Seminar, TU Wien

An Eyring–Kramers formula for parabolic SPDEs with space-time white noise

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Joint work with Barbara Gentz (Bielefeld)

Deterministic Allen–Cahn PDE

[Chafee & Infante 74, Allen & Cahn 75]

$$\partial_t u(x,t) = \partial_{xx} u(x,t) + f(u(x,t))$$

- $x \in [0, L]$
- $u(x,t) \in \mathbb{R}$
- Either periodic or zero-flux Neumann boundary conditions
- In this talk: $f(u) = u - u^3$ (results more general)

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Energy function:

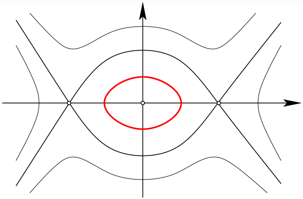
$$V[u] = \int_0^L \Big[\frac{1}{2}u'(x)^2 - \frac{1}{2}u(x)^2 + \frac{1}{4}u(x)^4\Big]\,\mathrm{d}x \qquad \to \ \text{min}$$

Stationary solutions: $u_0''(x) = -u_0(x) + u_0(x)^3$ critical points of V

Stationary solutions

$u_0''(x) = -f(u_0(x)) = -u_0(x) + u_0(x)^3$

- $u_\pm(x) \equiv \pm 1$
- $u_0(x) \equiv 0$
- Nonconstant solutions satisfying b.c.
  (expressible in terms of Jacobi elliptic fcts)



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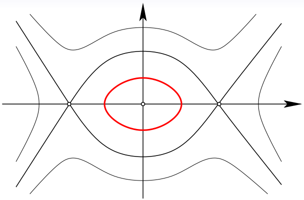
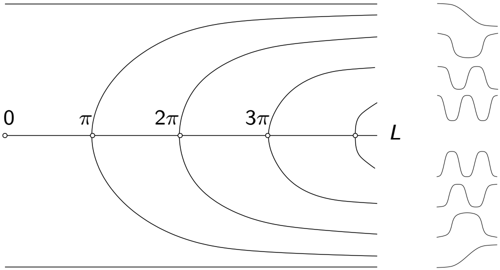


- $u_\pm(x) \equiv \pm 1$
- $u_0(x) \equiv 0$
- Nonconstant solutions satisfying b.c. (expressible in terms of Jacobi elliptic fcts)
- Neumann b.c: k nonconstant solutions when $L > k\pi$



- Periodic b.c: k families when $L > 2k\pi$

Stability of stationary solutions

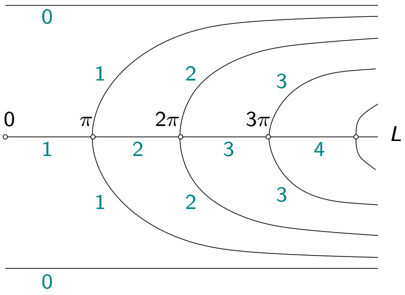
$u_0''(x) = -u_0(x) + u_0(x)^3$

Variational eq around u_0: $\partial_t v_t(x) = v_t''(x) + [1 - u_0(x)]v_t(x)$

Sturm–Liouville spectrum of RHS determines stability of u_0

- $u_\pm \equiv \pm 1$: always stable (global minima of V)
- $u_0 \equiv 0$: always unstable, eigenvalues $\lambda_k = \left(\frac{\beta k \pi}{L}\right)^2 - 1$ (Neumann b.c.: $\beta = 1$, periodic b.c.: $\beta = 2$)

Neumann b.c.:
Number of positive eigenvalues
(= unstable directions)



Coarsening dynamics

[Carr & Pego 89, Chen 04] (Link to simulation)

Stochastic partial differential equations

$$\dot{u}_t(x) = \Delta u_t(x) + f(u_t(x)) \quad + \sqrt{2\varepsilon}\,\ddot{W}_{tx} \qquad (\Delta \equiv \partial_{xx}, f(u) = u - u^3)$$

$\ddot{W}_{tx}$: space–time white noise (formal derivative of Brownian sheet)

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Construction of mild solution via Duhamel formula:

- $\dot{u}_t = \Delta u_t \qquad \Rightarrow \qquad u_t = \mathrm{e}^{\Delta t}\, u_0$
  where $\mathrm{e}^{\Delta t}\cos\bigl(\frac{k\pi x}{L}\bigr) = \mathrm{e}^{-(k\pi/L)^2 t}\cos\bigl(\frac{k\pi x}{L}\bigr)$

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- $\dot{u}_t = \Delta u_t + \sqrt{2\varepsilon}\,\ddot{W}_{tx} \quad \Rightarrow \quad u_t = \mathrm{e}^{\Delta t}\, u_0 + \sqrt{2\varepsilon}\,\underbrace{\int_0^t \mathrm{e}^{\Delta(t-s)}\,\dot{W}_x(\mathrm{d}s)}_{=:w_t(x)}$

$$w_t(x) = \sum_k \int_0^t \mathrm{e}^{-(k\pi/L)^2(t-s)}\,\mathrm{d}W_s^{(k)} \cos\big(\tfrac{k\pi x}{L}\big) \in H^s \cap \mathcal{C}^\alpha \;\forall s, \alpha < \tfrac{1}{2}$$

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- $\dot{u}_t = \Delta u_t + \sqrt{2\varepsilon}\,\ddot{W}_{tx} + f(u_t)$

$$\Rightarrow \qquad u_t = e^{\Delta t}\, u_0 + \sqrt{2\varepsilon}\, w_t + \int_0^t e^{\Delta(t-s)} f(u_s)\, \mathrm{d}s =: F_t[u]$$

$\Rightarrow$ Existence and a.s. uniqueness [Faris & Jona-Lasinio 1982]

Stochastic partial differential equations

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Fourier variables: $u_t(x) = \dfrac{1}{\sqrt{L}} \displaystyle\sum_{k=-\infty}^{\infty} z_k(t)\, \mathrm{e}^{\mathrm{i}\,\pi k x/L}$

$$\Rightarrow \qquad \mathrm{d}z_k = -\lambda_k z_k\, \mathrm{d}t - \frac{1}{L} \sum_{k_1+k_2+k_3=k} z_{k_1} z_{k_2} z_{k_3}\, \mathrm{d}t + \sqrt{2\varepsilon}\, \mathrm{d}W_t^{(k)}$$

Itō SDE, $\mathrm{d}W_t^{(k)}$: independent Wiener processes
$\lambda_k = -1 + (\pi k/L)^2$

Energy functional:

$$V[u] = \frac{1}{2} \sum_{k=-\infty}^{\infty} \lambda_k |z_k|^2 + \frac{1}{4L} \sum_{k_1+k_2+k_3+k_4=0} z_{k_1} z_{k_2} z_{k_3} z_{k_4}$$

$$\Rightarrow \qquad \mathrm{d}z_t = -\nabla V(z_t)\, \mathrm{d}t + \sqrt{2\varepsilon}\, \mathrm{d}W_t$$

Coarsening dynamics with noise

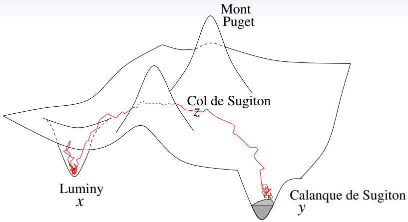
(Link to simulation)

Metastability and Eyring–Kramers law

$$\mathrm{d}x_t = -\nabla V(x_t)\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t$$

$V : \mathbb{R}^d \to \mathbb{R}$ confining potential

$\tau_y^x = \inf\{t > 0 \colon x_t \in \mathcal{B}_\varepsilon(y)\}$
first-hitting time of small ball $\mathcal{B}_\varepsilon(y)$, when starting in x



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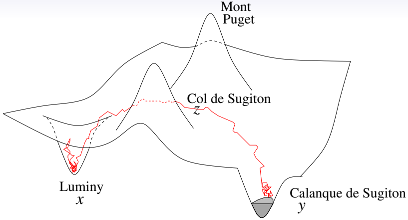

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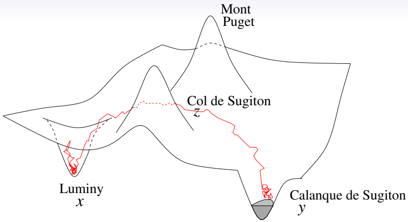
Arrhenius' law (1889): $\mathbb{E}[\tau_y^x] \simeq \mathrm{e}^{[V(z)-V(x)]/\varepsilon}$

Eyring–Kramers law (1935, 1940):
Eigenvalues of Hessian of V at minimum x: $0 < \nu_1 \leqslant \nu_2 \leqslant \dots \leqslant \nu_d$
Eigenvalues of Hessian of V at saddle z: $\lambda_1 < 0 < \lambda_2 \leqslant \dots \leqslant \lambda_d$

$$\mathbb{E}[\tau_y^x] = 2\pi\sqrt{\frac{\lambda_2 \dots \lambda_d}{|\lambda_1|\nu_1 \dots \nu_d}}\,\mathrm{e}^{[V(z)-V(x)]/\varepsilon}\big[1 + \mathcal{O}_\varepsilon(1)\big]$$

Metastability and Eyring–Kramers law

$$dx_t = -\nabla V(x_t)\, dt + \sqrt{2\varepsilon}\, dW_t$$



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Arrhenius' law: proved by Freidlin, Wentzell (1979) using large deviations
Eyring–Kramers law: Bovier, Eckhoff, Gayrard, Klein (2004) using potential theory, Helffer, Klein, Nier (2004) using Witten Laplacian, ...

Potential theory and Eyring–Kramers law

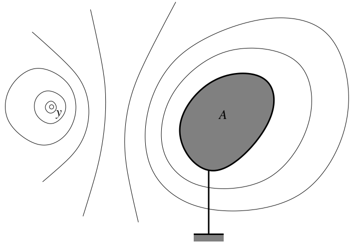
Consider first Brownian motion $W_t^x = x + W_t$

Fact 1: $w_A(x) = \mathbb{E}[\tau_A^x]$ satisfies

$$\begin{cases} -\frac{1}{2}\Delta w_A(x) = 1 & x \in A^c \\ w_A(x) = 0 & x \in A \end{cases}$$

$G_{A^c}(x,y)$ Green's function

$\Rightarrow w_A(x) = \int_{A^c} G_{A^c}(x,y)\,\mathrm{d}y$



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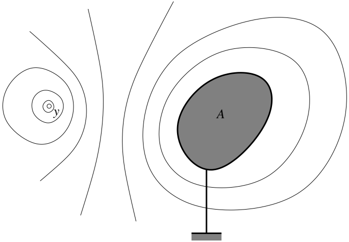
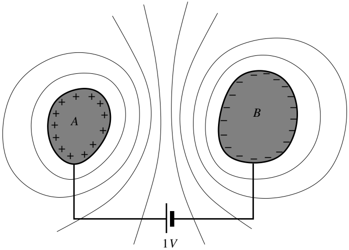


$G_{A^c}(x,y)$ Green's function

$$\Rightarrow w_A(x) = \int_{A^c} G_{A^c}(x,y)\,\mathrm{d}y$$

Fact 2: $h_{A,B}(x) = \mathbb{P}[\tau_A^x < \tau_B^x]$ satisfies

$$\begin{cases} -\frac{1}{2}\Delta h_{A,B}(x) = 0 & x \in (A \cup B)^c \\ h_{A,B}(x) = 1 & x \in A \\ h_{A,B}(x) = 0 & x \in B \end{cases}$$



$\rho_{A,B}$: "surface charge density" on ∂A

$$\Rightarrow h_{A,B}(x) = \int_{\partial A} G_{B^c}(x,y)\,\rho_{A,B}(\mathrm{d}y)$$

Potential theory and Eyring–Kramers law

Capacity: $\text{cap}_A(B) = \int_{\partial A} \rho_{A,B}(\text{d}y) \;\Rightarrow\; \nu_{A,B}(\text{d}y) = \frac{\rho_{A,B}(\text{d}y)}{\text{cap}_A(B)}$ prob measure

Variational representation: Dirichlet form

$$\text{cap}_A(B) = \int_{(A\cup B)^c} \|\nabla h_{A,B}(x)\|^2 \,\text{d}x = \inf_{h\in\mathcal{H}_{A,B}} \int_{(A\cup B)^c} \|\nabla h(x)\|^2 \,\text{d}x$$

($\mathcal{H}_{A,B}$: set of sufficiently smooth functions satisfying b.c.)

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Key observation: let $C = \mathcal{B}_\varepsilon(x)$, then (using $G(y,z) = G(z,y)$)

$$\begin{aligned}\int_{A^c} h_{C,A}(y)\,\text{d}y &= \int_{A^c}\int_{\partial C} G_{A^c}(y,z)\rho_{C,A}(\text{d}z)\,\text{d}y \\ &= \int_{\partial C} w_A(z)\rho_{C,A}(\text{d}z) = \text{cap}_C(A)\mathbb{E}^{\nu_{C,A}}[w_A]\end{aligned}$$

Harnack inequalities $\Rightarrow \mathbb{E}^x[\tau_A] \simeq \mathbb{E}^{\nu_{C,A}}[w_A] = \frac{1}{\text{cap}_C(A)} \int_{A^c} h_{C,A}(y)\,\text{d}y$

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Harnack inequalities $\Rightarrow$ $\mathbb{E}^x[\tau_A] \simeq \mathbb{E}^{\nu_{C,A}}[w_A] = \dfrac{1}{\mathrm{cap}_C(A)}\displaystyle\int_{A^c} h_{C,A}(y)\,\mathrm{d}y$

General case: replace $\frac{1}{2}\Delta$ by $\varepsilon\Delta - \nabla V(x)\cdot\nabla$ and $\mathrm{d}x$ by $\mathrm{e}^{-V(x)/\varepsilon}\,\mathrm{d}x$

Eyring–Kramers law for SPDEs: heuristics

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Initial condition: u_{in} near $u_- \equiv -1$ with eigenvalues ν_k

Target: $u_+ \equiv 1$, $\tau_+ = \inf\{t > 0 \colon \|u_t - u_+\|_{L^\infty}\} < \rho$

Transition state: ($\beta = 1$ for Neumann b.c., $\beta = 2$ for periodic b.c.)

$$u_{\mathrm{ts}}(x) = \begin{cases} u_0(x) \equiv 0 & \text{if } L \leqslant \beta\pi \\ u_1(x) \ \beta\text{-kink stationary sol.} & \text{if } L > \beta\pi \end{cases}$$

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[Faris & Jona-Lasinio 82]: large-deviation principle

$\Rightarrow$ Arrhenius law: $\mathbb{E}^{u_{\rm in}}[\tau_+] \simeq e^{(V[u_{\rm ts}] - V[u_-])/\varepsilon}$

[Maier & Stein 01]: formal computation; for Neumann b.c.

$$\Rightarrow \mathbb{E}^{u_{\rm in}}[\tau_+] \simeq 2\pi \sqrt{\frac{1}{|\lambda_0|\nu_0} \prod_{k=1}^{\infty} \frac{\lambda_k}{\nu_k}}\, e^{(V[u_{\rm ts}] - V[u_-])/\varepsilon}$$

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- ▷ Rigorous proof?
- ▷ What happens when $L \to \beta\pi$ as then $\lambda_1 \to 0$?

Eyring–Kramers law for SPDEs: main result

Theorem 1: Neumann b.c. [B & Gentz, Elec J Proba 2013]

- If $L < \pi - c$, then

$$\mathbb{E}^{u_{\rm in}}[\tau_+] = 2\pi \sqrt{\frac{1}{|\lambda_0|\nu_0} \prod_{k=1}^{\infty} \frac{\lambda_k}{\nu_k}}\, \mathrm{e}^{(V[u_{\rm ts}]-V[u_-])/\varepsilon} \bigl[1 + \mathcal{O}(\varepsilon^{1/2}|\log\varepsilon|^{3/2})\bigr]$$

- If $L > \pi + c$, then same formula with extra factor $\frac{1}{2}$ (since 2 saddles)

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- If $L > \pi + c$, then same formula with extra factor $\frac{1}{2}$ (since 2 saddles)
- If $\pi - c \leqslant L \leqslant \pi$, then

$$\mathbb{E}^{u_{\text{in}}}[\tau_+] = 2\pi \sqrt{\frac{\lambda_1 + \sqrt{3\varepsilon/2L}}{|\lambda_0|\nu_0\nu_1} \prod_{k=2}^{\infty} \frac{\lambda_k}{\nu_k}}\, \frac{\mathrm{e}^{(V[u_{\text{ts}}]-V[u_-])/\varepsilon}}{\Psi_+(\lambda_1/\sqrt{3\varepsilon/2L})} \big[1 + R(\varepsilon)\big]$$

  where Ψ_+ explicit, involves Bessel function $K_{1/4}$, $\lim_{\alpha\to\infty}\Psi_+(\alpha) = 1$
- If $\pi \leqslant L \leqslant \pi + c$, then same formula, with another function Ψ_-, involving Bessel functions $I_{\pm 1/4}$, $\lim_{\alpha\to\infty}\Psi_-(\alpha) = 2$

Sketch of proof: Spectral Galerkin approximation

$u_t^{(d)}(x) = \frac{1}{\sqrt{L}} \sum_{k=-d}^{d} z_k(t)\,\mathrm{e}^{\mathrm{i}\,\pi k x/L} \quad \Rightarrow \quad \mathrm{d}z_t = -\nabla V(z_t)\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t$

Theorem [Blömker & Jentzen 13]

For all $\gamma \in (0, \frac{1}{2})$ there exists an a.s. finite r.v. $Z : \Omega \to \mathbb{R}_+$ s.t. $\forall \omega \in \Omega$

$$\sup_{0 \leqslant t \leqslant T} \|u_t(\omega) - u_t^{(d)}(\omega)\|_{L^\infty} < Z(\omega) d^{-\gamma} \qquad \forall d \in \mathbb{N}$$

Sketch of proof: Spectral Galerkin approximation

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$$\sup_{0 \leqslant t \leqslant T} \|u_t(\omega) - u_t^{(d)}(\omega)\|_{L^\infty} < Z(\omega) d^{-\gamma} \qquad \forall d \in \mathbb{N}$$

Proposition (using potential theory)

$\exists \varepsilon_0 > 0 \colon \forall \varepsilon < \varepsilon_0\ \exists d_0(\varepsilon) < \infty \colon \forall d \geqslant d_0\ \exists \nu_d$ proba measure on $\partial\mathcal{B}_r(u_-)$

$$\int_{\partial\mathcal{B}_r(u_-)} \mathbb{E}^{v_0}[\tau_+^{(d)}]\nu_d(dv_0) = C(d,\varepsilon)\, e^{H(d)/\varepsilon}\big[1 + R(\varepsilon)\big]$$

Sketch of proof: Spectral Galerkin approximation

$u_t^{(d)}(x) = \frac{1}{\sqrt{L}} \sum_{k=-d}^{d} z_k(t) \, e^{i\pi kx/L} \quad \Rightarrow \quad dz_t = -\nabla V(z_t) \, dt + \sqrt{2\varepsilon} \, dW_t$

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Proposition (using large deviations and lots of other stuff)

$H_0 := V[u_{\text{ts}}] - V[u_-]$. $\forall \eta > 0 \; \exists \varepsilon_0, T_1, H_1 \colon \forall \varepsilon < \varepsilon_0 \; \exists d_0 < \infty$ s.t. $\forall d \geqslant d_0$

$$\sup_{v_0 \in \mathcal{B}_r(u_-)} \mathbb{E}^{v_0}[\tau_+^2] \leqslant T_1^2 \, e^{2(H_0 + \eta)/\varepsilon} \; , \; \sup_{d \geqslant d_0} \sup_{v_0 \in \mathcal{B}_r(u_-)} \mathbb{E}^{v_0}[(\tau_+^{(d)})^2] \leqslant T_1^2 \, e^{2H_1/\varepsilon}$$

Main step of the proof

Set $T_{\mathrm{Kr}} = C(\infty, \varepsilon)\, \mathrm{e}^{H_0/\varepsilon}$

Let $B = \mathcal{B}_\rho(u_+)$ and define nested sets $B_- \subset B \subset B_+$ at L^∞-distance δ

$$\Omega_{K,d} = \left\{ \sup_{t \in [0, KT_{\mathrm{Kr}}]} \|v_t - v_t^{(d)}\|_{L^\infty} \leqslant \delta, \tau_{B_-}^{(d)} \leqslant KT_{\mathrm{Kr}} \right\}$$

$$\mathbb{P}(\Omega_{K,d}^c) \leqslant \mathbb{P}\{Z > \delta d^\gamma\} + \frac{\mathbb{E}^{v_0^{(d)}}[\tau_{B_-}^{(d)}]}{KT_{\mathrm{Kr}}} \quad \overset{\text{Cauchy–Schwarz}}{\Rightarrow} \quad \limsup_{d \to \infty} \mathbb{P}(\Omega_{K,d}^c) = \frac{M(\varepsilon)}{K}$$

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Integrate against ν_d and use Cauchy–Schwarz to bound error terms:
$\mathbb{E}^{v_0}[\tau_B 1_{\{\Omega_{K,d}^c\}}] \leqslant \sqrt{\mathbb{E}^{v_0}[\tau_B^2]\mathbb{P}(\Omega_{K,d}^c)}$, take $d\to\infty$ and finally K large

Eyring–Kramers law for SPDEs: main result

Theorem 2: periodic b.c. [B & Gentz, Elec J Proba 2013]

- If $L < 2\pi - c$, then

$$\mathbb{E}^{u_{\rm in}}[\tau_+] = \frac{2\pi}{\sqrt{|\lambda_0|\nu_0}} \prod_{k=1}^{\infty} \frac{\lambda_k}{\nu_k} \, \mathrm{e}^{(V[u_{\rm ts}]-V[u_-])/\varepsilon} \Bigl[1 + \mathcal{O}(\varepsilon^{1/2}|\log\varepsilon|^{3/2})\Bigr]$$

- If $2\pi - c \leqslant L \leqslant 2\pi$, then

$$\mathbb{E}^{u_{\rm in}}[\tau_+] = \frac{2\pi}{\sqrt{|\lambda_0|\nu_0}} \frac{\lambda_1 + \sqrt{3\varepsilon/L}}{\nu_1} \prod_{k=2}^{\infty} \frac{\lambda_k}{\nu_k} \frac{\mathrm{e}^{(V[u_{\rm ts}]-V[u_-])/\varepsilon}}{\Theta_+(\lambda_1/\sqrt{3\varepsilon/L})} \Bigl[1 + R(\varepsilon)\Bigr]$$

where Θ_+ explicit, involves error function, $\lim_{\alpha\to\infty}\Theta_+(\alpha) = 1$

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  where Θ_+ explicit, involves error function, $\lim_{\alpha\to\infty}\Theta_+(\alpha) = 1$
- Similar expression for $2\pi \leqslant L \leqslant 2\pi + c$ with $\lambda_1 = 0$
- If $L \geqslant 2\pi + c$, then

$$\mathbb{E}^{u_{\rm in}}[\tau_+] = \frac{2\pi}{\sqrt{|\lambda_0|\nu_0}} \frac{\sqrt{2\pi\varepsilon\lambda_1}}{\nu_1} \prod_{k=2}^{\infty} \frac{\sqrt{\lambda_k\lambda_{-k}}}{\nu_k} \frac{\mathrm{e}^{(V[u_{\rm ts}]-V[u_-])/\varepsilon}}{L\|u'_{\rm ts}\|_{L^2}} \left[1 + R(\varepsilon)\right]$$

  $L\|u'_{\rm ts}\|_{L^2}$ = "length of the saddle"

Outlook

- ▷ Potentials with more than two wells: essentially understood
- ▷ Potentials invariant under symmetry group: done for SDEs
- ▷ Allen–Cahn in dimension $d \geqslant 2$: only possible since Martin Hairer's theory of regularity structures

[Link]

- ▷ Open problem: non-gradient systems

References

- N. B. and Barbara Gentz, *Anomalous behavior of the Kramers rate at bifurcations in classical field theories*, J. Phys. A: Math. Theor. **42**, 052001 (2009)
- ______, *The Eyring–Kramers law for potentials with nonquadratic saddles*, Markov Processes Relat. Fields **16**, 549–598 (2010)
- ______, **Sharp estimates for metastable lifetimes in parabolic SPDEs: Kramers' law and beyond**, Electronic J. Probability **18**, (24):1–58 (2013)
- N. B., *Kramers' law: Validity, derivations and generalisations*, Markov Processes Relat. Fields **19**, 459–490 (2013)
- N. B. and Sébastien Dutercq, *The Eyring–Kramers law for Markovian jump processes with symmetries*, preprint `arXiv:1312.0835` (2013)

Thanks for your attention