High frequency analysis of the dissipative Helmholtz equation

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The Helmholtz equation

We study on $\mathbb{R}^n$ the following Helmholtz equation:

$$(-h^2\Delta + V_1(x) - ihV_2(x) - E)u \;= S.$$

This equation models accurately the propagation of the electromagnetic field of a laser in material medium.

$$\begin{aligned} V_1(x) - E &: \text{refraction index,} \\ V_2(x) &: \text{absorption index,} \\ S &: \text{source term,} \\ h &: \text{wave length,} \end{aligned}$$

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We consider the high frequency approximation $h \to 0$.

The non-selfadjoint Schrödinger operator

- When V_2 is constant, it can be put in the spectral parameter:

$$(H_1^h - z_h)u_h = S,$$

with

$$H_1^h = -h^2\Delta + V_1(x) \quad \text{and} \quad z_h = E + ihV_2.$$

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- When V_2 is variable, it has to be in the operator itself:

$$(H_h - E)u_h = S,$$

with

$$H_h = -h^2\Delta + V_1(x) - ihV_2(x).$$

$\rightsquigarrow$ we have to work with a non-selfadjoint operator.

Dissipative operators

The operator H on the Hilbert space $\mathcal{H}$ is said to be dissipative if

$$\forall \varphi \in \mathcal{D}(H), \quad \operatorname{Im} \langle H\varphi, \varphi \rangle \leqslant 0.$$

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- The resolvent $(H - z)^{-1}$ is well-defined if $\operatorname{Im} z > 0$ and

$$\left\| (H - z)^{-1} \right\|_{\mathcal{L}(\mathcal{H})} \leqslant \frac{1}{\operatorname{Im} z}.$$

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- H generates a contractions semi-group

$$t \in \mathbb{R}_+ \mapsto e^{-itH}, \quad \left\| e^{-itH} \right\|_{\mathcal{L}(\mathcal{H})} \leqslant 1,$$

and for $\varphi \in \mathcal{D}(H)$:

$$\frac{d}{dt} \left\| e^{-itH} \varphi \right\|_{\mathcal{H}}^2 = 2 \operatorname{Im} \left\langle He^{-itH} \varphi, e^{-itH} \varphi \right\rangle \leqslant 0.$$

Questions

1. We first look for uniform resolvent estimates:

$$\sup_{\substack{\operatorname{Re} z \sim E \\ \operatorname{Im} z > 0}} \left\| (H \quad - z)^{-1} \right\|_{\mathcal{L}(\mathcal{H}_1, \mathcal{H}_1^*)} \leqslant c$$

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$$\lim_{\mu \to 0^+} (H - (E + i\mu))^{-1} \text{ exists in } \mathcal{L}(\mathcal{H}_1, \mathcal{H}_1^*),$$

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$$\|u\|_{\mathcal{H}_1^*} = \left\| (H - (E + i0))^{-1} S \right\|_{\mathcal{H}_1^*} \leqslant c \|S\|_{\mathcal{H}_1}.$$

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$$\forall h \in]0, h_0], \qquad \sup_{\substack{\operatorname{Re} z \sim E \\ \operatorname{Im} z > 0}} \left\| (H_h - z)^{-1} \right\|_{\mathcal{L}(\mathcal{H}_1, \mathcal{H}_1^*)} \leqslant c(h)$$

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$$\left\| u_h \right\|_{\mathcal{H}_1^*} = \left\| (H_h - (E + i0))^{-1} S_h \right\|_{\mathcal{H}_1^*} \leqslant c(h) \left\| S_h \right\|_{\mathcal{H}_1} .$$

We study these estimates in an abstract setting, and then for the dissipative Schrödinger operator.

Questions

2. We study the semiclassical measures for the solution u_h of the Helmholtz equation for a particular term source S_h:

$$\left\langle \mathrm{Op}^w_{h_m}(q) u_{h_m}, u_{h_m} \right\rangle \xrightarrow[m\to\infty]{} \int_{\mathbb{R}^{2n}} q \, d\mu,$$

where $h_m \to 0$ and

$$\mathrm{Op}^w_h(q)u(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} q\left(\frac{x+y}{2}, \xi\right) u(y) \, dy \, d\xi$$

(Weyl quantization of q).

Mourre's commutators method

Theorem (E.Mourre 81,...)

Let H_1 be a self-adjoint operator on the Hilbert space $\mathcal{H}$.

The self-adjoint operator A on $\mathcal{H}$ is said to be conjugate to H_1 on the open set $J \subset \mathbb{R}$ if

- *some conditions about the commutators $[H_1, iA]$ and $[[H_1, iA], iA]$ are satisfied,*
- *and for some $\alpha > 0$:*

$$\mathbb{1}_J(H_1)[H_1, iA]\mathbb{1}_J(H_1) \geqslant \alpha \mathbb{1}_J(H_1).$$

In this case, for $\delta > \frac{1}{2}$ and a compact $I \subset J$ there exists $c > 0$ such that for $\operatorname{Re} z \in I$ and $\operatorname{Im} z \neq 0$

$$\left\| \langle A \rangle^{-\delta} (H_1 - z)^{-1} \langle A \rangle^{-\delta} \right\|_{\mathcal{L}(\mathcal{H})} \leqslant c.$$

$\langle \lambda \rangle = \left(1 + |\lambda|^2\right)^{\frac{1}{2}}$

Mourre's commutators method

Theorem

Let $H = H_1 - iV$ *be a dissipative operator on the Hilbert space* $\mathcal{H}$, *where* H_1 *is self-adjoint and* $V \geqslant 0$ *is self-adjoint and* H_1*-bounded with relative bound <1.*

The self-adjoint operator A *on* $\mathcal{H}$ *is said to be conjugate to* H *on the open set* $J \subset \mathbb{R}$ *if*

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Two words about the assumption

$$\mathbb{1}_J(H_1)[H_1, iA]\mathbb{1}_J(H_1) \geqslant \alpha \mathbb{1}_J(H_1).$$

- We do not have a functionnal calculus for the non-selfadjoint operator H.

  We use functionnal calculus for the self-adjoint part H_1, and the assumption that the dissipative part V is "smaller" than H_1.

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Lemma (Quadratic estimates)

Let $T = T_R - iT_I$ where T_R is self-adjoint and $T_I \geqslant 0$ is self-adjoint and T_R-bounded with relative bound <1.
If $B^*B \leqslant T_I$, Q is bounded and $\operatorname{Im} z > 0$ we have

$$\left\| B(T-z)^{-1}Q \right\| \leqslant \left\| Q^*(T-z)^{-1}Q \right\|^{\frac{1}{2}}.$$

- We use the quadratic estimates with

$$T = H_1 - i\varepsilon\phi(H_1)[H_1, iA]\phi(H_1), \quad \operatorname{supp}\phi \subset J,$$

and $B = \sqrt{\varepsilon}\sqrt{\alpha}\phi(H_1)$.

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Mourre's commutators method

Theorem (J.R. 10)

Let $H = H_1 - iV$ be a dissipative operator on the Hilbert space $\mathcal{H}$, where H_1 is self-adjoint and $V \geqslant 0$ is self-adjoint and H_1-bounded with relative bound <1.

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In this case, for $\delta > \frac{1}{2}$ and a compact $I \subset J$ there exists $c > 0$ such that for $\operatorname{Re} z \in I$ and $\operatorname{Im} z > 0$

$$\left\| \langle A \rangle^{-\delta} (H - z)^{-1} \langle A \rangle^{-\delta} \right\|_{\mathcal{L}(\mathcal{H})} \leqslant c.$$

More about abstract resolvent estimates

- Limiting absorption principle: for $\lambda \in J$ the limit

$$\lim_{\mu \to 0^+} \langle A \rangle^{-\delta} (H - (\lambda + i\mu))^{-1} \langle A \rangle^{-\delta}$$

exists in $\mathcal{L}(\mathcal{H})$ and defines a continuous function of λ.

- Estimate in Besov spaces.

- Estimates for the powers of the resolvent and regularity of the limit.

Classical flow

- Let

$$H_1^h = -h^2 \Delta + V_1(x)$$

with

$$|\partial^\alpha V_1(x)| \leqslant c_\alpha \langle x \rangle^{-\rho - |\alpha|}, \quad \rho > 0.$$

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- Let

$$p(x,\xi) = |\xi|^2 + V_1(x).$$

We denote by $\phi^t(x_0,\xi_0) = (\overline{x}(t,x_0,\xi_0), \overline{\xi}(t,x_0,\xi_0))$ the solution of the hamiltonian system

$$\begin{cases} \partial_t \overline{x}(t) = 2\overline{\xi}(t), \\ \partial_t \overline{\xi}(t) = -\nabla V_1(\overline{x}(t)), \\ \overline{x}(0) = x_0, \quad \overline{\xi}(0) = \xi_0. \end{cases}$$

Resolvent estimates

Theorem (D.Robert-H.Tamura 87, X.P.Wang 87)

Let $\delta > \frac{1}{2}$ and $E > 0$.
Then we can find $h_0 > 0$, a neighborhood I of E and $c \geqslant 0$ such that for $h \in]0, h_0]$ and $\operatorname{Re} z \in I$ and $\operatorname{Im} z \neq 0$ we have

$$\left\| \langle x \rangle^{-\delta} (H_1^h - z)^{-1} \langle x \rangle^{-\delta} \right\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leqslant \frac{c}{h}$$

if and only if E is non-trapping:

$$p(x,\xi) = E \implies |\overline{x}(t,x,\xi)| \xrightarrow[t \to \pm\infty]{} +\infty.$$

Resolvent estimates

Theorem (J.R. 10)

Let $\delta > \frac{1}{2}$ *and* $E > 0$. *Suppose that* $V_2 \geqslant 0$ *is of long range. Then we can find* $h_0 > 0$, *a neighborhood* I *of* E *and* $c \geqslant 0$ *such that for* $h \in]0, h_0]$ *and* $\operatorname{Re} z \in I$ *and* $\operatorname{Im} z > 0$ *we have*

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if and only if for $(x, \xi) \in p^{-1}(\{E\})$

$$\sup_{t \in \mathbb{R}} |\overline{x}(t, x, \xi)| < \infty \quad \Longrightarrow \quad \exists T \in \mathbb{R}, \;\; V_2(\overline{x}(T, x, \xi)) > 0.$$

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We have the limiting absorption principle and the limit of the resolvent

$$(H_h - (E + i0))^{-1} : L^{2,\delta}(\mathbb{R}^n) \to L^{2,-\delta}(\mathbb{R}^n)$$

gives the unique outgoing solution for the equation

$$(H_h - E)u = S.$$

C. Gérard and A. Martinez (88) constructed a conjugate operator to H_1^h, using pseudo-differential calculus.

- We look for a conjugate operator of the form

$$A_h = \mathrm{Op}_h^w(x \cdot \xi + r(x,\xi)), \quad r \in C_0^\infty(\mathbb{R}^{2n})$$

(if $V_1 = 0$ we can choose $r = 0$).

- In order to have

$$\mathbb{1}_J(H_1^h)[H_1^h, iA_h]\mathbb{1}_J(H_1^h) \geqslant c_0 h \mathbb{1}_J(H_1^h), \quad c_0 > 0,$$

after quantization, we construct r such that

$$\{p, x \cdot \xi + r(x,\xi)\} \geqslant c_0 \quad \text{on } p^{-1}(J).$$

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$$\mathbb{1}_J(H_1^h)\big([H_1^h, iA_h] + \beta V_2\big)\mathbb{1}_J(H_1^h) \geqslant c_0 h \mathbb{1}_J(H_1^h), \quad c_0 > 0,$$

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Let

$$u_h = (H_h - (E + i0))^{-1} S_h$$

where

$$H_h = -h^2 \Delta + V_1(x) - ihV_2(x)$$

and S_h is an explicit source term which concentrates on a bounded submanifold of $\mathbb{R}^n$:

- Γ bounded submanifold of $\mathbb{R}^n$ of dimension $d \in [\![0, n-1]\!]$, σ_Γ Lebesgue measure on Γ,
- $A \in C_0^\infty(\Gamma)$,
- $S \in \mathcal{S}(\mathbb{R}^n)$,

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$$\forall \delta > \frac{1}{2}, \quad \|S_h\|_{L^{2,\delta}(\mathbb{R}^n)} = O\big(\sqrt{h}\big)$$

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We have:

$$\forall \delta > \frac{1}{2}, \quad \|S_h\|_{L^{2,\delta}(\mathbb{R}^n)} = O\big(\sqrt{h}\big) \quad \text{and} \quad \|u_h\|_{L^{2,-\delta}(\mathbb{R}^n)} = O\left(\frac{1}{\sqrt{h}}\right).$$

Known results for a constant absorption index

- J.D.Benamou-F.Castella-T.Katsaounis-B.Perthame-02: $\Gamma = \{0\}$, semiclassical measure as the limit of the Wigner transform
(see also F.Castella (05)).
- F.Castella-B.Perthame-O.Runborg-02: Γ affine subspace of $\mathbb{R}^n$, $V_1 = 0$.
- X.P.Wang-P.Zhang-06: $V_1 \neq 0$.
- E.Fouassier-06: two source points.
- E.Fouassier-07: V_1 discontinuous along a hyperplane.
- J.-F.Bony-09: $\Gamma = \{0\}$, microlocal point of view.

The Assumptions

- V_1 of long range.
- V_2 of short range:

$$\left|\partial^{\alpha} V_2(x)\right| \leqslant c_{\alpha} \left\langle x \right\rangle^{-1-\rho-|\alpha|}, \quad \rho > 0.$$

- E satisfies the damping assumption on trapped trajectories:

$$\sup_{t \in \mathbb{R}} \left|\overline{x}(t,x,\xi)\right| < \infty \quad \Longrightarrow \quad \exists\, T \in \mathbb{R},\ V_2(\overline{x}(T,x,\xi)) > 0.$$

- $\forall z \in \Gamma, \quad V_1(z) < E$.
- if $N_E\Gamma = \left\{(z,\xi) \in N\Gamma \,:\, \left|\xi\right|^2 + V_1(z) = E\right\}$ then

$$\sigma_{N_E\Gamma}\left(\{w \in N_E\Gamma \,:\, \exists t > 0, \phi^t(w) \in N_E\Gamma\}\right) = 0.$$

Theorem (J.R. 10)

- *There exists a non-negative Radon measure μ on $\mathbb{R}^{2n}$ such that*

$$\forall q \in C_0^\infty(\mathbb{R}^{2n}), \quad \langle \mathrm{Op}_h^w(q) u_h, u_h \rangle \xrightarrow[h \to 0]{} \int_{\mathbb{R}^{2n}} q \, d\mu.$$

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- *μ is characterized by the following three properties:*
  a. $\operatorname{supp} \mu \subset p^{-1}(\{E\})$.
  b. *$\mu = 0$ on the incoming region $\{|x| \gg 1, x \cdot \xi \leqslant -\frac{1}{2} |x| |\xi|\}$.*
  c. *μ satisfies the Liouville equation*

$$\{p, \mu\} + 2 V_2 \mu = \underbrace{\pi (2\pi)^{d-n} |A(z)|^2 |\xi|^{-1} \left|\hat{S}(\xi)\right|^2}_{\kappa(z,\xi)} \sigma_{N_E \Gamma}.$$

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- *These three properties imply that for all $q \in C_0^\infty(\mathbb{R}^{2n})$ the integral of q is given by*

$$\int_0^\infty \int_{N_E \Gamma} \kappa(z, \xi) q(\phi^t(z, \xi)) e^{-2 \int_0^t V_2(\phi^s(z,\xi)) \, ds} \, d\sigma_{N_E \Gamma}(z, \xi) dt$$

The three new difficulties:

- Non-selfadjointness of H_h.
- Geometry of Γ (and $N_E\Gamma$).
- Trapped trajectories.

Time-dependant approach

Let $w \in \mathbb{R}^{2n}$ and $q \in C_0^\infty(\mathbb{R}^{2n})$ supported close to w.

$$\mathrm{Op}_h^w(q) u_h = \frac{i}{h} \int_0^{T_0} \mathrm{Op}_h^w(q) e^{-\frac{it}{h}(H_h - E)} S_h \, dt + \mathrm{Op}_h^w(q) e^{-\frac{iT_0}{h}(H_h - E)} u_h.$$

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$$\begin{aligned}\mathrm{Op}_h^w(q) u_h = \frac{i}{h}\int_0^{T_0} \mathrm{Op}_h^w(q) e^{-\frac{it}{h}(H_h - E)} S_h \, dt \\ + \mathrm{Op}_h^w(q) e^{-\frac{iT_0}{h}(H_h - E)} u_h.\end{aligned}$$

Let

$$U_1^h(t) = e^{-\frac{it}{h} H_1^h}, \quad U_h(t) = e^{-\frac{it}{h} H_h}$$

Proposition

Let $t \geqslant 0$ and $a \in C_b^\infty(\mathbb{R}^{2n})$. We have

$$U_1^h(t)^* \mathrm{Op}_h^w(a) U_1^h(t) = \mathrm{Op}_h^w\left(a \circ \phi^t\right) + \underset{h \to 0}{O}(h).$$

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Partial semiclassical measures

To avoid large times, we first study

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- $\forall \varepsilon > 0, \exists T_0 > 0, \forall T \geqslant T_0,$

$$\limsup_{h \to 0} \left| \langle \mathrm{Op}_h^w(q) u_h, u_h \rangle - \left\langle \mathrm{Op}_h^w(q) u_h^T, u_h^T \right\rangle \right| \leqslant \varepsilon,$$

- and

$$\int_{\mathbb{R}^{2n}} q \, d\mu_T \xrightarrow[T \to +\infty]{} \int_{\mathbb{R}^{2n}} q \, d\mu,$$

for some non-negative Radon measure μ on $\mathbb{R}^{2n}$.