Desynchronisation of Coupled Bistable Oscillators Perturbed by Additive Noise

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Gradient System: $\mathrm{d}x^\sigma(t) = -\nabla V_\gamma(x^\sigma(t))\,\mathrm{d}t + \sqrt{N}\sigma\,\mathrm{d}B(t)$

Potential: $V_\gamma(x) = \sum_{i\in\Lambda} U(x_i) + \frac{\gamma}{4}\sum_{i\in\Lambda}(x_{i+1} - x_i)^2$

Gradient system $\qquad \mathrm{d}x^\sigma(t) = -\nabla V(x^\sigma(t))\,\mathrm{d}t + \sigma\,\mathrm{d}B(t)$

τ: First-passage time from one potential well to another

- Large deviations (Wentzell & Freidlin): $\lim_{\sigma\to 0} \sigma^2 \log(\mathbb{E}\{\tau\})$
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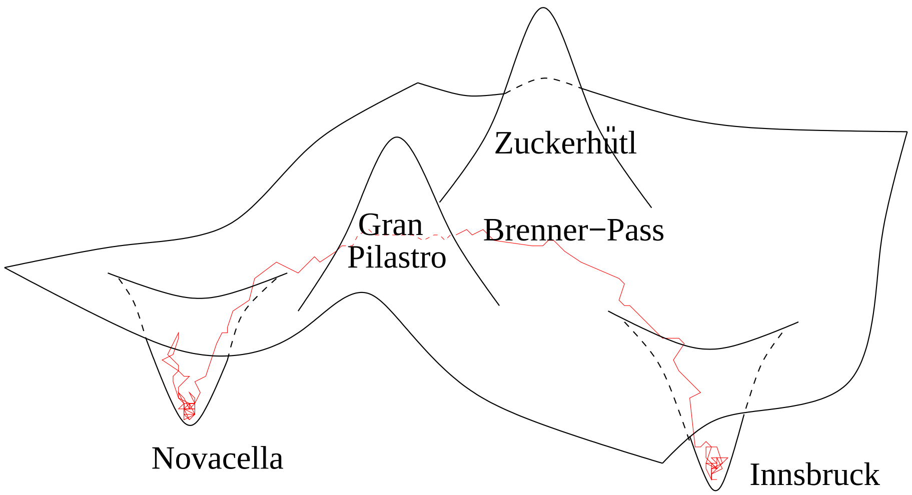

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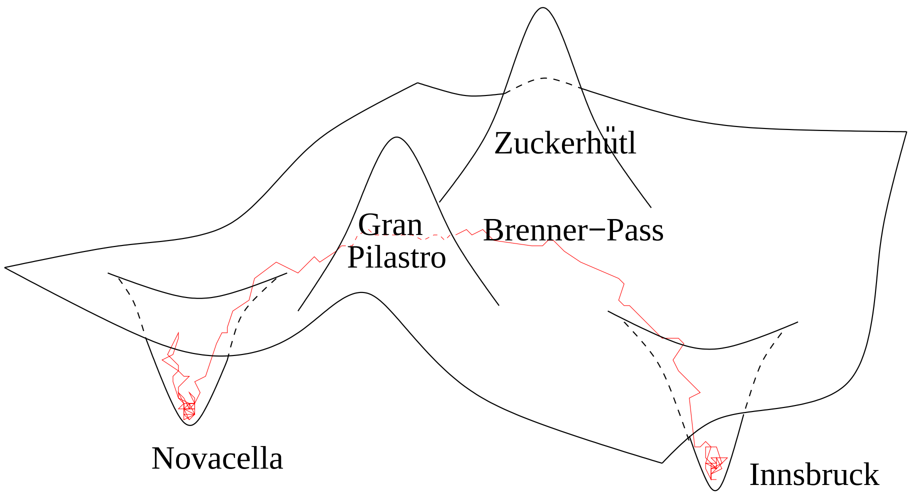

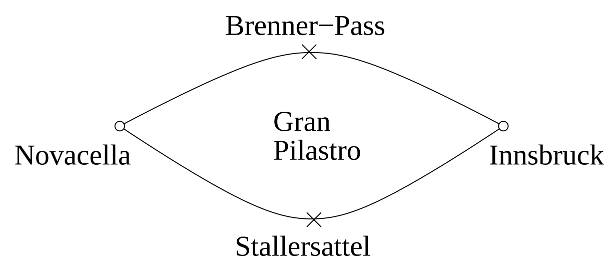

▷ Stationary pts: $\mathcal{S} = \{x : \nabla V(x) = 0\}$

▷ Saddles of index k: $\mathcal{S}_k = \{x \in \mathcal{S} : \text{Hess}\, V(x)$ a k v.p. $> 0\}$

▷ Graph $\mathcal{G} = (\mathcal{S}_0, \mathcal{E})$, $x \leftrightarrow y$ si $x, y \in$ unst. manif. of $s \in \mathcal{S}_1$

▷ $x_t \sim$ markovian jump process on $\mathcal{G}$

Weak coupling

- $\triangleright$ $\gamma = 0$: $\mathcal{S} = \{-1, 0, 1\}^\Lambda$, $\mathcal{S}_0 = \{-1, 1\}^\Lambda$, $\mathcal{G} =$ hypercube.

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Theorem: $\forall N$, $\exists \gamma^\star(N) > 0$ s.t. points of each $S_k(\gamma)$ continuous in γ for $0 \leqslant \gamma < \gamma^\star(N)$

$\frac{1}{4} \leqslant \inf_{N \geqslant 2} \gamma^\star(N) \leqslant \gamma^\star(3) = \frac{1}{3}\Big(\sqrt{3 + 2\sqrt{3}} - \sqrt{3}\Big) = 0.2701\ldots$

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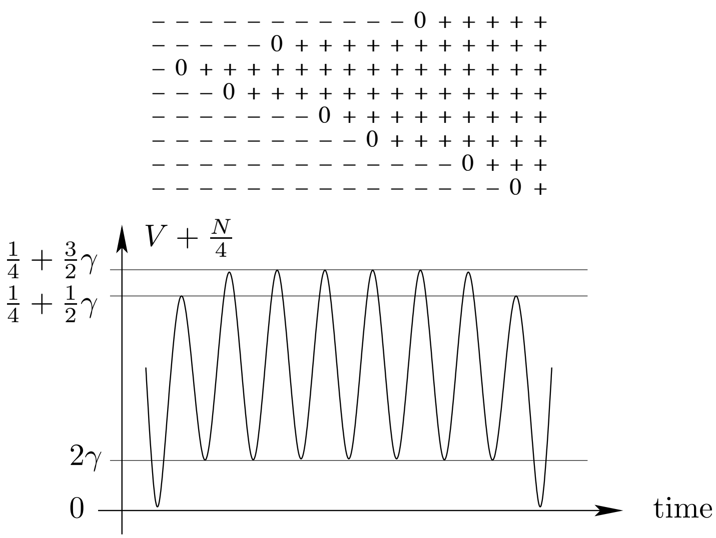
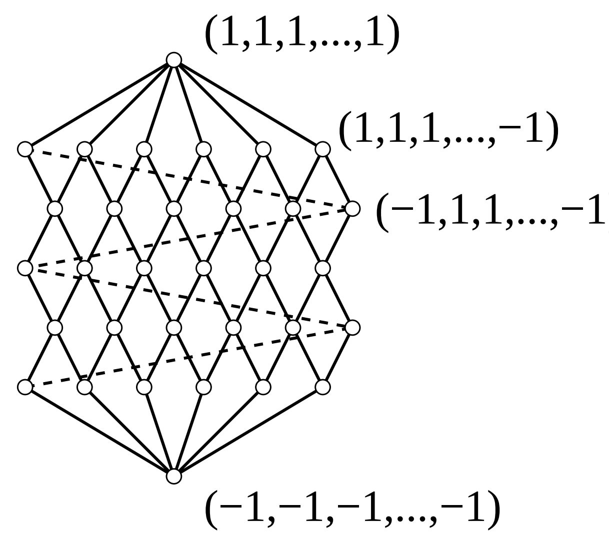
▷ $0 < \gamma \ll 1$:

$$V_\gamma(x^\star(\gamma)) = V_0(x^\star(0)) + \frac{\gamma}{4} \sum_{i \in \Lambda} (x^\star_{i+1}(0) - x^\star_i(0))^2 + \mathcal{O}(\gamma^2)$$





Strong coupling: Synchronisation

Remarks:
- $I^{\pm} = \pm(1, 1, \ldots 1) \in \mathcal{S}_0 \; \forall\gamma$
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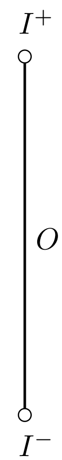

Theorem:

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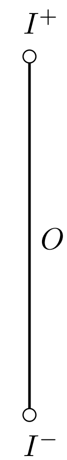

Theorem:

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Corollary: $\forall N$, $\forall\gamma > \gamma_1(N)$, $\forall 0 < r < R \leqslant \frac{1}{2}$, $\forall x_0 \in \mathcal{B}(I^-, r)$:

- Let $\tau_+ = \tau^{\text{hit}}(\mathcal{B}(I^+, r))$. Then $\forall\delta > 0$,

$$\lim_{\sigma\to 0} \mathbb{P}^{x_0}\Bigl\{\mathrm{e}^{(1/2-\delta)/\sigma^2} \leqslant \tau_+ \leqslant \mathrm{e}^{(1/2+\delta)/\sigma^2}\Bigr\} = 1$$

- Let $\tau_O = \tau^{\text{hit}}(\mathcal{B}(O, r))$,
and $\tau_- = \inf\{t > \tau^{\text{exit}}(\mathcal{B}(I^-, R)) \colon x_t \in \mathcal{B}(I^-, r)\}$. Then

$$\lim_{\sigma\to 0} \mathbb{P}^{x_0}\Bigl\{\tau_O < \tau_+ \Bigm| \tau_+ < \tau_-\Bigr\} = 1$$

Symmetry groups

Potential V_γ invariant by

- $R(x_1, \ldots, x_N) = (x_2, \ldots, x_N, x_1)$
- $S(x_1, \ldots, x_N) = (x_N, x_{N-1}, \ldots, x_1)$
- $C(x_1, \ldots, x_N) = -(x_1, \ldots, x_N)$

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$\Rightarrow V_\gamma$ invariant by group $G = D_N \times \mathbb{Z}_2$ generated by R,S,C
G acts as group of transformations on $\mathcal{X}$, $\mathcal{S}$, $\mathcal{S}_k$ $\forall k$

- Orbit of $x \in \mathcal{X}$: $O_x = \{gx \colon g \in G\}$
- Isotropy group of $x \in \mathcal{X}$: $C_x = \{g \in G \colon gx = x\}$
- Fixed-point space of $H \subset G$: $\mathrm{Fix}(H) = \{x \in \mathcal{X} \colon hx = x \; \forall h \in H\}$

Properties:
$|C_x||O_x| = |G|$
$C_{gx} = gC_xg^{-1}$
$\mathrm{Fix}(gHg^{-1}) = g\,\mathrm{Fix}(H)$

$N = 2$

$z^\star$	$O_{z^\star}$	$C_{z^\star}$	$\mathsf{Fix}(C_{z^\star})$
$(0,0)$	$\{(0,0)\}$	G	$\{(0,0)\}$
$(1,1)$	$\{(1,1),(-1,-1)\}$	$D_2 = \{\mathsf{id}, S\}$	$\{(x,x)\}_{x\in\mathbb{R}} = \mathcal{D}$
$(1,-1)$	$\{(1,-1),(-1,1)\}$	$\{\mathsf{id}, CS\}$	$\{(x,-x)\}_{x\in\mathbb{R}}$
$(1,0)$	$\{\pm(1,0),\pm(0,1)\}$	$\{\mathsf{id}\}$	$\{(x,y)\}_{x,y\in\mathbb{R}} = \mathcal{X}$

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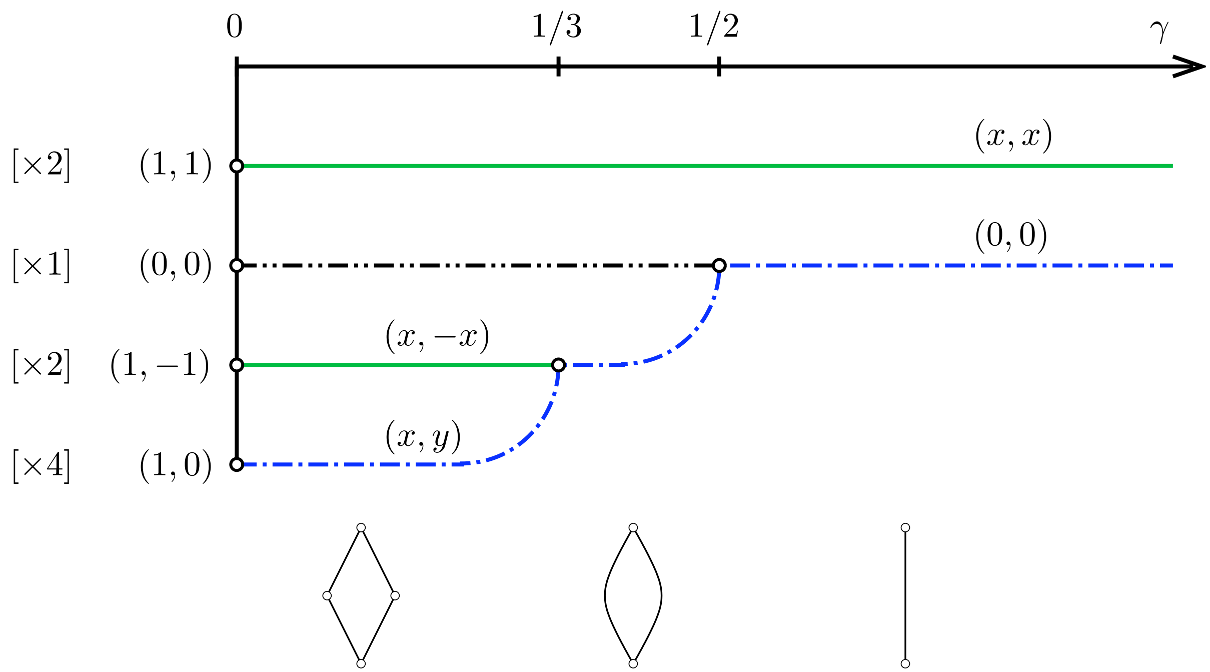
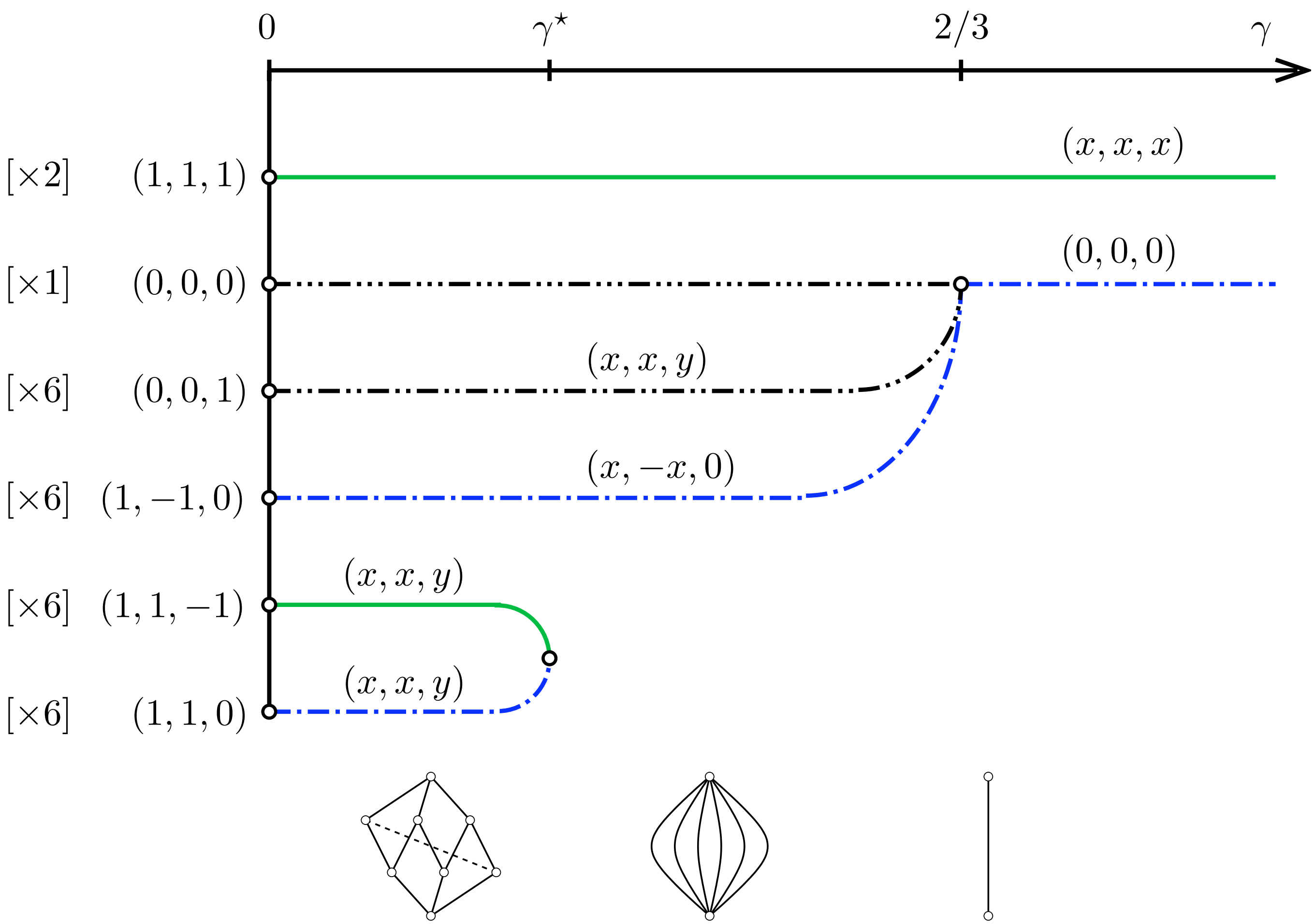
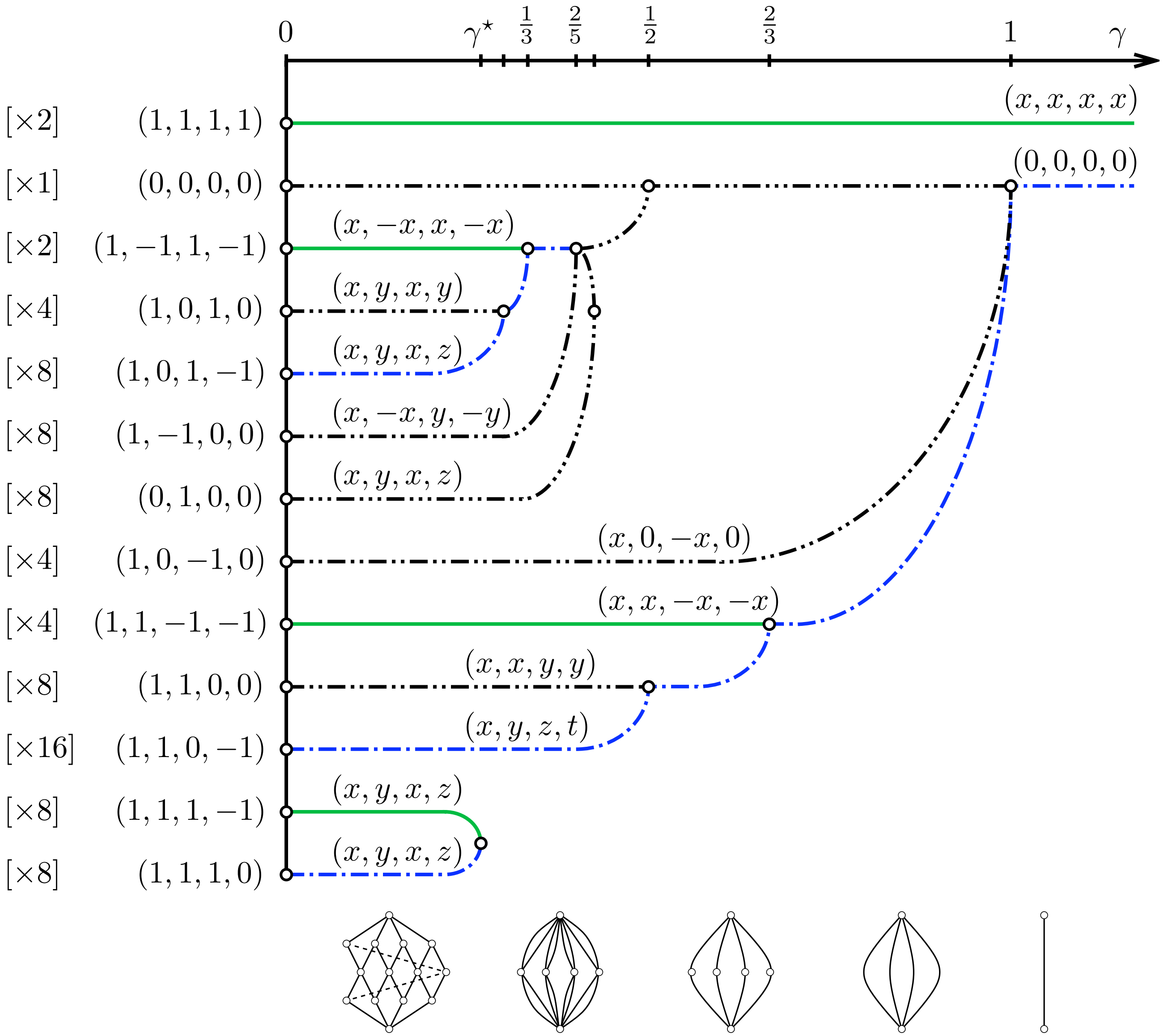


$N = 3$



0
$\gamma^\star$
2/3
γ
$[\times 2]$ $(1,1,1)$
$[\times 1]$ $(0,0,0)$
$[\times 6]$ $(0,0,1)$
$[\times 6]$ $(1,-1,0)$
$[\times 6]$ $(1,1,-1)$
$[\times 6]$ $(1,1,0)$
(x,x,x)
$(0,0,0)$
(x,x,y)
$(x,-x,0)$
(x,x,y)
(x,x,y)

$N = 4$



0
γ*
1/3
2/5
1/2
2/3
1
γ
[×2] (1, 1, 1, 1)
[×1] (0, 0, 0, 0)
[×2] (1, −1, 1, −1)
[×4] (1, 0, 1, 0)
[×8] (1, 0, 1, −1)
[×8] (1, −1, 0, 0)
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[×4] (1, 0, −1, 0)
[×4] (1, 1, −1, −1)
[×8] (1, 1, 0, 0)
[×16] (1, 1, 0, −1)
[×8] (1, 1, 1, −1)
[×8] (1, 1, 1, 0)
(x, x, x, x)
(0, 0, 0, 0)
(x, −x, x, −x)
(x, y, x, y)
(x, y, x, z)
(x, −x, y, −y)
(x, y, x, z)
(x, 0, −x, 0)
(x, x, −x, −x)
(x, x, y, y)
(x, y, z, t)
(x, y, x, z)
(x, y, x, z)

Desynchronisation

Theorem: $\forall$ even N, $\exists \delta(N) > 0$ s.t. for $\gamma_1 - \delta(N) < \gamma < \gamma_1$, $|\mathcal{S}| = 2N + 3$, and can be decomposed as

$$\begin{aligned} \mathcal{S}_0 &= O_{I^+} = \{I^+, I^-\} \\ \mathcal{S}_1 &= O_A = \{A, RA, \ldots, R^{N-1}A\} \\ \mathcal{S}_2 &= O_B = \{B, RB, \ldots, R^{N-1}B\} \\ \mathcal{S}_3 &= O_O = \{O\} \end{aligned}$$

with

$A_j(\gamma) = \frac{2}{\sqrt{3}} \sqrt{1 - \frac{\gamma}{\gamma_1}} \sin\left(\frac{2\pi}{N}\left(j - \frac{1}{2}\right)\right) + \mathcal{O}\left(1 - \frac{\gamma}{\gamma_1}\right)$

$V_\gamma(A) = -\frac{1}{3}\left(\gamma_1 - \gamma\right)^2 + \mathcal{O}\left((\gamma_1 - \gamma)^3\right)$

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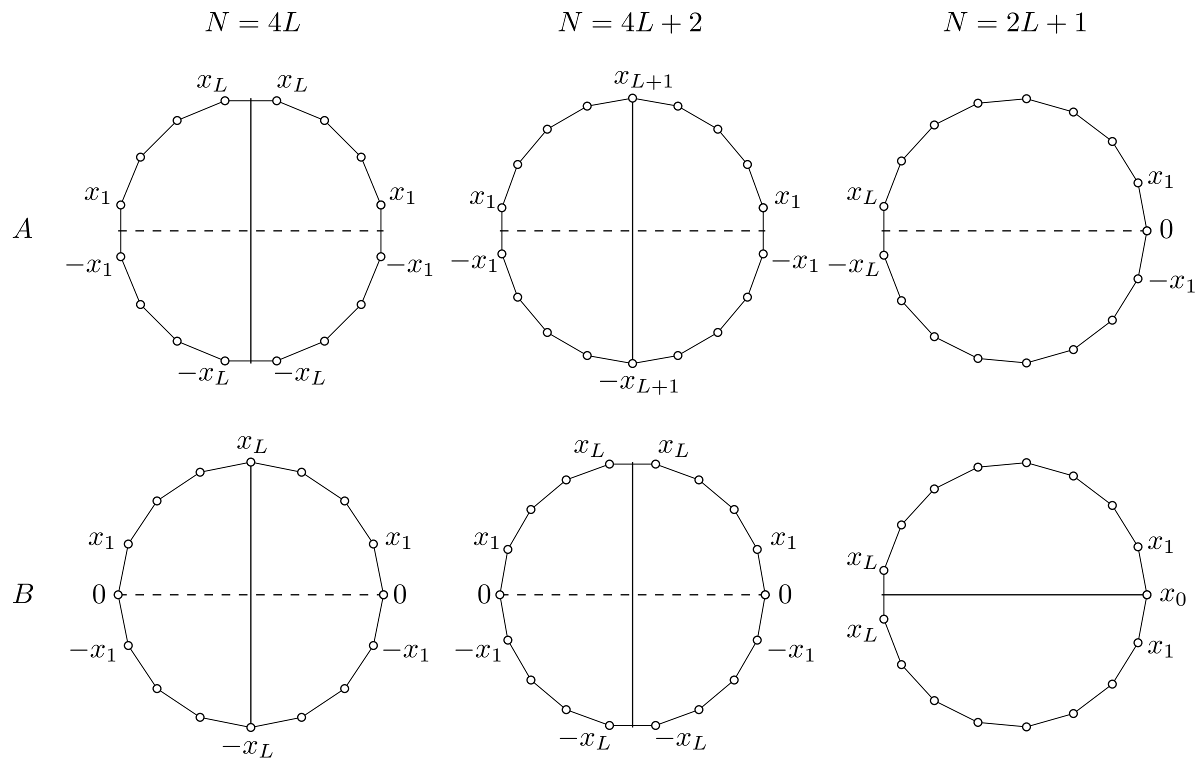
- $\triangleright$ N odd: similar result, $|\mathcal{S}| \geqslant 4N + 3$
- $\triangleright$ Similar corollary τ, with $\tau_0 \mapsto \tau_{\cup gA}$
- $\triangleright$ A and B have particular symmetries

Symmetries



N	x	Fix(C_x)
$4L$	A	$(x_1, \ldots, x_L, x_L, \ldots, x_1, -x_1, \ldots, -x_L, -x_L, \ldots, -x_1)$
	B	$(x_1, \ldots, x_L, \ldots, x_1, 0, -x_1, \ldots, -x_L, \ldots, -x_1, 0)$
$4L + 2$	A	$(x_1, \ldots, x_{L+1}, \ldots, x_1, -x_1, \ldots, -x_{L+1}, \ldots, -x_1)$
	B	$(x_1, \ldots, x_L, x_L \ldots, x_1, 0, -x_1, \ldots, -x_L, -x_L, \ldots, -x_1, 0)$
$2L + 1$	A	$(x_1, \ldots, x_L, -x_L, \ldots, -x_1, 0)$
	B	$(x_1, \ldots, x_L, x_L, \ldots, x_1, x_0)$

Case N large

Let $\tilde{\gamma} = \frac{\gamma}{\gamma_1} = \gamma(1 - \cos(2\pi/N))$,

$\tilde{\gamma}_M = \frac{1-\cos(2\pi/N)}{1-\cos(2\pi M/N)} \quad \left(= \frac{1}{M^2} + \mathcal{O}\left(\frac{1}{N^2}\right)\right)$

Theorem: $\forall M \geqslant 1$, $\exists N_M < \infty$ s.t. for $N \geqslant N_M$ and $\tilde{\gamma}_{M+1} < \tilde{\gamma} < \tilde{\gamma}_M$, $\mathcal{S}$ can be decomposed as

$$
\begin{aligned}
\mathcal{S}_0 &= O_{I^+} = \{I^+, I^-\} \\
\mathcal{S}_{2m-1} &= O_{A^{(m)}} && m = 1, \dots, M \\
\mathcal{S}_{2m} &= O_{B^{(m)}} && m = 1, \dots, M\ , \\
\mathcal{S}_{2M+1} &= O_O = \{O\}
\end{aligned}
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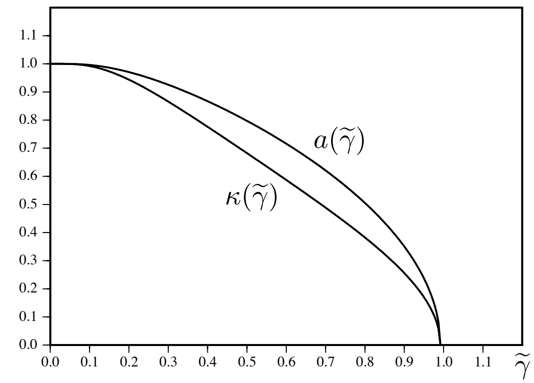
with $A_j^{(m)}(\tilde{\gamma}) = a(m^2\tilde{\gamma})\,\mathrm{sn}\left(\frac{4\,\mathrm{K}(\kappa(m^2\tilde{\gamma}))}{N} m\left(j - \frac{1}{2}\right), \kappa(m^2\tilde{\gamma})\right) + \mathcal{O}\left(\frac{M}{N}\right)$

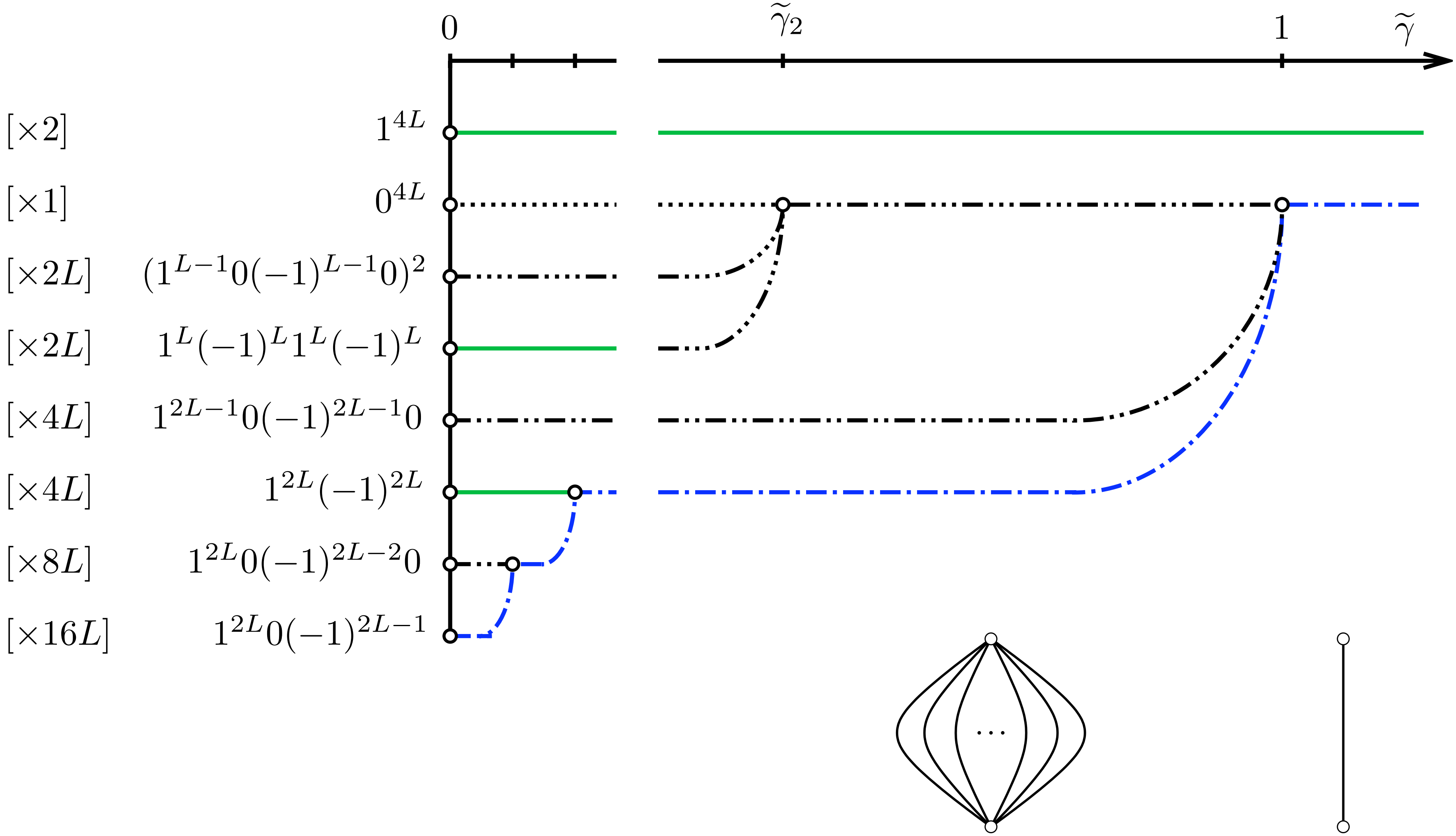
and $\kappa(\tilde{\gamma})$, $a(\tilde{\gamma})$ implicitly defined by

$$\tilde{\gamma} = \frac{\pi^2}{4\,\mathrm{K}(\kappa(\tilde{\gamma}))^2(1+\kappa(\tilde{\gamma})^2)}$$

$$a(\tilde{\gamma})^2 = \frac{2\kappa(\tilde{\gamma})^2}{1+\kappa(\tilde{\gamma})^2}$$



Case N large: bifurcation diagram (N=4L)



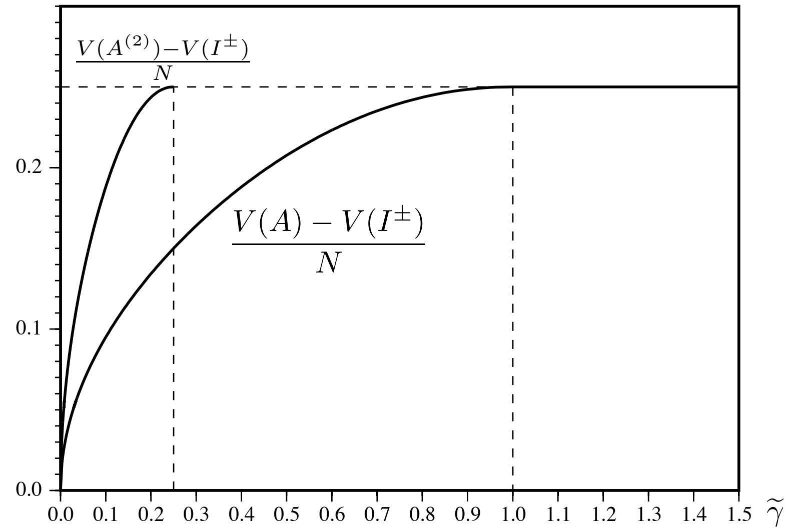
Potential difference:

$$\frac{V(A)-V(I^{\pm})}{N} = H(\widetilde{\gamma}) + \mathcal{O}\left(\frac{\kappa^2}{N}\right)$$

with

$$H(\widetilde{\gamma}) = \frac{1}{4} - \frac{1}{3(1+\kappa^2)}\left[\frac{2+\kappa^2}{1+\kappa^2} - 2\frac{\mathsf{E}(\kappa)}{\mathsf{K}(\kappa)}\right]$$

$$\kappa = \kappa(\widetilde{\gamma})$$



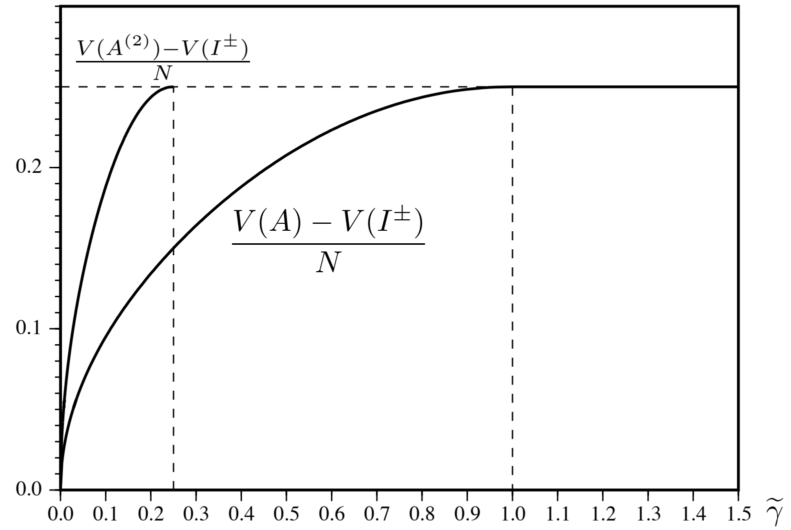
Potential difference:

$$\frac{V(A)-V(I^{\pm})}{N} = H(\widetilde{\gamma}) + \mathcal{O}\Bigl(\frac{\kappa^2}{N}\Bigr)$$

with

$$H(\widetilde{\gamma}) = \frac{1}{4} - \frac{1}{3(1+\kappa^2)}\Bigl[\frac{2+\kappa^2}{1+\kappa^2} - 2\frac{\mathsf{E}(\kappa)}{\mathsf{K}(\kappa)}\Bigr]$$

$$\kappa = \kappa(\widetilde{\gamma})$$



Corollaire: $\forall 0 < \widetilde{\gamma} \leqslant 1$, $\exists N_0(\widetilde{\gamma})$ s.t. $\forall N \geqslant N_0(\widetilde{\gamma})$, $\forall 0 < r < R \leqslant \frac{1}{2}$, $\forall x_0 \in \mathcal{B}(I^-, r)$:

- Let $\tau_+ = \tau^{\text{hit}}(\mathcal{B}(I^+, r))$. Then $\forall \delta > 0$,

  $$\lim_{\sigma\to 0} \mathbb{P}^{x_0}\Bigl\{\mathrm{e}^{(H(\widetilde{\gamma})-\delta)/\sigma^2} \leqslant \tau_+ \leqslant \mathrm{e}^{(H(\widetilde{\gamma})+\delta)/\sigma^2}\Bigr\} = 1$$

- Let $\tau_A = \tau^{\text{hit}}(\bigcup_{g\in G}\mathcal{B}(gA, r))$, and $\tau_- = \inf\{t > \tau^{\text{exit}}(\mathcal{B}(I^-, R)) : x_t \in \mathcal{B}(I^-, r)\}$. Then

  $$\lim_{\sigma\to 0} \mathbb{P}^{x_0}\Bigl\{\tau_A < \tau_+ \Bigm| \tau_+ < \tau_-\Bigr\} = 1$$

Techniques of proofs

- Weak coupling: Construction of symbolic dynamics
- Synchronisation: Lyapunov functions
- Control of set $\mathcal{S}$ of stationary points:

$$x \in \mathcal{S} \quad \Leftrightarrow \quad f(x_n) + \frac{\gamma}{2}\Big[x_{n+1} - 2x_n + x_{n-1}\Big] = 0$$

$$\Leftrightarrow \quad \begin{cases} x_{n+1} = x_n + \varepsilon w_n - \frac{1}{2}\varepsilon^2 f(x_n) \\ w_{n+1} = w_n - \frac{1}{2}\varepsilon\Big[f(x_n) + f(x_{n+1})\Big] \end{cases}$$

$$\varepsilon = \frac{2\pi}{N\sqrt{\gamma}} \ll 1$$

$$C(x,w) = \frac{1}{2}(x^2 + w^2) - \frac{1}{4}x^4$$

$$\Rightarrow C(x_{n+1}, w_{n+1}) = C(x_n, w_n) + \mathcal{O}(\varepsilon^3)$$