Irreversible diffusions and non-selfadjoint operators: Results and open problems

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Stochastic differential equations

$$\mathrm{d}x_t = f(x_t)\,\mathrm{d}t + \sqrt{2\varepsilon}g(x_t)\,\mathrm{d}W_t \qquad x \in \mathbb{R}^n$$

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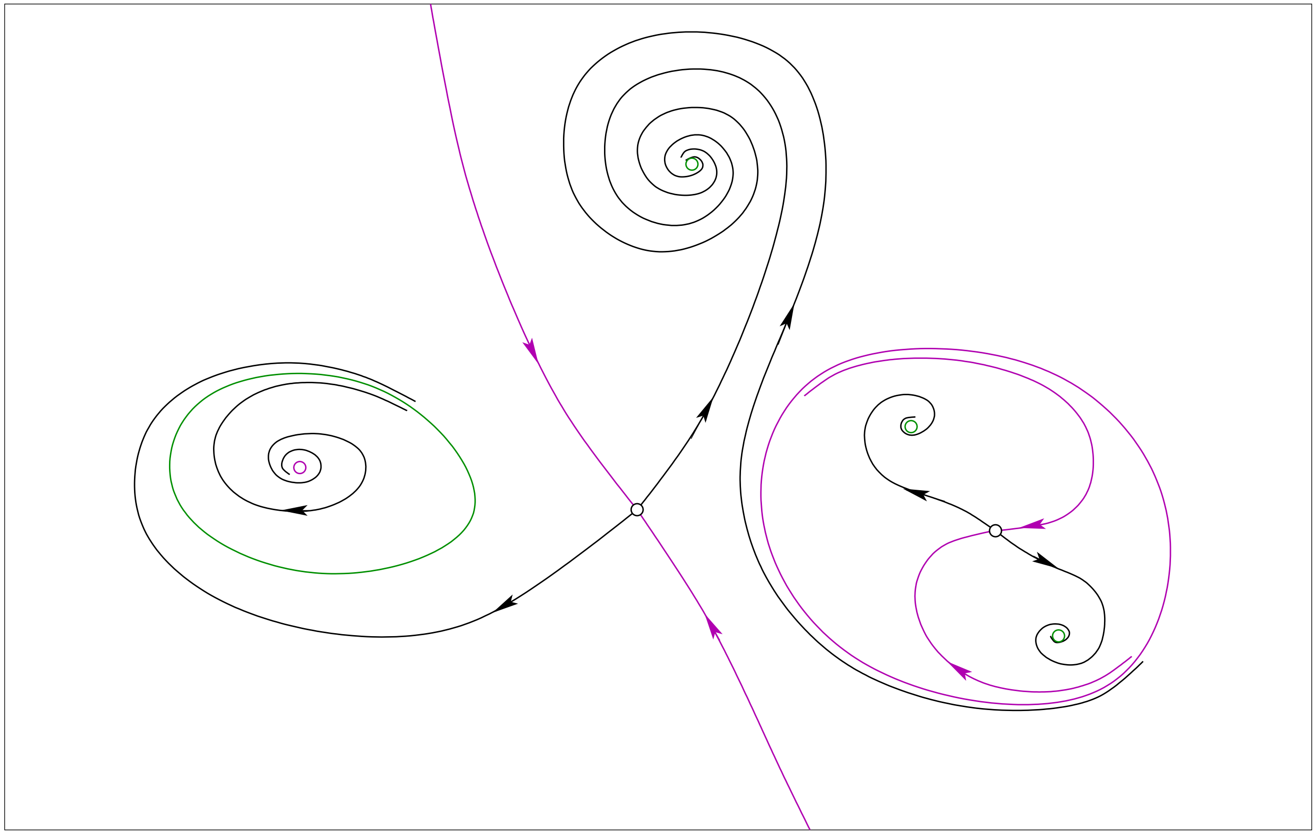

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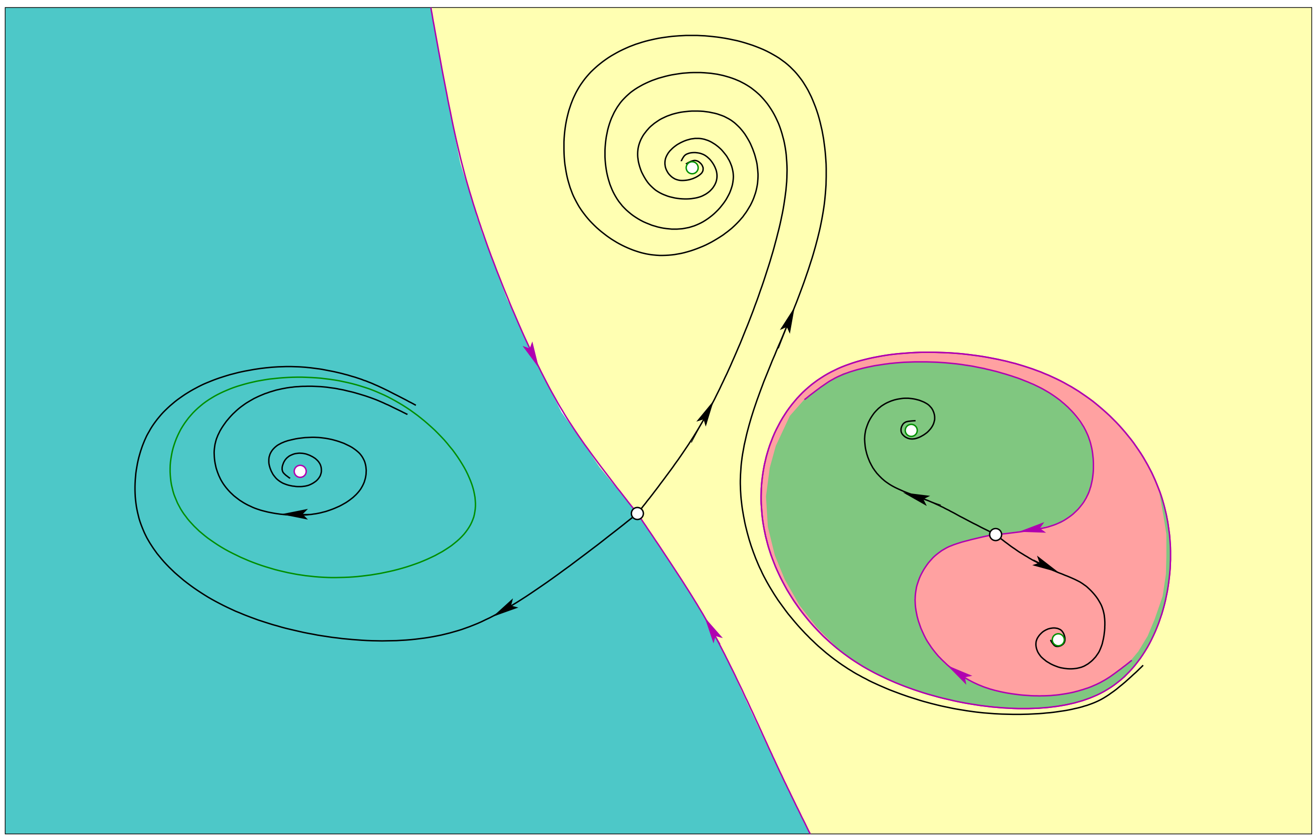

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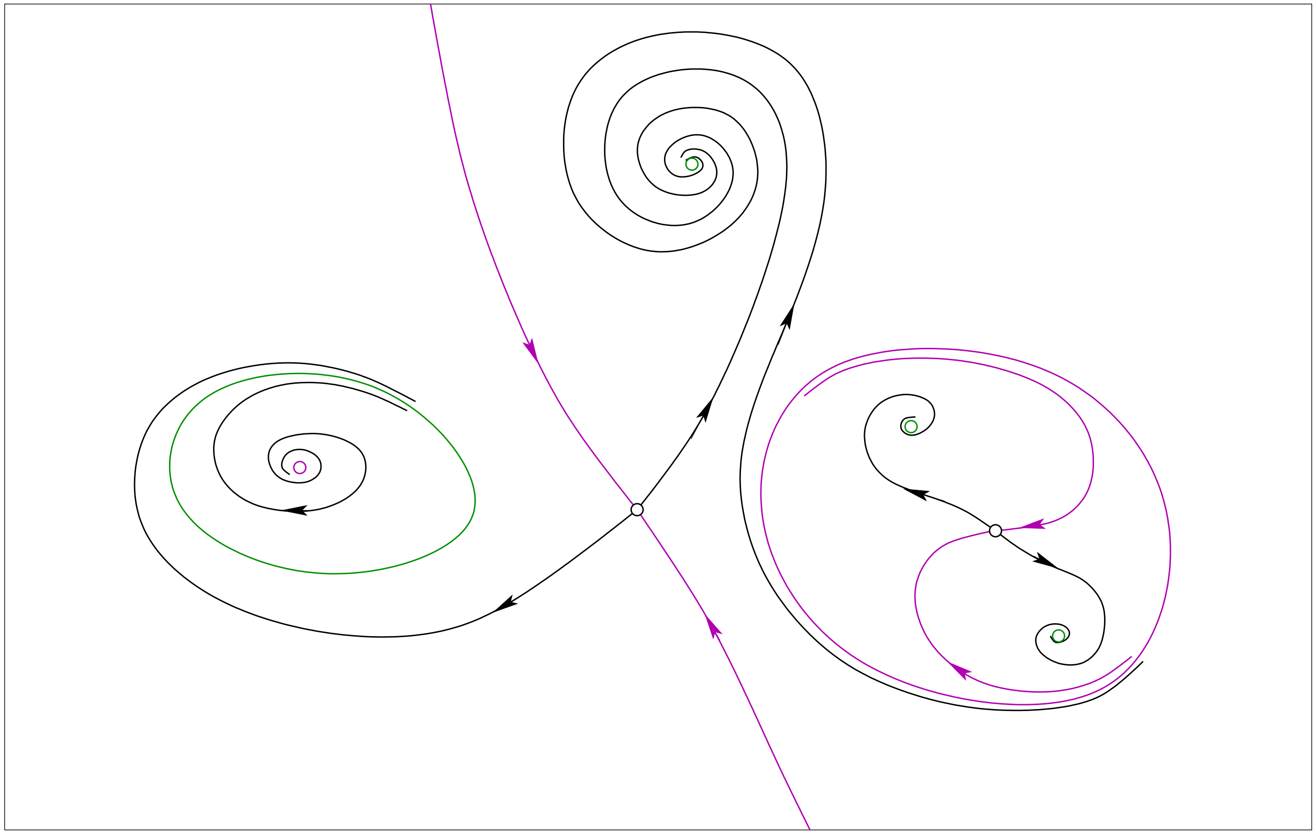

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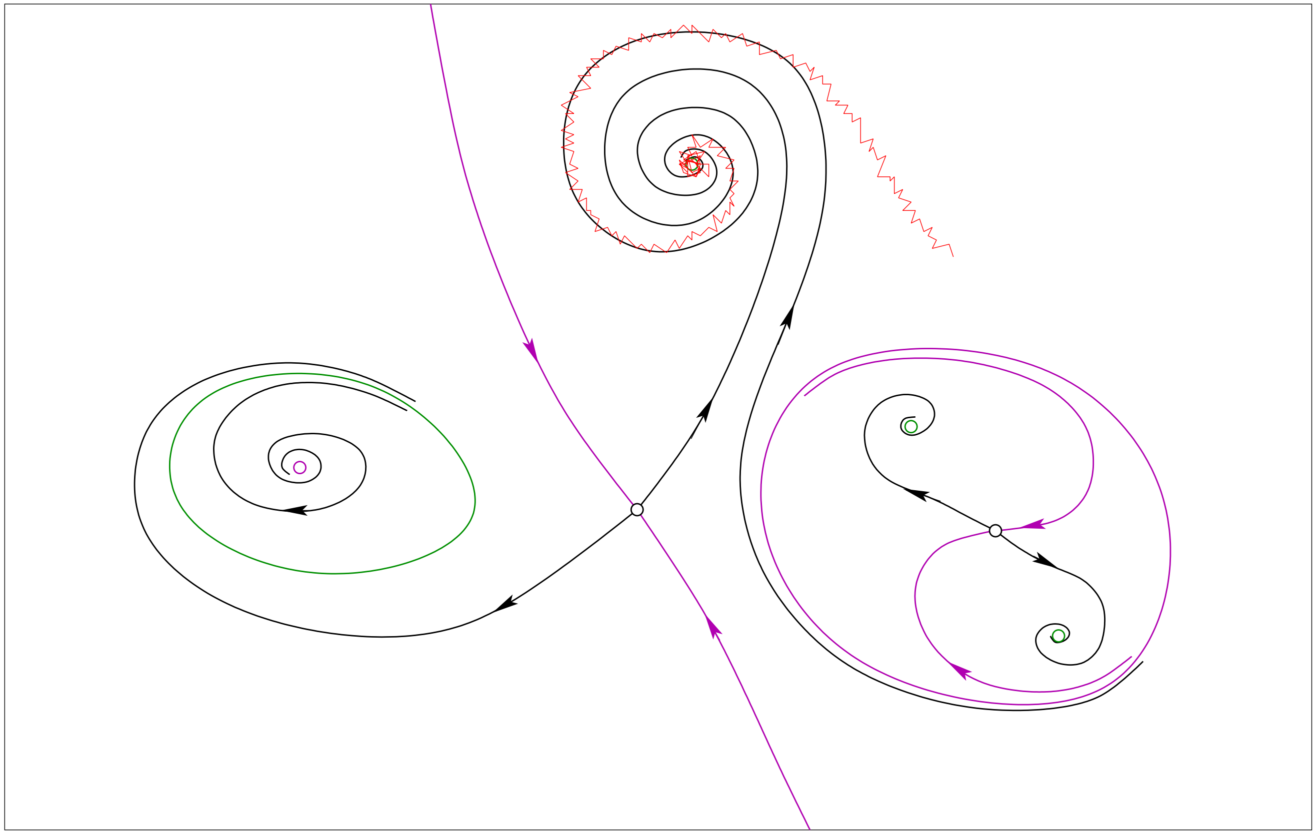

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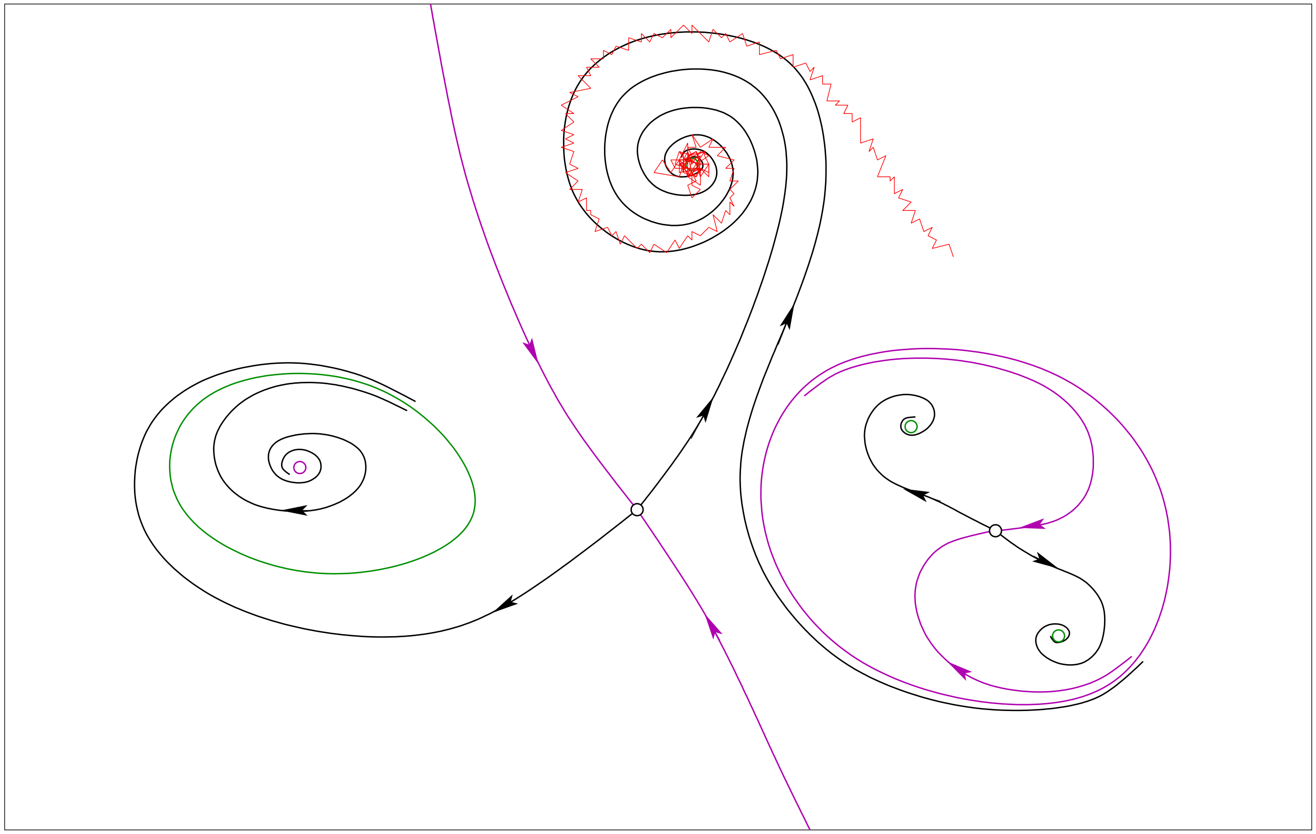

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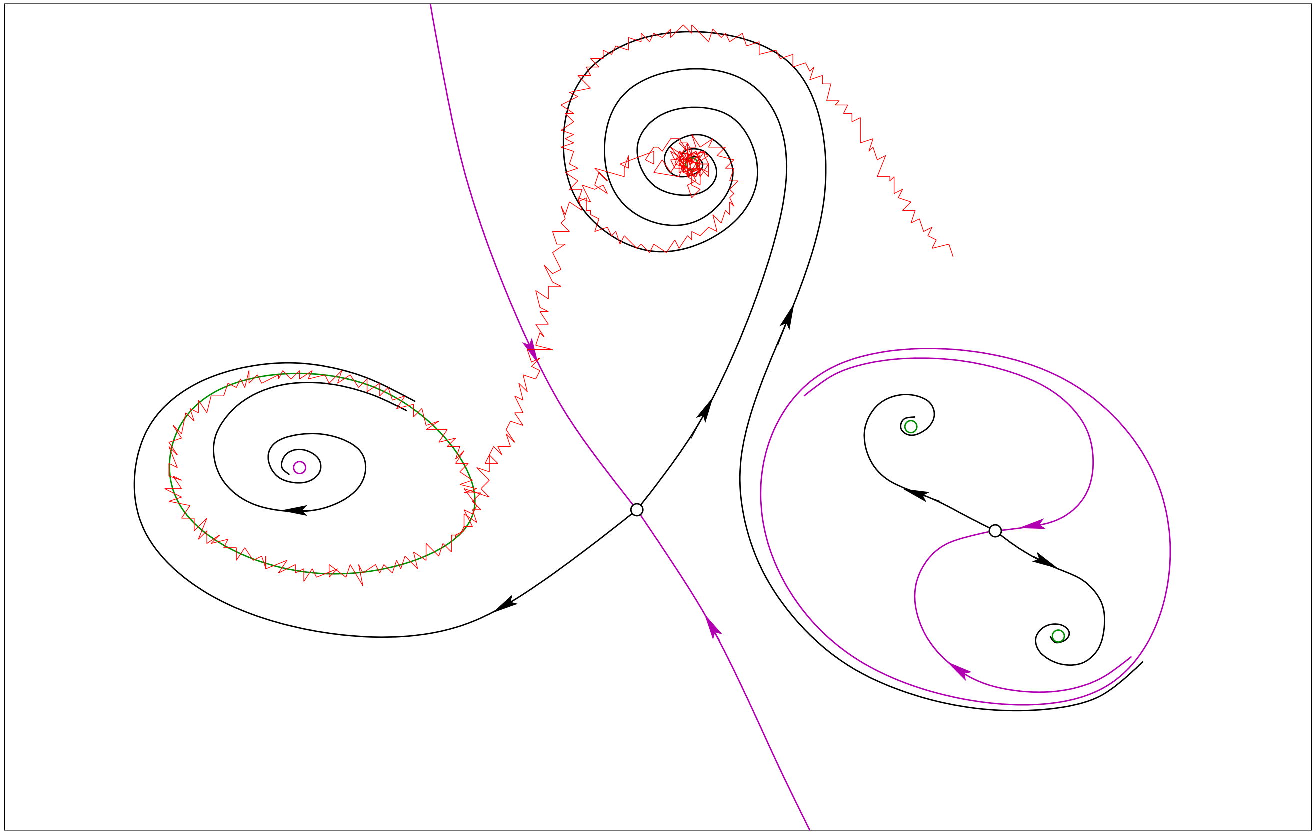

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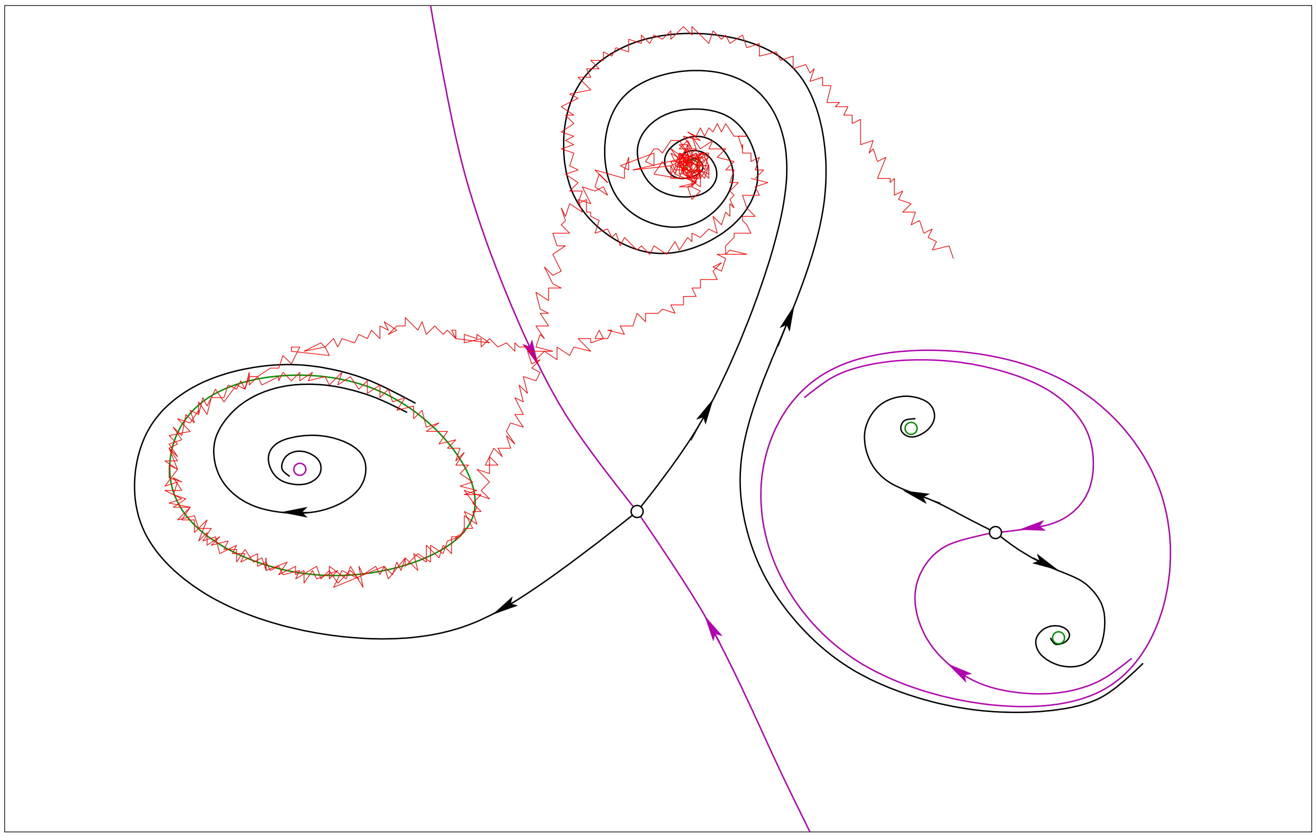

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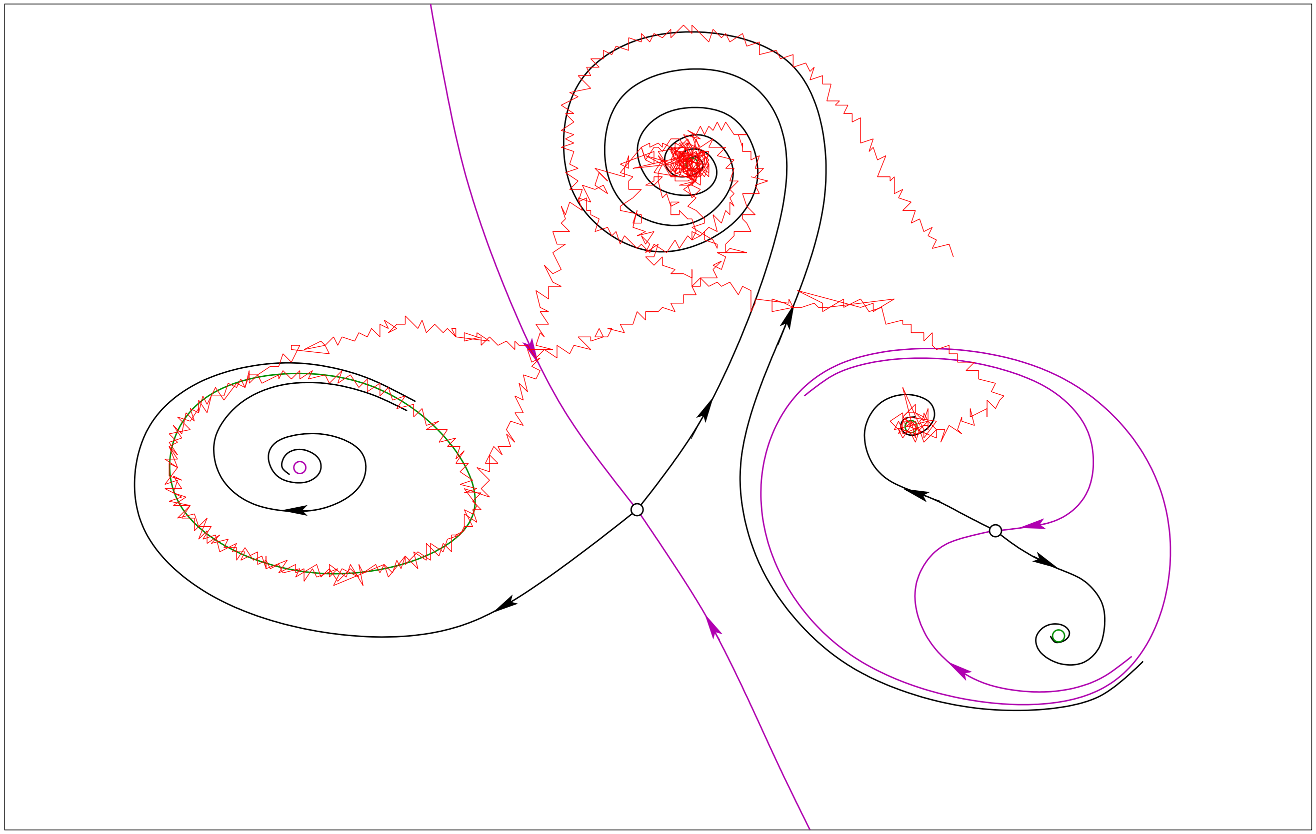

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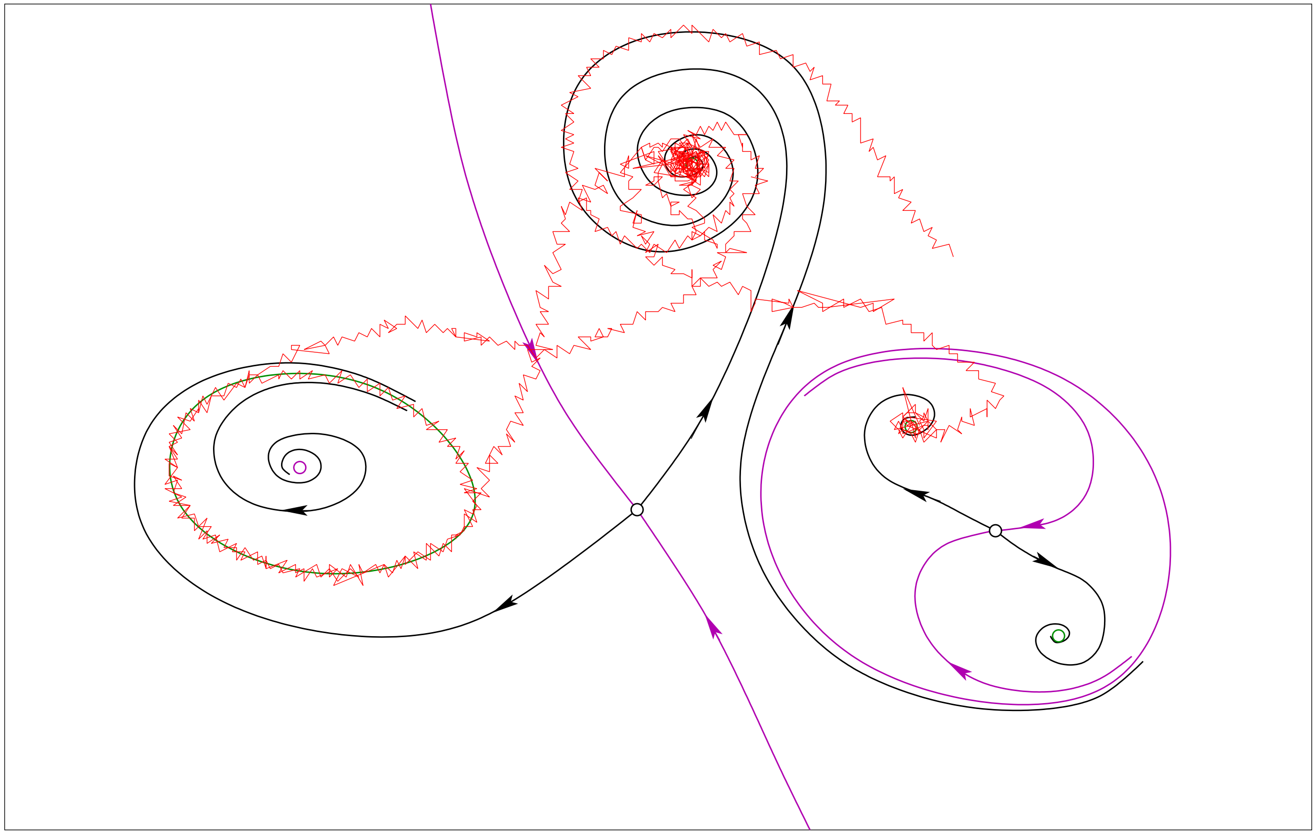

Can jumps between attractors be described as a Markov process?

Transition probability, generator, and Kolmogorov equations

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▷ Transition probability density: $p_t(x,y)$

▷ Markov semigroup T_t: for $\varphi \in L^\infty$,

$$(T_t\varphi)(x) = \mathbb{E}^x[\varphi(x_t)] = \int p_t(x,y)\varphi(y)\,\mathrm{d}y$$

Generator: $L\varphi = \frac{\mathrm{d}}{\mathrm{d}t}T_t\varphi|_{t=0}$

$$(L\varphi)(x) = \sum_i f_i(x)\frac{\partial\varphi}{\partial x_i} + \varepsilon\sum_{i,j}(gg^T)_{ij}(x)\frac{\partial^2\varphi}{\partial x_i\partial x_j}$$

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▷ Adjoint semigroup: for $\mu \in L^1$,

$$(\mu T_t)(y) = \mathbb{P}^\mu\{x_t = \mathrm{d}y\} = \int \mu(x)p_t(x,y)\,\mathrm{d}x$$

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▷ Kolmogorov equations: $\dfrac{\mathrm{d}}{\mathrm{d}t}p_t(x,y) = L_x p_t(x,y)$

$$\frac{\mathrm{d}}{\mathrm{d}t}p_t(x,y) = L_y^* p_t(x,y) \qquad \text{(Fokker–Planck)}$$

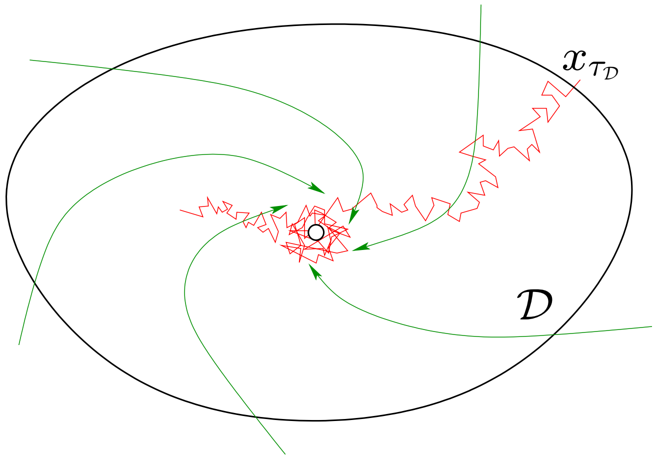
Stochastic exit problem

Given $\mathcal{D} \subset \mathbb{R}^n$, define first-exit time

$$\tau_{\mathcal{D}} = \inf\{t > 0 \colon x_t \notin \mathcal{D}\}$$

First-exit location $x_{\tau_{\mathcal{D}}} \in \partial\mathcal{D}$
defines harmonic measure

$$\mu(A) = \mathbb{P}^x\{x_{\tau_{\mathcal{D}}} \in A\}$$



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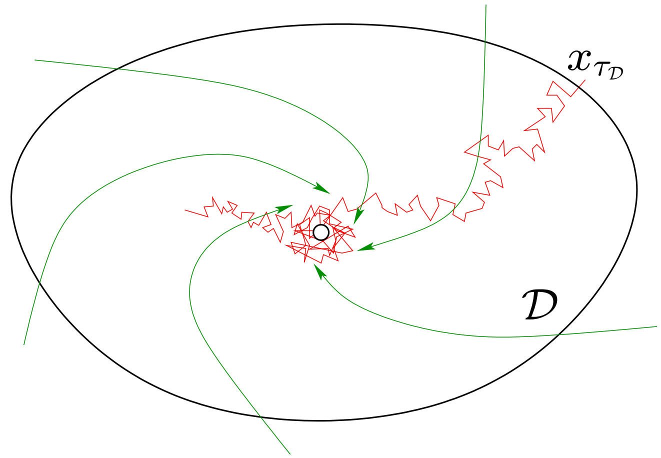

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Facts (following from Dynkin's formula):

▷ $u(x) = \mathbb{E}^x[\tau_{\mathcal{D}}]$ satisfies

$$\begin{cases} Lu(x) = -1 & x \in \mathcal{D} \\ u(x) = 0 & x \in \partial\mathcal{D} \end{cases}$$

▷ For $\varphi \in L^\infty(\partial\mathcal{D}, \mathbb{R})$, $h(x) = \mathbb{E}^x[\varphi(x_{\tau_{\mathcal{D}}})]$ satisfies

$$\begin{cases} Lh(x) = 0 & x \in \mathcal{D} \\ h(x) = \varphi(x) & x \in \partial\mathcal{D} \end{cases}$$

Wentzell–Freidlin theory

$$\mathrm{d}x_t = f(x_t)\,\mathrm{d}t + \sqrt{2\varepsilon} g(x_t)\,\mathrm{d}W_t \qquad x \in \mathbb{R}^n$$

Large-deviation principle with rate function

$$I(\gamma) = \frac{1}{2}\int_0^T (\dot{\gamma}_t - f(\gamma_t))^T D(\gamma_t)^{-1}(\dot{\gamma}_t - f(\gamma_t))\,\mathrm{d}t \qquad D = gg^T$$

For a set Γ of paths $\gamma : [0,T] \to \mathbb{R}^n$

$$\mathbb{P}\{(x_t)_{0\leqslant t\leqslant T} \in \Gamma\} \simeq \mathrm{e}^{-\inf_\Gamma I/2\varepsilon}$$

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Consider domain $\mathcal{D}$ contained in basin of attraction of attractor $\mathcal{A}$

Quasipotential: $V(y) = \inf\{I(\gamma) : \gamma : \mathcal{A} \to y \in \partial\mathcal{D}$ in arbitrary time$\}$

- $\lim_{\varepsilon\to 0} 2\varepsilon \log \mathbb{E}[\tau_{\mathcal{D}}] = \overline{V} = \inf_{y\in\partial\mathcal{D}} V(y)$ [Wentzell, Freidlin '69]
- If inf reached at a single point $y^* \in \mathcal{D}$ then
  $\lim_{\varepsilon\to 0} \mathbb{P}\{\|x_{\tau_{\mathcal{D}}} - y^*\| > \delta\} = 0 \quad \forall\delta > 0$ [Wentzell, Freidlin '69]
- Exponential distr of $\tau_{\mathcal{D}}$: $\lim_{\varepsilon\to 0} \mathbb{P}\{\tau_{\mathcal{D}} > s\mathbb{E}[\tau_{\mathcal{D}}]\} = \mathrm{e}^{-s}$ [Day '83]

Reversible case

$$\mathrm{d}x_t = -\nabla V(x_t)\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t \qquad x \in \mathbb{R}^n$$

▷ $L = \varepsilon\Delta - \nabla V(x)\cdot\nabla = \varepsilon\,\mathrm{e}^{V/\varepsilon}\,\nabla\cdot\mathrm{e}^{-V/\varepsilon}\,\nabla$ self-adjoint in $L^2(\mathbb{R}^n, \mathrm{e}^{-V/\varepsilon}\,\mathrm{d}x)$

▷ Reversibility (detailed balance): $\mathrm{e}^{-V(x)/\varepsilon}\,p_t(x,y) = \mathrm{e}^{-V(y)/\varepsilon}\,p_t(y,x)$

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- ▷ Reversibility (detailed balance): $\mathrm{e}^{-V(x)/\varepsilon}\,p_t(x,y) = \mathrm{e}^{-V(y)/\varepsilon}\,p_t(y,x)$
- ▷ $\widetilde{L} = \mathrm{e}^{-V/2\varepsilon}\,L\,\mathrm{e}^{V/2\varepsilon}$ self-adjoint in $L^2(\mathbb{R}^n, \mathrm{d}x)$

  In fact, it is the Schrödinger operator $\widetilde{L} = \varepsilon\Delta + \frac{1}{\varepsilon}U(x)$
  where $U(x) = \frac{1}{2}\varepsilon\Delta V(x) - \frac{1}{4}\|\nabla V(x)\|^2$

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Results: Assume V has N local minima

- ▷ $-L$ has N exp small ev $0 = \lambda_0 < \dots < \lambda_{N-1}$ + spectral gap
- ▷ Precise expressions of the λ_i (Kramers' law)
- ▷ $\lambda_i^{-1}, i = 1,\dots,N-1$ are expected transition times between neighbourhoods of minima (in specific order)

Methods:
- ▷ Large deviations [Freidlin, Wentzell, Sugiura,...]
- ▷ Semiclassical analysis [Mathieu, Miclo, Kolokoltsov,...]
- ▷ Potential theory [Bovier, Gayrard, Eckhoff, Klein]
- ▷ Witten Laplacian [Helffer, Nier, Le Peutrec, Viterbo]

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- ▷ Results exist on the Kramers–Fokker–Planck operator

$$L = \varepsilon y \frac{\partial}{\partial x} - \varepsilon V'(x) \frac{\partial}{\partial y} + \frac{\gamma}{2}\Big(y - \varepsilon \frac{\partial}{\partial y}\Big)\Big(y + \varepsilon \frac{\partial}{\partial y}\Big)$$

[Hérau, Hitrik, Sjöstrand,. . .]

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▷ Here we consider two questions involving periodic orbits, namely

⋈ What is the harmonic measure for the exit through an unstable periodic orbit?

⋈ What can we say on exponentially small eigenvalues for systems admitting N stable periodic orbits?

Random Poincaré map

Near a periodic orbit, in appropriate coordinates

$$\begin{aligned} \mathrm{d}\varphi_t &= f(\varphi_t, x_t)\,\mathrm{d}t + \sigma F(\varphi_t, x_t)\,\mathrm{d}W_t && \varphi \in \mathbb{R} \\ \mathrm{d}x_t &= g(\varphi_t, x_t)\,\mathrm{d}t + \sigma G(\varphi_t, x_t)\,\mathrm{d}W_t && x \in E \subset \mathbb{R}^{n-1} \end{aligned}$$

▷ all functions periodic in φ (say period 1)
▷ $f \geqslant c > 0$ and σ small $\Rightarrow$ φ_t likely to increase
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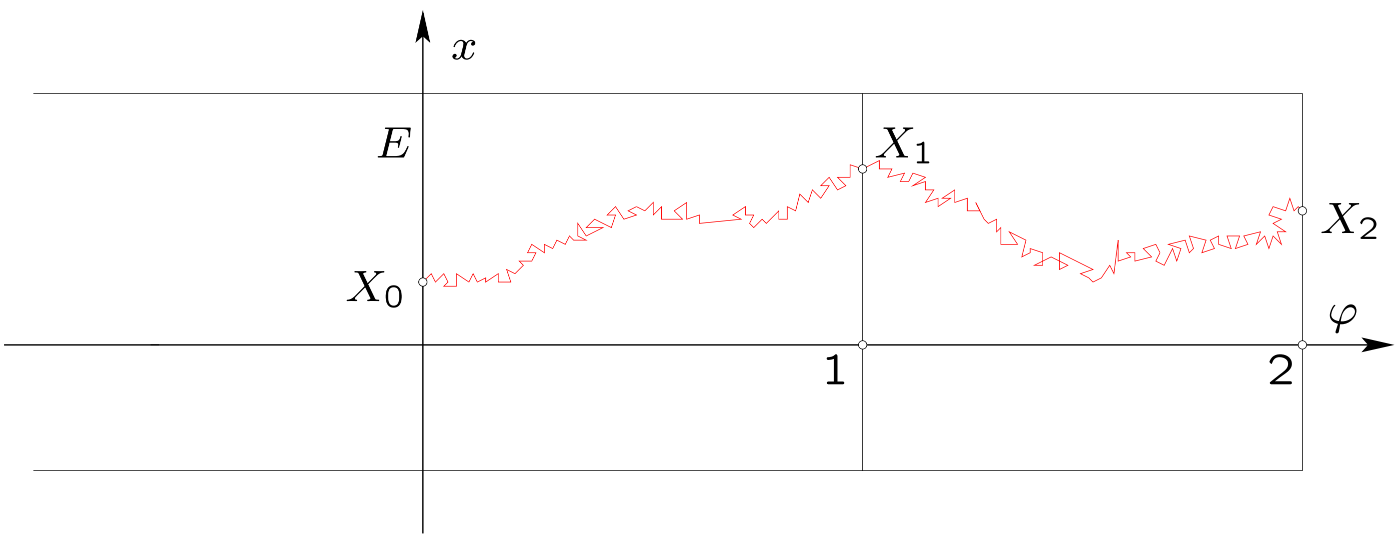
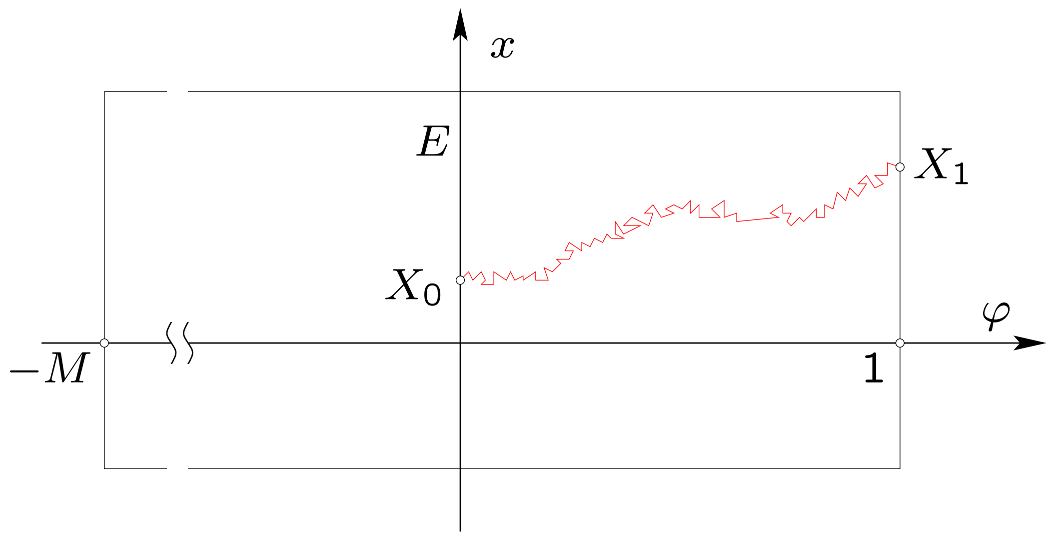


$X_0, X_1, \dots$ form (substochastic) Markov chain

Random Poincaré map and harmonic measure



- τ: first-exit time of $z_t = (\varphi_t, x_t)$ from $\mathcal{D} = (-M, 1) \times E$
- $\mu_z(A) = \mathbb{P}^z\{z_\tau \in A\}$: harmonic measure (wrt generator $\mathcal{L}$)
- [Ben Arous, Kusuoka, Stroock '84]: under hypoellipticity cond, μ_z admits (smooth) density $h(z, y)$ wrt arclength on $\partial\mathcal{D}$

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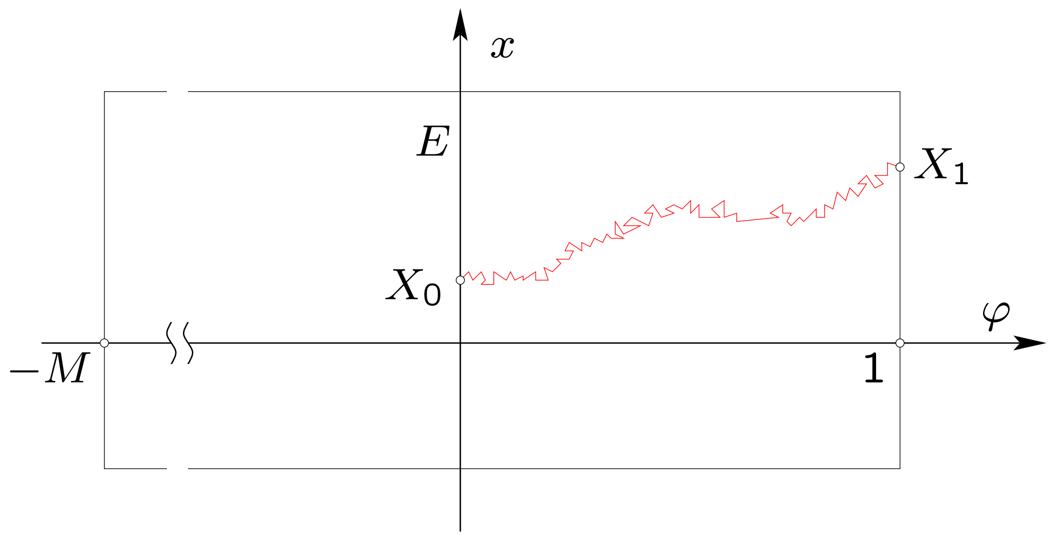

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- Remark: $\mathcal{L}_z h(z, y) = 0$ (kernel is harmonic)
- For $B \subset E$ Borel set

$$\mathbb{P}^{X_0}\{X_1 \in B\} = K(X_0, B) := \int_B K(X_0, \mathrm{d}y)$$

where $K(x, \mathrm{d}y) = h((0, x), y)\,\mathrm{d}y =: k(x, y)\,\mathrm{d}y$

Fredholm theory

Consider integral operator K acting

- ▷ on L^∞ via $f \mapsto (Kf)(x) = \int_E k(x,y) f(y)\, \mathrm{d}y = \mathbb{E}^x[f(X_1)]$
- ▷ on L^1 via $m \mapsto (mK)(A) = \int_E m(x) k(x,y)\, \mathrm{d}x = \mathbb{P}^\mu\{X_1 \in A\}$

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[Fredholm 1903]:

▷ If $k \in L^2$, then K has eigenvalues λ_n of finite multiplicity

▷ Eigenfcts $Kh_n = \lambda_n h_n$, $h_n^* K = \lambda_n h_n^*$ form complete ON basis

[Perron, Frobenius, Jentzsch 1912, Krein–Rutman '50, Birkhoff '57]:

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$$\Rightarrow \mathbb{P}^x\{X_n \in \mathrm{d}y | X_n \in E\} = \pi_0(\mathrm{d}x) + \mathcal{O}((|\lambda_1|/\lambda_0)^n)$$

where $\pi_0 = h_0^* / \int_E h_0^*$ is quasistationary distribution (QSD)
[Yaglom '47, Bartlett '57, Vere-Jones '62, ...]

How to estimate the principal eigenvalue

▷ "Trivial" bounds: $\forall A \subset E$ with $\text{Lebesgue}(A) > 0$,

$$\inf_{x \in A} K(x, A) \leqslant \lambda_0 \leqslant \sup_{x \in E} K(x, E)$$

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Proof: $x^* = \operatorname{argmax} h_0 \Rightarrow \lambda_0 = \int_E k(x^*, y) \frac{h_0(y)}{h_0(x^*)} \, \mathrm{d}y \leqslant K(x^*, E)$

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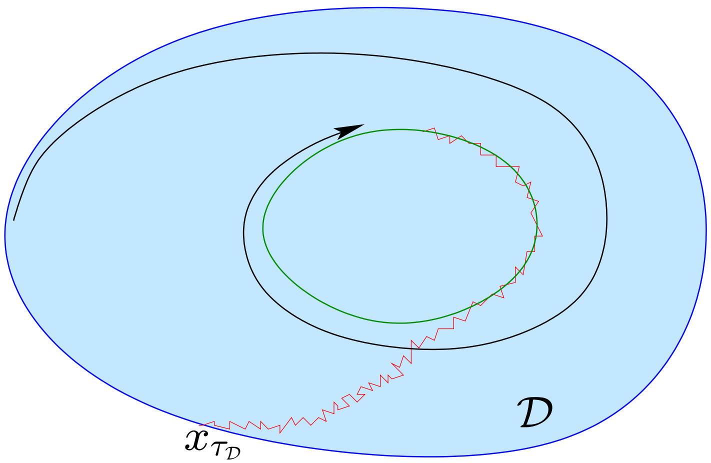
▷ Bounds using Laplace transforms (see below)

Application: Exit through an unstable periodic orbit

Planar SDE

$\mathrm{d}x_t = f(x_t)\,\mathrm{d}t + \sigma g(x_t)\,\mathrm{d}W_t$



$\mathcal{D} \subset \mathbb{R}^2$: int of unstable periodic orbit

First-exit time: $\tau_{\mathcal{D}} = \inf\{t > 0 \colon x_t \notin \mathcal{D}\}$

Law of first-exit location $x_{\tau_{\mathcal{D}}} \in \partial\mathcal{D}$?

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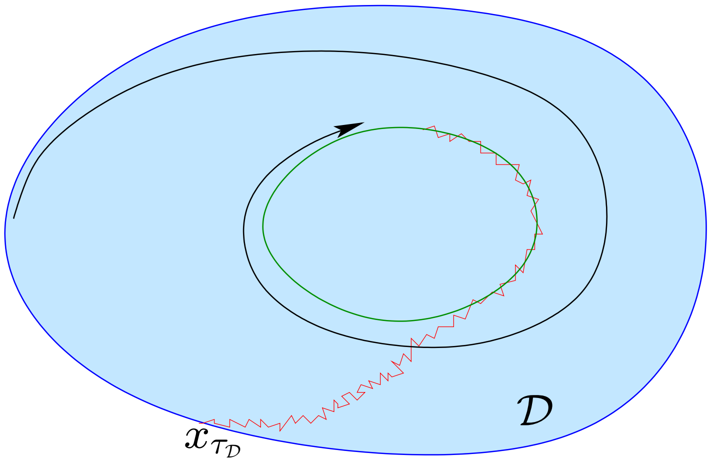

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Large-deviation principle with rate function

$$I(\gamma) = \frac{1}{2}\int_0^T (\dot\gamma_t - f(\gamma_t))^T D(\gamma_t)^{-1}(\dot\gamma_t - f(\gamma_t))\,\mathrm{d}t \qquad D = gg^T$$

Quasipotential:

$$V(y) = \inf\{I(\gamma) \colon \gamma : \text{stable orbit} \to y \in \partial\mathcal{D} \text{ in arbitrary time}\}$$

Theorem [Freidlin, Wentzell '69]: If V reaches its min at a unique $y^* \in \partial\mathcal{D}$, then $x_{\tau_{\mathcal{D}}}$ concentrates in y^* as $\sigma \to 0$

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$dx_t = f(x_t)\,dt + \sigma g(x_t)\,dW_t$

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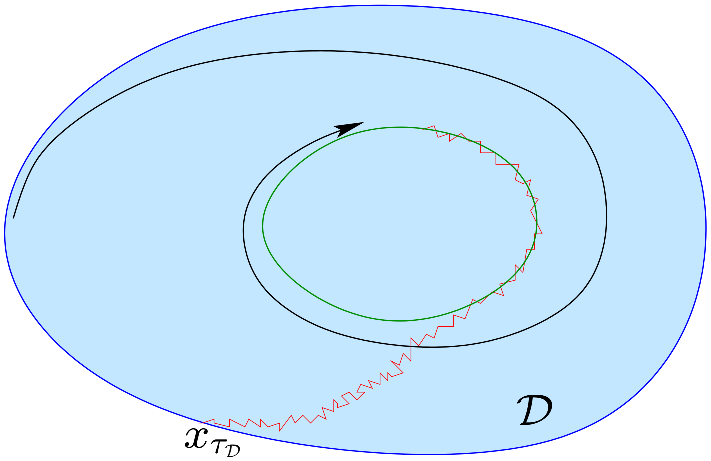

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Problem: V is constant on $\partial\mathcal{D}$!

Most probable exit paths

Minimisers of I obey Hamilton equations with Hamiltonian

$H(\gamma, \psi) = \frac{1}{2}\psi^T D(\gamma)\psi + f(\gamma)^T \psi$

where $\psi = D(\gamma)^{-1}(\dot{\gamma} - f(\gamma))$

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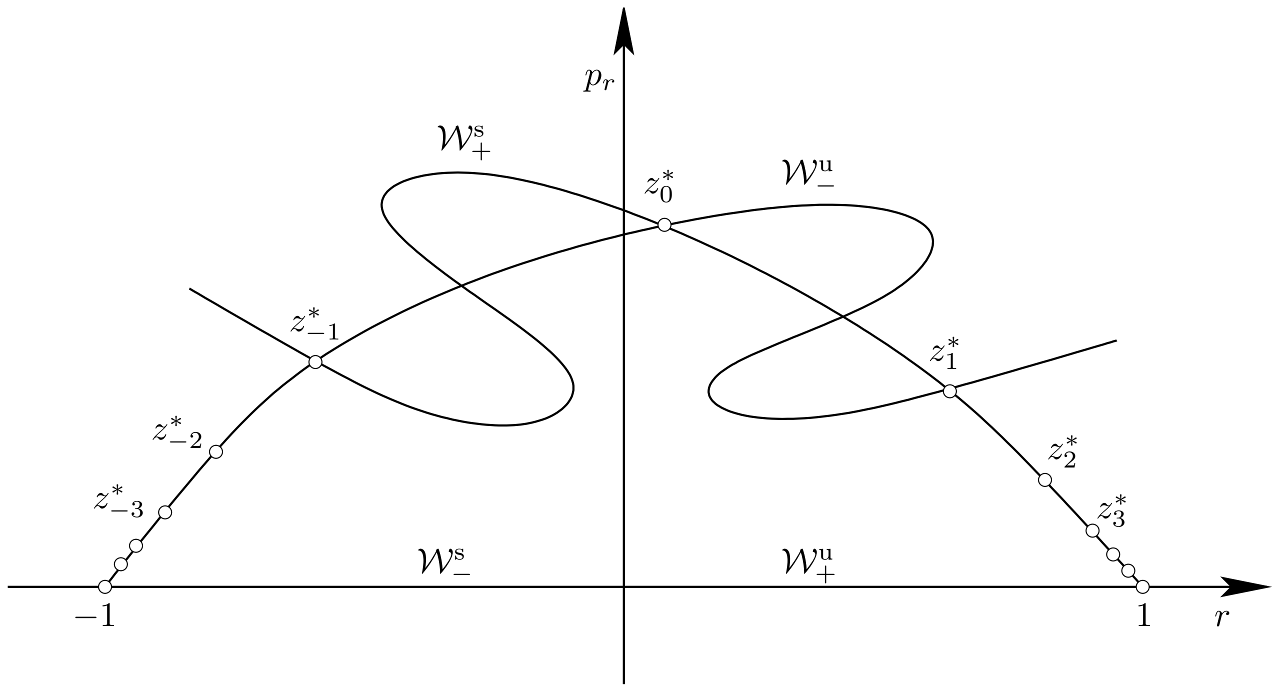
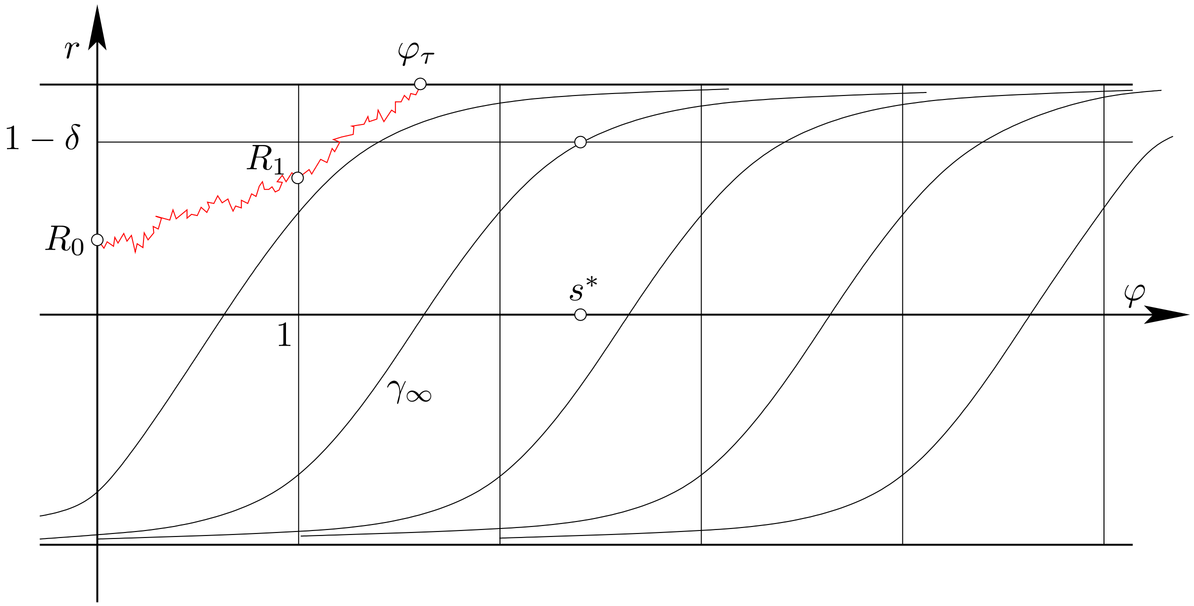


Generically optimal path (for infinite time) is isolated

Random Poincaré map

In polar-type coordinates (r, φ):



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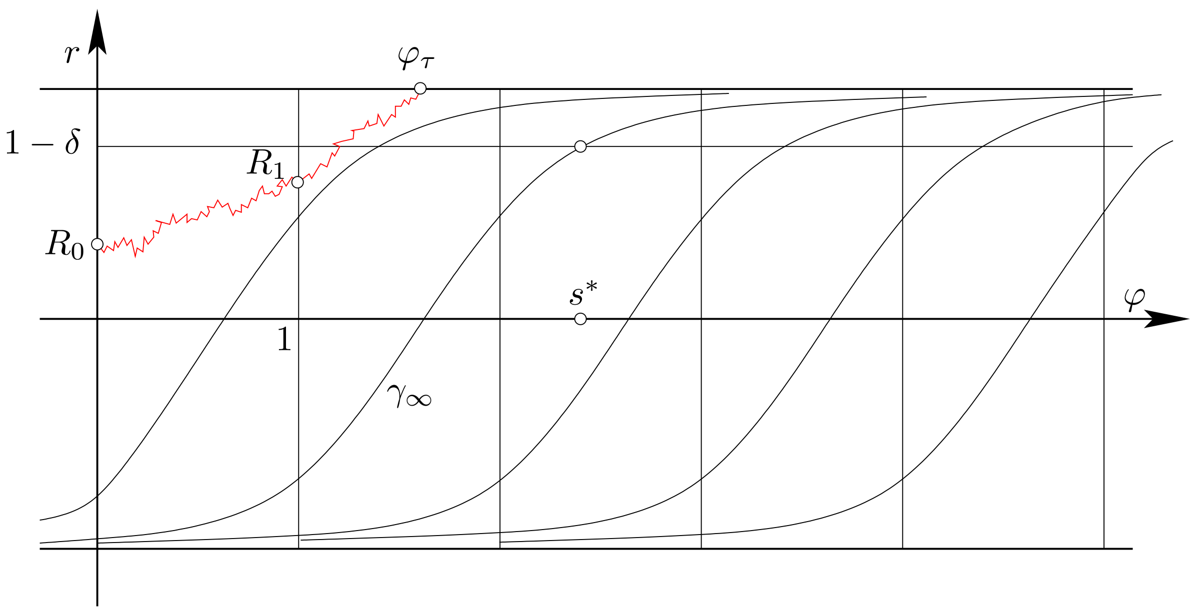

$$\mathbb{P}^{R_0}\{R_n \in A\} = \lambda_0^n h_0(R_0) \int_A h_0^*(y)\,\mathrm{d}y \Big[1 + \mathcal{O}((|\lambda_1|/\lambda_0)^n)\Big]$$

If $t = n + s$,

$$\mathbb{P}^{R_0}\{\varphi_\tau \in \mathrm{d}t\} = \lambda_0^n h_0(R_0) \int h_0^*(y) \mathbb{P}^y\{\varphi_\tau \in \mathrm{d}s\}\,\mathrm{d}y \Big[1 + \mathcal{O}((|\lambda_1|/\lambda_0)^n)\Big]$$

Periodically modulated exponential distribution

Main result: cycling

Theorem [B & Gentz, 2012] $\forall\Delta, \delta > 0\ \exists\sigma_0 > 0 : \forall 0 < \sigma < \sigma_0$

$$\mathbb{P}^{r_0,0}\{\varphi_\tau \in [\varphi, \varphi + \Delta]\} = C(\sigma)(\lambda_0)^\varphi \chi_\Delta(\varphi) Q_{\lambda_+ T_+}\biggl(\frac{|\log\sigma| - \theta(\varphi) + \mathcal{O}(\delta)}{\lambda_+ T_+}\biggr) \times \Bigl[1 + \mathcal{O}(\mathrm{e}^{-c\varphi/|\log\sigma|}) + \mathcal{O}(\delta|\log\delta|)\Bigr]$$

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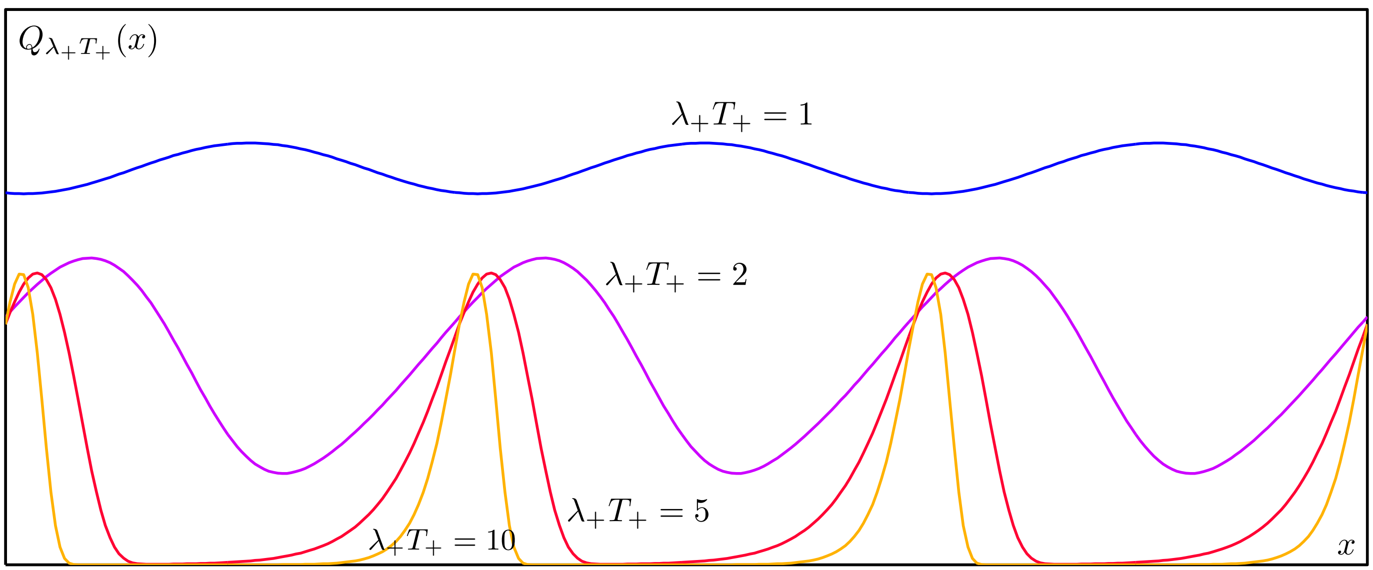
▷ $Q_{\lambda T}(x)=\displaystyle\sum_{n=-\infty}^{\infty}A(\lambda T(n-x))$ with $A(x)=\frac12\exp\{-2x-\frac12\mathrm{e}^{-2x}\}$

Cycling profile, periodicised Gumbel distribution



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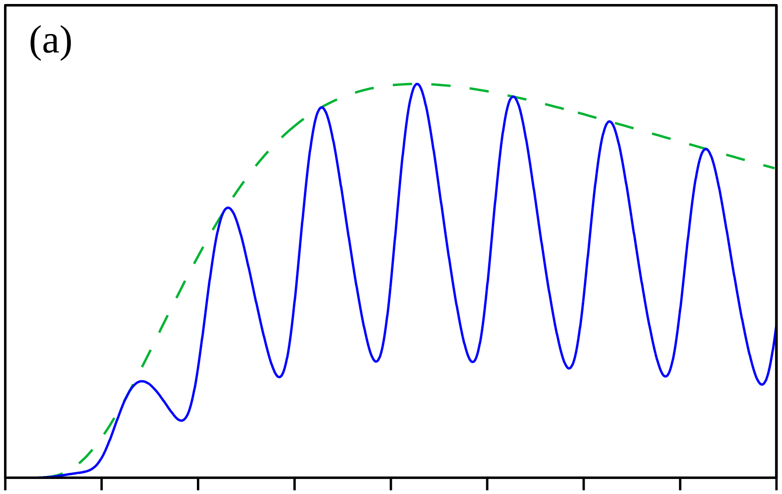
Cycling: periodic dependence on $|\log\sigma|$

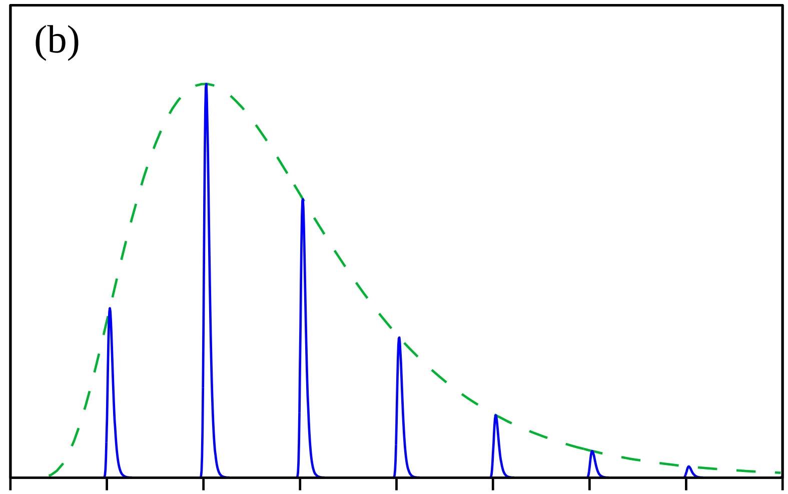
[Day'90, Maier & Stein '96, Getfert & Reimann '09]

Main result: cycling

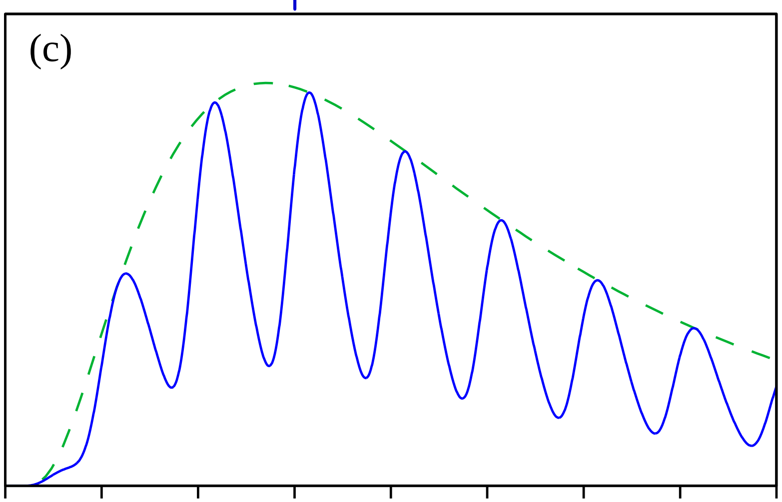
$V = 0.5,\ \lambda_+ = 1$



$\sigma = 0.4,\ T_+ = 2$



$\sigma = 0.4,\ T_+ = 20$



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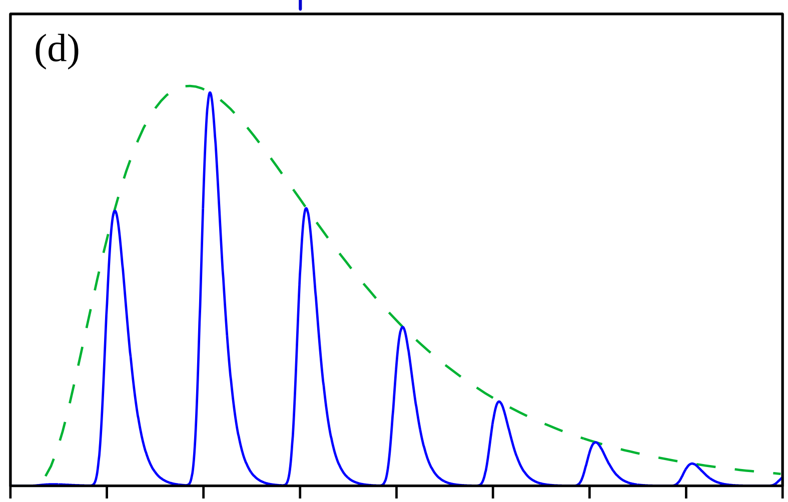
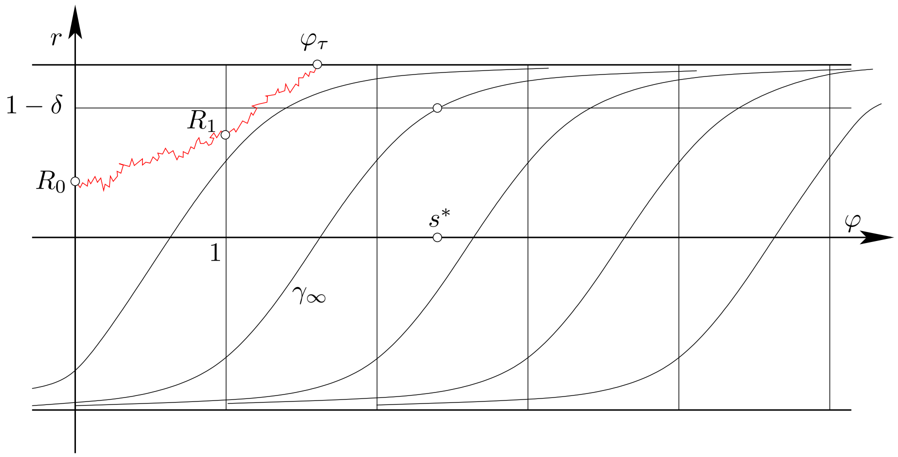


$\sigma = 0.5,\ T_+ = 5$

Sketch of proof



Split into two Markov chains:

- ▷ Chain killed upon r reaching $1-\delta$ in $\varphi = \varphi_{\tau_-}$

$$\mathbb{P}^0\{\varphi_{\tau_-} \in [\varphi_1, \varphi_1 + \Delta]\} \simeq (\lambda_0^{\mathrm{s}})^{\varphi_1} \, \mathrm{e}^{-J(\varphi_1)/\sigma^2}$$

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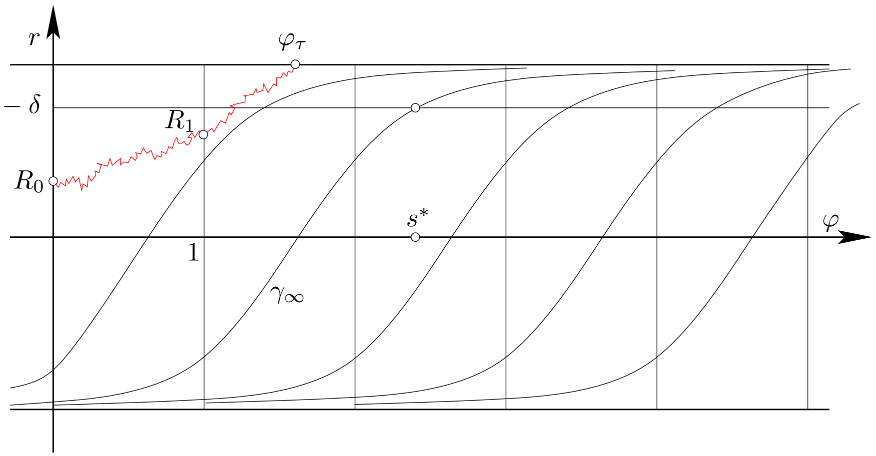

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▷ Chain killed at $r = 1 - 2\delta$ and on unstable orbit $r = 1$

– Principal eigenvalue: $\lambda_0^{\mathsf{u}} = \mathrm{e}^{-2\lambda_+ T_+}(1 + \mathcal{O}(\delta))$

$\lambda_+ =$ Lyapunov exponent, $T_+ =$ period of unstable orbit

– Using LDP:

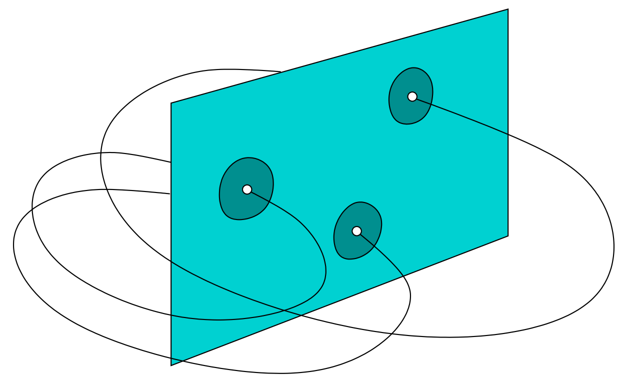
$$\mathbb{P}^{\varphi_1}\{\varphi_\tau \in [\varphi, \varphi + \Delta]\} \simeq (\lambda_0^{\mathsf{u}})^{\varphi - \varphi_1} \, \mathrm{e}^{-[I_\infty + c(\mathrm{e}^{-2\lambda_+ T_+(\varphi - \varphi_1)})]/\sigma^2}$$

Open questions

- ▷ Proof involving only spectral theory, without large deviations
  Note: linearisation around unstable orbit $\simeq$ harmonic oscillator
- ▷ More precise estimates on spectrum and eigenfunctions of K
- ▷ Link between spectrum of K and of L (with Dirichlet b.c.)
- ▷ Origin of Gumbel distribution

Systems with several stable periodic orbits

[Joint work with Barbara Gentz, Christian Kuehn, in progress]



- ▷ Consider system of dim$\geqslant 3$ with several stable periodic orbits
- ▷ We want to quantify transitions between these orbits
- ▷ Define again a Poincaré section and associated Markov process
- ▷ Exponentially small eigenvalues of this process?

Laplace transforms

Given $A \subset E$, $B \subset E \cup \{\Delta\}$, $A \cap B = \emptyset$, $x \in E$ and $u \in \mathbb{C}$, define

$$\tau_A = \inf\{n \geqslant 1 \colon X_n \in A\} \qquad G^u_{A,B}(x) = \mathbb{E}^x[\mathrm{e}^{u\tau_A} 1_{\{\tau_A < \tau_B\}}]$$
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▷ $G^u_{A,B}(x)$ is analytic for $|\mathrm{e}^u| < \Big[\sup_{x \in (A\cup B)^c} K(x,(A \cup B)^c)\Big]^{-1}$

▷ $G^u_{A,B} = H^u_{A,B}$ in $(A \cup B)^c$, $H^u_{A,B} = 1$ in A and $H^u_{A,B} = 0$ in B

▷ Feynman–Kac-type relation

$$KH^u_{A,B} = \mathrm{e}^{-u} G^u_{A,B}$$

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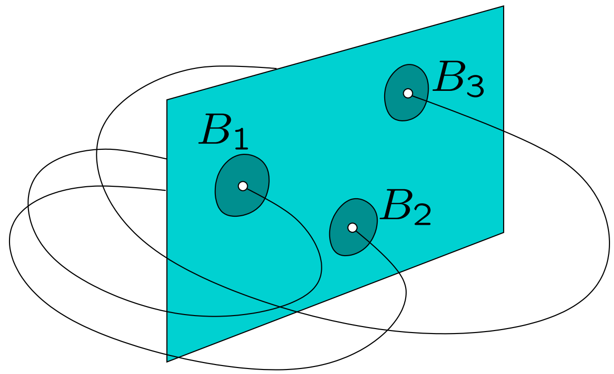
Proof:

$$\begin{aligned}
(KH^u_{A,B})(x) &= \mathbb{E}^x\Big[\mathbb{E}^{X_1}\big[\mathrm{e}^{u\sigma_A} 1_{\{\sigma_A<\sigma_B\}}\big]\Big] \\
&= \mathbb{E}^x\Big[1_{\{X_1\in A\}}\mathbb{E}^{X_1}\big[\mathrm{e}^{u\sigma_A} 1_{\{\sigma_A<\sigma_B\}}\big]\Big] + \mathbb{E}^x\Big[1_{\{X_1\in A^c\}}\mathbb{E}^{X_1}\big[\mathrm{e}^{u\sigma_A} 1_{\{\sigma_A<\sigma_B\}}\big]\Big] \\
&= \mathbb{E}^x\big[1_{\{1=\tau_A<\tau_B\}}\big] + \mathbb{E}^x\big[\mathrm{e}^{u(\tau_A-1)} 1_{\{1<\tau_A<\tau_B\}}\big] \\
&= \mathbb{E}^x\big[\mathrm{e}^{u(\tau_A-1)} 1_{\{\tau_A<\tau_B\}}\big] = \mathrm{e}^{-u} G^u_{A,B}(x)
\end{aligned}$$

⇒ if $G^u_{A,B}$ varies little in $A \cup B$, it is close to an eigenfunction

Heuristics (inspired by [Bovier, Eckhoff, Gayrard, Klein '04])

- ▷ Stable periodic orbits in $x_1, \dots, x_N$
- ▷ B_i small ball around x_i, $B = \bigcup_{i=1}^N B_i$
- ▷ Eigenvalue equation
  $(Kh)(x) = \mathrm{e}^{-u} h(x)$
- ▷ Assume $h(x) \simeq h_i$ in B_i



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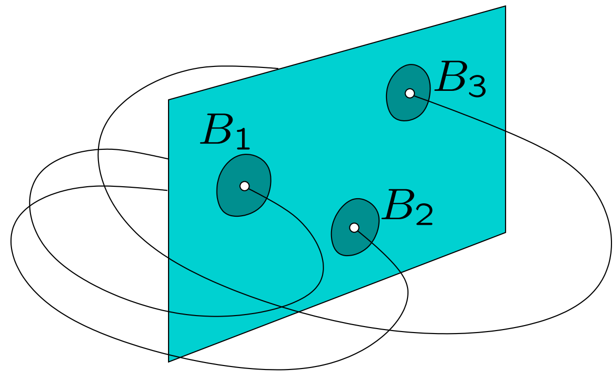

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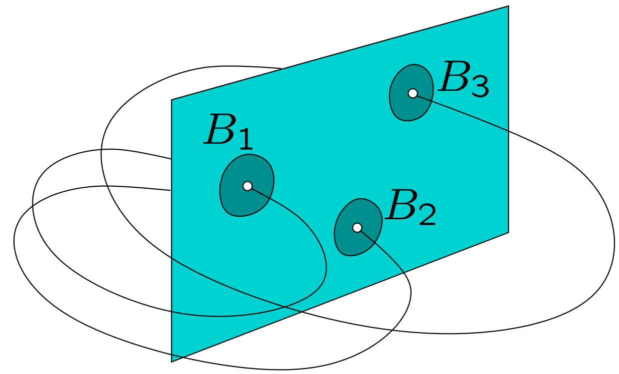
Ansatz: $h(x) = \sum_{j=1}^{N} h_j H^u_{B_j, B \setminus B_j}(x) + r(x)$

Heuristics (inspired by [Bovier, Eckhoff, Gayrard, Klein '04])



- ▷ Stable periodic orbits in $x_1, \dots, x_N$
- ▷ B_i small ball around x_i, $B = \bigcup_{i=1}^N B_i$
- ▷ Eigenvalue equation
  $(Kh)(x) = \mathrm{e}^{-u} h(x)$
- ▷ Assume $h(x) \simeq h_i$ in B_i

Ansatz: $h(x) = \sum_{j=1}^N h_j H^u_{B_j, B \setminus B_j}(x) + r(x)$

- ▷ $x \in B_i$: $h(x) = h_i + r(x)$
- ▷ $x \in B^c$: eigenvalue equation is satisfied (by Feynman–Kac)
- ▷ $x = x_i$: eigenvalue equation yields by Feynman–Kac

$$h_i = \sum_{j=1}^N h_j M_{ij}(u) \qquad M_{ij}(u) = G^u_{B_j, B \setminus B_j}(x_i) = \mathbb{E}^{x_i}[\mathrm{e}^{u\tau_B} 1_{\{\tau_B = \tau_{B_j}\}}]$$

$\Rightarrow$ condition $\det(M - \mathbb{1}) = 0 \Rightarrow N$ eigenvalues exp close to 1

If $\mathbb{P}\{\tau_B > 1\} \ll 1$ then $M_{ij}(u) \simeq \mathrm{e}^u \, \mathbb{P}^{x_i}\{\tau_B = \tau_{B_j}\} =: \mathrm{e}^u P_{ij}$ and $Ph \simeq \mathrm{e}^{-u} h$

Control of the error term

The error term satisfies the boundary value problem

$$(Kr)(x) = \mathrm{e}^{-u}\, r(x) \qquad x \in B^c$$

$$r(x) = h(x) - h_i \qquad x \in B_i$$

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Lemma: For u s.t. G^u_{B,E^c} exists, the unique solution of

$$\begin{aligned} (K\psi)(x) &= \mathrm{e}^{-u}\, \psi(x) && x \in B^c \\ \psi(x) &= \theta(x) && x \in B \end{aligned}$$

is given by $\psi(x) = \mathbb{E}^x[\mathrm{e}^{u\tau_B}\, \theta(X_{\tau_B})]$.

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Proof:

- ▷ Show that $\mathcal{T} f(x) = \mathbb{E}^x[\mathrm{e}^u \theta(X_1) 1_{\{X_1 \in B\}}] + \mathbb{E}^x[\mathrm{e}^u f(X_1) 1_{\{X_1 \in B^c\}}]$ is a contraction on $L^\infty(B^c)$
- ▷ Set $\psi_0(x) = 0$, $\psi_{n+1}(x) = \mathcal{T}\psi_n(x) \quad \forall n \geqslant 0$
- ▷ Show by induction that $\psi_n(x) = \mathbb{E}^x[\mathrm{e}^{u\tau_B} \theta(X_{\tau_B}) 1_{\{\tau_B \leqslant n\}}]$
- ▷ $\psi(x) = \lim_{n\to\infty} \psi_n(x)$ is fixed point of $\mathcal{T}$ ⇒ satisfies the bndry value problem

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$\Rightarrow r(x) = \mathbb{E}^x[\mathrm{e}^{u\tau_B} \theta(X_{\tau_B})]$ where $\theta(x) = \sum_j [h(x) - h_j] 1_{\{x \in B_j\}}$

To show that $h(x) - h_j$ is small in B_j: use Harnack inequalities

Conclusions

- ▷ Reduction to an N-state process in the sense that

$$\mathbb{P}^x\{X_n \in B_i\} = \sum_{j=1}^{N} \lambda_j^n h_j(x) h_j^*(B_i) + \mathcal{O}(|\lambda_{N+1}|^n)$$

- ▷ Residence times are approx exponential (provided system can relax to QSD)
- ▷ Generically, eigenvalues λ_j are determined by "metastable hierarchy" of periodic orbits

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▷ Residence times are approx exponential (provided system can relax to QSD)

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Open problems

▷ How to determine efficiently the M_{ij} or $P_{ij} = \mathbb{P}^{x_i}\{\tau_B = \tau_{B_j}\}$?
Large deviations – but not easy to implement and not very precise

▷ How to approximate left eigenfunctions (QSDs)

▷ Chaotic orbits?

Further reading

▷ N.B. and Barbara Gentz, *On the noise-induced passage through an unstable periodic orbit I: Two-level model*, J. Stat. Phys. **114**:1577–1618 (2004).

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▷ N.B., *Kramers' law: Validity, derivations and generalisations*, preprint `arXiv:1106.5799` (to appear in MPRF)

▷ Gérard Ben Arous, Shigeo Kusuoka, and Daniel W. Stroock, *The Poisson kernel for certain degenerate elliptic operators*, J. Funct. Anal. **56**:171–209 (1984).

▷ F. Hérau, J. Sjöstrand and M. Hitrik, *Tunnel effect and symmetries for Kramers Fokker-Planck type operators*, Journal of the Inst. of Math. Jussieu **10**:567–634 (2011).

▷ Martin V. Day, *Cycling and skewing of exit measures for planar systems*, Stoch. Stoch. Rep. **48**:227–247 (1994).

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