Noise-induced phenomena in slow–fast dynamical systems

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Joint work with Barbara Gentz, Universität Bielefeld

Macsdiem07, University of Sussex, February 2007

Deterministic slow-fast system

Fast variables: $x \in \mathbb{R}^n$ (e.g. light particles, prey, atmosphere)
Slow variables: $y \in \mathbb{R}^m$ (e.g. heavy particles, predator, ocean)

$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= \varepsilon g(x,y) \end{aligned} \qquad \overset{t \mapsto \varepsilon t}{\Longleftrightarrow} \qquad \begin{aligned} \varepsilon \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{aligned}$$

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\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= 0 \end{aligned} & \not\Longleftrightarrow & \begin{aligned} 0 &= f(x,y) \\ \dot{y} &= g(x,y) \end{aligned}
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Fast system

$\dot{x} = f_y(x)$

y: parameter

Slow system

$f(x^\star(y), y) = 0$

$\dot{y} = g(x^\star(y), y)$

Example: the Van der Pol oscillator

$$\ddot{x} + \gamma(x^2 - 1)\dot{x} + x = 0$$

$$\begin{aligned} \dot{x} &= y + x - \tfrac{1}{3}x^3 \\ \dot{y} &= -\varepsilon x \end{aligned} \quad \overset{t \mapsto \varepsilon t}{\Longleftrightarrow} \quad \begin{aligned} \varepsilon\dot{x} &= y + x - \tfrac{1}{3}x^3 \\ \dot{y} &= -x \end{aligned}$$

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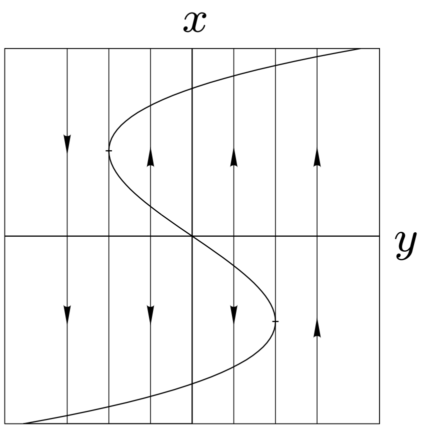

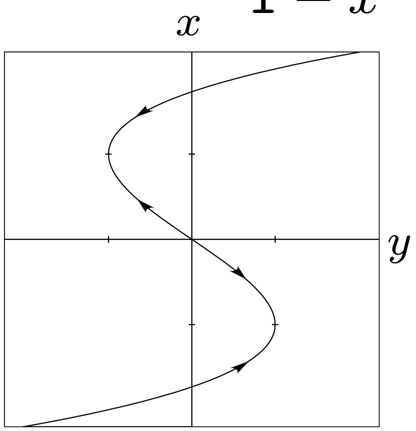

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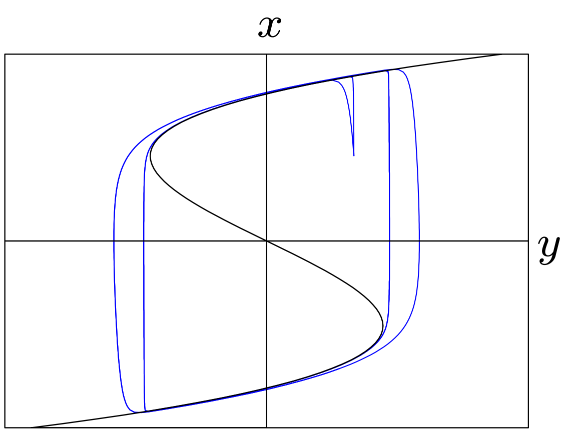
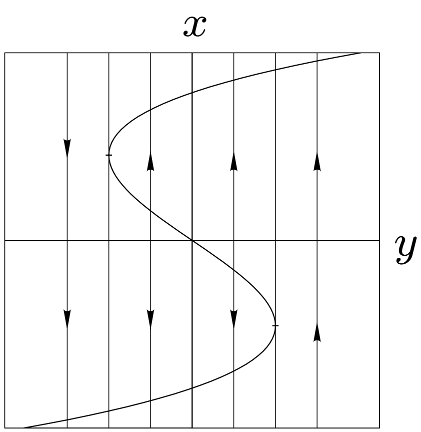
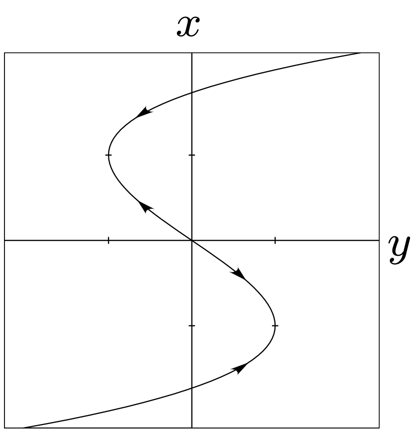


Relaxation oscillations





Geometric singular perturbation theory

$$
\begin{aligned}
\varepsilon \dot{x} &= f(x, y) \qquad && x \in \mathbb{R}^n, \text{ fast variables} \\
\dot{y} &= g(x, y) \qquad && y \in \mathbb{R}^m, \text{ slow variables}
\end{aligned}
$$

- Slow manifold: $f(x^\star(y), y) = 0$ (for all y in some domain)
- Stability: Eigenvalues of $\partial_x f(x^\star(y), y)$ have negative real parts

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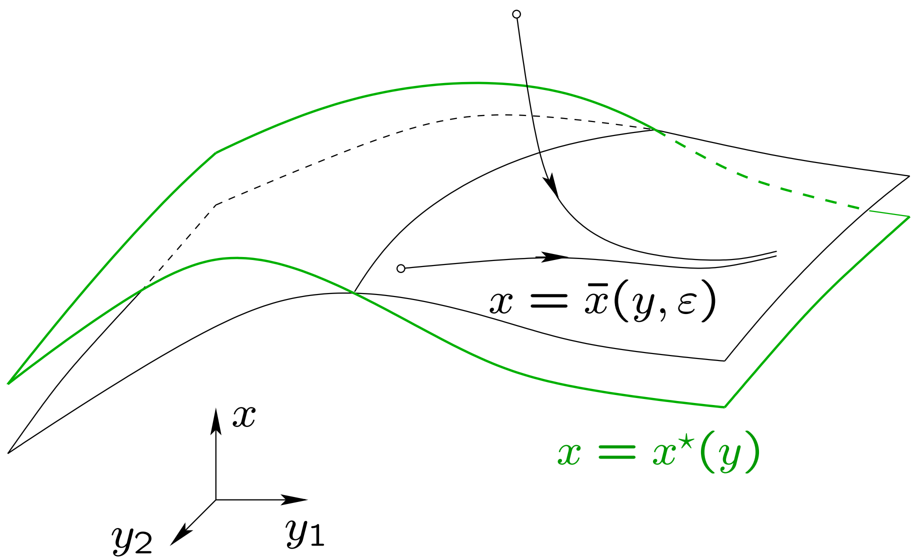
Theorem [Tihonov '52, Fenichel '79]
$\exists$ *adiabatic manifold* $x = \bar{x}(y,\varepsilon)$
s.t.

- $\bar{x}(y,\varepsilon)$ is invariant
- $\bar{x}(y,\varepsilon)$ attracts nearby solutions
- $\bar{x}(y,\varepsilon) = x^\star(y) + \mathcal{O}(\varepsilon)$

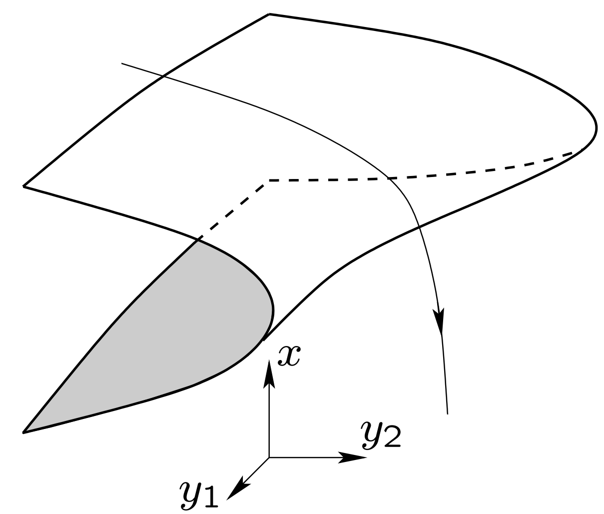


Bifurcations

k eigenvalues of $\partial_x f(x^\star(y), y)$ have vanishing real part for some y

Reduced system on centre manifold:

$$\varepsilon \dot{x} = f(x,y) \qquad x \in \mathbb{R}^k$$
$$\dot{y} = g(x,y)$$



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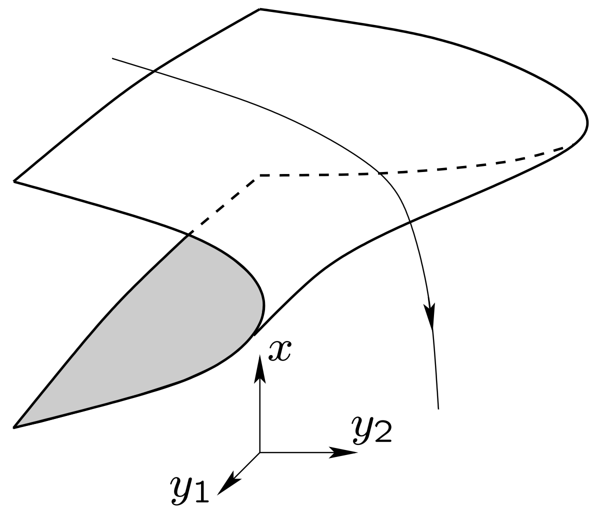

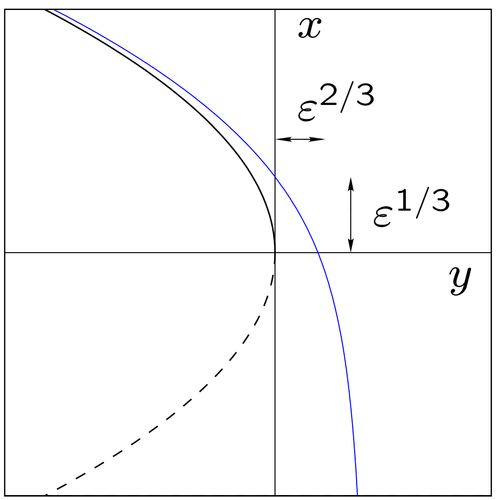

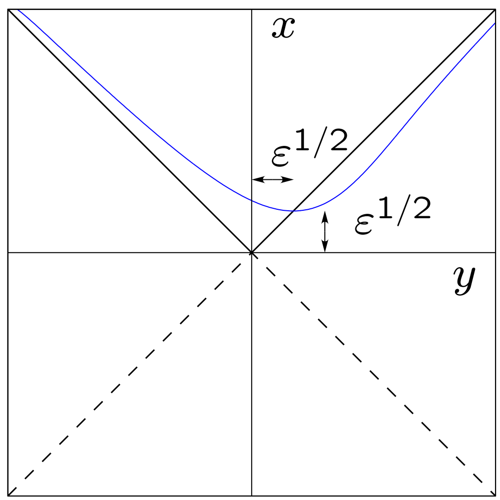

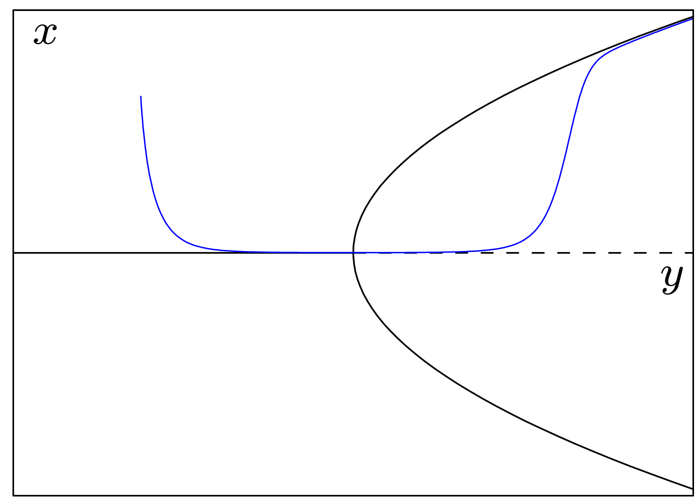

Saddle–node

$f(x, y) = -x^2 - y + \dots$

Transcritical

$f(x, y) = -x^2 + y^2 + \dots$

Pitchfork

$f(x, y) = yx - x^3 + \dots$

Stochastic perturbation: one-dimensional case

$$\mathrm{d}x_t = \frac{1}{\varepsilon} f(x_t, t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

Slow–fast system with $y_t = t$

If $\exists$ stable slow manif: $f(x^\star(t), t) = 0$,

$$a^\star(t) = \partial_x f(x^\star(t), t) \leqslant -a_0$$

then $\exists$ adiabatic solution: $\bar{x}(t, \varepsilon) = x^\star(t) + \mathcal{O}(\varepsilon)$ of $\varepsilon\dot{x} = f(x, t)$

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Observation: Let $\bar{a}(t,\varepsilon) = \partial_x f(\bar{x}(t,\varepsilon), t) = a^\star(t) + \mathcal{O}(\varepsilon)$

Consider linearised equation at $\bar{x}(t,\varepsilon)$:

$$\mathrm{d}\xi_t = \frac{1}{\varepsilon}\bar{a}(t,\varepsilon)\xi_t\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

ξ_t: gaussian process with variance $\sigma^2 v(t)$, s.t. $\varepsilon\dot{v} = 2\bar{a}(t,\varepsilon)v + 1$

Asymptotically, $v(t) \simeq v^\star(t) = 1/2|\bar{a}(t,\varepsilon)|$

$\mathcal{B}(h)$: strip of width $\simeq h\sqrt{v^\star(t,\varepsilon)}$ around $\bar{x}(t,\varepsilon)$

Stochastic perturbation: one-dimensional case

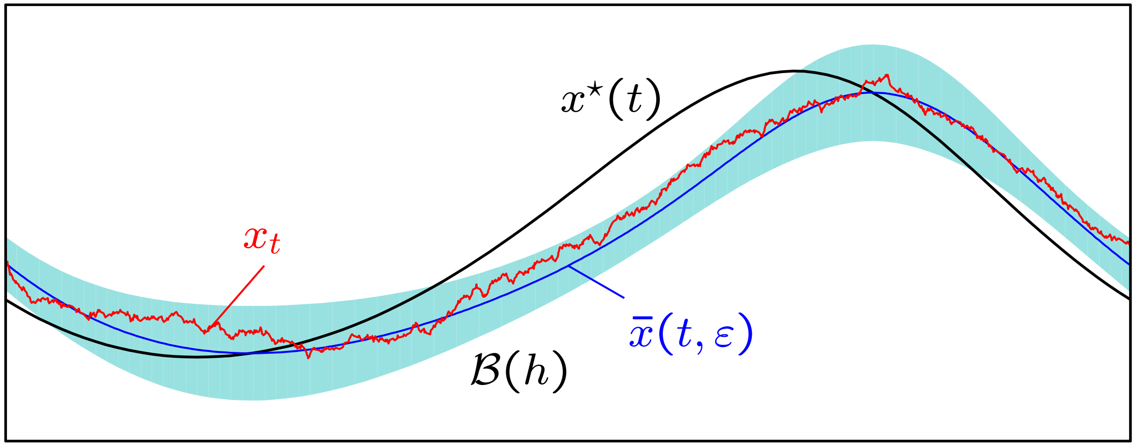
$$\mathrm{d}x_t = \frac{1}{\varepsilon} f(x_t, t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

Theorem: [B. & G., PTRF 2002]

$$C(t,\varepsilon)\mathrm{e}^{-\kappa_- h^2/2\sigma^2} \leqslant \mathbb{P}\Big\{\text{leaving } \mathcal{B}(h) \text{ before time } t\Big\} \leqslant C(t,\varepsilon)\mathrm{e}^{-\kappa_+ h^2/2\sigma^2}$$

$$\kappa_\pm = 1 \mp \mathcal{O}(h)$$

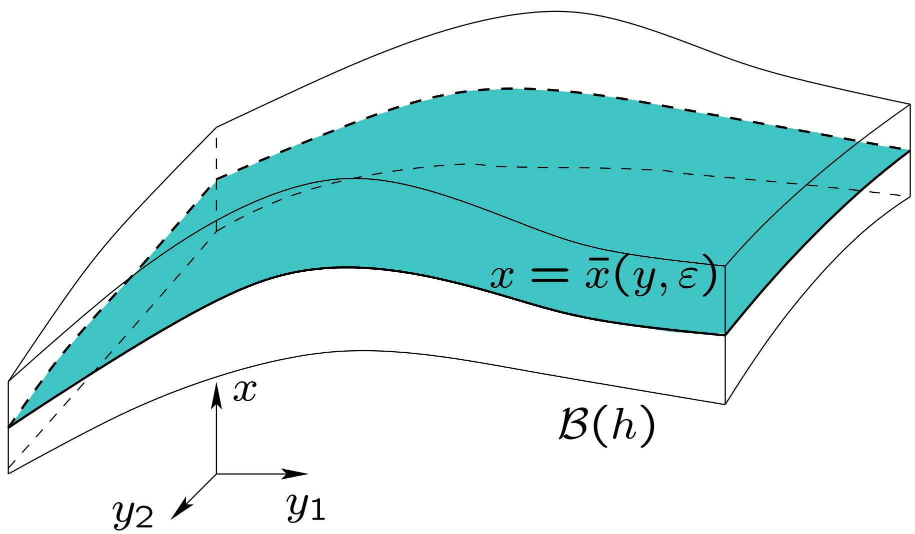
$$C(t,\varepsilon) = \sqrt{\frac{2}{\pi}} \frac{1}{\varepsilon} \left| \int_0^t \bar{a}(s,\varepsilon)\,\mathrm{d}s \right| \frac{h}{\sigma} \left[1 + \text{error of order } \mathrm{e}^{-h^2/\sigma^2}\, t/\varepsilon \right]$$



Stochastic perturbation: n-dimensional case

$$\begin{cases} \mathrm{d}x_t = \dfrac{1}{\varepsilon} f(x_t, y_t)\,\mathrm{d}t + \dfrac{\sigma}{\sqrt{\varepsilon}} F(x_t, y_t)\,\mathrm{d}W_t & \text{(fast variables } \in \mathbb{R}^n) \\ \mathrm{d}y_t = \quad g(x_t, y_t)\,\mathrm{d}t + \sigma' \; G(x_t, y_t)\,\mathrm{d}W_t & \text{(slow variables } \in \mathbb{R}^m) \end{cases}$$

Stable slow manifold: $f(x^\star(y), y) = 0$, $A(y) = \partial_x f(x^\star(y), y)$ stable



$$\mathcal{B}(h) := \Big\{ (x,y) \colon \Big\langle \big[x - \bar{x}(y,\varepsilon)\big], X^\star(y)^{-1} \big[x - \bar{x}(y,\varepsilon)\big] \Big\rangle < h^2 \Big\}$$

$X^\star(y)$ solution of $A(y)X^\star + X^\star A(y)^{\mathsf{T}} + F(x^\star, y)F(x^\star, y)^{\mathsf{T}} = 0$

Stochastic perturbation: n-dimensional case

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Theorem [B. & G., JDE 2003]

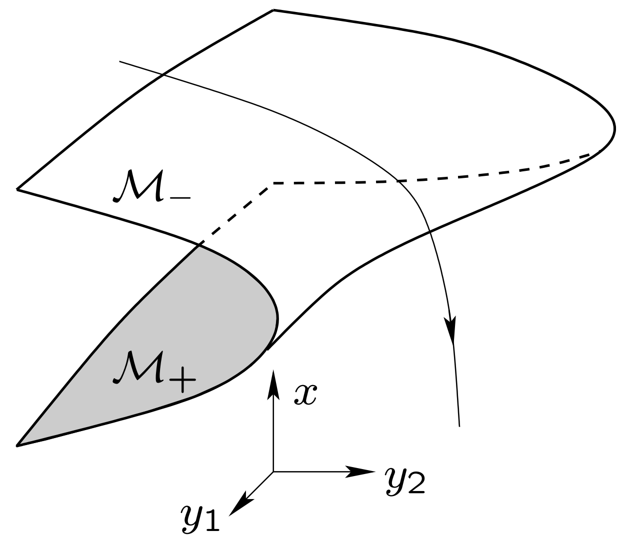
- $\mathbb{P}\Big\{\text{leaving } \mathcal{B}(h) \text{ before time } t\Big\} \simeq C(t,\varepsilon)\,\mathrm{e}^{-\kappa h^2/2\sigma^2}$
  $\kappa = 1 - \mathcal{O}(h) - \mathcal{O}(\varepsilon)$.
- Projection on $\bar{x}(y,\varepsilon)$:

$$\mathrm{d}y_t^0 = g(\bar{x}(y_t^0,\varepsilon), y_t^0)\,\mathrm{d}t + \sigma' G(\bar{x}(y_t^0,\varepsilon), y_t^0)\,\mathrm{d}W_t$$

  y_t^0 approximates y_t to order $\sigma\sqrt{\varepsilon}$ up to Lyapunov time of $\dot{y} = g(\bar{x}(y,\varepsilon)y)$.

Bifurcations

$x^\star(y)$ slow manifold for $y \in \mathcal{D}_0$
$A(y) = \partial_x f(x^\star(y), y)$
Some ev of $A(y)$ cross imaginary axis as y approaches $\partial\mathcal{D}_0$



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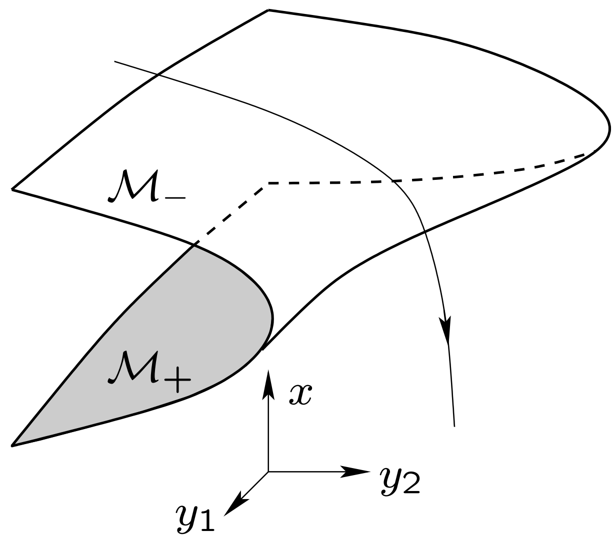


Theorem [B. & G., JDE 2003]
System can be approximated by projection on centre manifold.

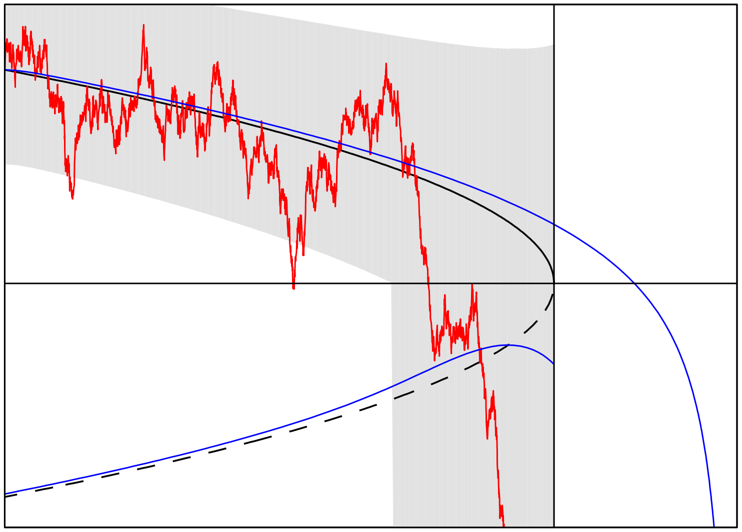
- Saddle–node bifurcation: transitions through unstable manifold, relaxation oscillations, hysteresis
- Pitchfork bifurcation: decrease of bifurcation delay
- (Avoided) transcritical bifurcation: stochastic resonance

Saddle–node bifurcation

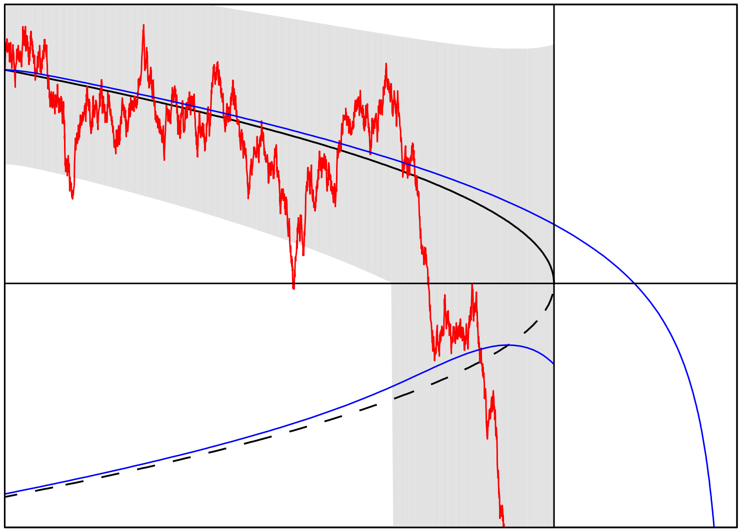
Consider

$$\mathrm{d}x_t = \frac{1}{\varepsilon} f(x_t, t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

$$f(x,t) = -x^2 - t$$
$$+ \text{higher order terms}$$



Saddle–node bifurcation



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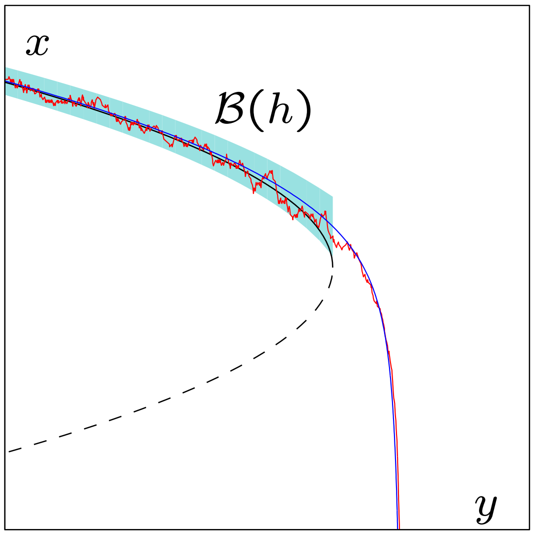
New effects:

- Linearisation at slow solution of order $\varepsilon^{-1/3}$ near $t = 0$
  $\Rightarrow \mathcal{B}(h)$ has width of order $h\varepsilon^{-1/6}$
  $\Rightarrow$ typical fluctuations of order $\sigma\varepsilon^{-1/6}$
- If $\sigma \ll \varepsilon^{1/2}$: $\sigma\varepsilon^{-1/6} \ll \varepsilon^{1/3} =$ distance to origin
- If $\sigma \gg \varepsilon^{1/2}$: sample paths reach unstable manifold already at times of order $-\sigma^{4/3}$
  $\Rightarrow$ new technique: count excursions towards unstable manifold

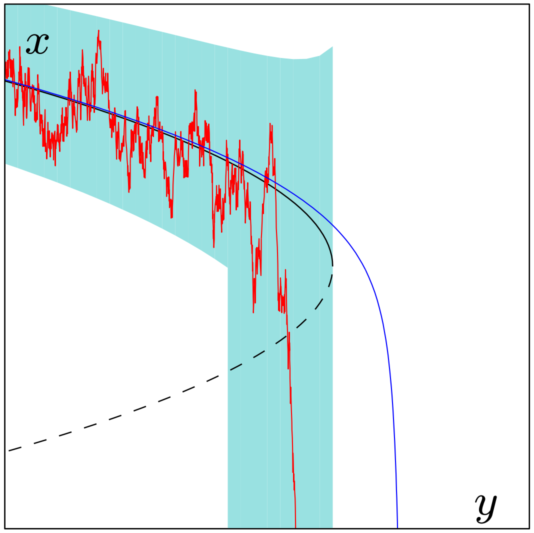
Saddle–node bifurcation

e.g. $f(x,y) = -y - x^2$

$\sigma \ll \sigma_c = \varepsilon^{1/2}$



$\sigma \gg \sigma_c = \varepsilon^{1/2}$



Deterministic case $\sigma = 0$: Solutions stay at distance $\varepsilon^{1/3}$ above bifurcation point until time $\varepsilon^{2/3}$ after bifurcation.

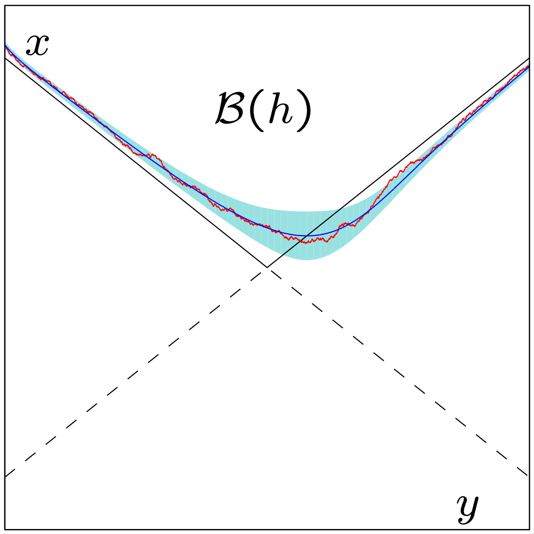
Theorem: [B. & G., Nonlinearity 2002]

1. If $\sigma \ll \sigma_c$: Paths likely to stay in $\mathcal{B}(h)$ until time $\varepsilon^{2/3}$ after bifurcation, maximal spreading $\sigma/\varepsilon^{1/6}$.
2. If $\sigma \gg \sigma_c$: Transition typically for $t \asymp -\sigma^{4/3}$ transition probability $\geqslant 1 - \mathrm{e}^{-c\sigma^2/\varepsilon|\log\sigma|}$

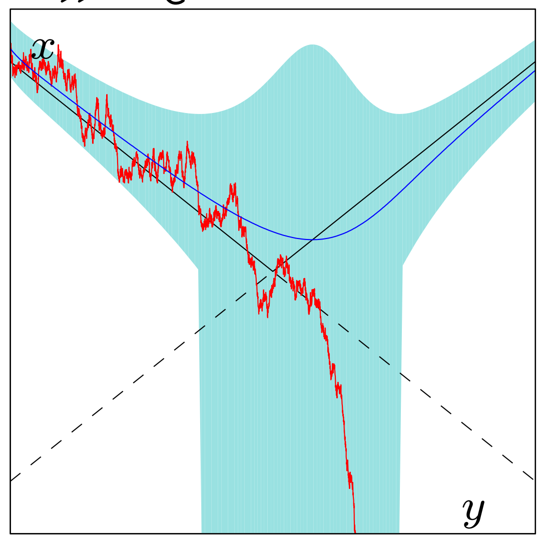
Transcritical bifurcation

e.g. $f(x, y) = y^2 - x^2$

$\sigma \ll \sigma_c = \varepsilon^{3/4}$



$\sigma \gg \sigma_c = \varepsilon^{3/4}$



Deterministic case $\sigma = 0$: Solutions stay at distance $\varepsilon^{1/2}$ above bifurcation point

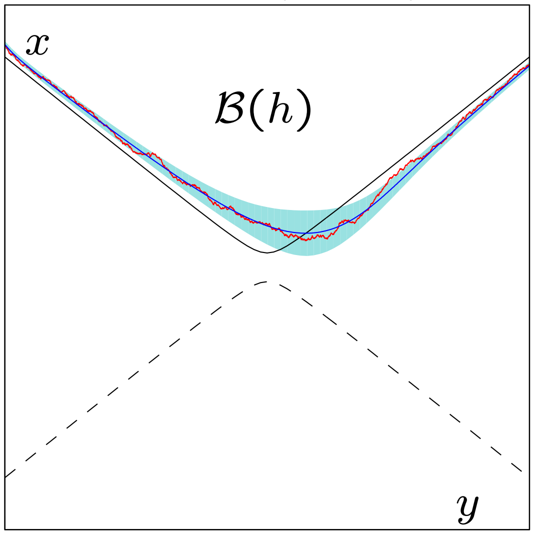
Theorem: [B. & G., Ann. App. Probab. 2002]

1. If $\sigma \ll \sigma_c$: Paths likely to stay in $\mathcal{B}(h)$, transition probability $\leqslant e^{-c\sigma_c^2/\sigma^2}$.
2. If $\sigma \gg \sigma_c$: Transition typically for $t \asymp -\sigma^{2/3}$ transition probability $\geqslant 1 - e^{-c\sigma^{4/3}/\varepsilon|\log \sigma|}$

Avoided transcritical bifurcation

e.g. $f(x, y) = y^2 + \delta - x^2$

$\sigma \ll \sigma_{\text{c}} = (\delta \vee \varepsilon)^{3/4}$



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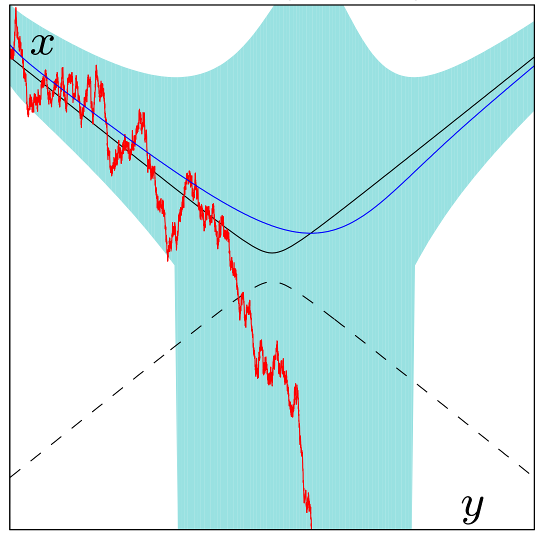


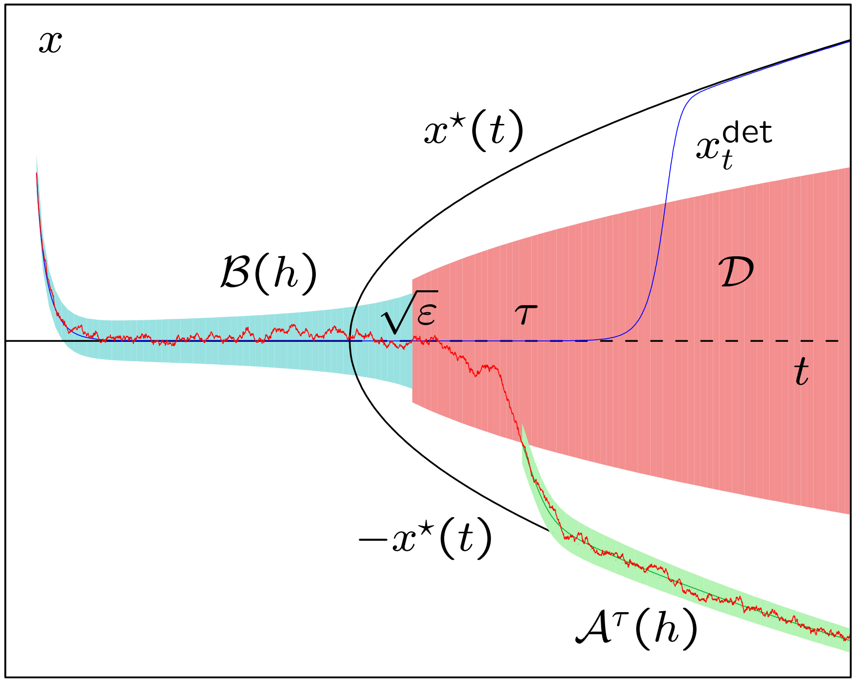
Minimal distance between branches $= \delta^{1/2}$
Det. case $\sigma = 0$: Solutions stay $(\delta \vee \varepsilon)^{1/2}$ above bif. point

Theorem: [B. & G., Ann. App. Probab. 2002]

1. If $\sigma \ll \sigma_{\text{c}}$: Paths likely to stay in $\mathcal{B}(h)$, transition probability $\leqslant \text{e}^{-c\sigma_{\text{c}}^2/\sigma^2}$.
2. If $\sigma \gg \sigma_{\text{c}}$: Transition typically for $t \asymp -\sigma^{2/3}$ transition probability $\geqslant 1 - \text{e}^{-c\sigma^{4/3}/\varepsilon|\log \sigma|}$

Pitchfork bifurcation

e.g. $f(x,y) = yx - x^3$



Theorem [B. & G., PTRF 2002]

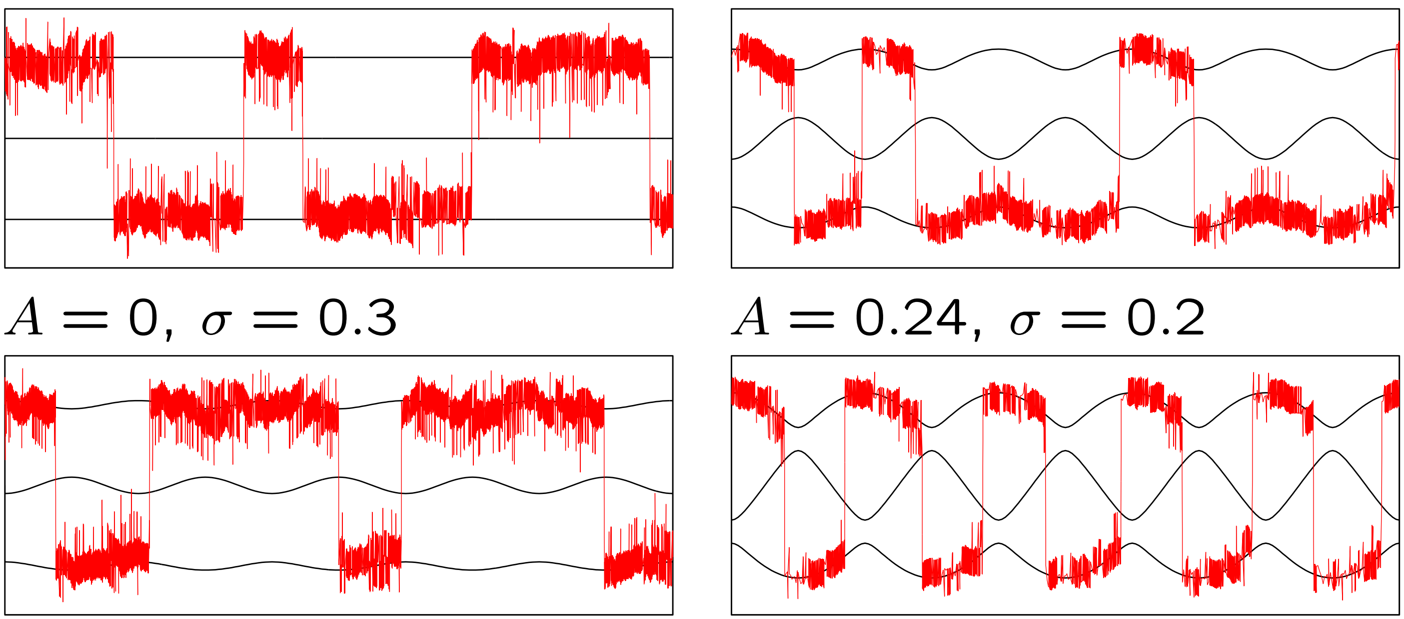
- Paths concentrated in $\mathcal{B}(h)$ up to time $\sqrt{\varepsilon}$
  Typical spreading $\sigma\varepsilon^{-1/4}$
- Paths likely to leave $\mathcal{D}$ at time $\sqrt{\varepsilon|\log\sigma|}$
- Paths likely to stay in $\mathcal{A}^\tau(h)$ after leaving $\mathcal{D}$

Stochastic resonance

$$dx_t = [-x^3 + x + A\cos \varepsilon t]\,dt + \sigma\,dW_t$$

- deterministically bistable climate (Croll, Milankovitch)
- random perturbations due to weather (Benzi/Sutera/Vulpiani, Nicolis/Nicolis)

Sample paths $\{x_t\}_t$ for $\varepsilon = 0.001$:



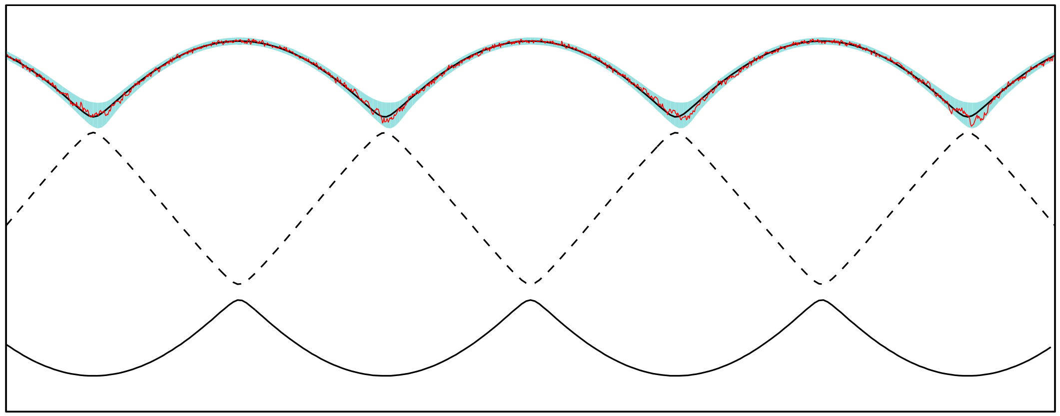
$A = 0$, $\sigma = 0.3$ $A = 0.24$, $\sigma = 0.2$

$A = 0.1$, $\sigma = 0.27$ $A = 0.35$, $\sigma = 0.2$

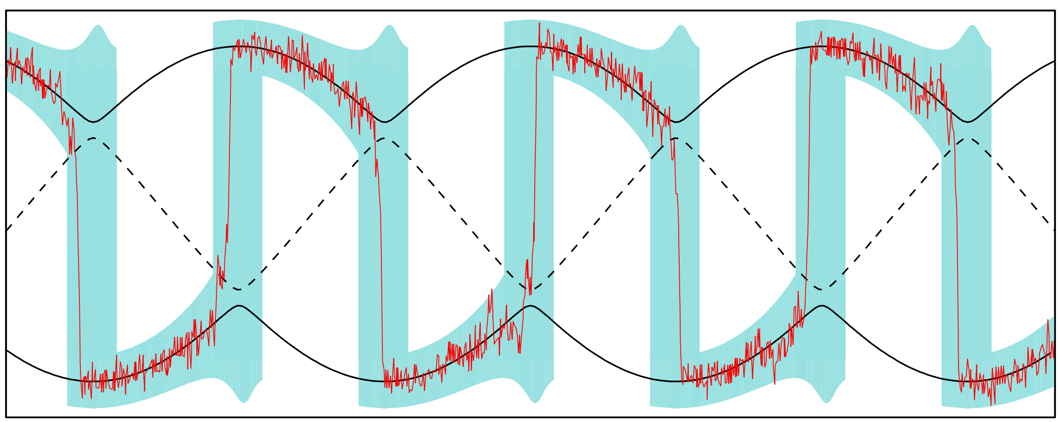
Stochastic resonance

Critical noise intensity: $\sigma_c = (\delta \vee \varepsilon)^{3/4}$, $\delta = A_c - A$

$\sigma \ll \sigma_c$: transitions unlikely



$\sigma \gg \sigma_c$: synchronisation



Stochastic resonance: Residence-time distribution

$q(t)$: probability density of time between transitions

Without forcing ($A = 0$): $q(t) \sim$ exponential.

With forcing ($A \gg \sigma^2$):

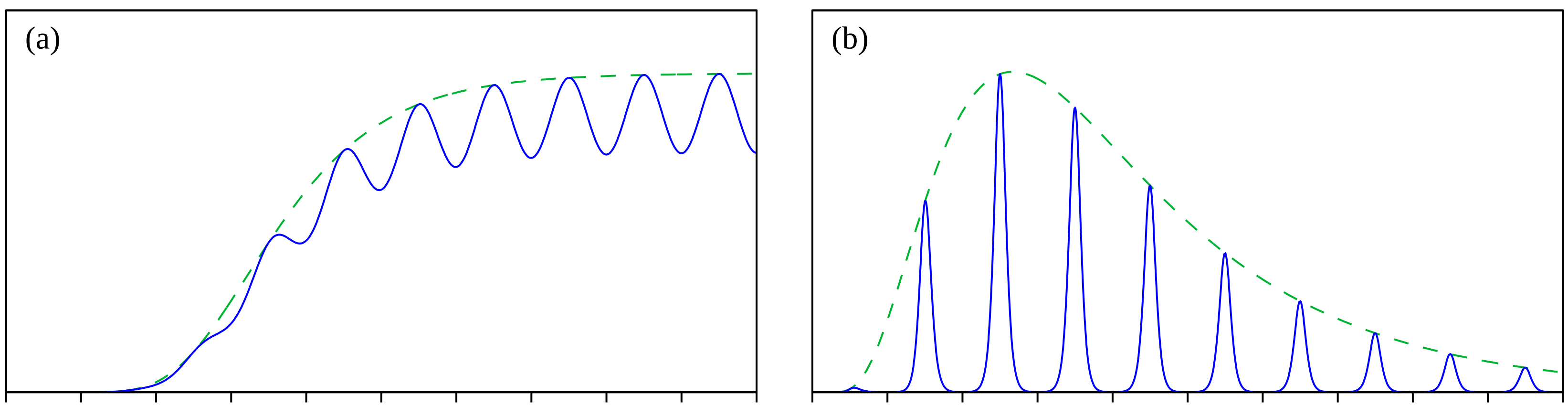
Theorem: [B. & G., Europhys Letters 2005]

$$q(t) \simeq f_{\text{trans}}(t) \frac{\mathrm{e}^{-t/T_{\mathsf{K}}}}{T_{\mathsf{K}}} \frac{\lambda T}{2} \sum_{k=-\infty}^{\infty} \frac{1}{\cosh^2(\lambda(t + T/2 - kT))}$$

T: forcing period

T_{K}: Kramers' time, $T_{\mathsf{K}} \simeq \frac{1}{\sigma} \mathrm{e}^{2H/\sigma^2}$

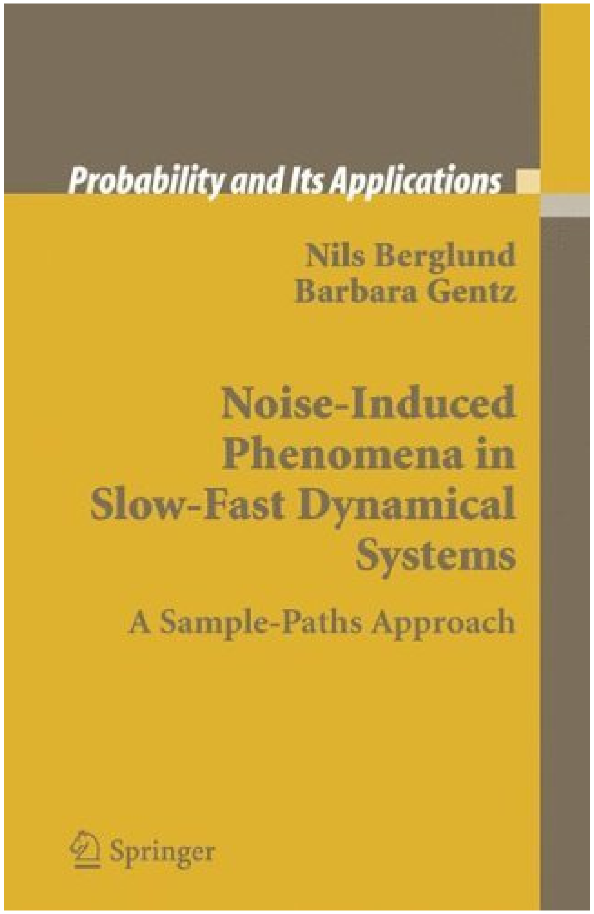
λ: Lyapunov exponent



$\sigma = 0.2$, $T = 2$ $\qquad$ $\sigma = 0.4$, $T = 10$

References

- N. B. & B. G, *Pathwise description of dynamic pitchfork bifurcations with additive noise*, Probab. Theory Related Fields **122**, 341–388 (2002)
- ______, *A sample-paths approach to noise-induced synchronization: Stochastic resonance in a double-well potential*, Ann. Appl. Probab. **12**, 1419-1470 (2002)
- ______, *The effect of additive noise on dynamical hysteresis*, Nonlinearity **15**, 605–632 (2002)
- ______, *Universality of first-passage and residence-time distributions in non-adiabatic stochastic resonance*, Europhys. Letters **70**, 1–7 (2005)
- ______, *Noise-Induced Phenomena in Slow-Fast Dynamical Systems. A Sample-Paths Approach.* Springer, Probability and its Applications, 276+xvi pages (2005)



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