The relativistic mean-field equations of the atomic nucleus [1]

Simona Rota Nodari

Ceremade, Université Paris-Dauphine & Università degli Studi di Milano

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The relativistic mean-field theory (RMF)

Approximations of the relativistic mean-field theory :

- Mean-field approximation : the nucleons behave as noninteracting particles moving in a mean field generated by mesons and photons.
- *No-sea approximation* : we neglect the vacuum polarization (Dirac sea).

Fields generated by mesons and photons :

- σ meson : medium range attractive interaction ;
- ω meson : short range repulsive interaction ;
- ρ meson : description of isospin-dependent effects ;
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The Lagrangian of the RMF theory can be written as

$$\mathcal{L} = \mathcal{L}_{nucleons} + \mathcal{L}_{mesons} + \mathcal{L}_{coupling}. \tag{1.1}$$

$$\mathcal{L}_{nucleons} = \sum_{\alpha=1}^{\Omega} w_\alpha \bar{\psi}_\alpha (i\gamma^\mu \partial_\mu - m_b)\psi_\alpha \tag{1.2}$$

$$\begin{aligned}\mathcal{L}_{mesons} &= \frac{1}{2}(\partial^\mu \sigma \partial_\mu \sigma - m_\sigma^2 \sigma^2) - \frac{1}{2}(\overline{\partial^\mu \omega^\nu} \partial_\mu \omega_\nu - m_\omega^2 \omega^\mu \omega_\mu) \\ &\quad - \frac{1}{2}(\overline{\partial^\mu \mathbf{R}^\nu} \partial_\mu \mathbf{R}_\nu - m_\rho^2 \mathbf{R}^\mu \mathbf{R}_\mu) - \frac{1}{2}\overline{\partial^\mu A^\nu} \partial_\mu A_\nu\end{aligned} \tag{1.3}$$

$$\mathcal{L}_{coupling} = -g_\sigma \sigma \rho_s - g_\omega \omega^\mu \rho_\mu - g_\rho \mathbf{R}^\mu \cdot \boldsymbol{\rho}_\mu - eA^\mu \rho_\mu^c - U(\sigma) \tag{1.4}$$

with $U(\sigma) = \frac{1}{3} b_2 \sigma^3 + \frac{1}{4} b_3 \sigma^4$.

The densities are

$$\begin{aligned}\rho_s &= \textstyle\sum_{\alpha=1}^{\Omega} w_\alpha \bar{\psi}_\alpha \psi_\alpha, & \rho_\mu &= \textstyle\sum_{\alpha=1}^{\Omega} w_\alpha \bar{\psi}_\alpha \gamma_\mu \psi_\alpha, \\ \boldsymbol{\rho}_\mu &= \textstyle\sum_{\alpha=1}^{\Omega} w_\alpha \bar{\psi}_\alpha \boldsymbol{\tau} \gamma_\mu \psi_\alpha, & \rho_\mu^c &= \textstyle\sum_{\alpha=1}^{\Omega} w_\alpha \bar{\psi}_\alpha \frac{1}{2}(1+\tau_0)\gamma_\mu \psi_\alpha.\end{aligned} \tag{1.5}$$

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The mean-field equations

Simplification of the model :

- the single-particle states are eigenstates of $\tau_0 \to$ only $R_{0\mu}$ and $\rho_{0\mu}$ appear ;
- stationarity : all time derivatives and spatial components of densities and fields vanish $\to$ only the fields σ, ω_0, R_{00} and A_0 remain ;
- $$\psi_\alpha(\mathbf{x}, t) = e^{-i\varepsilon_\alpha t}\psi_\alpha(\mathbf{x}). \tag{1.6}$$

We obtain

$$\varepsilon_\alpha\gamma_0\psi_\alpha = \Big[-i\gamma\cdot\nabla + m_b + g_\sigma\sigma + g_\omega\omega_0\gamma_0 + g_\rho R_{00}\gamma_0\tau_0 + \frac{1}{2}eA_0\gamma_0(1+\tau_0)\Big]\psi_\alpha, \tag{1.7}$$

$$(-\Delta + m_\sigma^2)\sigma + U'(\sigma) = -g_\sigma\rho_s, \tag{1.8}$$

$$(-\Delta + m_\omega^2)\omega_0 = g_\omega\rho_0, \tag{1.9}$$

$$(-\Delta + m_\rho^2)R_{00} = g_\rho\rho_{00}, \tag{1.10}$$

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We consider $b_2 = b_3 = 0$ and we choose a fixed occupation of the orbitals, that means

$$w_\alpha = \begin{cases} 1 & \alpha = 1, \ldots, A \\ 0 & \text{otherwise} \end{cases} \tag{1.12}$$

where A is the nucleon number.

In this case, the equations (1.8-1.11) can be solved explicitly and we obtain

$$\begin{aligned} \varepsilon_\alpha \psi_\alpha &= \Big[H_0 - \beta \frac{g_\sigma^2}{4\pi} \left(\frac{e^{-m_\sigma |\cdot|}}{|\cdot|} \star \rho_s \right) + \frac{g_\omega^2}{4\pi} \left(\frac{e^{-m_\omega |\cdot|}}{|\cdot|} \star \rho_0 \right) \\ &\quad + \tau_0 \frac{g_\rho^2}{4\pi} \left(\frac{e^{-m_\rho |\cdot|}}{|\cdot|} \star \rho_{00} \right) + \frac{1}{2}(1+\tau_0) \frac{e^2}{4\pi} \left(\frac{1}{|\cdot|} \star \rho_0^c \right) \Big] \psi_\alpha \end{aligned} \tag{1.13}$$

where $H_0 = -i\alpha \cdot \nabla + \beta m_b$.

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where $H_0 = -i\boldsymbol{\alpha} \cdot \nabla + \beta m_b$.

Using the convention $\tau_0 = 1$ for the protons and $\tau_0 = -1$ for the neutrons, the nonlinear Dirac equations are given by

$$H_{p,\Psi}\psi_i := \Big[H_0 - \beta\frac{g_\sigma^2}{4\pi}\left(\frac{e^{-m_\sigma|\cdot|}}{|\cdot|}\star\rho_s\right) + \frac{g_\omega^2}{4\pi}\left(\frac{e^{-m_\omega|\cdot|}}{|\cdot|}\star\rho_0\right) + \frac{g_\rho^2}{4\pi}\left(\frac{e^{-m_\rho|\cdot|}}{|\cdot|}\star\rho_{00}\right) + \frac{e^2}{4\pi}\left(\frac{1}{|\cdot|}\star\rho_0^c\right)\Big]\psi_i = \varepsilon_i\psi_i \quad (1.14)$$

if $1 \le i \le Z$, and

$$H_{n,\Psi}\psi_i := \Big[H_0 - \beta\frac{g_\sigma^2}{4\pi}\left(\frac{e^{-m_\sigma|\cdot|}}{|\cdot|}\star\rho_s\right) + \frac{g_\omega^2}{4\pi}\left(\frac{e^{-m_\omega|\cdot|}}{|\cdot|}\star\rho_0\right) - \frac{g_\rho^2}{4\pi}\left(\frac{e^{-m_\rho|\cdot|}}{|\cdot|}\star\rho_{00}\right)\Big]\psi_i = \varepsilon_i\psi_i \quad (1.15)$$

if $Z+1 \le i \le A$, under the constraints $\int_{\mathbb{R}^3}\psi_i^*\psi_j = \delta_{ij}$ for $1 \le i,j \le Z$ and $Z+1 \le i,j \le A$.

The minimization problem

The nonlinear Dirac equations are the Euler-Lagrange equations of the energy functional

$$\begin{aligned}
\mathcal{E}(\Psi) \quad = \quad & \sum_{j=1}^{A} \int_{\mathbb{R}^3} \psi_j^* H_0 \psi_j - \frac{g_\sigma^2}{8\pi} \int\int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\rho_s(x)\rho_s(y)}{|x-y|} e^{-m_\sigma |x-y|}\, dxdy \\
& + \frac{g_\omega^2}{8\pi} \int\int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\rho_0(x)\rho_0(y)}{|x-y|} e^{-m_\omega |x-y|}\, dxdy \\
& + \frac{g_\rho^2}{8\pi} \int\int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\rho_{00}(x)\rho_{00}(y)}{|x-y|} e^{-m_\rho |x-y|}\, dxdy \\
& + \frac{e^2}{8\pi} \int\int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\rho_0^c(x)\rho_0^c(y)}{|x-y|}\, dxdy \qquad (1.16)
\end{aligned}$$

under the constraints $\int_{\mathbb{R}^3} \psi_i^* \psi_j = \delta_{ij}$ for $1 \le i, j \le Z$ and for $Z+1 \le i, j \le A$.

Since this functional is not bounded from below under the constraints $\int_{\mathbb{R}^3} \psi_i^* \psi_j = \delta_{ij}$, we introduce the following minimization problem ([1])

$$
\begin{aligned}
I \quad = \quad & \inf \Big\{ \mathcal{E}(\Psi); \int_{\mathbb{R}^3} \psi_i^* \psi_j = \delta_{ij}, 1 \leq i,j \leq Z, Z+1 \leq i,j \leq A, \\
& \Lambda^-_{p,\Psi}(\psi_1, \dots, \psi_Z) = 0, \Lambda^-_{n,\Psi}(\psi_{Z+1}, \dots, \psi_A) = 0 \Big\} \qquad (1.17)
\end{aligned}
$$

together with its extension

$$
\begin{aligned}
I(\lambda_1, \dots, \lambda_A) \quad = \quad & \inf \Big\{ \mathcal{E}(\Psi); \int_{\mathbb{R}^3} \psi_i^* \psi_j = \lambda_i \delta_{ij}, 1 \leq i,j \leq Z, \\
& Z+1 \leq i,j \leq A, \Lambda^-_{p,\Psi}(\psi_1, \dots, \psi_Z) = 0, \\
& \Lambda^-_{n,\Psi}(\psi_{Z+1}, \dots, \psi_A) = 0 \Big\} \qquad (1.18)
\end{aligned}
$$

where, for $\mu = p, n$, $\Lambda^-_{\mu,\Psi} = \chi_{(-\infty,0)}(H_{\mu,\Psi})$.

Theorem 1

If $g_\sigma, g_\omega, g_\rho$ and e are sufficiently small, a minimizer of (1.17) is a solution of the equations (1.14) and (1.15).

Theorem 2

If $g_\sigma, g_\omega, g_\rho$ and e are sufficiently small, any minimizing sequence of (1.17) is relatively compact up to a translation if and only if the following condition holds

$$I < I(\lambda_1, \ldots, \lambda_A) + I(1-\lambda_1, \ldots, 1-\lambda_A) \tag{1.19}$$

for all $\lambda_k \in [0,1]$, $k = 1, \ldots, A$, such that $\sum_{k=1}^{A} \lambda_k \in (0, A)$.
In particular, if (1.19) holds, there exists a minimum of (1.17).

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Properties of the potential

If $g_\sigma, g_\omega, g_\rho$ and e are sufficiently small,

- $H_{\mu,\Psi}$ is a self-adjoint isomorphism between $H^{1/2}$ and its dual $H^{-1/2}$, whose inverse is bounded independently of Ψ
- any minimizing sequence $\Psi^k = (\psi_1^k, \dots, \psi_A^k)$ is bounded in $\left(H^{1/2}(\mathbb{R}^3)\right)^A$ and I is bounded from below.

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Concentration-compactness lemma

Lemma 3 ([2],[3])

Let $(P_k)_k$ be a sequence of probability measures on $\mathbb{R}^N$. Then there exists a subsequence that we still denote by P_k such that one of the following properties holds :

1. *(compactness up to a translation)* $\exists y^k \in \mathbb{R}^N$, $\forall \varepsilon > 0$, $\exists R < \infty$
$$P_k\left(B\left(y^k, R\right)\right) \geq 1 - \varepsilon;$$
2. *(vanishing)* $\forall R < \infty$
$$\sup_{y \in \mathbb{R}^N} P_k\left(B\left(y, R\right)\right) \underset{k}{\rightarrow} 0;$$
3. *(dichotomy)* $\exists \alpha \in (0,1)$, $\forall \varepsilon > 0$, $\forall M < \infty$, $\exists R_0 \geq M$, $\exists y^k \in \mathbb{R}^N$, $\exists R_k \underset{k}{\rightarrow} +\infty$ *such that*
$$\left|P_k\left(B\left(y^k, R_0\right)\right) - \alpha\right| \leq \varepsilon, \quad \left|P_k\left(B\left(y^k, R_k\right)^c\right) - (1-\alpha)\right| \leq \varepsilon.$$

Dichotomy does not occur

Let P_k be a probability measure in $\mathbb{R}^3$ whose density is $\frac{1}{A}\sum_{i=1}^{A}|\psi_i^k|^2$.

If dichotomy occurs (case iii.), then Ψ^k can be split into two parts Ψ_1^k and Ψ_2^k. More precisely,

$$\begin{aligned}\psi_{i,1}^k &= \xi_{R_0}(\cdot - y^k)\psi_i^k\\ \psi_{i,2}^k &= \zeta_{R_k}(\cdot - y^k)\psi_i^k\end{aligned}$$

with $R_k \underset{k}{\rightarrow} +\infty$, $\xi_\mu = \xi\left(\frac{\cdot}{\mu}\right)$, $\zeta_\mu = \zeta\left(\frac{\cdot}{\mu}\right)$ and

$$\xi(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| \geq 2 \end{cases} \qquad \zeta(x) = \begin{cases} 0 & |x| \leq 1 \\ 1 & |x| \geq 2 \end{cases}$$

with $\xi, \zeta \in \mathcal{D}(\mathbb{R}^3)$. We remind that $\mathrm{dist}\left(\mathrm{supp}\ \psi_{i,1}^k, \mathrm{supp}\ \psi_{i,2}^k\right) \underset{k}{\rightarrow} +\infty$ and $\left\|\psi_i^k - \left(\psi_{i,1}^k + \psi_{i,2}^k\right)\right\|_{L^p} \underset{k}{\rightarrow} 0$ for $2 \leq p < 3$. Next, we may assume that

$$\int_{\mathbb{R}^3} \psi_{i,1}^{k^*}\psi_{j,1}^k = \lambda_i\delta_{ij}, \quad \int_{\mathbb{R}^3} \psi_{i,2}^{k^*}\psi_{j,2}^k = (1-\lambda_i)\delta_{ij} \tag{2.1}$$

for $1 \leq i,j \leq Z$, $Z+1 \leq i,j \leq A$ and $0 \leq \lambda_i \leq 1$.

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$\Psi_1^k = (\psi_{1,1}^k, \dots, \psi_{A,1}^k)$ and $\Psi_2^k = (\psi_{1,2}^k, \dots, \psi_{A,2}^k)$ do not necessarily satisfy the constraints of $I(\lambda_1, \dots, \lambda_A)$ and $I(1-\lambda_1, \dots, 1-\lambda_A)$ respectively.
First of all, we show that, for $\mu = p, n$,

$$\Lambda^-_{\mu,\Psi_1^k} \Psi^k_{\mu,1} \underset{k}{\to} 0 \quad \text{et} \quad \Lambda^-_{\mu,\Psi_2^k} \Psi^k_{\mu,2} \underset{k}{\to} 0 \tag{2.2}$$

in $H^{1/2}(\mathbb{R}^3)$.
Second, using the implicit function theorem, we construct $\Phi_1^k = (\Phi^k_{p,1}, \Phi^k_{n,1}), \Phi_2^k = (\Phi^k_{p,2}, \Phi^k_{n,2}) \in \left(H^{1/2}(\mathbb{R}^3)\right)^Z \times \left(H^{1/2}(\mathbb{R}^3)\right)^N$, small perturbations of Ψ_1^k, Ψ_2^k in $\left(H^{1/2}(\mathbb{R}^3)\right)^A$, such that

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and

$$\mathrm{Gram}_{L^2}(\Phi^k_{\mu,i}) = \mathrm{Gram}_{L^2}(\Psi^k_{\mu,i}) \tag{2.4}$$

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in $H^{1/2}(\mathbb{R}^3)$.
Second, using the implicit function theorem, we construct $\Phi_1^k = (\Phi_{p,1}^k, \Phi_{n,1}^k), \Phi_2^k = (\Phi_{p,2}^k, \Phi_{n,2}^k) \in \left(H^{1/2}(\mathbb{R}^3)\right)^Z \times \left(H^{1/2}(\mathbb{R}^3)\right)^N$, small perturbations of Ψ_1^k, Ψ_2^k in $\left(H^{1/2}(\mathbb{R}^3)\right)^A$, such that

$$\Lambda^-_{\mu,\Phi_1^k}\Phi_{\mu,1}^k = 0 \quad \text{et} \quad \Lambda^-_{\mu,\Phi_2^k}\Phi_{\mu,2}^k = 0 \tag{2.3}$$

and

$$\mathrm{Gram}_{L^2}(\Phi_{\mu,i}^k) = \mathrm{Gram}_{L^2}(\Psi_{\mu,i}^k) \tag{2.4}$$

for $\mu = p, n$ and $i = 1, 2$.

Finally, thanks to the continuity of $\mathcal{E}$ in $H^{1/2}(\mathbb{R}^3)$, we obtain

$$\begin{aligned} I &= \lim_{k\to\infty} \mathcal{E}(\Psi^k) \geq \underline{\lim}_{k\to\infty} \mathcal{E}(\Psi_1^k) + \underline{\lim}_{k\to\infty} \mathcal{E}(\Psi_2^k) \\ &= \underline{\lim}_{k\to\infty} \mathcal{E}(\Phi_1^k) + \underline{\lim}_{k\to\infty} \mathcal{E}(\Phi_2^k) \\ &\geq I(\lambda_1, \ldots, \lambda_A) + I(1-\lambda_1, \ldots, 1-\lambda_A) \end{aligned}$$

that clearly contradicts (1.19).

Vanishing does not occur

If vanishing occurs (case ii.), then $\forall R < \infty$

$$\sup_{y\in\mathbb{R}^3} \int_{B(y,R)} \left|\psi_j^k\right|^2 \underset{k}{\rightarrow} 0$$

for $j = 1, \ldots, A$ and $\psi_1^k, \ldots, \psi_A^k$ converge strongly in $L^p(\mathbb{R}^3)$ to 0 for $2 < p < 3$. As a consequence,

$$\lim_{k\to\infty} \mathcal{E}(\Psi^k) = \sum_{j=1}^{A} \lim_{k\to\infty} \int_{\mathbb{R}^3} \psi_j^{k^*} H_0 \psi_j^k,$$

and

$$I\left(\lambda_1, \ldots, \lambda_A\right) = m_b \sum_{j=1}^{A} \lambda_j$$

thanks to the constraints of the problem.

This contradicts (1.19) because we have

$$I = m_b A = m_b \sum_{j=1}^{A} \lambda_j + m_b \sum_{j=1}^{A} (1-\lambda_j) = I\left(\lambda_1, \ldots, \lambda_A\right) + I\left(1-\lambda_1, \ldots, 1-\lambda_A\right).$$

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Let $\tilde{\Psi}^k = \Psi^k(\cdot + y^k)$; $\tilde{\Psi}^k$ is a minimizing sequence.

$$\{\tilde{\Psi}^k\}_k \text{ bounded in } \left(H^{1/2}(\mathbb{R}^3)\right)^A \Rightarrow \begin{cases} \tilde{\Psi}^k \xrightarrow[k]{(H^{1/2})^A} \tilde{\Psi} & \\ \tilde{\Psi}^k \underset{k}{\rightarrow} \tilde{\Psi} & a.e. \\ \tilde{\Psi}^k \xrightarrow[k]{L^p_{loc}} \tilde{\Psi} & 2 \leq p < 3 \end{cases}$$

+ concentration-compactness argument

$$\tilde{\Psi}^k \xrightarrow[k]{L^p} \tilde{\Psi} \quad 2 \leq p < 3$$

Since $\|\tilde{\psi}_j - \tilde{\psi}_j^k\|_{L^2} \to 0$ for $k \to +\infty$,

$$\int_{\mathbb{R}^3} \tilde{\psi}_i^* \tilde{\psi}_j = \lim_{k \to +\infty} \int_{\mathbb{R}^3} \psi_i^{k^*} \psi_j^k = \delta_{ij}$$

for $1 \leq i, j \leq Z$ et $Z + 1 \leq i, j \leq A$. Moreover, $\Lambda^-_{\mu, \tilde{\Psi}} \tilde{\Psi}_\mu = 0$ for $\mu = p, n$ and

$$\mathcal{E}(\tilde{\Psi}) \leq \liminf_{k \to +\infty} \mathcal{E}(\Psi^k) \leq \mathcal{E}(\tilde{\Psi}).$$

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Solutions of the relativistic mean-field equations

$$X = \left\{ \gamma \in \mathcal{B}(\mathcal{H}); \gamma = \gamma^*, (m_b^2 - \Delta)^{1/4} \gamma (m_b^2 - \Delta)^{1/4} \in \sigma_1(\mathcal{H}) \right\}. \quad (3.1)$$

$$\Gamma_P = \left\{ \gamma \in X; \gamma^2 = \gamma, \mathrm{tr}(\gamma) = P \right\}. \quad (3.2)$$

Given $\gamma = (\gamma_p, \gamma_n) \in X \times X$, we define

$$H_{p,\gamma}\gamma_p := \left[H_0 - \beta \frac{g_\sigma^2}{4\pi} \left(\frac{e^{-m_\sigma|\cdot|}}{|\cdot|} \star \rho_s \right) + \frac{g_\omega^2}{4\pi} \left(\frac{e^{-m_\omega|\cdot|}}{|\cdot|} \star \rho_0 \right) + \frac{g_\rho^2}{4\pi} \left(\frac{e^{-m_\rho|\cdot|}}{|\cdot|} \star \rho_{00} \right) + \frac{e^2}{4\pi} \left(\frac{1}{|\cdot|} \star \rho_p \right) \right] \gamma_p \quad (3.3)$$

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where

$$\begin{aligned} \rho_s(x) &= \bar{\rho}_p(x) + \bar{\rho}_n(x) \\ \rho_0(x) &= \rho_p(x) + \rho_n(x) \\ \rho_{00}(x) &= \rho_p(x) - \rho_n(x) \end{aligned}$$

with $\bar{\rho}_p(x) = \mathrm{tr}(\beta\gamma_p(x,x))$, $\bar{\rho}_n(x) = \mathrm{tr}(\beta\gamma_n(x,x))$, $\rho_p(x) = \mathrm{tr}(\gamma_p(x,x))$ et $\rho_n(x) = \mathrm{tr}(\gamma_n(x,x))$.
Finally, for $\mu = p, n$, we define

$$\Lambda^{\pm}_{\mu,\gamma} = \chi_{\mathbb{R}^{\pm}}(H_{\mu,\gamma}).$$

Let $\tilde{\Psi} = (\tilde{\Psi}_p, \tilde{\Psi}_n)$ be a minimizer of the problem (1.17) and consider $\tilde{\gamma}_p$ and $\tilde{\gamma}_n$ the orthogonal projectors defined by

$$\tilde{\gamma}_p = \sum_{i=1}^{Z} |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i| \quad \text{and} \quad \tilde{\gamma}_n = \sum_{i=Z+1}^{A} |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i| \qquad (3.5)$$

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Then, we show that, for $\mu = p, n$,

$$[H_{\mu,\tilde{\gamma}}, \tilde{\gamma}_\mu] = 0. \tag{3.6}$$

In fact, (3.6) implies

$$H_{p,\tilde{\Psi}}\tilde{\psi}_i = \varepsilon_i \tilde{\psi}_i \quad \text{for } 1 \le i \le Z,$$
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First, we remark that $\tilde{\gamma} = (\tilde{\gamma}_p, \tilde{\gamma}_n)$ minimizes the energy

$$\begin{aligned}\mathcal{E}(\gamma_p,\gamma_n) \quad = \quad & \mathrm{tr}(H_0\gamma_p) + \mathrm{tr}(H_0\gamma_n) - \frac{g_\sigma^2}{8\pi}\int\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_s(x)\rho_s(y)}{|x-y|} e^{-m_\sigma|x-y|}\,dxdy \\ & + \frac{g_\omega^2}{8\pi}\int\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_0(x)\rho_0(y)}{|x-y|} e^{-m_\omega|x-y|}\,dxdy \\ & + \frac{g_\rho^2}{8\pi}\int\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_{00}(x)\rho_{00}(y)}{|x-y|} e^{-m_\rho|x-y|}\,dxdy \\ & + \frac{e^2}{8\pi}\int\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_p(x)\rho_p(y)}{|x-y|}\,dxdy \end{aligned} \tag{3.7}$$

on $\Gamma^+_{Z,N} = \Gamma^+_Z \times \Gamma^+_N$

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$\Gamma_{Z,N}^+$ is a subset of

$$\bar{\Gamma}_{Z,N} = \left\{\gamma = (\gamma_p, \gamma_n) \in \Gamma_Z \times \Gamma_N; \left[H_{p,\gamma}^-, \gamma_p\right] = 0, \left[H_{n,\gamma}^-, \gamma_n\right] = 0\right\}$$

and, since $\tilde{\gamma} \in \Gamma_{Z,N}^+$, we have $\left[H_{\mu,\tilde{\gamma}}^-, \tilde{\gamma}_\mu\right] = 0$ for $\mu = p, n$.
Thus, we have to prove that $\left[H_{\mu,\tilde{\gamma}}^+, \tilde{\gamma}_\mu\right] = 0$ for $\mu = p, n \rightarrow$ by contradiction
For $\mu = p, n$, we assume that $\left[H_{\mu,\tilde{\gamma}}^+, \tilde{\gamma}_\mu\right] \neq 0$ and we define

$$\tilde{\gamma}_\mu^\varepsilon = \mathcal{U}_\mu^\varepsilon \tilde{\gamma}_\mu \left(\mathcal{U}_\mu^\varepsilon\right)^{-1} \qquad (3.8)$$

with $\mathcal{U}_\mu^\varepsilon = \exp\left(-\varepsilon\left[H_{\mu,\tilde{\gamma}}^+, \tilde{\gamma}_\mu\right]\right)$.

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$\Gamma^+_{Z,N}$ is a subset of

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$$\bar{\Gamma}_{Z,N} = \left\{\gamma = (\gamma_p, \gamma_n) \in \Gamma_Z \times \Gamma_N; \left[H_{p,\gamma}^-, \gamma_p\right] = 0, \left[H_{n,\gamma}^-, \gamma_n\right] = 0\right\}$$

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Thus, we have to prove that $\left[H_{\mu,\tilde{\gamma}}^+, \tilde{\gamma}_\mu\right] = 0$ for $\mu = p, n$ $\rightarrow$ by contradiction
For $\mu = p, n$, we assume that $\left[H_{\mu,\tilde{\gamma}}^+, \tilde{\gamma}_\mu\right] \neq 0$ and we define

$$\tilde{\gamma}_\mu^\varepsilon = \mathcal{U}_\mu^\varepsilon \tilde{\gamma}_\mu \left(\mathcal{U}_\mu^\varepsilon\right)^{-1} \quad (3.8)$$

with $\mathcal{U}_\mu^\varepsilon = \exp\left(-\varepsilon\left[H_{\mu,\tilde{\gamma}}^+, \tilde{\gamma}_\mu\right]\right)$.

with

$$\Gamma_Z^+ = \left\{\gamma_p \in \Gamma_Z; \gamma_p = \Lambda_{p,\gamma}^+ \gamma_p \Lambda_{p,\gamma}^+\right\},$$
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$\Gamma_{Z,N}^+$ is a subset of

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and, since $\tilde{\gamma} \in \Gamma^+_{Z,N}$, we have $\left[H^-_{\mu,\tilde{\gamma}}, \tilde{\gamma}_\mu\right] = 0$ for $\mu = p, n$.
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As before, we construct $\gamma^\varepsilon = (\gamma_p^\varepsilon, \gamma_n^\varepsilon)$, small perturbation of $\tilde{\gamma}^\varepsilon = (\tilde{\gamma}_p^\varepsilon, \tilde{\gamma}_n^\varepsilon)$ such that $\gamma^\varepsilon \in \Gamma_{Z,N}^+$. Next, we want to prove that

$$\mathcal{E}(\gamma_p^\varepsilon, \gamma_n^\varepsilon) < \mathcal{E}(\tilde{\gamma}_p, \tilde{\gamma}_n).$$

Since $(\gamma_p^\varepsilon, \gamma_n^\varepsilon)$ is a small perturbation of $(\tilde{\gamma}_p, \tilde{\gamma}_n)$, we can write

$$\mathcal{E}(\gamma_p^\varepsilon, \gamma_n^\varepsilon) - \mathcal{E}(\tilde{\gamma}_p, \tilde{\gamma}_n) = \operatorname{tr}\left(H_{p,\tilde{\gamma}}(\gamma_p^\varepsilon - \tilde{\gamma}_p)\right) + \operatorname{tr}\left(H_{n,\tilde{\gamma}}(\gamma_n^\varepsilon - \tilde{\gamma}_n)\right) + o(\varepsilon). \quad (3.9)$$

More precisely,

$$\begin{aligned}\mathcal{E}(\gamma_p^\varepsilon, \gamma_n^\varepsilon) - \mathcal{E}(\tilde{\gamma}_p, \tilde{\gamma}_n) &= \operatorname{tr}\left(H_{p,\tilde{\gamma}}^+ \Lambda_{p,\tilde{\gamma}}^+ (\gamma_p^\varepsilon - \tilde{\gamma}_p) \Lambda_{p,\tilde{\gamma}}^+\right) \\ &+ \operatorname{tr}\left(H_{p,\tilde{\gamma}}^- \Lambda_{p,\tilde{\gamma}}^- (\gamma_p^\varepsilon - \tilde{\gamma}_p) \Lambda_{p,\tilde{\gamma}}^-\right) + \operatorname{tr}\left(H_{n,\tilde{\gamma}}^+ \Lambda_{n,\tilde{\gamma}}^+ (\gamma_n^\varepsilon - \tilde{\gamma}_n) \Lambda_{n,\tilde{\gamma}}^+\right) \\ &+ \operatorname{tr}\left(H_{n,\tilde{\gamma}}^- \Lambda_{n,\tilde{\gamma}}^- (\gamma_n^\varepsilon - \tilde{\gamma}_n) \Lambda_{n,\tilde{\gamma}}^-\right) + o(\varepsilon) \\ &:= T_p^+ + T_p^- + T_n^+ + T_n^- + o(\varepsilon). \quad (3.10)\end{aligned}$$

- $T_\mu^- = o(\varepsilon)$ for $\mu = p, n$
- $T_\mu^+ = \operatorname{tr}\left(H_{\mu,\tilde{\gamma}}^+ \Lambda_{\mu,\tilde{\gamma}}^+ (\tilde{\gamma}_\mu^\varepsilon - \tilde{\gamma}_\mu) \Lambda_{\mu,\tilde{\gamma}}^+\right) + o(\varepsilon)$

As before, we construct $\gamma^\varepsilon = (\gamma_p^\varepsilon, \gamma_n^\varepsilon)$, small perturbation of $\tilde{\gamma}^\varepsilon = (\tilde{\gamma}_p^\varepsilon, \tilde{\gamma}_n^\varepsilon)$ such that $\gamma^\varepsilon \in \Gamma_{Z,N}^+$. Next, we want to prove that

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- $T_\mu^- = o(\varepsilon)$ for $\mu = p, n$
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By definition,

$$\begin{aligned}\tilde{\gamma}_\mu^\varepsilon - \tilde{\gamma}_\mu &= \mathcal{U}_\mu^\varepsilon \tilde{\gamma}_\mu (\mathcal{U}_\mu^\varepsilon)^{-1} - \tilde{\gamma}_\mu \\ &= \left(1 - \varepsilon \left[H^+_{\mu,\tilde{\gamma}}, \tilde{\gamma}_\mu\right]\right) \tilde{\gamma}_\mu \left(1 + \varepsilon \left[H^+_{\mu,\tilde{\gamma}}, \tilde{\gamma}_\mu\right]\right) - \tilde{\gamma}_\mu + o(\varepsilon) \\ &= -\varepsilon \left[\left[H^+_{\mu,\tilde{\gamma}}, \tilde{\gamma}_\mu\right], \tilde{\gamma}_\mu\right] + o(\varepsilon).\end{aligned}$$

Then

$$T^+_\mu = -\varepsilon \operatorname{tr}\left(H^+_{\mu,\tilde{\gamma}} \left[\left[H^+_{\mu,\tilde{\gamma}}, \tilde{\gamma}_\mu\right], \tilde{\gamma}_\mu\right]\right) + o(\varepsilon)$$

for $\mu = p, n$ and

$$\begin{aligned}\mathcal{E}(\gamma_p^\varepsilon, \gamma_n^\varepsilon) - \mathcal{E}(\tilde{\gamma}_p, \tilde{\gamma}_n) &= -\varepsilon \sum_{\mu=p,n} \operatorname{tr}\left(H^+_{\mu,\tilde{\gamma}} \left[\left[H^+_{\mu,\tilde{\gamma}}, \tilde{\gamma}_\mu\right], \tilde{\gamma}_\mu\right]\right) + o(\varepsilon) \\ &= 2\varepsilon \sum_{\mu=p,n} \operatorname{tr}\left((H^+_{\mu,\tilde{\gamma}} \tilde{\gamma}_\mu)^2 - (H^+_{\mu,\tilde{\gamma}})^2 \tilde{\gamma}_\mu^2\right) + o(\varepsilon) \\ &= 2\varepsilon \sum_{\mu=p,n} \langle (H^+_{\mu,\tilde{\gamma}} \tilde{\gamma}_\mu)^*, H^+_{\mu,\tilde{\gamma}} \tilde{\gamma}_\mu \rangle - \langle H^+_{\mu,\tilde{\gamma}} \tilde{\gamma}_\mu, H^+_{\mu,\tilde{\gamma}} \tilde{\gamma}_\mu \rangle + o(\varepsilon) \quad (3.11)\end{aligned}$$

where $\langle A, B \rangle = \operatorname{tr}(A^* B)$ is the Hilbert–Schmidt inner product.

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&= 2\varepsilon \sum_{\mu=p,n} \mathrm{tr}\left((H^+_{\mu,\tilde{\gamma}} \tilde{\gamma}_\mu)^2 - (H^+_{\mu,\tilde{\gamma}})^2 \tilde{\gamma}_\mu^2\right) + o(\varepsilon) \\
&= 2\varepsilon \sum_{\mu=p,n} \langle (H^+_{\mu,\tilde{\gamma}} \tilde{\gamma}_\mu)^*, H^+_{\mu,\tilde{\gamma}} \tilde{\gamma}_\mu \rangle - \langle H^+_{\mu,\tilde{\gamma}} \tilde{\gamma}_\mu, H^+_{\mu,\tilde{\gamma}} \tilde{\gamma}_\mu \rangle + o(\varepsilon) \qquad (3.11)
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Thanks to the Cauchy-Schwarz inequality, we obtain

$$\mathcal{E}(\gamma_p^\varepsilon, \gamma_n^\varepsilon) - \mathcal{E}(\tilde{\gamma}_p, \tilde{\gamma}_n) \leq 0\,;$$

the equality holds $\Leftrightarrow (H^+_{\mu,\tilde{\gamma}}\tilde{\gamma}_\mu)^* = \pm H^+_{\mu,\tilde{\gamma}}\tilde{\gamma}_\mu$.

$(H^+_{\mu,\tilde{\gamma}}\tilde{\gamma}_\mu)^* = \pm H^+_{\mu,\tilde{\gamma}}\tilde{\gamma}_\mu \Leftrightarrow \left[H^+_{\mu,\tilde{\gamma}}, \tilde{\gamma}_\mu\right] = 0$; then, if $\left[H^+_{\mu,\tilde{\gamma}}, \tilde{\gamma}_\mu\right] \neq 0$ for $\mu = p, n$, there exists $\gamma^\varepsilon \in \Gamma^+_{Z,N}$ such that

$$\mathcal{E}(\gamma_p^\varepsilon, \gamma_n^\varepsilon) - \mathcal{E}(\tilde{\gamma}_p, \tilde{\gamma}_n) < 0,$$

$\rightarrow$ contradiction : $\tilde{\gamma}$ minimizes the energy on $\Gamma^+_{Z,N}$.

As a conclusion, $[H_{\mu,\tilde{\gamma}}, \tilde{\gamma}_\mu] = 0$ for $\mu = p, n$ and $\tilde{\Psi}$ is a solution of the equations (1.14) and (1.15).

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As a conclusion, $[H_{\mu,\tilde{\gamma}}, \tilde{\gamma}_\mu] = 0$ for $\mu = p, n$ and $\tilde{\Psi}$ is a solution of the equations (1.14) and (1.15).

The nonrelativistic limit

By a change of physical units, we introduce the speed of light c and we obtain

$$\left[-ic\alpha\nabla + \beta(m_b c^2 + S) + V\right]\psi_j = (m_b c^2 - \mu_j)\psi_j \tag{4.1}$$

$$\left[-\Delta + m_\sigma^2 c^2\right] S = -g_\sigma^2 c \rho_s \tag{4.2}$$

$$\left[-\Delta + m_\omega^2 c^2\right] V = g_\omega^2 c \rho_0 \tag{4.3}$$

with $\mu_j \geq 0$. Writing $\psi_j = \begin{pmatrix} \varphi_j \\ \chi_j \end{pmatrix}$, the densities are given by

$$\rho_s = \sum_{j=1}^{A} \left(|\varphi_j|^2 - |\chi_j|^2\right) \quad \rho_0 = \sum_{j=1}^{A} \left(|\varphi_j|^2 + |\chi_j|^2\right) \tag{4.4}$$

and the equation (4.1) becomes

$$\begin{cases} -ic\sigma\nabla\chi_j + (S+V)\varphi_j = -\mu_j\varphi_j \\ -ic\sigma\nabla\varphi_j - (2m_b c^2 + S - V - \mu_j)\chi_j = 0 \end{cases} \tag{4.5}$$

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$$\begin{aligned} \left[-ic\alpha\nabla + \beta(m_b c^2 + S) + V\right]\psi_j &= (m_b c^2 - \mu_j)\psi_j &(4.1)\\ \left[-\Delta + m_\sigma^2 c^2\right] S &= -g_\sigma^2 c \rho_s &(4.2)\\ \left[-\Delta + m_\omega^2 c^2\right] V &= g_\omega^2 c \rho_0 &(4.3)\end{aligned}$$

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From the system (4.5), we obtain

$$\chi_j = \frac{-ic\sigma\nabla\varphi_j}{2m_b c^2 + S - V - \mu_j}. \tag{4.6}$$

For $c \to \infty$, we can write

$$S = -\frac{1}{c}\left(\frac{g_\sigma}{m_\sigma}\right)^2 \left[\frac{-\Delta}{m_\sigma^2 c^2} + 1\right]^{-1} \rho_s = -\frac{1}{c}\left(\frac{g_\sigma}{m_\sigma}\right)^2 \rho_s + O\left(\frac{1}{c^3}\right) \tag{4.7}$$

$$V = \frac{1}{c}\left(\frac{g_\omega}{m_\omega}\right)^2 \left[\frac{-\Delta}{m_\omega^2 c^2} + 1\right]^{-1} \rho_0 = \frac{1}{c}\left(\frac{g_\omega}{m_\omega}\right)^2 \rho_0 + O\left(\frac{1}{c^3}\right) \tag{4.8}$$

and, in accord with the physical values of the meson masses and of the coupling constants (see [4],[5]), we can suppose

$$\left(\frac{g_\sigma}{m_\sigma}\right)^2 = \left(\frac{g_\omega}{m_\omega}\right)^2 + ac \tag{4.9}$$

$$\frac{1}{c}\left(\frac{g_\sigma}{m_\sigma}\right)^2 = \vartheta m_b c^2 \tag{4.10}$$

with $a > 0$ small and $\theta > 0$.

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with $a > 0$ small and $\theta > 0$.

As a consequence,

$$S + V = 2\vartheta m_b c^2 \sum_{j=1}^{A} |\chi_j|^2 - a\rho_0 + O\left(\frac{1}{c^3}\right), \tag{4.11}$$

$$S - V = -2\vartheta m_b c^2 \sum_{j=1}^{A} |\varphi_j|^2 + a\rho_0 + O\left(\frac{1}{c^3}\right). \tag{4.12}$$

As a conclusion,

$$\chi_j = \frac{-i\sigma\nabla\varphi_j}{2m_b c\left(1 - \vartheta \sum_{j=1}^{A} |\varphi_j|^2\right)} + O\left(\frac{1}{c^2}\right). \tag{4.13}$$

As a consequence,

$$S + V = 2\vartheta m_b c^2 \sum_{j=1}^{A} |\chi_j|^2 - a\rho_0 + O\left(\frac{1}{c^3}\right), \tag{4.11}$$

$$S - V = -2\vartheta m_b c^2 \sum_{j=1}^{A} |\varphi_j|^2 + a\rho_0 + O\left(\frac{1}{c^3}\right). \tag{4.12}$$

As a conclusion,

$$\chi_j = \frac{-i\sigma\nabla\varphi_j}{2m_b c\left(1 - \vartheta \sum_{j=1}^{A} |\varphi_j|^2\right)} + O\left(\frac{1}{c^2}\right). \tag{4.13}$$

Then, we obtain

$$-\frac{1}{2m_b}\sigma\nabla\left(F(\Phi)\sigma\nabla\varphi_k\right)+\frac{\vartheta}{2m_b}F(\Phi)^2\sum_{j=1}^{A}\left|\sigma\nabla\varphi_j\right|^2\varphi_k-a\rho_\Phi\varphi_k=-\mu_k\varphi_k \quad (4.14)$$

with $F(\Phi)=\frac{1}{(1-\vartheta\rho_\Phi)}$ and $\rho_\Phi=\sum_{j=1}^{A}|\varphi_j|^2$. Using the formula

$$\sigma_k\sigma_l=\delta_{kl}\mathbb{1}+i\varepsilon_{klm}\sigma_m$$

where ε_{klm} is the Levi-Civita symbol and δ_{kl} is the Kronecker delta, we get

$$\begin{aligned}-\sigma\nabla\left(F(\Phi)\sigma\nabla\right) &= -\nabla\cdot\left(F(\Phi)\nabla\right)-i\sigma\cdot\left(\nabla F(\Phi)\times\nabla\right)\\ &= \mathbf{p}\cdot\left(F(\Phi)\mathbf{p}\right)+\underbrace{\nabla F(\Phi)\cdot(\mathbf{p}\times\sigma)}_{\text{spin-orbit term}}.\end{aligned}$$

The equation (4.14) can be seen as the Euler-Lagrange equations of the energy functional

$$J(\Phi)=\frac{1}{2m_b}\sum_{i=1}^{A}\int_{\mathbb{R}^3}\frac{|\sigma\nabla\varphi_i|^2}{(1-\vartheta\rho_\Phi)_+}-\frac{a}{2}\int_{\mathbb{R}^3}\rho_\Phi^2. \quad (4.15)$$

Then, we obtain

$$-\frac{1}{2m_b}\sigma\nabla\left(F(\Phi)\sigma\nabla\varphi_k\right)+\frac{\vartheta}{2m_b}F(\Phi)^2\sum_{j=1}^{A}|\sigma\nabla\varphi_j|^2\varphi_k-a\rho_\Phi\varphi_k=-\mu_k\varphi_k \quad (4.14)$$

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with $F(\Phi)=\frac{1}{(1-\vartheta\rho_\Phi)}$ and $\rho_\Phi=\sum_{j=1}^{A}|\varphi_j|^2$. Using the formula

$$\sigma_k\sigma_l=\delta_{kl}\mathbb{1}+i\varepsilon_{klm}\sigma_m$$

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