Random Fuzzy Galois Lattices
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Abstract: Given a random fuzzy context, the probability of a closed set and the probability of a pair be a concept are computed. This yields the mean size of a fuzzy random concept lattice which can be useful for the choice of which fuzzy structure is appropriate to a context.

1 Introduction

1.1 Preliminaries on fuzzy logic

The fuzzy setting is obtained by replacing the two-element Boolean algebra on which the classical concept lattices are based by complete residuated lattices.

Residuated lattices, basic properties and examples—see [2]

Notation:
- \( L = \langle L, \otimes, \rightarrow, \wedge, \vee, 0, 1 \rangle \) ... complete residuated lattice
- \( A \in L^U \) ... \( A \) is an \( L \)-set (fuzzy set) in universe \( U \), \( A(u) \) is a degree to which element \( u \in U \) belongs to \( A \)
- \( \langle X, Y, I \rangle \) ... \( L \)-context (fuzzy context), i.e. \( I : X \times Y \rightarrow L \);
- \( I(x, y) \) ... degree to which object \( x \) has attribute \( y \)
- for \( A \in L^X \), \( B \in L^Y \), concept forming operators
  \[ A^\uparrow(y) = \bigwedge_{x \in X} (A(x) \rightarrow I(x, y)), B^\downarrow(y) = \bigwedge_{y \in Y} (B(y) \rightarrow I(x, y)) \]
- \( B(X, Y, I) = B(I) = \{ \langle A, B \rangle \in L^X \times L^Y \mid A^\uparrow = B, B^\downarrow = A \} \) ... \( L \)-concept lattice (fuzzy concept lattice) of \( \langle X, Y, I \rangle \);

Pairs \( \langle A, B \rangle \in L^X \times L^Y \) with \( A^\uparrow = B, B^\downarrow = A \) are called \( L \)-concepts (fuzzy concepts, or just concepts).

For simplicity, we assume that \( L \) is finite and linearly ordered, i.e.

\[ L = \{ 0, \frac{1}{n}, \ldots, \frac{n-2}{n-1}, 1 \} \]

Then \( \wedge \) and \( \vee \) are min and max. Examples of \( \otimes \) and \( \rightarrow \) include

- Lukasiewicz: \( a \otimes b = \max(0, b + 1 - 1), a \rightarrow b = \min(1, 1 - a + b) \),
- Gödel: \( a \otimes b = \min(a, b), a \rightarrow b = 1 \) if \( a \leq b \), \( a \rightarrow b = b \) if \( a > b \).
1.2 Preliminaries on probabilistic setting

A random fuzzy context \ldots as in [4, 5], but generalized Bernoulli model.

Let for \( y \in Y \), \( \langle p_{y,0}, \ldots, p_{y,n-1} \rangle \) be a distribution on \( L \). That is, for every \( x \in X, y \in Y \), \( I(x, y) \) has value \( \frac{i}{n-1} \) with probability \( p_{y,i} \).

Furthermore, for \( K \subseteq L \), put
\[
p_y(K) = \sum_{i \in K} p_{y,i},
\]
the associated probability measure.

In Boolean case, we thus have \( \langle p_{y,0}, p_{y,1} \rangle \) with \( p_{y,0} = 1 - p_{y,1} \). In such case, we also denote \( p_{y,1} \) simply by \( p_y \).

2 Problem statement

**Problem 1** (main problem): What is the size of a random fuzzy concept lattice?

Inspired by [4, 5]

**Problem 2** (subproblem): Given a random fuzzy context \( \langle X, Y, I \rangle \) and a pair \( \langle A, B \rangle \in L^X \times L^Y \), what is the probability, \( P(A, B) = P(\langle A, B \rangle \in B(I)) \), that \( \langle A, B \rangle \) is a concept.

2.1 General observations

It seems crucial is to find an appropriate reformulation of the definition of a concept which is suitable for assessing \( P(A, B) \).

Perhaps three possible views may be considered.

2.2 View 1

2.2.1 Suitable conditions for \( \langle A, B \rangle \) being a concept

Find necessary and sufficient conditions for \( \langle A, B \rangle \), possibly independent, whose conjunction is equivalent to \( \langle A, B \rangle \) being a concept.

In fact, the Boolean case is based on the following conditions:

\[
\begin{align*}
C_1 &: A \otimes B \subseteq I \text{(which is equivalent to } A \subseteq B^\downarrow \text{, and to } B \subseteq A^\uparrow), \\
C_2 &: A^\uparrow \subseteq B, \\
C_3 &: B^\downarrow \subseteq A.
\end{align*}
\]

One has
\[
(\langle A, B \rangle \in B(I) \text{ if and only if } C_1 \& C_2 \& C_3.
\]

These conditions are seemingly not independent, which is also true in a general case. However, in the Boolean case \( (L = \{0, 1\}) \), they are independent. With independence we obtain
\[
P(A, B) = P(A \otimes B \subseteq I) \cdot P(A^\uparrow \subseteq B) \cdot P(B^\downarrow \subseteq A).
\]
2.2.2 Getting the probabilities $P(A \otimes B \subseteq I), P(A^\uparrow \subseteq B), P(B^\downarrow \subseteq A)$

Here we show, how the above probabilities may be computed in the general setting (general $L$). It will turn out that for $L = \{0,1\}$ (Boolean case), the conditions are independent and the computation leads to the formulas known for the Boolean case.

**First** $P(A \otimes B \subseteq I)$:

$A \otimes B \subseteq I$ iff for each $x \in X, y \in Y: A(x) \otimes B(y) \leq I(x, y)$. Therefore,

$$
P(A \otimes B \subseteq I) = \prod_{x \in X} \prod_{y \in Y} P(A(x) \otimes B(y) \leq I(x, y)) =
$$

$$
= \prod_{x \in X} \prod_{y \in Y} p_y([A(x) \otimes B(y), 1]),
$$

because the interval $[A(x) \otimes B(y), 1]$ contains just the values $a$ in $L$ for which $(A(x) \otimes B(y)) \leq a$.

**Remark 1** In Boolean case:

$$
\prod_{x \in X} \prod_{y \in Y} p_y([A(x) \otimes B(y), 1]) = \prod_{A(x) = 1} \prod_{B(y) = 1} p_y([A(x) \otimes B(y), 1]) =
$$

$$
= \prod_{A(x) = 1} \prod_{B(y) = 1} p_y([1]) = \prod_{x \in X} \prod_{y \in Y} p_{y,1} = \prod_{y \in B} p_B^{[A]},
$$

where $p_B = \prod_{y \in B} p_{y,1}$, because if $A(x) = 0$ or $B(y) = 0$, then $p_y([A(x) \otimes B(y), 1]) = p_y([0, 1]) = 1$.

**Second** $P(A^\uparrow \subseteq B)$:

$A^\uparrow \subseteq B$ iff for each $y \in Y: A^\uparrow(y) \leq B(y)$. Since $L$ is linearly ordered, we have

$$
P(A^\uparrow(y) \leq B(y)) = 1 - P(A^\uparrow(y) > B(y)).
$$

Hence,

$$
P(A^\uparrow \subseteq B) = \prod_{y \in Y} P(A^\uparrow(y) \leq B(y)) = \prod_{y \in Y} [1 - P(A^\uparrow(y) > B(y))],
$$

so the problem reduces to determining $P(A^\uparrow(y) > B(y))$. We have $A^\uparrow(y) > B(y)$ iff $\bigwedge_{x \in X} A(x) > B(y)$). Since $L$ is linearly ordered and finite, the last condition is equivalent to: for each $x \in X: A(x) \rightarrow I(x, y) > B(y)$. Hence,

$$
P(A^\uparrow(y) > B(y)) = \prod_{x \in X} P(A(x) \rightarrow I(x, y) > B(y)).
$$

Now denote by

$$
R(a,b) = \{ l \in L \mid b \rightarrow l > a \} 
$$

(1)

$$
V(a,b,c) = \{ l \in L \mid l \land a \rightarrow b > c \} 
$$

(2)
\( R(a, b) \) is easily computable; it has explicit form for particular structures such as Gödel or Łukasiewicz. Then
\[
P(P(A(x) \rightarrow I(x, y)) > B(y))) = p_y(R(B(y), A(x))).
\]

Altogether,
\[
P(A^\uparrow \subseteq B) = \prod_{y \in Y} P(A^\uparrow(y) \leq B(y)) = \prod_{y \in Y} [1 - P(A^\uparrow(y) > B(y))] = \prod_{y \in Y} (1 - \prod_{x \in X} p_y(R(B(y), A(x)))�.
\]

**Remark 2** In Boolean case, the above formula reduces to
\[
P(A^\uparrow \subseteq B) = \prod_{y \in Y} P(A^\uparrow(y) \leq B(y)) = \prod_{B(y) = 0} [1 - P(A^\uparrow(y) > B(y))]
\]
(because for \( B(y) = 1 \): \( P(A^\uparrow(y) > B(y)) = 1 \))
\[
\prod_{B(y) = 0} (1 - \prod_{x \in X} p_y(R(B(y), A(x)))) = \prod_{B(y) = 0} (1 - \prod_{A(x) = 1} p_y(R(B(y), A(x)))) = \prod_{y \in Y - B} (1 - \prod_{x \in A} p_y,1) = \prod_{y \in Y - B} (1 - p_y(B, A) = 1).
\]
(because \( p_y(R(B(y), A(x))) = p_y(R(0,1)) = p_y(\{1\}) = p_y,1\)), which is the classical formula.

### 3 Probability that a fuzzy set is closed

#### 3.1 Closed fuzzy sets

**Definition 1** Let \( A \in L^X \) (resp. \( B \in L^Y \)) be a fuzzy set, the closure of \( A \) (resp. of \( B \)), denoted by \( \bar{A} \) (resp. by \( \bar{B} \)), is the fuzzy set \( (A^\uparrow)^\downarrow \in L^X \) (resp. \( (B^\downarrow)^\uparrow \in L^Y \)). \( A \) (resp. \( B \)) is said closed if \( \bar{A} = A \) (resp. \( \bar{B} = B \)).

Let \( C(X, I) \) (resp. \( C(Y, I) \)) denote the set of closed sets in \( L^X \) (resp. in \( L^Y \)). It is easily seen that \( A \rightarrow (A, A^\uparrow) \) (resp. \( B \rightarrow (B^\downarrow, B) \)) is a one-to-one onto mapping from \( C(X, I) \) (resp. from \( C(Y, I) \)) to \( B(X, Y, I) \). In particular the random number of concepts \( |B(X, Y, I)| \) is equal to the random number \( |C(X, I)| \) (resp. \( |C(Y, I)| \)) of closed sets in \( L^X \) (resp. in \( L^Y \)). Our goal, in this section, is to compute \( P(A \in C(X, I)) \), the probability of the fuzzy set \( A \) to be closed. The following property will simplify the computation.

#### 3.2 Extensivety

**Definition 2** The closure operator is said extensive if
\[
\forall A \in L^X, \forall B \in L^Y, A \leq \bar{A} \text{ and } B \leq \bar{B}.
\] (3)

Two examples of extensive closures are given by the following structures
Proposition 1  Lukasiewicz and Gödel structures define extensive closure operators

Proof: \( A(x) \leq \bar{A}(x) = \bigwedge_{y \in Y} A^\uparrow y \rightarrow I(x, y) \) holds \( \forall x \in X \) if and only if we have
\[
\forall x \in X, \forall y \in Y, A(x) \leq A^\uparrow y \rightarrow I(x, y).
\] (4)

In the Lukasiewicz case \( A^\uparrow y \rightarrow I(x, y) = \min(1, 1 - A^\uparrow y + I(x, y)) \). As \( A(x) \leq 1 \), (4) holds if \( A(x) \leq 1 - A^\uparrow y + I(x, y) \), that is \( A^\uparrow y \leq 1 - A(x) + I(x, y) \). But this inequality holds because \( A^\uparrow y = \bigwedge_{t \in X} A(t) \rightarrow I(t, y) \leq A(x) \rightarrow I(x, y) \leq 1 - A(x) + I(x, y) \);

In the Gödel structure case, if \( A^\uparrow y \leq I(x, y) \) then \( A^\uparrow y \rightarrow I(x, y) = 1 \) and (4) trivially holds. If \( I(x, y) < A^\uparrow y = \bigwedge_{t \in X} A(t) \rightarrow I(t, y) \leq A(x) \rightarrow I(x, y) \), then we necessarily have \( A(x) \leq I(x, y) \) otherwise we would obtain \( I(x, y) < I(x, y) \), a contradiction. In that case \( A^\uparrow y \rightarrow I(x, y) = I(x, y) \) and the inequality \( A(x) \leq I(x, y) \) is nothing but (4). □

3.3 Notations

For computations convenience, we will adopt the following notations.

First of all, as \( L \) is assumed finite and linearly ordered we will denote by \( l^- \) the predecessor of \( l \) for any \( l \in L : l > 0 \).

Next, for any \( A \in L^X, x \in X, y \in Y, S \subseteq X \) such that \( |S| \geq 1 \), consider the random variable
\[
F_{S,A}(y) = \bigwedge_{x \in S} (A(x) \rightarrow I(x, y))
\] (5)

and the following events
\[
G_{x,A} = (\bar{A}(x) > A(x))
\] (6)
\[
G_{x,y,A} = (A^\uparrow y \rightarrow I(x, y) > A(x))
\] (7)
\[
G_{S,A} = \bigcap_{x \in S} G_{x,A}
\] (8)
\[
G_{S,y,A} = \bigcap_{x \in S} G_{x,y,A}
\] (9)

Observe that \( G_{\{x\},A} = G_{x,A} \) and that
\[
A^\uparrow(y) = F_{X,A}(y)
\] (10)
\[
F_{S_1 \cup S_2,A}(y) = F_{S_1,A}(y) \land F_{S_2,A}(y)
\] (11)

for any nonempty \( S_1, S_2 \subseteq X \).
3.4 Inclusion-Exclusion principle

The following result hinges on the inclusion-exclusion principle

Lemma 1 Assuming extensively, we have for any \( A \in L^X \)

\[
P(A \in C(X, I)) = 1 - \sum_{x \in X} \prod_{y \in Y} P(G_{x,y,A}) + \sum_{k=2}^{\lfloor |X| \rfloor} \left[ (-1)^k \sum_{|K|=k} \prod_{y \in Y} P(G_{K,y,A}) \right] \tag{12}
\]

Proof: First, by definition \( P(A \in C(X, I)) = P(\bigcap_{x \in X} (\bar{A}(x) = A(x))) \). Then, extensively implies that

\[
P(A \in C(X, I)) = P(\bigcap_{x \in X} (\bar{A}(x) \leq A(x))) = 1 - P(\bigcup_{x \in X} G_{x,A})
\]

and inclusion-exclusion principle yields

\[
P(A \in C(X, I)) = 1 - \sum_{x \in X} P(G_{x,A}) + \sum_{k=2}^{\lfloor |X| \rfloor} \left[ (-1)^k \sum_{|K|=k} P(G_{K,A}) \right]. \tag{13}
\]

Now

\[
G_{x,A} = (\bar{A}(x) > A(x)) = \left( \bigwedge_{y \in Y} (A^\top(y) \rightarrow I(x,y)) > A(x) \right)
\]

shows that

\[
G_{x,A} = \bigcap_{y \in Y} G_{x,y,A}. \tag{14}
\]

Observing that

\[
G_{x,y,A} = \left( \bigwedge_{t \in X} (A(t) \rightarrow I(t,y)) \rightarrow I(x,y) > A(x) \right)
\]

only depends on column \( y \) random values of \( I \), independence of columns of \( I \) and (14) then imply

\[
P(G_{x,A}) = \prod_{y \in Y} P(G_{x,y,A}). \tag{16}
\]

Using (8), (14) and (9), we now observe that

\[
G_{K,A} = \bigcap_{x \in K} \bigcap_{y \in Y} G_{x,y,A} \tag{17}
\]

\[
= \bigcap_{y \in Y} \bigcap_{x \in K} G_{x,y,A} \tag{18}
\]

\[
= \bigcap_{y \in Y} G_{K,y,A} \tag{19}
\]
and as $G_{K,y,A}$ is the intersection of the events $G_{x,y,A}$, $x \in K$, each of them only depending on column $y$ values of $I$, $G_{K,y,A}$ itself only depends on column $y$ values of $I$, so that independence of columns yields

$$P(G_{K,A}) = \prod_{y \in Y} P(G_{K,y,A}).$$  \hfill (20)

Finally (13), (14) and (20) clearly imply the result announced in lemma 1 formula (12). \hfill \Box

### 3.5 Distribution of $F_{S,A}(y)$

According to lemma 1 formula (12) it remains to compute $P(G_{x,y,A})$ and more generally $P(G_{K,y,A})$. As it will be seen this will require the computation of $P(F_{S,A}(y) = l)$ for $l \in L, \emptyset \subset S \subseteq X$.

**Lemma 2** Let $y \in Y$ and let $S : \emptyset \subset S \subseteq X$, be a nonempty finite subset of $X$. For any $l \in L$, let

$$q(l, y, S, A) = P(F_{S,A}(y) > l) \quad \text{and} \quad D(l, y, S, A) = P(F_{S,A}(y) = l).$$

Then, we have

$$q(l, y, S, A) = \prod_{s \in S} p_y(R(A(s), l))$$ \hfill (21)

and

$$D(l, y, S, A) = \begin{cases} 1 - q(0, y, S, A) & \text{if } l = 0 \\ q(l^-, y, S, A) - q(l, y, S, A) & \text{if } 0 < l < 1 \\ q(1^-, y, S, A) & \text{if } l = 1. \end{cases}$$ \hfill (22)

**Proof:**

$$q(l, y, S, A) = P(\bigwedge_{s \in S} (A(s) \rightarrow I(s, y)) > l)$$

$$= P(\bigwedge_{s \in S} (A(s) \rightarrow I(s, y)) > l)$$

$$= \prod_{s \in S} P(A(s) \rightarrow I(s, y) > l)$$

$$= \prod_{s \in S} p_y(R(l, A(s))). \quad \Box$$

and the three equalities in (22) are obvious consequences of the definitions of $q$ and $D$.  


3.6 Computation of $P(G_{x,y,A})$

Lemma 3 If $\{x\} \subsetneq X$, we have

$$P(G_{x,y,A}) = \sum_{r \in L} p_y(r) \sum_{l \in L : (l \land (A(x) \rightarrow r)) \rightarrow r > A(x)} D(l, y, X \setminus \{x\}, A)$$

(23)

where $D$ is defined in (22).

If $X = \{x\}$ then

$$P(G_{x,y,A}) = \sum_{r \in L : (A(x) \rightarrow r) \rightarrow r > A(x)} p_y(r).$$

(24)

Proof: First of all, due to (10), we have

$$P(G_{x,y,A}) = P(A^y \rightarrow I(x,y) > A(x)) = P(F_{x,A}(y) \rightarrow I(x,y) > A(x)).$$

As $F_{x,A}(y) = F_{x \setminus \{x\},A}(y) \land F_{\{x\},A}(y)$ due to (11) and as $F_{\{x\},A}(y) = A(x) \rightarrow I(x,y)$, we get

$$P(G_{x,y,A}) = P([F_{x \setminus \{x\},A}(y) \land (A(x) \rightarrow I(x,y))] \rightarrow I(x,y) > A(x))$$

$$= \sum_{r \in L} P([F_{x \setminus \{x\},A}(y) \land (A(x) \rightarrow r)] \rightarrow r > A(x), I(x,y) = r)$$

$$= \sum_{r \in L} P([F_{x \setminus \{x\},A}(y) \land (A(x) \rightarrow r)] \rightarrow r > A(x)) p_y(r)$$

because $P(I(x,y) = r) = p_y(r)$, and $F_{x \setminus \{x\},A}(y)$ only depends on $(I(t,y))_{t \neq x}$ which is independent of $I(x,y)$.

Applying (22) with $S = X \setminus \{x\}$, we get

$$P(G_{x,y,A}) = \sum_{r \in L} p_y(r) \sum_{l \in L : (l \land (A(x) \rightarrow r)) \rightarrow r > A(x), F_{x \setminus \{x\},A}(y) = l} D(l, y, X \setminus \{x\}, A)$$

as announced.

If $X = \{x\}$, the computation is straightforward:

$$P(G_{x,y,A}) = P([A(x) \rightarrow I(x,y)] \rightarrow I(x,y) > A(x)) = \sum_{r \in L : (A(x) \rightarrow r) \rightarrow r > A(x)} p_y(r).$$

□

3.7 Computation of $P(G_{K,y,A})$

Lemma 4 Let $y \in Y$. Let $K = \{x_1, \ldots, x_k\}$ with distinct $x_i \in X$, where $k = |K| \geq 2$. For any $\ul{l} = (l_1, \ldots, l_k) \in L^k$ let

$$\beta_{A,K,\ul{l}} = \bigwedge_{i=1}^k (A(x_i) \rightarrow l_i).$$
Then, if \( K \subseteq X \)

\[
P(G_{K,y,A}) = \sum_{l = (l_1, \ldots, l_k) \in L^k} \prod_{i=1}^{k} p_y(l_i) \sum_{\alpha \in L, \forall i = 1, \ldots, k} D(\alpha, y, X \setminus K, A) \tag{25}
\]

and if \( K = X \)

\[
P(G_{K,y,A}) = \sum_{l = (l_1, \ldots, l_k) \in L^k} \prod_{i=1}^{k} p_y(l_i) \tag{26}
\]

where \( D \) is defined in (22).

**Remark:** Applying this result to \( K = \{x\} \) provides the result of the preceding section.

**Proof:** Denoting \((I(K, y) = l)\) the event \((I(x_1, y) = l_1, \ldots, I(x_k, y) = l_k)\), we first have

\[
P(G_{K,y,A}) = P(\bigcap_{i=1}^{k} (A^\uparrow(y) \rightarrow I(x_i, y) > A(x_i)))
\]

\[
= \sum_{l = (l_1, \ldots, l_k) \in L^k} P(\bigcap_{i=1}^{k} (A^\uparrow(y) \rightarrow I(x_i, y) > A(x_i)), I(K, y) = l).
\]

If \( K \subseteq X \), we have \( A^\uparrow(y) = F_{X,A}(y) = F_{X \setminus K,A}(y) \land F_{K,A}(y) \) due to (11) and on \((I(K, y) = l)\), we have \( F_{K,A}(y) = \bigwedge_{i=1}^{k} A(x_i) \rightarrow I(x_i, y) = \beta_{A,K,l} \), so that

\[
P(G_{K,y,A}) = \sum_{l = (l_1, \ldots, l_k) \in L^k} P(\bigcap_{i=1}^{k} (F_{X \setminus K,A}(y) \land \beta_{A,K,l} \rightarrow l_i > A(x_i)), I(K, y) = l)
\]

\[
= \sum_{l = (l_1, \ldots, l_k) \in L^k} P(\bigcap_{i=1}^{k} (F_{X \setminus K,A}(y) \land \beta_{A,K,l} \rightarrow l_i > A(x_i)))P(I(K, y) = l)
\]

because the event \( \bigcap_{i=1}^{k} (F_{X \setminus K,A}(y) \land \beta_{A,K,l} \rightarrow l_i > A(x_i)) \) which only depends on \((I(t, y))_{t \in X \setminus K}\) is independent of the event \((I(K, y) = l)\) which only depends on \((I(t, y))_{t \in K}\).

Now, as the entries of \( I \) are independent, we have

\[
P(I(K, y) = l) = P(I(x_1, y) = l_1, \ldots, I(x_k, y) = l_k) = \prod_{i=1}^{k} p_y(l_i)
\]
so that
\[
P(G_{K,y,A}) = \sum_{l=(l_1,\ldots,l_k) \in L_k} \prod_{i=1}^k p_y(l_i) \sum_{\alpha \in L : \alpha \land \beta_{A,K,l} \Rightarrow l_i > A(x_i)} D(\alpha, y, X \setminus K, A)
\]
as announced.

If \( K = X \) then \( k = |X| \) and
\[
P(G_{K,y,A}) = \sum_{l=(l_1,\ldots,l_k) \in L_k} P(l) \prod_{i=1}^k p_y(l_i) \sum_{\alpha \in L : \alpha \land \beta_{A,X,l} \Rightarrow l_i > A(x_i)} D(\alpha, y, X \setminus K, A)
\]

Let us summarize the results of the present section in an algorithmic form:

**Theorem 2** To compute \( P(A \in C(X,I)) \) assuming extensivity
- Compute \( R(a,b) \) (1).
- Compute \( V(a,b,c) \) using (2).
- Compute \( q(l,y,S,A) \) using (21).
- Compute \( D(l,y,S,A) \) using (22).
- Compute \( P(G_{x,y,A}) \) using (23) and (24).
- Compute \( P(G_{K,y,A}) \) using (25) and (26).
- Compute \( P(A \in C(X,I)) \) using (12).

### 4 Probability that a pair is a concept

Recall that \((A,B) \in B(X,Y,I)\) iff \( A^\uparrow = B \) and \( B^\downarrow = A \).

**Lemma 5** \((A,B) \in B(X,Y,I)\) if and only if
\[
\forall x \in X, \forall y \in Y, A(x) \rightarrow I(x,y) \geq B(y) \tag{27}
\]
\[
\forall x \in X, \forall y \in Y, B(y) \rightarrow I(x,y) \geq A(x) \tag{28}
\]
\[
\forall y \in Y, \exists x \in X, A(x) \rightarrow I(x,y) = B(y) \tag{29}
\]
\[
\forall x \in X, \exists y \in Y, B(y) \rightarrow I(x,y) = A(x) \tag{30}
\]

**Remark 3** If the inequality appearing in (27) holds for a \( y \in Y \) such that \( B(y) = 1 \) then the equality appearing in (29) holds for that \( y \). Similarly if (28) holds for a \( x \in X \) such that \( A(x) = 1 \) then the equality appearing in (30) holds for that \( x \). This holds because \( \rightarrow \) cannot be > 1.
Remark 4 Note that conditions (27) and (28) concern each entry $I(x, y)$ while condition (29) (resp. condition (30)) concern each column $y$ (resp. each row $x$).

Remark 5 In the Boolean case, (27) and (28) mean that rectangle $A \times B$ is full of ones, while condition (29) and (30) mean that it is a maximal one.

Proof: $A^\uparrow = B \iff \forall y \in Y \bigwedge_{x \in X} A(x) \rightarrow I(x, y) = B(y)$

$\iff \begin{cases} 
\forall y \in Y, \forall x \in X, A(x) \rightarrow I(x, y) \geq B(y) \\
\forall y \in Y, \exists x \in X : A(x) \rightarrow I(x, y) = B(y) 
\end{cases}$

$\iff$ (27) and (29)

the second equivalence being due to the fact that $X$ is finite.

In the same way $B^\downarrow = A$ is equivalent to (28) and (30). □

4.1 Gödel structure case

Our purpose is now to get a deeper insight of conditions (27) to (30) in the Gödel structure case in which $a \rightarrow l = 1$ if $a \leq l$, $a \rightarrow l = l$ if $l < a$.

Lemma 6 Let $a, b \in L$, then we have \{ $l \in L : a \rightarrow l \geq b$ \} = [a \land b, 1].

Proof: Let $K = \{ l \in L : a \rightarrow l \geq b \}$. If $a \leq l$ then $(a \rightarrow l) = 1 \geq b$ and $l \in K$. If $l < a$ then, as $(a \rightarrow l) = l$, $l \in K$ if and only if $b \leq l < a$. Therefore $K = [a \land b, 1]$. □

As a consequence of lemma 6, we have

Lemma 7 $I$ is a solution of (27) (resp. of (28)), (resp. of (27) and (28)) if and only if $\forall x \in X, \forall y \in Y, I(x, y) \in [A(x) \land B(y), 1]$.

Concerning (29) and (30), first observe the following:

Lemma 8 Let $y \in Y : B(y) < 1$, then $x \in X$ verifies $A(x) \rightarrow I(x, y) = B(y)$ if and only if $B(y) < A(x)$ and $I(x, y) = B(y)$

Lemma 9 Let $x \in X : A(x) < 1$, then $y \in Y$ verifies $B(y) \rightarrow I(x, y) = A(x)$ if and only if $A(x) < B(y)$ and $I(x, y) = A(x)$

Proof: It suffices to prove lemma 8 as it yields lemma 9 by just exchanging $(X, A)$ and $(Y, B)$. If $A(x) \rightarrow I(x, y) = B(y)$, we should have $I(x, y) < A(x)$ otherwise $A(x) \rightarrow I(x, y) = 1$ and the equality is impossible since $B(y) < 1$. Then, as $A(x) \rightarrow I(x, y) = I(x, y)$, we should have $I(x, y) = B(y)$ and therefore $B(y) < A(x)$. The converse clearly holds by definition of $\rightarrow$. □

As $(A, B) \in B(X, Y, I)$ only if (29) and (30) hold, a consequence of lemma 8 and lemma 9 is
Lemma 10 In the Gödel structure case, \((A, B) \in \mathcal{B}(X, Y, I)\) only if the following conditions on \((A, B)\) first hold:

\[
\forall y \in Y \text{ such that } B(y) < 1, \text{ we have } |S(y)| \geq 1 \tag{31}
\]

\[
\forall x \in X \text{ such that } A(x) < 1, \text{ we have } |T(x)| \geq 1 \tag{32}
\]

where

\[
S(y) = \{ x \in X : B(y) < A(x) \} \tag{34}
\]

\[
T(x) = \{ y \in Y : A(x) < B(y) \} \tag{35}
\]

and \(| |\) stands for cardinality.

Also, it will be convenient to introduce the following set \(U(y)\)

\[
\forall y \in Y, \text{ let } U(y) = \{ x \in X : A(x) = B(y) \}. \tag{36}
\]

Lemmas 8 and 9 motivate us to consider the following three disjoint subsets of \(X \times Y\) which cover it:

\[
C_1 = \{ (x, y) \in X \times Y : B(y) < A(x) \} \tag{37}
\]

\[
C_2 = \{ (x, y) \in X \times Y : A(x) < B(y) \} \tag{38}
\]

\[
C_3 = \{ (x, y) \in X \times Y : A(x) = B(y) \}. \tag{39}
\]

Notice that if \((x, y) \in C_1\) (resp. \(C_2\)) then necessarily \(B(y) < 1\) (resp. \(A(x) < 1\)) so that we can write \(C_1\) and \(C_3\) (resp. \(C_2\)) as a disjoint union of subset of columns (resp. of rows), as follows:

\[
C_1 = \bigcup_{y \in Y : B(y) < 1} \{ y \} \times S(y) \tag{40}
\]

\[
C_2 = \bigcup_{x \in X : A(x) < 1} \{ x \} \times T(x) \tag{41}
\]

\[
C_3 = \bigcup_{y \in Y} \{ y \} \times U(y). \tag{42}
\]

Note that each subset \(\{ y \} \times S(y)\) (resp. \(\{ x \} \times T(x)\)) is nonempty when \((A, B)\) satisfies condition (31) (resp. (32)).

The following is a key characterization of \(I\) for \((A, B)\) to be a Gödel-concept:

Lemma 11 Assume that \((A, B)\) satisfies conditions (31) and (32). Then, \((A, B) \in \mathcal{B}(X, Y, I)\) if and only if it satisfies the three following conditions:

\[
\forall y \in Y \text{ such that } B(y) < 1, \exists V \subseteq S(y) \text{ such that } |V| \geq 1, I = B(y) \text{ on } \{ y \} \times V, \text{ and } I > B(y) \text{ on } \{ y \} \times (S(y) \setminus V) \tag{43}
\]

\[
\forall x \in X \text{ such that } A(x) < 1, \exists W \subseteq T(x) \text{ such that } |W| \geq 1; I = A(x) \text{ on } \{ x \} \times W, \text{ and } I > A(x) \text{ on } \{ x \} \times (T(x) \setminus W) \tag{44}
\]

\[
\forall y \in Y, I \geq B(y) \text{ on } \{ y \} \times U(y). \tag{45}
\]
Proof: First notice that, due to (40), (41), (42), conditions (43), (44), (45) concern the random values of I on the three disjoint sets C₁, C₂, C₃, respectively.

The ‘if’ part
If condition (43) holds, then either I = B(y) or I > B(y) on \{y\} × S(y), ∀y : B(y) < 1 so that I ≥ B(y) = A(x) ∧ B(y) because by definition (37) we have B(y) < A(x) on the set C₁. Therefore the condition required by lemma 7 is satisfied on the set C₁. Similarly, it is also satisfied on the set C₂ if condition (44) holds. Moreover, if condition (45) holds then I ≥ B(y) on \{y\} × U(y) ∀y ∈ Y so that I ≥ A(x) ∧ B(y) because by (39), we have A(x) = B(y) on the set C₃. Thus, the condition required by lemma (7) is satisfied on X × Y so that (27) and (28) hold.

Moreover, if condition (43) holds, then ∀y : B(y) < 1 there exists a set V ⊆ S(y) and an element x ∈ V : I(x, y) = B(y) because |V| ≥ 1. As, B(y) < A(x) on C₁ by definition (37), we also have I(x, y) < A(x) and lemma 8 yields equality (29) for any y ∈ Y : B(y) < 1. This equality also holds for any y ∈ Y : B(y) = 1 because (27) holds on X × Y and, as mentioned in Remark 3, inequality in (27) for a y ∈ Y such that B(y) = 1 implies equality (29) for this y. Hence, if condition (43) holds, condition (29) also holds. Similarly condition (44) implies (30).

Thus, (43), (44), (45) imply (27), (28), (29), and (30), that is (A, B) ∈ B(X, Y, I). This proves the ‘if’ part.

The ‘only if’ part
If (A, B) ∈ B(X, Y, I), then (27) holds and lemma 7 yields ∀x ∈ X, ∀y ∈ Y, I(x, y) ≥ A(x) ∧ B(y). As on S(y) we have B(y) < A(x), we get I ≥ B(y) on \{y\} × S(y). Let V = \{x ∈ S(y) : I(x, y) = B(y)\}. If (A, B) ∈ B(X, Y, I), then (29) holds and lemma 8 yields |V| ≥ 1. Obviously, I > B(y) on \{y\} × (S(y) \ V). Clearly \{x ∈ S(y) : I(x, y) = B(y)\} is the only subset V of S(y) that can satisfy (43). Thus (A, B) ∈ B(X, Y, I) implies (43).

Similarly (A, B) ∈ B(X, Y, I) implies (44) by using (28), lemma 7, (30) and lemma 9. It also implies (45) because of (28), lemma (7) and equality A(x) ∧ B(y) = B(y) on \{y\} × U(y). This proves the ‘only if’ part and completes the proof of lemma 11 □

We can now get the main result of this section:

Theorem 3 Let P_{(A,B)} denote the probability that (A, B) ∈ B(X, Y, I).
In the Gödel structure case, if (A, B) does not satisfy conditions (31) and (32),
then $P_{(A,B)} = 0$ otherwise

$$
P_{(A,B)} = \prod_{y \in Y : B(y) < 1} \sum_{r=1}^{\lfloor S(y) \rfloor} \left( \sum_{r \leq T(x)} \prod_{y \in Y} p_y(B(y)) \prod_{y \in T(x) \setminus W} p_y((A(x), 1)) \prod_{y \in T(x) \setminus W} p_y((B(y), 1)) \right)
$$

where $S(y)$, $T(x)$ and $U(y)$ are defined in (34), (35) and (36), respectively.

**Proof:** Due to lemma 11, we have $P_{(A,B)} = P((43), (44) and (45))$. As the three conditions (43), (44), (45) concern the random values of $I$ on the three disjoint sets $C_1, C_2, C_3$, respectively, we have $P_{(A,B)} = P((43))P((44))P((45))$ due to independence of $I$ on disjoint sets.

Now, since the columns $y$ of $I$ are independent and the entries of each column $y$ too, it is seen that

$$
P((43)) = \prod_{y \in Y : B(y) < 1} \sum_{r=1}^{\lfloor S(y) \rfloor} \prod_{V \subseteq T(x) : |V| = r} \prod_{x \in V} p_y(B(y)) \prod_{y \in T(x) \setminus V} p_y((B(y), 1))$$

Similarly, we have

$$
P((44)) = \prod_{x \in X : A(x) < 1} \sum_{r=1}^{\lfloor T(x) \rfloor} \prod_{W \subseteq T(x) : |W| = r} \prod_{x \in W} p_y(A(x)) \prod_{y \in T(x) \setminus W} p_y((A(x), 1)).$$

Also,

$$
P((45)) = \prod_{y \in Y} p_y([B(y), 1])^{U(y)}.$$

These equalities imply the result announced in Theorem 3 and complete its proof. □

### 4.2 Lukasiewicz structure case

Similarly, let us examine conditions (27) to (30) in the Lukasiewicz structure case where $a \rightarrow l = 1$ if $a \leq l$ and $a \rightarrow l = 1 + l - a$ if $l < a$. For any real number $r \in \mathbb{R}$, let $r^+ = \max(0, r)$ denote its positive part.

**Lemma 12** Let $a, b \in L$, then we have $\{ l \in L : a \rightarrow l \geq b \} = [(a + b - 1)^+, 1]$. 

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Proof: Let \( K = \{ l \in L : a \to l \geq b \} \).
If \( b = 1 \) then clearly \([a, 1] \subseteq K \) and if \( l < a \) then \((a \to l) = 1 + l - a \) cannot be \( \geq 1 \). Therefore \( K = [a, 1] = [(a + b - 1)^+, 1] \).
If \( b < 1 \) we still have \([a, 1] \subseteq K \). Note that this implies that \( K = [a, 1] = [0, 1] = [(a + b - 1)^+, 1] \) if \( a = 0 \). However if \( a > 0 \), \( l < a \) belongs to \( K \) if and only if \( 1 + l - a \geq b \), that is \( l \geq a + b - 1 \). As \( l \geq 0 \), this holds if and only if \((a + b - 1)^+ \leq l < a \). Therefore \( K = [(a + b - 1)^+, 1] \).

As a consequence of lemma 12, we have

**Lemma 13** I is a solution of (27) (resp. of (28)), (resp. of (27) and (28)) if and only if \( \forall x \in X, \forall y \in Y, I(x, y) \in [(A(x) + B(y) - 1)^+, 1] \).

We now observe that the equalities in (29) and in (30) hold under the same conditions:

**Lemma 14** Let \( y \in Y : B(y) < 1 \), then \( x \in X \) verifies \( A(x) \to I(x, y) = B(y) \) if and only if \( A(x) + B(y) - 1 \geq 0 \) and \( I(x, y) = A(x) + B(y) - 1 \).

**Lemma 15** Let \( x \in X : A(x) < 1 \), then \( y \in Y \) verifies \( B(y) \to I(x, y) = A(x) \) if and only if \( A(x) + B(y) - 1 \geq 0 \) and \( I(x, y) = A(x) + B(y) - 1 \).

Proof: It suffices to prove lemma 14 as it yields lemma 15 by just exchanging \((X, A)\) and \((Y, B)\). If \( A(x) \to I(x, y) = B(y) \), we should have \( I(x, y) < A(x) \) otherwise \( A(x) \to I(x, y) = 1 \) and the equality is impossible since \( B(y) < 1 \). Then, as \( A(x) \to I(x, y) = 1 + I(x, y) - A(x) \), we should have \( I(x, y) = A(x) + B(y) - 1 \) and \( A(x) + B(y) - 1 \geq 0 \). The converse clearly holds by definition of \( \to \) because \( A(x) + B(y) - 1 < A(x) \).

We will have to consider subsets \( V \subseteq X \times Y \) verifying

\[
\forall y \in Y \exists x \in X : (x, y) \in V \\
\forall x \in X \exists y \in Y : (x, y) \in V.
\]

(46)

Note that such subsets are necessarily nonempty. Also let

\[
H = \{(x, y) \in X \times Y : A(x) + B(y) - 1 \geq 0\}.
\]

(47)

As \((A, B) \in \mathcal{B}(X, Y, I)\) only if (29) and (30) hold, a consequence of lemma 14 and lemma 15 is

**Lemma 16** In the Lukasiewicz structure case, \((A, B) \in \mathcal{B}(X, Y, I)\) only if \( H \) verifies (46).

Proof: The first condition of (46) should hold \( \forall y \in Y : B(y) < 1 \) due to (29) and lemma 14 and it trivially holds \( \forall y \in Y : B(y) = 1 \). Similarly the second condition of (46) should hold \( \forall x \in X : A(x) < 1 \) due to (30) and lemma 15 and it trivially holds \( \forall x \in X : A(x) = 1 \).

The characterization of \( I \) for \((A, B)\) to be a Lukasiewicz-concept is then given by the following
Lemma 17 Assume that \((A,B)\) is such that \(H\) verifies \((46)\). Then, \((A,B) \in B(X,Y,I)\) if and only if \(I\) satisfies the following condition:

\[
\exists W \subseteq H \text{ verifying } (46) : I(x,y) = A(x) + B(y) - 1 + \forall (x,y) \in W \
and I(x,y) > (A(x) + B(y) - 1)^+ \quad \forall (x,y) \in (X \times Y) \setminus W
\]

Proof:

The 'if' part
If condition \((48)\) holds, then we have

\[
I(x,y) \geq (A(x) + B(y) - 1)^+ \quad \forall (x,y) \in X \times Y
\]

because this holds on \((X \times Y) \setminus W\) and this also holds on \(W\) since on \(W\) we have \((A(x) + B(y) - 1)^+ = A(x) + B(y) - 1\) as \(W \subseteq H\). Thus due to lemma 13, conditions \((27)\) and \((28)\) hold. Moreover, if condition \((48)\) holds, as \(W\) verifies \((46)\), conditions \((29)\) and \((30)\) hold due to lemma 14 and lemma 15. Hence \((A,B)\) is a concept. This proves the 'if' part.

The 'only if' part
Assume that \((A,B)\) is a Lukasiewicz-concept.
First, the only set that can verify \((48)\) is necessarily \(W = \{(x,y) \in H : I(x,y) = A(x) + B(y) - 1\} = \{(x,y) \in X \times Y : I(x,y) = A(x) + B(y) - 1\}\), the second set equality being due to the fact that \(I \geq 0\).
Due to lemma 13, we have \(I(x,y) \geq (A(x) + B(y) - 1)^+ \quad \forall (x,y) \in X \times Y\) so that \(I(x,y) > (A(x) + B(y) - 1)^+ \quad \forall (x,y) \in (X \times Y) \setminus W\).
Last, as \((A,B)\) verifies conditions \((29)\) and \((30)\), the subset \(W\) verifies \((46)\) due to lemma 14 and lemma 15. This proves the 'only if' part and completes the proof of lemma 17 \(\square\)

As the random vectors \((I(x,y))_{(x,y) \in W}\) and \((I(x,y))_{(x,y) \in (X \times Y) \setminus W}\) are independent and their components too, the main result of this section is therefore a consequence of lemma 17:

Theorem 4 Let \(A \in L^X, B \in L^Y\) and let

\[
H = \{(x,y) \in X \times Y : A(x) + B(y) - 1 \geq 0\}.
\]

Let \(P_{(A,B)}\) denote the probability that \((A,B) \in B(X,Y,I)\).
In the Lukasiewicz structure case, if \(H\) does not satisfy condition \((46)\) then \(P_{(A,B)} = 0\) otherwise

\[
P_{(A,B)} = \sum_{W \in \mathcal{L}_H} \prod_{(x,y) \in W} p_y(A(x) + B(y) - 1) \prod_{(x,y) \in (X \times Y) \setminus W} p_y(\lfloor (A(x) + B(y) - 1)^+, 1 \rfloor)
\]

where

\[
\mathcal{L}_H = \{W \subseteq H : W \text{ verifies } (46)\}.
\]

It can be notice that if \(W \subseteq H\) and \(W \notin \mathcal{L}_H\) then any subset of \(W\) neither belong to \(\mathcal{L}_H\). This remark speeds up the computation of \(P_{(A,B)}\).
4.3 General structures

Here are some remarks concerning the general case.

Let $W(a, b) = \{ l \in L : a \rightarrow l \geq b \}$ so that the solution of (27) and (28) is $I(x, y) \in W(A(x), B(y)) \cap W(B(y), A(x))$, $\forall (x, y) \in X \times Y$.

If the subsets $\{(x, y) \in X \times Y : B(y) < 1, A(x) \rightarrow I(x, y) = B(y)\}$ and $\{(x, y) \in X \times Y : A(x) < 1, B(y) \rightarrow I(x, y) = A(x)\}$ are disjoint as in lemma 8 and lemma 9, then, similar arguments to those used in the G"odel structure case might lead to the solution of (27) - (30). If these subsets are equal as in lemma 14 and lemma 15, then, similar arguments to those used in the Lukasiewicz structure case might lead to the solution of (27) - (30).

5 Concept lattices mean size

The computation of the concept lattice mean size $M = E(|B(X, Y, I)|)$ can be easily derived from the previous computations.

**Corollary 5**

$$M = \sum_{A \in L^X} P(A \in C(X, I))$$  \hspace{1cm} (49)

where $P(A \in C(X, I))$ is given by Theorem 2

**Proof:**

$$|B(X, Y, I)| = |C(X, I)| = \sum_{A \in L^X} 1_{A \in C(X, I)}$$

yields

$$M = \sum_{A \in L^X} E(1_{A \in C(X, I)}) = \sum_{A \in L^X} P(A \in C(X, I)). \Box$$

Similarly, it is seen that

**Corollary 6** In the G"odel structure case

$$M = \sum_{(A, B) \in L^X \times L^Y} P((A, B) \in B(X, Y, I))$$  \hspace{1cm} (50)

where $P(A \in B(X, Y, I))$ is given by Theorem 3

It should be interesting in that case to know whether the latter computation of $M$ is less complex than the former one or not.

6 Particular structures

Given $a, b, c \in L$ the set

$$V(a, b, c) = \{ l \in L : l \wedge a \rightarrow b > c \}$$  \hspace{1cm} (51)

that often appears in our analysis is easily computable. Let us describe it in the Lukasiewicz or G"odel structure cases.
Lemma 18 In the Lukasiewicz structure case, we have
\[ V(a, b, c) = \{ l \in L : \min(1, \max(1 + b - l, 1 + b - a)) > c \}. \] (52)

Lemma 19 In the Gödel structure case, we have
\[
V(a, b, c) = \begin{cases} 
\emptyset & \text{if } c = 1 \\
[0, b] & \text{if } c < 1, b \leq c, a > b \\
L & \text{if } c < 1, b \leq c, a \leq b \\
L & \text{if } c < 1, b > c.
\end{cases}
\] (53)

Proof: The first case is obvious. In the second one, as \( l \land a \rightarrow b \) is either 1 or \( b \), we need \( l \land a \rightarrow b = 1 \), that is \( V(a, b, c) = \{ l \in L : l \land a \leq b \} = [0, b] \). The third case is similar. The last case is obvious.

References


