PERISTOCH Days – **November 28 and 29, 2024**

Effects of noise on the dynamics of two types of fast–slow mechanical systems

Nonlinear passive vibration control and transient phenomena in reed musical instruments

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LaMé Laboratoire de Mécanique Gabriel Lamé

Plan

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Nonlinear Energy Sink (NES)

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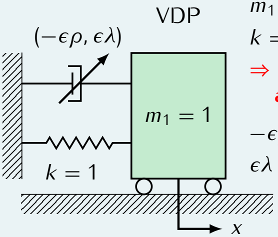
Targeted Energy Transfer (TET)

[Vakakis *et al.* (2006), Springer]

- Used for **passive** and **broadband** vibration mitigation in mechanical and acoustic systems:
  - Free vibrations
  - Forced vibrations
  - Self-sustained vibrations

Self-sustained oscillations: Van der Pol (VDP) oscillator

Van der Pol (VDP) oscillator



$m_1 = 1$: **mass**

$k = 1$: **stiffness**

$\Rightarrow \omega_0 = \sqrt{k/m_1} = 1$: **angular frequency**

$-\epsilon\rho$: **linear** damping

$\epsilon\lambda$: **nonlinear** damping

$$\ddot{x} - \epsilon\rho\dot{x} + \epsilon\lambda\dot{x}x^2 + x$$

ρ : **bifurcation parameter**

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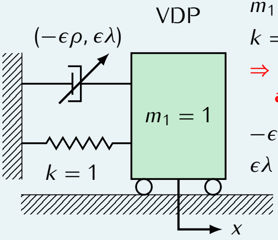

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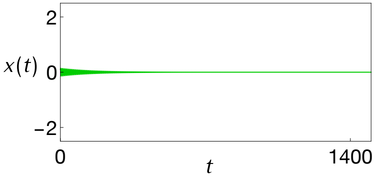
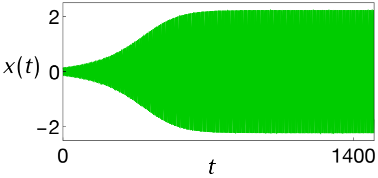
$\rho = 0$: **Hopf bifurcation point** of equilibrium $x^e = 0$

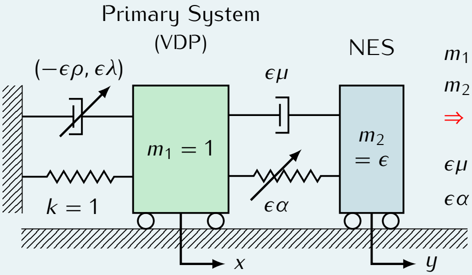
- $\rho < 0$: **Stable** equilibrium



- $\rho > 0$: **Unstable** equilibrium + **periodic solution**



Van der Pol oscillator coupled to an NES



$m_1 = 1$: **mass** of the VdP

$m_2 = \epsilon$: **mass** of the NES

$\Rightarrow \epsilon = m_2/m_1$: **mass ratio between NES and VDP**

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$\epsilon\alpha$: cubic stiffness of theNES

x: displacement of the VdP y: displacement of the NES

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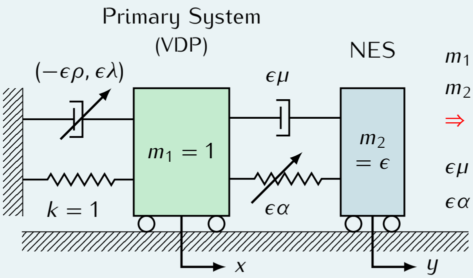

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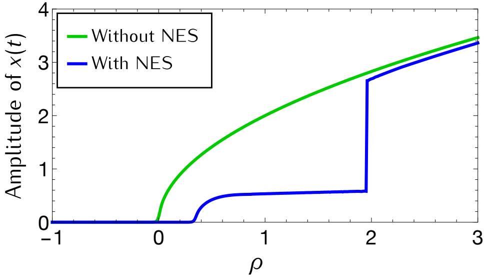
Assumption

Small-mass NES $\Rightarrow \boxed{0 < \epsilon \ll 1}$

Mitigation limit of the NES

Bifurcation diagram

Steady-state amplitude as a function of the **bifurcation parameter ρ**



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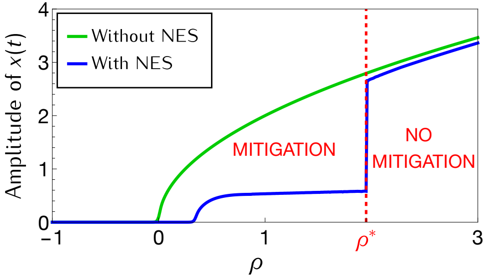

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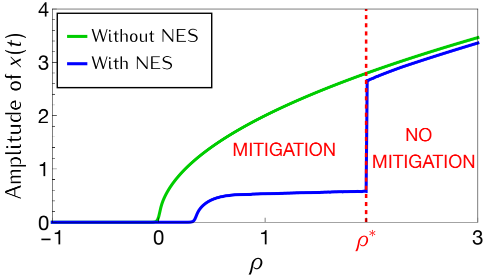
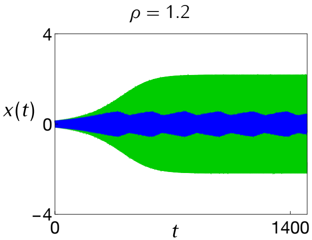
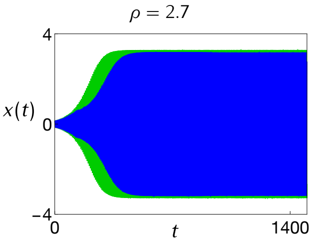


Quasi-periodic regime (SMR)



Periodic regime (with high amplitude)



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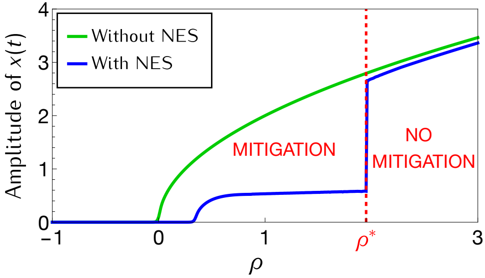

Zeroth-order global stability analysis [Gandelman & Bar (2012), Physica D]

Theoretical prediction of the mitigation limit when $\epsilon = 0$

Equations of the Amplitude-Phase Modulation Dynamics (APMD)

- Change of variable: x (VDP) and y (NES) $\Rightarrow$ $\boxed{u = x + \epsilon y}$ and $\boxed{v = x - y}$

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$\Rightarrow$ 1 : 1 **resonance capture** assumption

$\equiv$ u et v are **amplitude-** and **phase-modulated** $\Rightarrow$ $\boxed{u(t) = r(t)\sin(t + \theta_1(t))}$ et $\boxed{v(t) = s(t)\sin(t + \theta_2(t))}$

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$\hookrightarrow$ Computing the APMD using an **averaging** procedure:

$$\boxed{\begin{aligned} \dot{r} &= \epsilon f(r, s, \Delta) \\ \dot{s} &= g_1(r, s, \Delta, \epsilon) \\ \dot{\Delta} &= g_2(r, s, \Delta, \epsilon) \end{aligned}}$$

r et s: amplitudes of u and v
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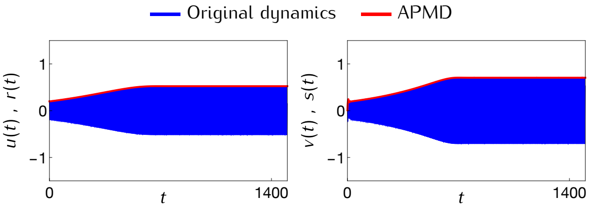
Original dynamics: **Periodic regime** $\equiv$ **APMD:** **Non-zero equilibrium**



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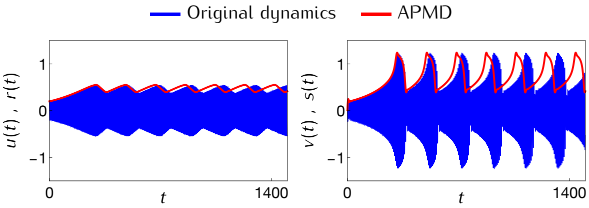

Original dynamics: SMR $\equiv$ **APMD:** **Periodic regime**

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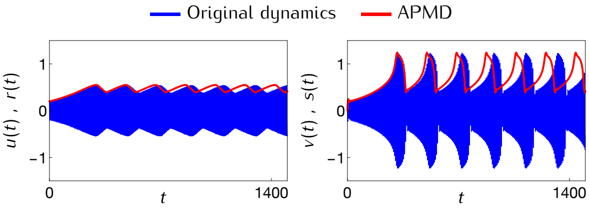

Original dynamics: SMR $\equiv$ **APMD:** **Periodic regime**

APMD $\equiv$ fast-slow dynamical system : **2 fast variables** s **and** Δ et **1 slow variable** r

$\Rightarrow$ Time evolution of the system = succession **fast epochs** and **slow epochs**

Zeroth-order fast-slow analysis of the APMD

APMD ≡ fast-slow dynamical system

- Time evolution of the system = succession **fast epochs** and **slow epochs**
- Theoretical analysis:
  - [Gandelman & Bar (2012), Physica D]: **multiple scales method**
  - [Bergeot *et al.* (2016), Int J Non Linear Mech]: **Geometric Singular Perturbation Theory (GSPT)**

ZEROTH-ORDER FAST-SLOW ANALYSIS OF THE APMD

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APMD
at the **fast time scale** t

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APMD
at the **slow time scale** $\tau = \epsilon t$

$$r' = f\,(r, s, \Delta)$$
$$\epsilon s' = g_1\,(r, s, \Delta, \epsilon)$$
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↪ **fast subsystem**
describes the fast epochs

We sate $\boxed{\epsilon = 0}$

Singularly perturbed system

APMD
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↪ **slow subsystem**
describes the slow epoch

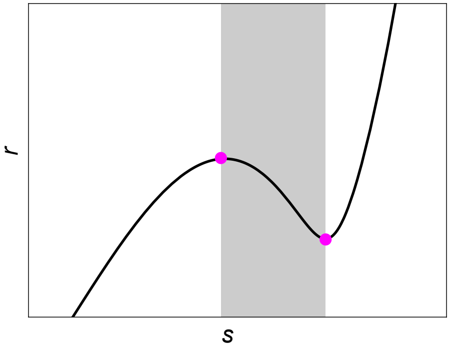
Zeroth-order fast-slow analysis of the APMD

Critical manifold (CM)

$$\mathcal{M}_0 = \left\{ (r, s, \Delta) \,\middle|\, g_1(r, s, \Delta, 0) = 0 \;,\; g_2(r, s, \Delta, 0) = 0 \right\}$$

$r = H(s)$ and $\Delta = G(s)$

Figure. — $r = H(s)$



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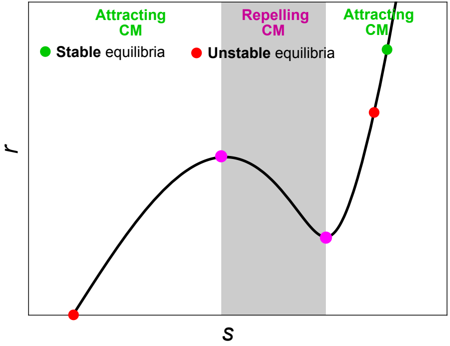

Figure. — $r = H(s)$

From the fast subsystem: Stability $\mathcal{M}_0 \Rightarrow$ 2 attracting branches et 1 repelling branch

From the slow subsystem: Equilibria (on $\mathcal{M}_0$) $\Rightarrow$ • **Stable** equilibria • **Unstable** equilibria

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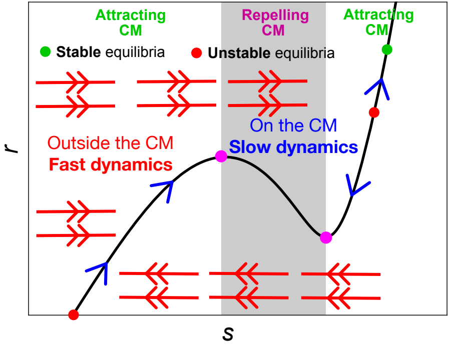

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Asymptotic behavior (when $\epsilon \to 0$) of APMD:

- During fast epochs: **horizontal trajectories outside $\mathcal{M}_0$ towards an attracting branch**
- During slow epochs: **on a attracting branch of $\mathcal{M}_0$ to a stable equilibrium** or **moving away from an unstable equilibrium in the slow subsystem**

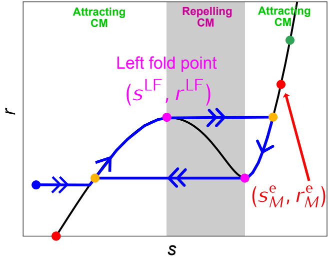
Zeroth-order fast-slow analysis of the APMD

Global stability analysis: theoretical prediction of the mitigation limit

• Initial condition • **Stable** equilibria • **Unstable** equilibria • Fold points • Zeroth-order arrival point

Original dynamics (OD): SMR

APMD: **Relaxation oscilllations**



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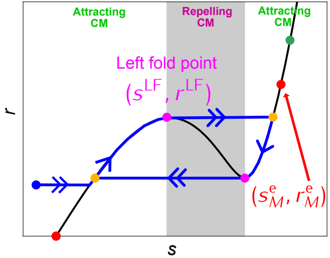
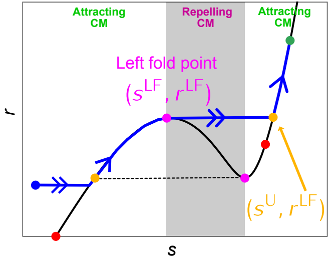


OD: No mitigation (periodic)
APMD: Stable equilibrium



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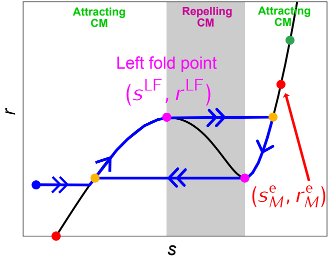

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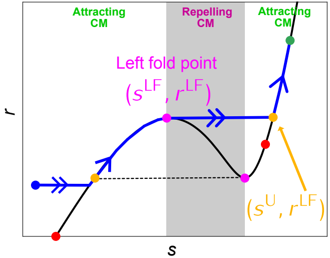

Zeroth-order arrival point

$$(s^{a}, r^{a}) = (s^{U}, r^{LF})$$

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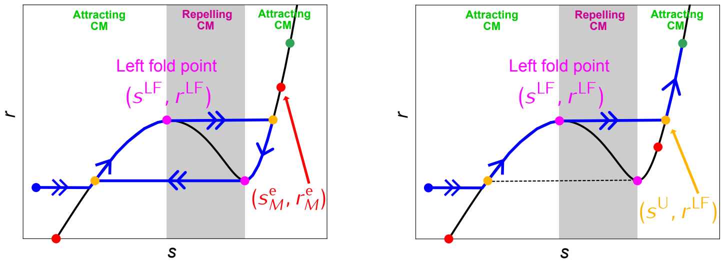

Zeroth-order arrival point

$$(s^a, r^a) = (s^U, r^{LF})$$

Zeroth-order theoretical prediction of the mitigation limit

Value of the bifurcation parameter ρ (denoted as ρ_0^*) solution of:

$$r^e_M = r^a = r^{LF}$$ $\Rightarrow$ Analytical expression of ρ_0^*

Zeroth-order fast-slow analysis of the APMD

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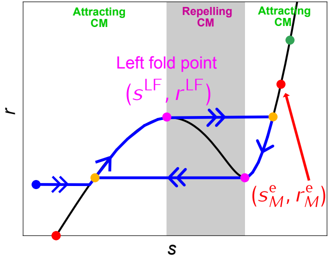

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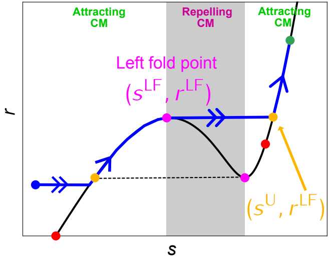


Zeroth-order arrival point

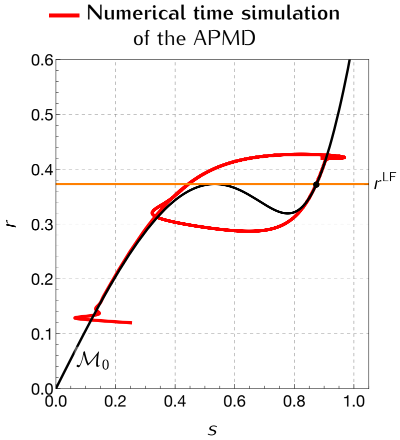
$$(s^a, r^a) = (s^U, r^{LF})$$

Today: presentation of 2 orignal results

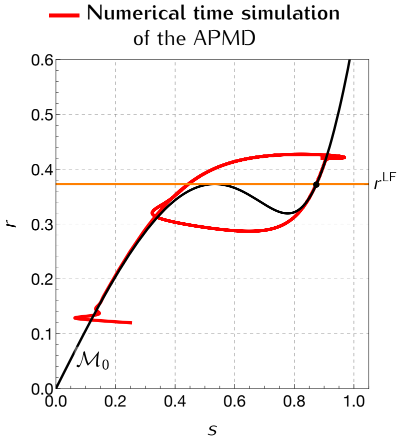
- **Result 1:** **scaling law** and new theoretical estimation of the mitigation limit [Bergeot (2021), J Sound Vib]
- **Result 2:** **effect of noise** on the mitigation limit of the NES [Bergeot (2023), Int. J. Non-Linear Mech.]

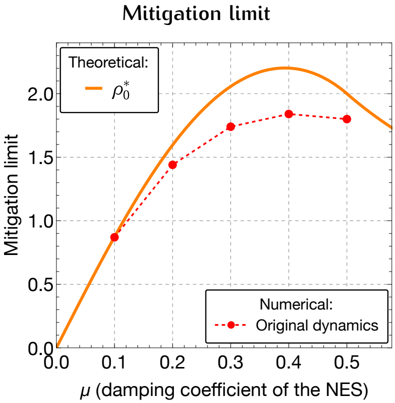
Plan

THE LIMITATIONS OF ZEROTH-ORDER ANALYSIS – THEORETICAL VS NUMERICAL RESULTS FOR $\epsilon = 0.015$



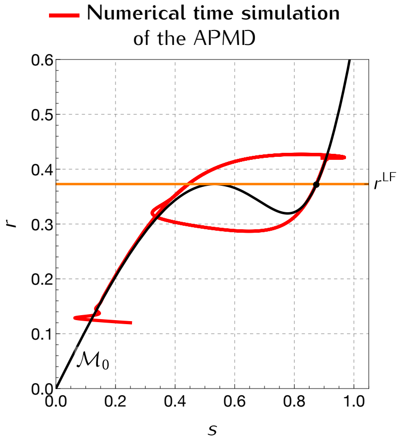
The limitations of zeroth-order analysis – theoretical vs numerical results for $\epsilon = 0.015$

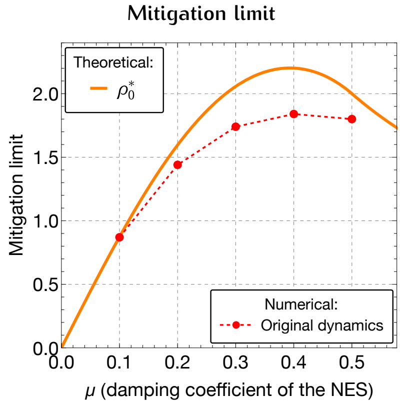




- For "large" values of ϵ: Underestimation of the arrival point ⇒ Overestimation of the mitigation limit

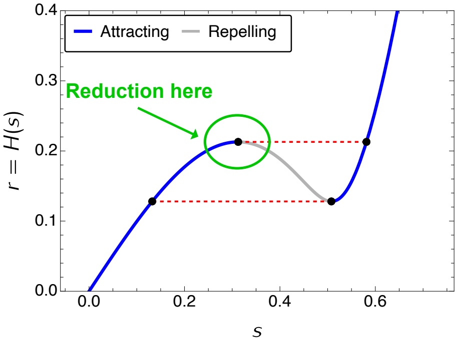
The limitations of zeroth-order analysis – theoretical vs numerical results for $\epsilon = 0.015$





- For "large" values of ϵ: Underestimation of the arrival point ⇒ Overestimation of the mitigation limit
- No description of the evolution of the mitigation limit as a function of ϵ.

Center manifold reduction of the APMD at the left fold point and scaling law (1/2)



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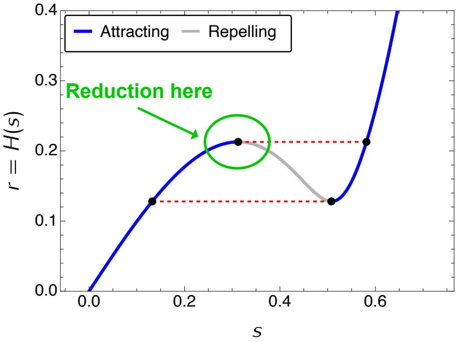

At the **left fold point** $(r^{\mathrm{LF}}, s^{\mathrm{LF}}, \Delta^{\mathrm{LF}})$ the APMD …

$$
\begin{aligned}
r' &= f(r, s, \Delta) \\
\epsilon s' &= g_1(r, s, \Delta, \epsilon) \\
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$$

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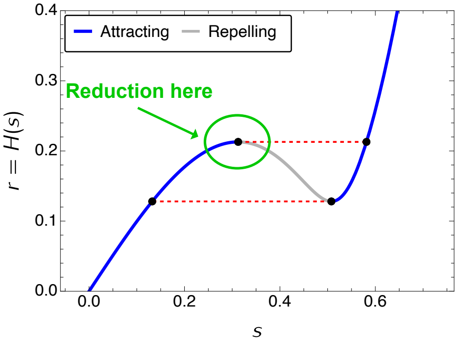

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… is reduced to the normal form of the **dynamic saddle-node bifurcation**:

$$\begin{aligned} \hat{\epsilon} x' &= x^2 + y \\ y' &= 1 \end{aligned}$$

y: new slow variable linked to r
x: new fast variable linked to s et Δ
$\hat{\epsilon}$: new small parameter linked to ϵ

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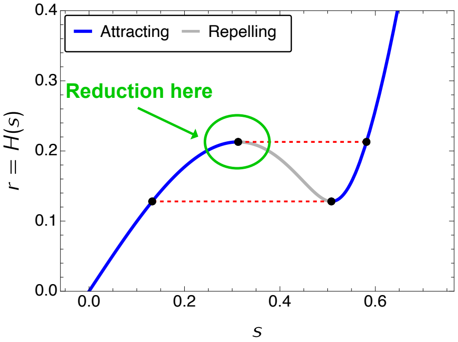

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⇒ Has a analytical solution:

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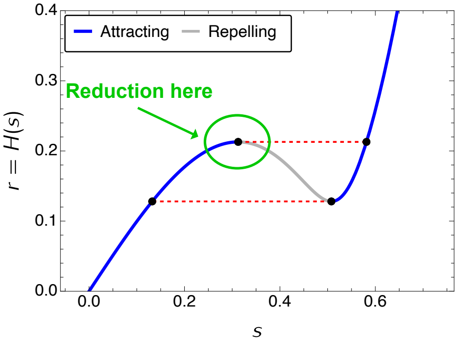

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$\Rightarrow$ **Has a analytical solution**:

Scaling law (normal form)

Analytical expression of **x as a function y and $\hat{\epsilon}$**:

$$x^\star(y, \hat{\epsilon}) = \hat{\epsilon}^{1/3} \frac{\mathrm{Ai}'\left(-\hat{\epsilon}^{-2/3} y\right)}{\mathrm{Ai}\left(-\hat{\epsilon}^{-2/3} y\right)}$$

Ai: Airy function

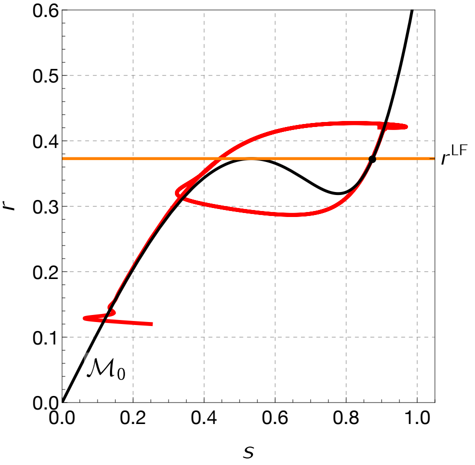
Center manifold reduction of the APMD at the left fold point and scaling law (2/2)

Scaling law (APMD)

Analytical expression of **s as a function of r and ϵ**:

$$s^\star(r,\epsilon) = s^{\mathrm{LF}} + \epsilon^{1/3} K_1 \frac{\mathrm{Ai}'\left(-\epsilon^{-2/3} K_2 \left(r - r^{\mathrm{LF}}\right)\right)}{\mathrm{Ai}\left(-\epsilon^{-2/3} K_2 \left(r - r^{\mathrm{LF}}\right)\right)}$$

- K_1 and K_2: constants depending on model parameters
- Ai and Ai′: Airy function and its derivative



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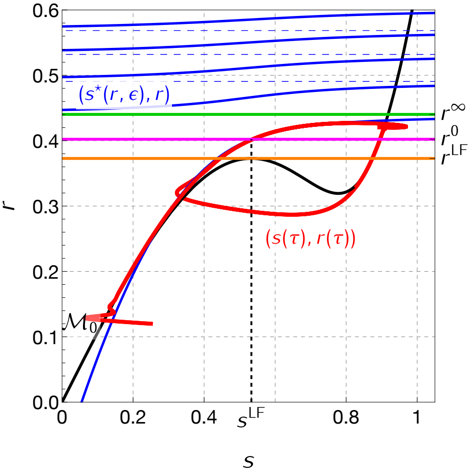

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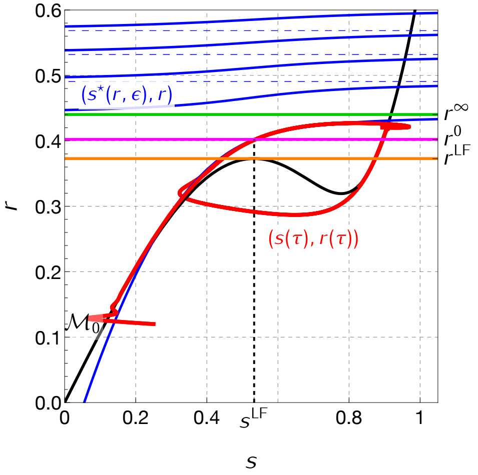
New estimation of the arrival point (s^A, r^A)

$$r^0 < r^a < r^\infty$$

r^0: defined as $s^\star(r) = s^{\mathrm{LF}}$ ⇒ first zero of Ai′

r^∞: defined as $s^\star(r) \to \infty$ ⇒ first zero of Ai



New theoretical estimation of the mitigation limit

From the zeroth-order analysis

Value of ρ (denoted as ρ_0^*) solution of:

$$r_M^{\mathrm{e}} = r^{\mathrm{a}} = r^{\mathrm{LF}}$$

From the scaling law

Lower bound: $\rho_{\epsilon,\inf}^*$ solution of:

$$r_M^{\mathrm{e}} = r^{\mathrm{a}} = r^{\infty}$$

Upper bound: $\rho_{\epsilon,\sup}^*$ solution of:

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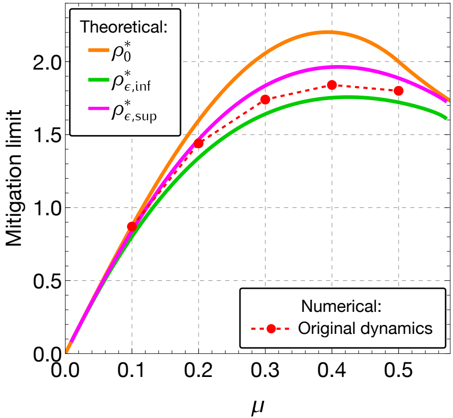
As a function of μ for $\epsilon = 0.015$:



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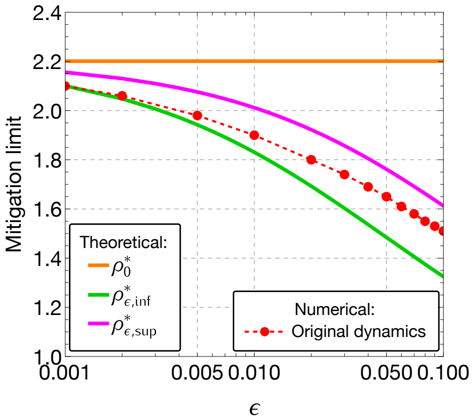
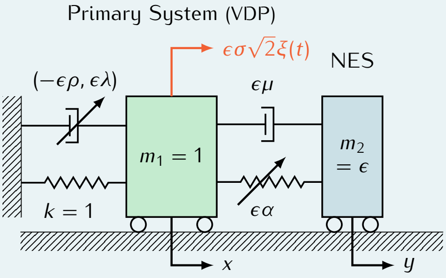


Plan

The stochastic system

VDP oscillator with stochastic forcing



- $\xi(\tau)$: **white noise** with $\xi(t) = dW(t)/dt$ and $W(t)$ the *Wiener process*
- **Assumption:** **small level of noise**, i.e., of order $\mathcal{O}(\epsilon)$

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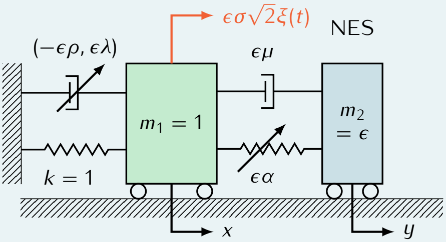

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Stochastic APMD

APMD ≡ **fast-slow dynamical system** with **white noise acting on the slow variable** r

$$\dot{r} = \epsilon f(r, s, \Delta) + \epsilon\sigma\xi(\tau)$$
$$\dot{s} = g_1(r, s, \Delta, \epsilon)$$
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Probability of being in a mitigation regime

- **Definition.** Denoted as $\boxed{p_{\mathrm{h},n}}$: probability for the system of being in a mitigation regime after a given number n of full cycles of relaxation oscillations.
- **In practice** $p_{\mathrm{h},n}$ computed as the proportion of samples for which we observe at least $n+1$ consecutive full cycles of relaxation oscillations from the beginning of the sample

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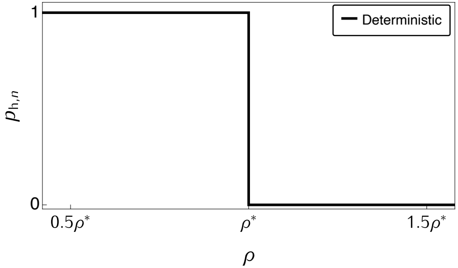

Figure: $p_{h,n}$ in the deterministic case

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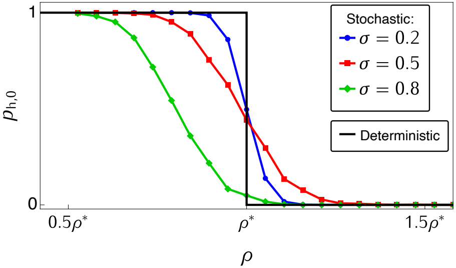

Figure: $p_{h,n}$ obtained using the **Monte Carlo method** with the stochastic APMD

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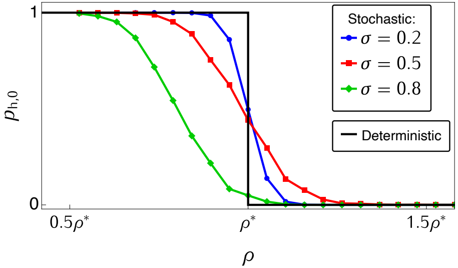

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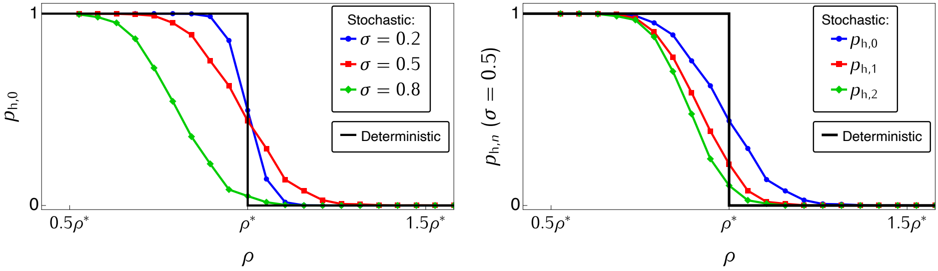

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Noise tends to promote the non mitigation regimes for high noise levels

Analytical results

PhD of Israa Zogheib (Nov. 2023- ; Dir. Nils Berglund and Baptiste Bergeot)

Objective

Prove the previous observations and **predict the probability of being in a mitigation regime**

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$$\hat{\epsilon}x' = x^2 + y$$
$$y' = 1 + \sqrt{\hat{\epsilon}}\hat{\sigma}\xi(\tau)$$

We define:

- The **first-passage time**, denoted as T, as follows

$$T = \inf\{t > 0 : x = X\}$$

- The **value of y for $t = T$** as y_T

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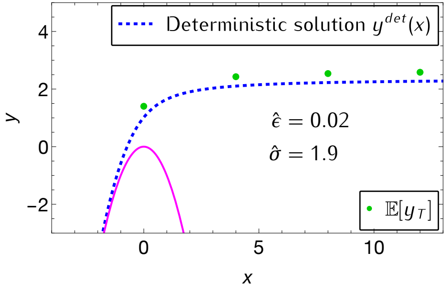
A first result: proof of $\mathbb{E}[y_T] > y^{det}(X)$ . . .



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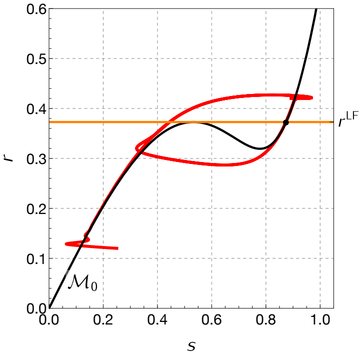

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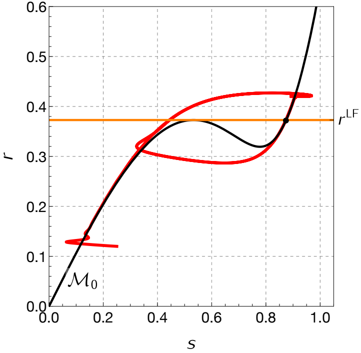

... which suggests that:

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... and gives a first element to prove that:

Noise promotes the non mitigation regimes

Plan

Plan

Single-reed musical instruments:

Saxophones

Clarinets

- Modeled by nonlinear dynamical systems linking **control parameters** (mouth pressure γ) to **output variables** (**acoustic pressure p inside the mouthpiece**)
- Previous theoretical studies on sound production performed with **control parameters constant in time** show that:
  - Appearance of sound = **Hopf bifurcation of the trivial equilibrium** (**silence, i.e., $p = 0$**) to a **stable periodic solution** (**musical note**)
  - Several stable solutions coexist in general = **Multistability**

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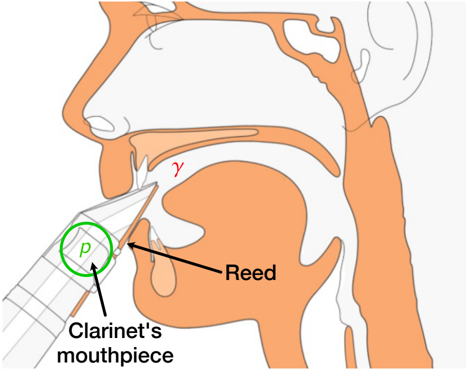

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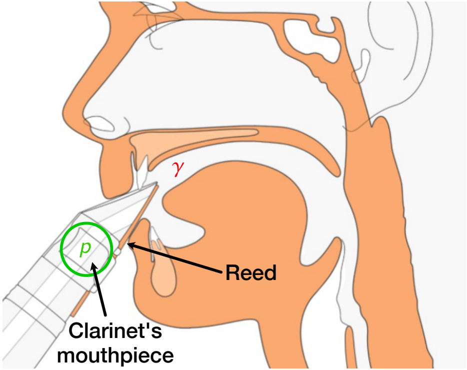

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Observation

During transients the musician **varies the control parameters in time**

Questions

- **In the context of musical acoustics:** during an **attack transient**, when the **mouth pressure increases**, what are the consequences:
  1. on the **appearance of sound**?
  2. on the **nature of the sound in case of multistability**? ⇒ silence? note? another note?
- **Open problems in nonlinear dynamics:** nonlinear dynamical systems with time-varying parameters when
  1. a bifurcation point is crossed ⇒ **bifurcation delay** [Benoit *et al.* (1991), Lect. Notes Math.]
  2. a multistability domain is crossed ⇒ **tipping phenomenon** [Ashwin *et al.* (2012), Philos Trans R Soc Lond, A]

Presented work

Predicting appearance of sound and the **nature of attack transient** in a **simple models** in the case of a **slow linear variation of the control parameter** **"mouth pressure" γ**

$$\dot{\gamma} = \epsilon \quad \text{with} \quad \boxed{0 < \epsilon \ll 1}$$

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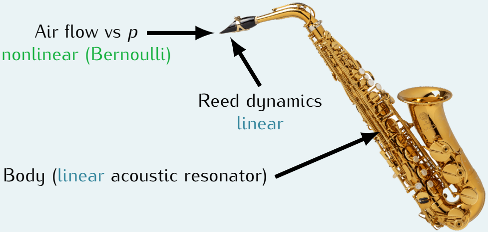

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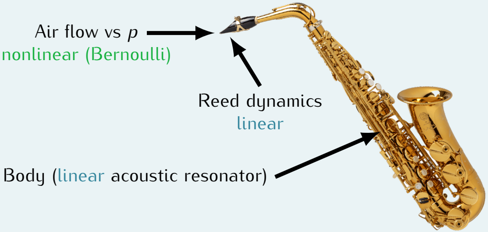

⇒ **System of coupled nonlinear ODEs**

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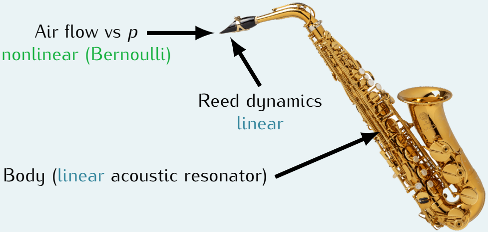

⇒ **System of coupled nonlinear ODEs**

Simple models having

⇒ **One-dimensional ODE:**

$$\boxed{\dot{x} = f(x, \gamma)}$$

x: amplitude of the mouthpiece pressure p
γ: control (or bifurcation) parameter

Model with a slowly time-varying γ = fast-slow system

$$\dot{x} = f(x, \gamma)$$
$$\dot{\gamma} = \epsilon$$

x: **fast variable**

γ: **slow variable**

Model with a slowly time-varying γ = fast-slow system

$$\begin{aligned} \dot{x} &= f(x, \gamma) \\ \dot{\gamma} &= \epsilon \end{aligned}$$

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γ: **slow variable**

Simple model at the fast time scale t

$$\begin{aligned} \dot{x} &= f(x, \gamma) \\ \dot{\gamma} &= \epsilon \end{aligned}$$

Simple model at the slow time scale $\tau = \epsilon t$

$$\begin{aligned} \epsilon \dot{x} &= f(x, \gamma) \\ \dot{\gamma} &= 1 \end{aligned}$$

Model with a slowly time-varying γ = fast-slow system

$$\begin{aligned}\dot{x} &= f(x, \gamma)\\ \dot{\gamma} &= \epsilon\end{aligned}$$

x: **fast variable**

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Simple model at the fast time scale t

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$$\begin{aligned}\dot{x} &= f(x, \gamma)\\ \dot{\gamma} &= 0\end{aligned}$$

$\hookrightarrow$ fast subsystem

We state

$$\epsilon = 0$$

Simple model at the slow time scale $\tau = \epsilon t$

$$\begin{aligned}\epsilon\dot{x} &= f(x, \gamma)\\ \dot{\gamma} &= 1\end{aligned}$$

$$\begin{aligned}0 &= f(x, \gamma)\\ \gamma' &= 1\end{aligned}$$

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Model with a slowly time-varying γ = fast-slow system

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- Defined by:

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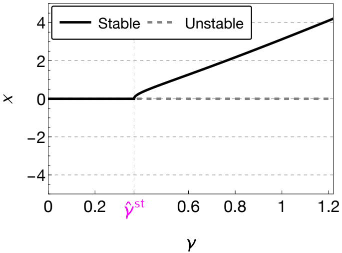
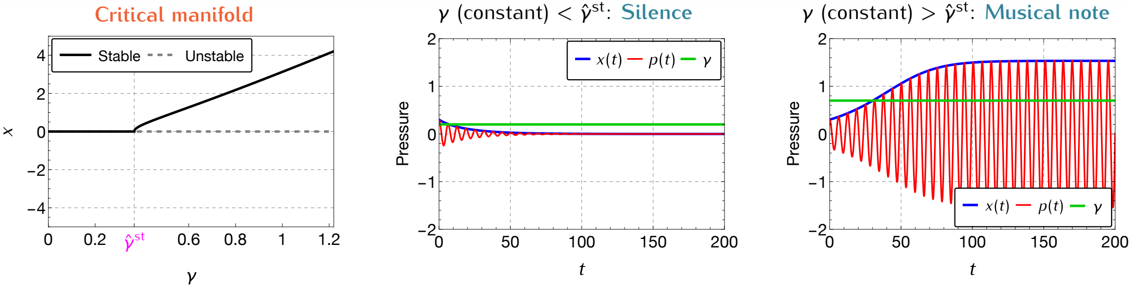
- = bifurcation diagram of the fast subsystem

Plan

Critical manifold



$\hat{\gamma}^{st}$: **Static** Pitchfork bifurcation point



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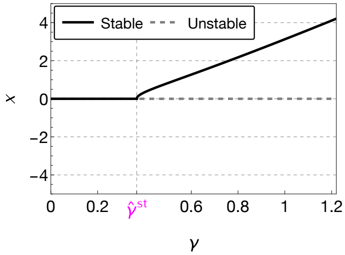

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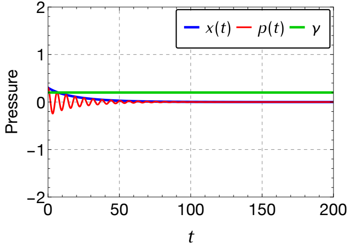
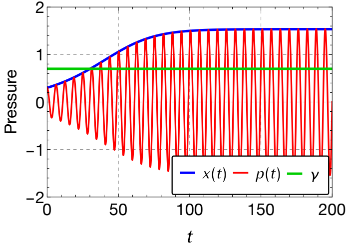
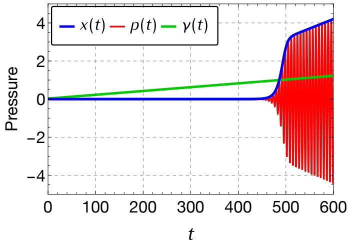
γ (constant) $< \hat{\gamma}^{st}$: **Silence**



γ (constant) $> \hat{\gamma}^{st}$: **Musical note**



γ slowly varies in time



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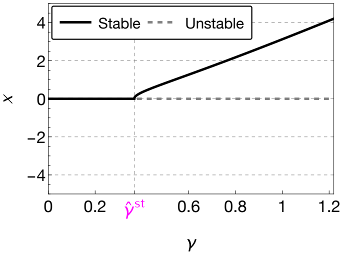

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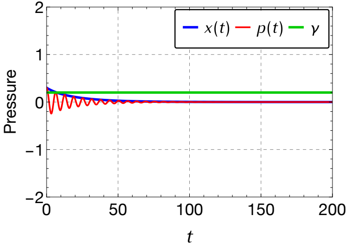

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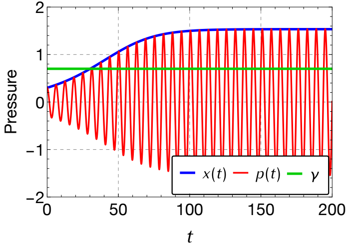

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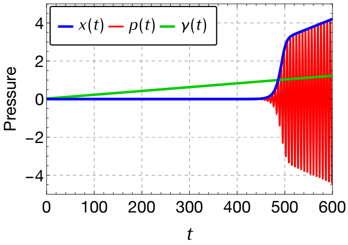
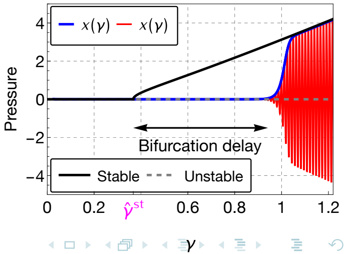


$x(t)$ as a function of $\gamma(t)$



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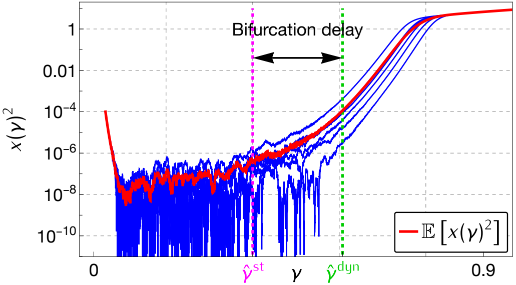

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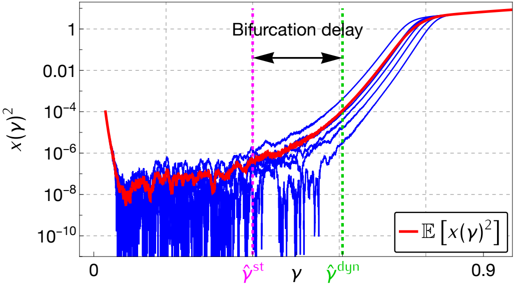

Definition: dynamic bifurcation point $\hat{\gamma}^{dyn}$

Value of γ such as $\mathbb{E}\left[x(\gamma)^2\right] = x(\gamma_0)^2$

Analytical prediction of bifurcation delay

Analytical solution of:

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[Bergeot & Vergez (2022), Nonlinear Dyn]

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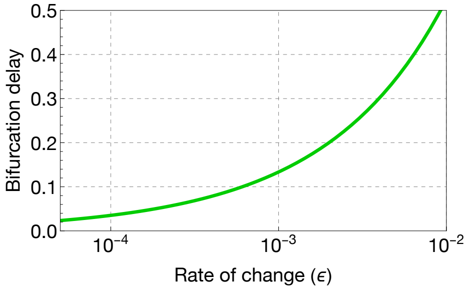

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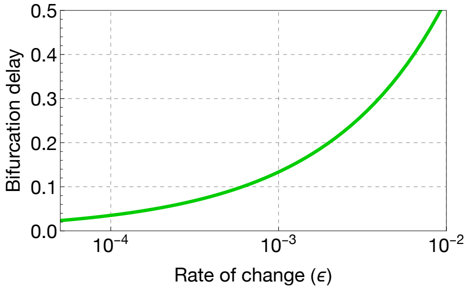
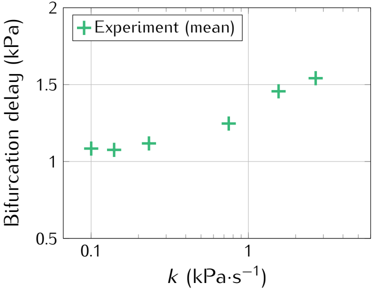


Experimental: as a function $k \propto \epsilon$:



[Bergeot *et al.* (2014), J Acoust Soc Am]

Plan

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Remark. $f(x, \gamma)$ now takes into account that **reed motion is limited by the instrument mouthpiece**

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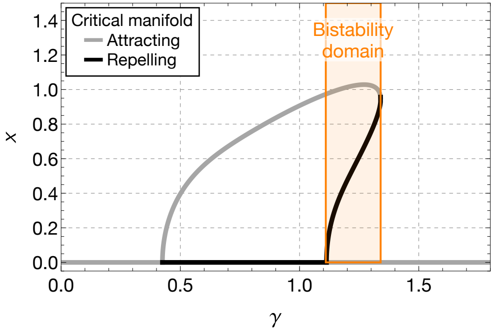

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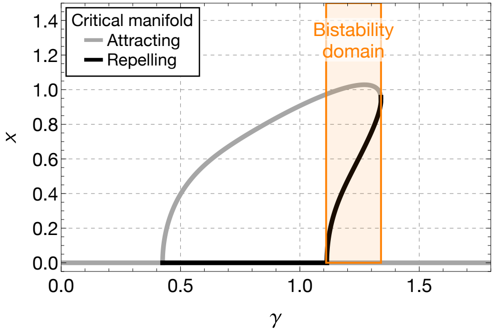

In the bistability domain $f(x, \gamma) = 0$ has **3 solutions**:

- **2 stable equilibria**
- **1 unstable equilibrium**

Problem statement

$$\begin{aligned}\dot{x} &= f(x,\gamma)\\ \dot{\gamma} &= \epsilon\end{aligned} \qquad (1)$$

- For a given initial condition, which attracting branch of the critical manifold will the trajectory of (1) follows when it crosses the bistability domain?

$\Rightarrow$ More concisely: **tipping of not tipping?**

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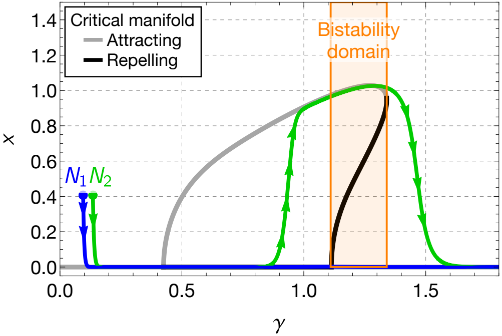

Figure. Numerical simulations of (1) with two close initial conditions N_1 and N_2

Observation

Although N_1 and N_2 are very close in the phase space, they lead to **qualitatively different behaviors during transient**:

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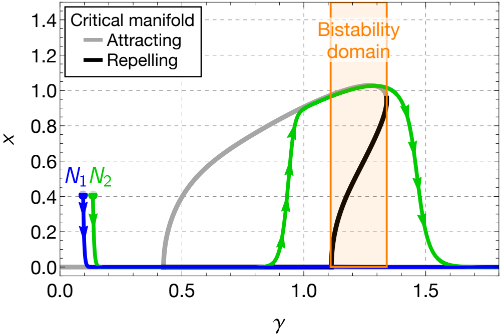

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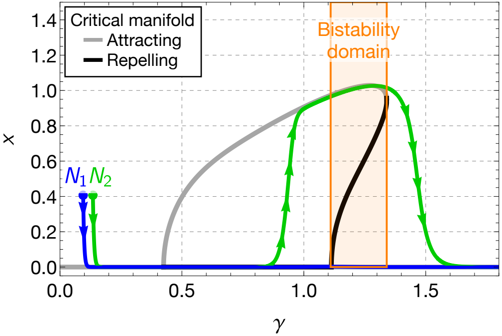

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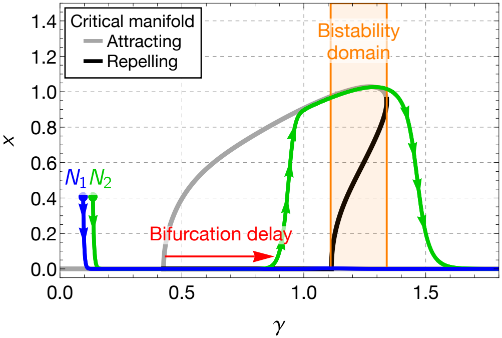

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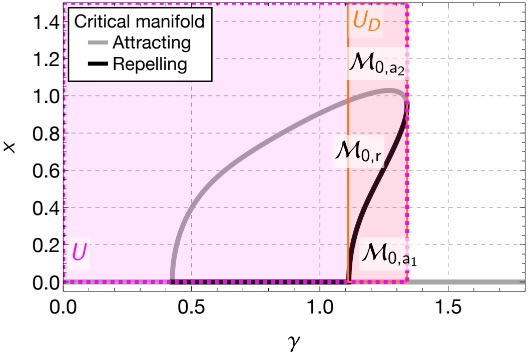
Remark

Bifurcation delay

$$\dot{x} = f(x, \gamma) \qquad \dot{\gamma} = \epsilon \tag{1}$$

$U_D = (\gamma_l, \gamma_u) \times \mathbb{R}^+$; $U = (0, \gamma_u) \times \mathbb{R}^+$



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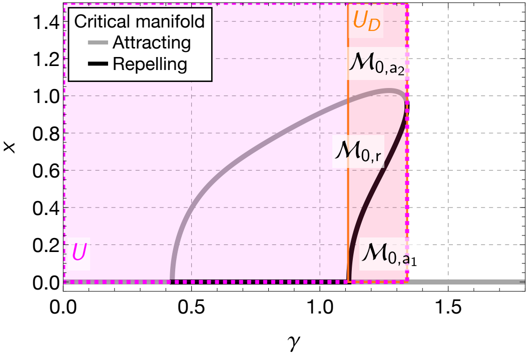

In U_D, $\mathcal{M}_0$ has **3 branches**:

$$\mathcal{M}_{0,a_i} = \{(x, \gamma) \in U_D \mid x = x_i^\star(\gamma)\}, \quad i = 1, 2$$

$$\mathcal{M}_{0,r} = \{(x, \gamma) \in U_D \mid x = x_3^\star(\gamma)\}$$

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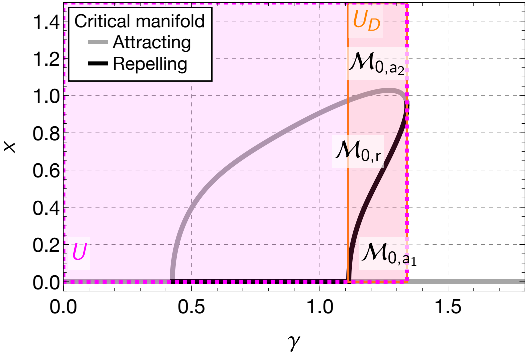

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Fenichel's theorem

$\Downarrow$

In U_D, one has **3 invariant manifolds**:

$$\mathcal{M}_{\epsilon,\mathrm{a}_i} = \{(x, \gamma) \in U_D \mid x = \bar{x}_i(\gamma, \epsilon)\}, \quad i = 1, 2$$

$$\mathcal{M}_{\epsilon,\mathrm{r}} = \{(x, \gamma) \in U_D \mid x = \bar{x}_3(\gamma, \epsilon)\}$$

with

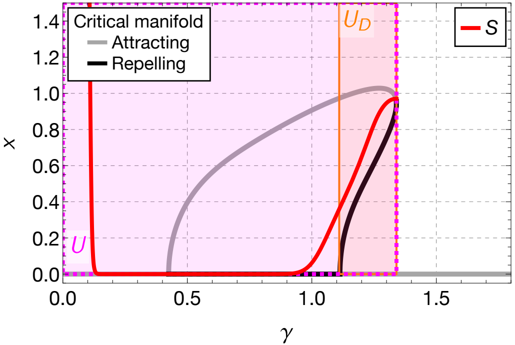
$$\bar{x}_i(\gamma, \epsilon) = x_i^\star(\gamma) + \mathcal{O}(\epsilon) \quad i = 1, 2, 3$$

Tipping separatrix

$$\begin{cases} \dot{x} = f(x, \gamma) \\ \dot{\gamma} = \epsilon \end{cases} \quad (1)$$

We define the **special solution** S of (1), called **tipping separatrix***, in U as

$$S = \{(x, \gamma) \in U \mid x = \bar{x}_3(\gamma, \epsilon)\}$$



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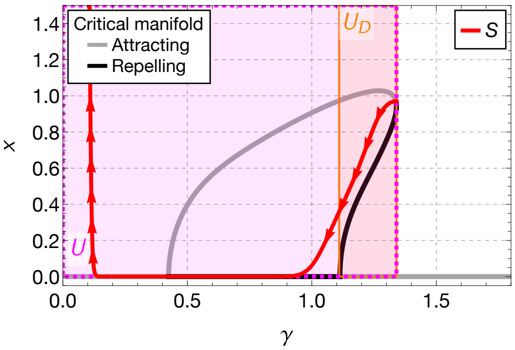
In practice

S is numerically approximated using a time reversal procedure since here $\mathcal{M}_{\epsilon,r}$ is attracting in reverse time



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RESULT [Bergeot *et al.* (2024), Chaos]

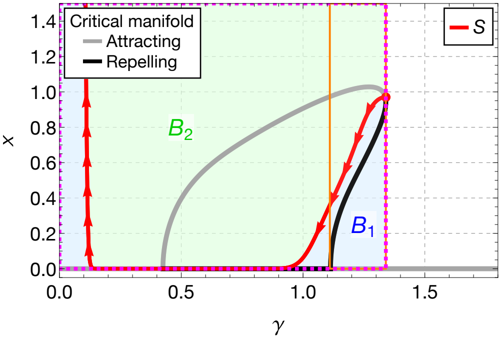
Tipping or not tipping?

The **tipping separatrix** S splits U into two subsets B_1 and B_2:

$$B_1 = \{(x, \gamma) \in U \mid x < \bar{x}_3(\gamma, \epsilon)\}$$ **NO TIPPING**

$$B_2 = \{(x, \gamma) \in U \mid x > \bar{x}_3(\gamma, \epsilon)\}$$ **TIPPING**

Orbits originating from initial conditions in B_1 (resp. B_2) follow $\mathcal{M}_{\epsilon,a_1}$ (resp. $\mathcal{M}_{\epsilon,a_2}$) when the slow variable γ crosses the bistability domain U_D



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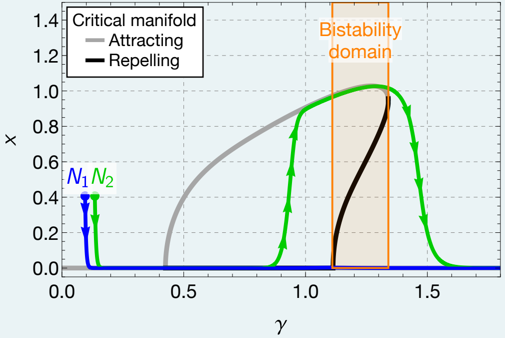

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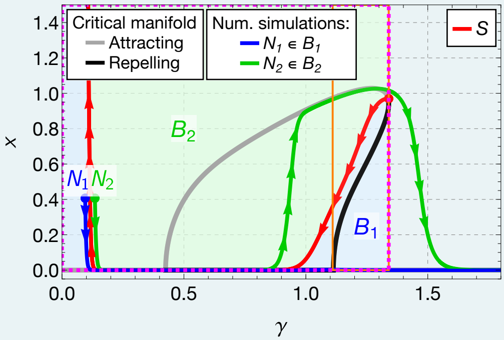
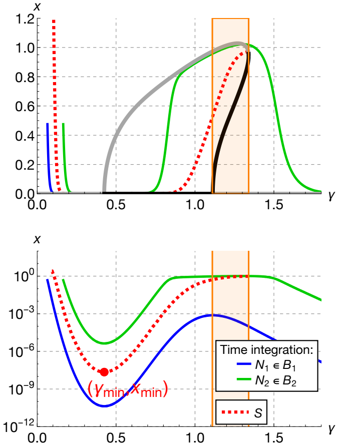


Explanation. Although N_1 and N_2 are very close in the phase space, **they are not in the same B subset**, that leads to **qualitatively different behavior during transient**

Effect of noise on tipping phenomenon

A **white noise of level σ** is **added to the fast variable x**

Deterministic: $\boxed{\sigma = 0}$



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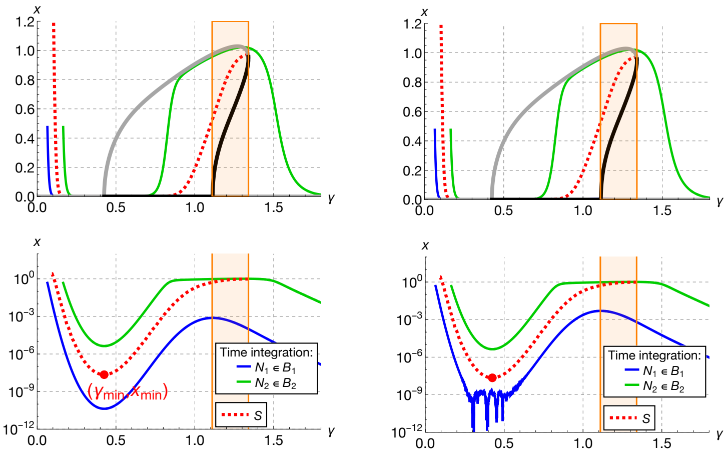

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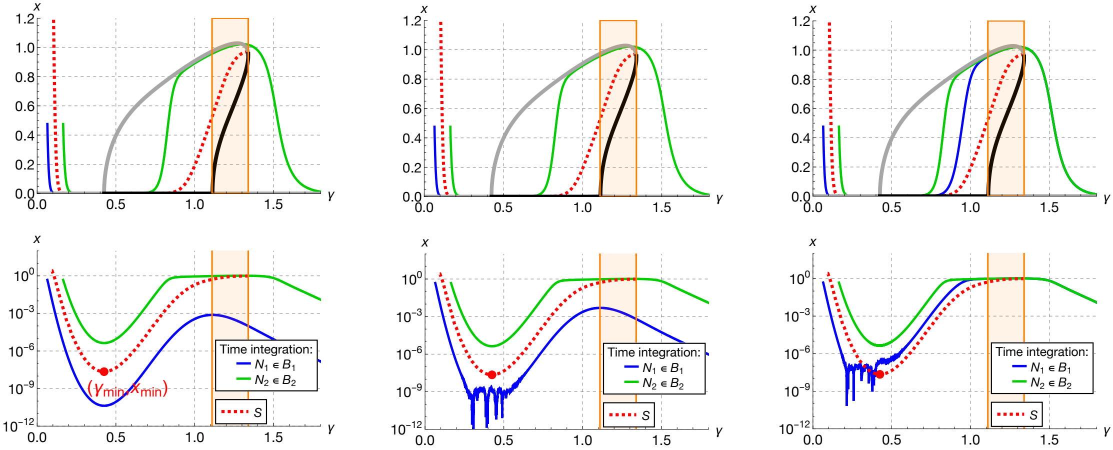

Probability of tipping P_{Tip}

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In practice, here P_{Tip} is the **probability that the trajectory of the stochastic system remains above the tipping (deterministic) separatrix** S **when** $y = y_l$ (the lower bound of the bistability domain U_D).

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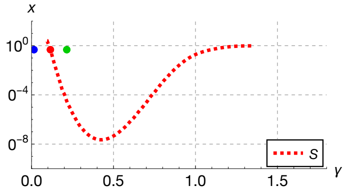
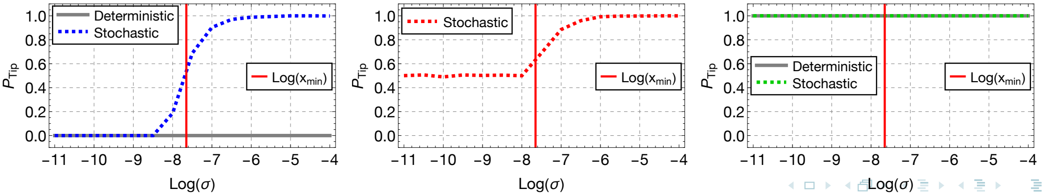
P_{Tip} computed numerically (Monte Carlo method with 2000 samples) for **3 initial conditions**:

● $\in B_1$ ● $\in S$ ● $\in B_2$





Thank you for your attention

Questions?