\ \dot{z} & = & \psi(\sqrt{w}, z) \end{array} \tag{5}$$

with the initial conditions $w(0) = 0$, $z(0) = z_0$. For small T, $\dot{w} > 0$ on $[0, T]$. Set now $x(t) = \sqrt{w(t)}$. Then (x, z) is a solution to (4) on $[0, T]$ with $x(T) > 0$. We may extend the solution to $[0, \infty)$ by using the above result with $x_0 > 0$.

We now prove uniqueness. Let (x, z) and (x', z') be two solutions of (4). Then

$$\begin{array}{rl} & (x(t) - x'(t))^2 + (z(t) - z'(t))^2 \\ = & 2\alpha \int_0^t (x(s) - x'(s))(\frac{1}{x(s)} - \frac{1}{x'(s)})ds + 2\int_0^t (x(s) - x'(s))(\phi(x(s), z(s)) - \phi(x'(s), z'(s)))ds \\ & +2\int_0^t (z(s) - z'(s))(\psi(x(s), z(s)) - \psi(x'(s), z'(s)))ds \\ \leq & 4L\int_0^t ((x(s) - x'(s))^2 + (z(s) - z'(s))^2)ds \end{array}$$

where L is the Lipschitz constant of ϕ and ψ. Uniqueness follows from Gronwall's inequality. ■

Lemma 4 *Let $\mathbf{u}(.) = (x(.), y(.))$ be a solution to (1) and let $\nu = (\lambda, \mu)$. Then the function $z(t) := \nu.\mathbf{u}(t) = \lambda x(t) + \mu y(t)$ is increasing on $[0, \infty)$ and we have $\mathbf{u}(t) \in S^\circ$ for any $t > 0$.*

Proof. Recall that condition (H) is in force. We easily check that $\dot{z}(t)$ is positive. ■

3 Existence. Case $x = 0, y = 0$.

There is an explicit solution to (1) when the starting point is the corner.

Proposition 5 *There is a solution to (1) with initial condition* $\mathbf{0}$ *given by*

$$\begin{array}{rcl} x(t) & = & c\sqrt{t} \\ y(t) & = & d\sqrt{t} \end{array} \tag{6}$$

where

$$\begin{array}{rcl} c & = & (2\alpha + \frac{\beta}{\delta}(\beta - \gamma + \sqrt{(\beta-\gamma)^2 + 4\alpha\delta}))^{1/2} \\ d & = & (2\delta + \frac{\gamma}{\alpha}(\gamma - \beta + \sqrt{(\beta-\gamma)^2 + 4\alpha\delta}))^{1/2}. \end{array} \tag{7}$$

Proof. Writing down $x(t) = c\sqrt{t}$ and $y(t) = d\sqrt{t}$ we have to solve the system

$$\begin{array}{rcl} \frac{c}{2} & = & \frac{\alpha}{c} + \frac{\beta}{d} \\ \frac{d}{2} & = & \frac{\gamma}{c} + \frac{\delta}{d}; \end{array}$$

We first compute

$$\frac{d}{c} = \frac{\gamma - \beta + \sqrt{(\beta-\gamma)^2 + 4\alpha\delta}}{2\alpha} \tag{8}$$

and then obtain (7) provided that

$$\begin{aligned} C &= 2\alpha + \tfrac{\beta}{\delta}(\beta - \gamma + \sqrt{(\beta-\gamma)^2 + 4\alpha\delta}) \\ D &= 2\delta + \tfrac{\gamma}{\alpha}(\gamma - \beta + \sqrt{(\beta-\gamma)^2 + 4\alpha\delta}) \end{aligned}$$

are positive. If $\beta \geq 0$, C is clearly positive. This is also true if $\beta < 0$ and $\beta\gamma < \alpha\delta$ since C may be written

$$C = \frac{4\alpha(\alpha\delta - \beta\gamma)}{2\alpha\delta - \beta\gamma + \beta^2 - \beta\sqrt{4(\alpha\delta - \beta\gamma) + (\beta+\gamma)^2}}.$$

The proof for D is similar. ■

Uniqueness in this case is more involved and will be treated in the last section. We only remark for the moment that the system (3) with $\alpha = \delta = 0$, $\beta > 0$, $\gamma > 0$ and initial value $\mathbf{0}$ has a one-parameter family of solutions.

4 Angular behavior.

We are now in a position to study the behavior of $\frac{y(t)}{x(t)}$. For any $\mathbf{u} = (x, y) \in S^\circ$ we set

$$\theta(\mathbf{u}) = \arctan \frac{y}{x}.$$

We also set

$$\mathbf{u}_* = (x_*, y_*) := \left(\frac{c}{\lambda c + \mu d}, \frac{d}{\lambda c + \mu d}\right).$$

Proposition 6 *Let $\mathbf{u}(.)$ be a solution to (1) starting at $\mathbf{u} = (x, y) \in S^\circ$. Then for any $t > 0$*

1.

$$\begin{array}{llll} \frac{d\theta(\mathbf{u}(t))}{dt} > 0 & \text{and} \quad \theta(\mathbf{u}(t)) < \theta(\mathbf{u}_*) & \text{if} & \theta(\mathbf{u}) < \theta(\mathbf{u}_*) \\ \frac{d\theta(\mathbf{u}(t))}{dt} = 0 & \text{and} \quad \theta(\mathbf{u}(t)) = \theta(\mathbf{u}_*) & \text{if} & \theta(\mathbf{u}) = \theta(\mathbf{u}_*) \\ \frac{d\theta(\mathbf{u}(t))}{dt} < 0 & \text{and} \quad \theta(\mathbf{u}(t)) > \theta(\mathbf{u}_*) & \text{if} & \theta(\mathbf{u}) > \theta(\mathbf{u}_*). \end{array}$$

2.

$$\begin{aligned} x(t) &\geq \min\left(x, c\tfrac{\lambda x + \mu y}{\lambda c + \mu d}\right) \\ y(t) &\geq \min\left(y, d\tfrac{\lambda x + \mu y}{\lambda c + \mu d}\right). \end{aligned} \tag{9}$$

Proof. From Lemma 4 we know that $\mathbf{u}(t) \in S^\circ$ for any $t \geq 0$.

1. We compute

$$\frac{d\theta(\mathbf{u}(t))}{dt} = \frac{1}{x^2(t) + y^2(t)}\left(\frac{d}{c} - \frac{y(t)}{x(t)}\right)\left[\alpha + \frac{x(t)(\beta - \gamma + \sqrt{(\beta-\gamma)^2 + 4\alpha\delta})}{2y(t)}\right] \tag{10}$$

   and the conclusion follows.

2. Let $\mathbf{a}, \mathbf{b} \in S^\circ$ with $0 \leq \theta(\mathbf{a}) < \theta(\mathbf{u}_*)$, $\theta(\mathbf{u}_*) < \theta(\mathbf{b}) \leq \frac{\pi}{2}$ and let $l > 0$. We set

$$\begin{aligned} A &= \{\mathbf{v} \in S^\circ : \theta(\mathbf{a}) \leq \theta(\mathbf{v}) \leq \theta(\mathbf{u}_*)\} \\ B &= \{\mathbf{v} \in S^\circ : \theta(\mathbf{b}) \geq \theta(\mathbf{v}) \geq \theta(\mathbf{u}_*)\}. \end{aligned} \tag{11}$$

It follows from above that any solution starting from A stays in A, and the same is true for B. If $\mathbf{u} \in A$,

$$x(t) \geq -\frac{\mu}{\lambda}y(t) + (x + \frac{\mu}{\lambda}y) \geq -\frac{\mu d}{\lambda c}x(t) + x + \frac{\mu}{\lambda}y$$

and therefore

$$x(t) \geq c\frac{\lambda x + \mu y}{\lambda c + \mu d}.$$

If $\mathbf{u} \in B$,

$$x(t) \geq \frac{x}{y}y(t) \geq \frac{x}{y}(-\frac{\lambda}{\mu}x(t) + \frac{\lambda x + \mu y}{\mu})$$

and therefore

$$x(t) \geq x.$$

Same estimations for $y(t)$. ■

Corollary 7 *Let* $\mathbf{u}(.)$ *be a solution to (1). Then*

$$\lim_{t\to\infty} \theta(\mathbf{u}(t)) = \theta(\mathbf{u}_*), \qquad i.e. \qquad \lim_{t\to\infty} \frac{y(t)}{x(t)} = \frac{d}{c}.$$

Proof. If $\mathbf{u} = (x, y) \in S^\circ$, this is an easy consequence of (10). If $\mathbf{u} = (0, 0)$ we may apply Lemma 4 and then (10). ■

5 Existence and uniqueness. Case $x > 0, y > 0$.

Proposition 8 *There exists a unique solution* $\mathbf{u}(.)$ *to (1) starting at* $\mathbf{u} = (x, y)$ *with* $x > 0, y > 0$. *It satisfies* $x(t) > 0, y(t) > 0$ *for any* $t \geq 0$.

Proof. We now assume $\theta(\mathbf{a}) > 0$ and $\theta(\mathbf{b}) < \frac{\pi}{2}$ in (11). Let $l > 0$ and $\nu = (\lambda, \mu)$. We set $L := \{\mathbf{v} \in S^\circ : \nu.\mathbf{v} \geq l\}$. From Lemma 4 and Proposition 6 we know that any solution starting from $A \cap L$ stays in $A \cap L$, and the same is true for $B \cap L$. As the system is Lipschitz in $A \cap L$ and in $B \cap L$, there is a unique global solution to (1) in both cases. ■

6 Existence and uniqueness. Case $x = 0, y > 0$.

Proposition 9 *There exists a unique solution* $\mathbf{u}(.)$ *to (1) starting at* $\mathbf{u} = (x, y)$ *with* $x = 0, y > 0$. *It satisfies* $x(t) > 0, y(t) > 0$ *for any* $t > 0$.

Proof. Let $\varepsilon \in (0, y\frac{\mu d}{\lambda c + \mu d})$. We define on $\mathbb{R}_+ \times \mathbb{R}$

$$\psi_\varepsilon(x, z) := \frac{1}{\max(\gamma x + z, \alpha\varepsilon)}.$$

We apply Lemma 3 to obtain a unique solution $x_\varepsilon(.), z_\varepsilon(.)$ to

$$\begin{aligned} x_\varepsilon(t) &= \alpha \int_0^t \frac{ds}{x_\varepsilon(s)} + \alpha^m \beta \int_0^t \psi_\varepsilon(x_\varepsilon(s), z_\varepsilon(s))ds \\ z_\varepsilon(t) &= \alpha y + \alpha(\alpha\delta - \beta\gamma) \int_0^t \psi_\varepsilon(x_\varepsilon(s), z_\varepsilon(s))ds. \end{aligned} \tag{12}$$

Let

$$\begin{aligned} y_\varepsilon(t) &= \tfrac{1}{\alpha}(\gamma x_\varepsilon(t) + z_\varepsilon(t)) \\ \tau_y(\varepsilon) &= \inf\{t > 0 : y_\varepsilon(t) < \varepsilon\}. \end{aligned}$$

On the interval $[0, \tau_y(\varepsilon)]$, $(x_\varepsilon(.), y_\varepsilon(.))$ is the unique solution to (1). From (9) we know that $y_\varepsilon(t) > \varepsilon$ on this interval. Thus $\tau_y(\varepsilon)) = \infty$ and $(x(.), y(.)) := (x_\varepsilon(.), y_\varepsilon(.))$ is the unique global solution to (1). ■

7 Path behavior.

Let us note $\mathbf{u}(t, \mathbf{u}_0)$ the solution to (1) starting at $\mathbf{u}_0 \in S^\circ$. Using Gronwall's inequality as in the proof of uniqueness, it is easily seen that for any $t > 0$ the solution $\mathbf{u}(t, \mathbf{u}_0)$ continuously depends on the initial condition $\mathbf{u}_0$. It has the Scaling Property:

$$\text{(SC)} \qquad \mathbf{u}(r^2 t, \mathbf{u}_0) = r\mathbf{u}(t, \frac{\mathbf{u}_0}{r})$$

for any $r > 0$, $t \geq 0$, $\mathbf{u}_0 \in S^\circ$. Using Lemma 4 we also note that any solution $\mathbf{u}(.)$ to (1) has the Semi-group Property:

$$\text{(SG)} \qquad \mathbf{u}(s + t) = \mathbf{u}(t, \mathbf{u}(s))$$

for any $s > 0$ and $t \geq 0$. With Proposition 8 and Proposition 9 this entails that $x(t) > 0$ and $y(t) > 0$ for any $t > 0$. We now set for any $r > 0$:

$$\begin{aligned} L_r &:= \{\mathbf{u} = (x, y) : x > 0, y > 0, \nu.\mathbf{u} = r\} \\ \overline{L_r} &:= \{\mathbf{u} = (x, y) : x \geq 0, y \geq 0, \nu.\mathbf{u} = r\} \end{aligned}$$

Lemma 10 *Let $\mathbf{u}(.)$ be a solution to (1) starting at $\mathbf{u}_0 \in S$ with $\nu.\mathbf{u}_0 \leq r$. We set*

$$\tau(r) := \inf\{t \geq 0 : \nu.\mathbf{u}(t) = r\}.$$

Then

$$\tau(r) \leq \frac{r^2}{2[\lambda(\lambda\alpha + \mu\gamma) + \mu(\lambda\beta + \mu\delta)]}.$$

Proof. Set

$$z(t) := \nu.\mathbf{u}(t).$$

As

$$z(t) = \nu.\mathbf{u}_0 + [\lambda(\lambda\alpha + \mu\gamma) + \mu(\lambda\beta + \mu\delta)] \int_0^t \frac{ds}{z(s)} + \int_0^t f(s)ds$$

with

$$f(s) = \mu(\lambda\alpha + \mu\gamma)\frac{y(s)}{x(s)} + \lambda(\lambda\beta + \mu\delta)\frac{x(s)}{y(s)} > 0$$

for $s > 0$, it follows from Lemma 2 that $z(t) \geq w(t)$ where

$$w(t) = \nu.\mathbf{u}_0 + [\lambda(\lambda\alpha + \mu\gamma) + \mu(\lambda\beta + \mu\delta)] \int_0^t \frac{ds}{w(s)},$$

and then

$$z^2(t) \geq w^2(t) = 2[\lambda(\lambda\alpha + \mu\gamma) + \mu(\lambda\beta + \mu\delta)]t + (\nu.\mathbf{u}_0)^2.$$

The conclusion follows. ■

We now define $q : \overline{L_1} \to L_1$ by

$$q(\mathbf{u}_1) = \frac{1}{2}\mathbf{u}(\tau(2), \mathbf{u}_1)$$

where

$$\tau(2) = \inf\{t \geq 0 : \nu.\mathbf{u}(t, \mathbf{u}_1) = 2\} \tag{13}$$

is finite from the above Lemma. Let now $r > 0$ and $\mathbf{u} \in \overline{L_r}$. From (SC), the geometric paths in S of $\mathbf{u}(., \mathbf{u})$ and $r\mathbf{u}(., \frac{\mathbf{u}}{r})$ are identical. Therefore

$$\mathbf{u}(\tau(2r), \mathbf{u}) = r\mathbf{u}(\tau(2), \frac{\mathbf{u}}{r})$$

where in this equality $\tau(2r)$ is relative to $\mathbf{u}(., \mathbf{u})$ and $\tau(2)$ is relative to $\mathbf{u}(., \frac{\mathbf{u}}{r})$. Thus

$$q(\frac{\mathbf{u}}{r}) = \frac{1}{2r}\mathbf{u}(\tau(2r), \mathbf{u}).$$

Iterating and using (SG), we get for any $n \geq 1$

$$q^n(\frac{\mathbf{u}}{r}) = \frac{1}{2^n r}\mathbf{u}(\tau(2^n r), \mathbf{u}). \tag{14}$$

Proposition 11 *There exists $k \in (0, 1)$ such that for any $\mathbf{u}_1 \in \overline{L_1}$*

$$|q(\mathbf{u}_1) - \mathbf{u}_*| \leq k\,|\mathbf{u}_1 - \mathbf{u}_*|.$$

Proof. From Proposition 6 we know that q has a unique invariant point u_*. We consider the solution $\mathbf{u}(t, \mathbf{u}_1) = (x(t), y(t))$ on the time interval $[0, \tau(2)]$, where $\tau(2)$ was defined in (13). We first assume that

$$\frac{y_1}{x_1} < \frac{y_*}{x_*} = \frac{d}{c}.$$

We note for further use that

$$\begin{array}{ccccccc} x_* & < & x_1 & \leq & \frac{1}{\lambda} & & \\ 0 & \leq & y_1 & < & y_* & < & \frac{1}{\mu}. \end{array}$$

We set

$$\mathbf{u}_2 = (x_2, y_2) := \mathbf{u}_1 + \frac{(\alpha y_1 + \beta x_1, \gamma y_1 + \delta x_1)}{\lambda(\alpha y_1 + \beta x_1) + \mu(\gamma y_1 + \delta x_1)}.$$

Then, $2\mathbf{u}_*$, $\mathbf{u}_* + \mathbf{u}_1$ and $\mathbf{u}_2 \in L_2$. Setting for $z \in [0, \infty]$

$$h(z) = \frac{\gamma z + \delta}{\alpha z + \beta}$$

we compute

$$\frac{dh}{dz}(z) = \frac{\beta\gamma - \alpha\delta}{(\alpha z + \beta)^2}\,. \tag{15}$$

From Proposition 6 we know that for any $t \in [0, \tau(2)]$

$$\frac{y(t)}{x(t)} < \frac{d}{c}. \tag{16}$$

When $\alpha\delta > \beta\gamma$, it follows from (15) and (16) that

$$\frac{\dot{y}(t)}{\dot{x}(t)} = h(\frac{y(t)}{x(t)}) > h(\frac{d}{c}) = \frac{d}{c}$$

and then $2q(\mathbf{u}_1)$ belongs to the open interval $(2\mathbf{u}_*, \mathbf{u}_* + \mathbf{u}_1)$ on L_2. Therefore,

$$|2q(\mathbf{u}_1) - 2\mathbf{u}_*| < |\mathbf{u}_1 - \mathbf{u}_*|.$$

When $\alpha\delta = \beta\gamma$, the path of the solution is a straight half-line with slope $\frac{d}{c}$ and

$$|2q(\mathbf{u}_1) - 2\mathbf{u}_*| = |\mathbf{u}_1 - \mathbf{u}_*|.$$

When $\alpha\delta < \beta\gamma$, $\frac{\dot{y}(t)}{\dot{x}(t)}$ is increasing on $[0, \tau(2)]$ and then

$$\frac{\gamma y_1 + \delta x_1}{\alpha y_1 + \beta x_1} \leq \frac{\dot{y}(t)}{\dot{x}(t)} < \frac{d}{c}.$$

As a result, $2q(\mathbf{u}_1)$ belongs to the open interval $(\mathbf{u}_* + \mathbf{u}_1, \mathbf{u}_2)$ on L_2. Moreover, using the relation $\lambda x_1 + \mu y_1 = 1$ twice, we get

$$\begin{array}{rcl} 2x_1 - x(\tau(2)) & > & 2x_1 - x_2 \\ & = & x_1 - \frac{\alpha y_1 + \beta x_1}{\lambda(\alpha y_1 + \beta x_1) + \mu(\gamma y_1 + \delta x_1)} \\ & = & \frac{\alpha\lambda x_1 y_1 + \beta\lambda x_1^2 + \gamma\mu x_1 y_1 + \delta\mu x_1^2 - \alpha y_1 - \beta x_1}{\lambda(\alpha y_1 + \beta x_1) + \mu(\gamma y_1 + \delta x_1)} \\ & = & \mu\frac{-\alpha y_1^2 + (\gamma - \beta)x_1 y_1 + \delta x_1^2}{\lambda(\alpha y_1 + \beta x_1) + \mu(\gamma y_1 + \delta x_1)} \\ & = & \frac{\alpha\mu}{\lambda(\alpha y_1 + \beta x_1) + \mu(\gamma y_1 + \delta x_1)}(x_1\frac{d}{c} - y_1)\left(y_1 + \frac{\beta - \gamma + \sqrt{(\beta-\gamma)^2 + 4\alpha\delta}}{2\alpha}x_1\right) \\ & = & \frac{\alpha(\lambda c + \mu d)}{c[\lambda(\alpha y_1 + \beta x_1) + \mu(\gamma y_1 + \delta x_1)]}(x_1 - x_*)\left(y_1 + \frac{\beta - \gamma + \sqrt{(\beta-\gamma)^2 + 4\alpha\delta}}{2\alpha}x_1\right). \end{array}$$

In the same way,

$$\begin{array}{rcl} y(\tau(2)) - 2y_1 & > & y_2 - 2y_1 \\ & = & \frac{\alpha(\lambda c + \mu d)}{c[\lambda(\alpha y_1 + \beta x_1) + \mu(\gamma y_1 + \delta x_1)]}(y_* - y_1)\left(y_1 + \frac{\beta - \gamma + \sqrt{(\beta-\gamma)^2 + 4\alpha\delta}}{2\alpha}x_1\right). \end{array}$$

Setting

$$k_1 \;=\; \frac{\lambda\mu(\beta - \gamma + \sqrt{(\beta-\gamma)^2 + 4\alpha\delta})}{4[\lambda(\lambda\alpha + \mu\gamma) + \mu(\lambda\beta + \mu\delta)]} \;>\; 0$$

we obtain

$$|2q(\mathbf{u}_1) - 2\mathbf{u}_1| > 2k_1\,|\mathbf{u}_1 - \mathbf{u}_*|$$

and then

$$\begin{array}{rcl} |q(\mathbf{u}_1) - \mathbf{u}_*| & = & |\mathbf{u}_1 - \mathbf{u}_*| - |q(\mathbf{u}_1) - \mathbf{u}_1| \\ & < & (1 - k_1)\,|\mathbf{u}_1 - \mathbf{u}_*|. \end{array}$$

If now $\frac{y_1}{x_1} > \frac{d}{c}$, in the same way there exists $k_2 > 0$ such that

$$|q(\mathbf{u}_1) - \mathbf{u}_*| < (1 - k_2)\,|\mathbf{u}_1 - \mathbf{u}_*|$$

We may take $k = 1 - \min(k_1, k_2)$ ■

8 Uniqueness. Case $x = 0, y = 0$.

Existence was proven in Section 3. We may now conclude the proof of Theorem 1.

Proposition 12 *The solution given by (6) is the unique solution to (1) starting at* $\mathbf{0}$*.*

Proof. Let $\mathbf{u}(.)$ be a solution to (1) starting at $\mathbf{0}$. For any $n \geq 1$ and $s > 0$,

$$\mathbf{u}(\tau(s)) = \mathbf{u}(\tau(s), \mathbf{u}(\tau(s2^{-n})))$$

where $\tau(s)$ in the l.h.s. is relative to $\mathbf{u}(.)$ and $\tau(s)$ in the r.h.s. is relative to $\mathbf{u}(., \mathbf{u}(\tau(s2^{-n})))$. We may apply (14) with $r = s2^{-n}$ and $\mathbf{u} = \mathbf{u}(\tau(s2^{-n}))$. We obtain

$$\frac{\mathbf{u}(\tau(s))}{s} = q^n(\frac{\mathbf{u}(\tau(s2^{-n}))}{s2^{-n}}).$$

From Proposition 11 (or directly from (10) it follows that the r.h.s. converges to $\mathbf{u}_*$ as $n \to \infty$. Thus for any $s > 0$

$$\frac{\mathbf{u}(\tau(s))}{s} = \mathbf{u}_*$$

and this implies

$$\frac{y(\tau(s))}{x(\tau(s))} = \frac{d}{c}.$$

From Lemma 10 we know that τ is one-to-one from $[0, \infty)$ to $[0, \infty)$, and thus for any $t > 0$,

$$\frac{y(t)}{x(t)} = \frac{d}{c}.$$

Going back to the system (1) we conclude that

$$\begin{aligned} x(t) &= c\sqrt{t} \\ y(t) &= d\sqrt{t}. \end{aligned} \qquad \blacksquare$$

References

[1] Fiedler P. and Pták V., *Some generalizations of positive definiteness and monotonicity*, Numerische Mathematik, 9 (1966), 163-172.