Adiabatic evolution of quantum observables driven by 1D shape resonances

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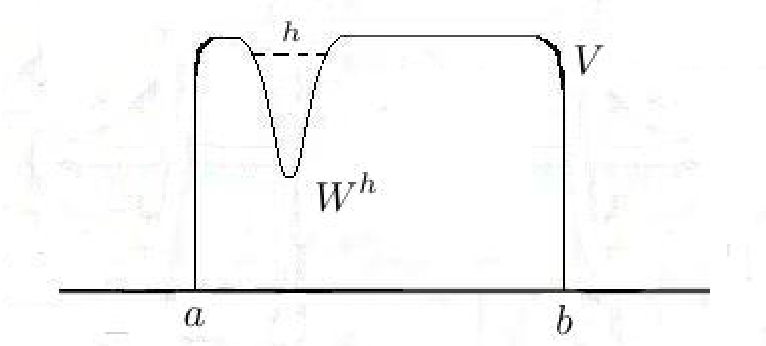
(joint work with: F.Nier, A. Faraj)

Motivations

$$H^h_{ref} = -h^2\Delta + V - W^h$$



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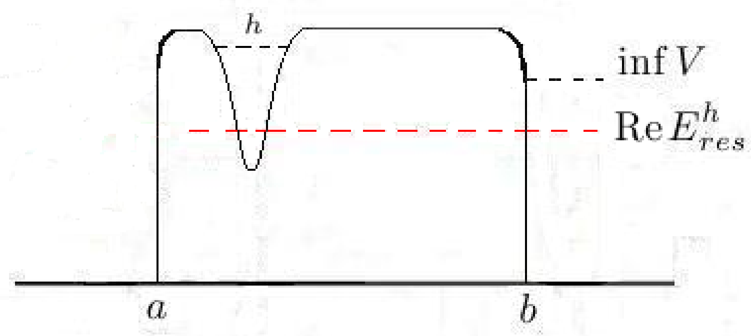

Resonances

$$E^h_{res} = E_R + \mathcal{O}\left(e^{-\frac{c}{h}}\right) , \quad \#\left\{E^h_{res}\right\} < C$$

$$\psi_{q,E^h_{res}}(t) = e^{-t\left|\mathrm{Im}\, E^h_{res}\right|} e^{-itE_R}\psi_{q,E^h_{res}}(0) + R(t)$$

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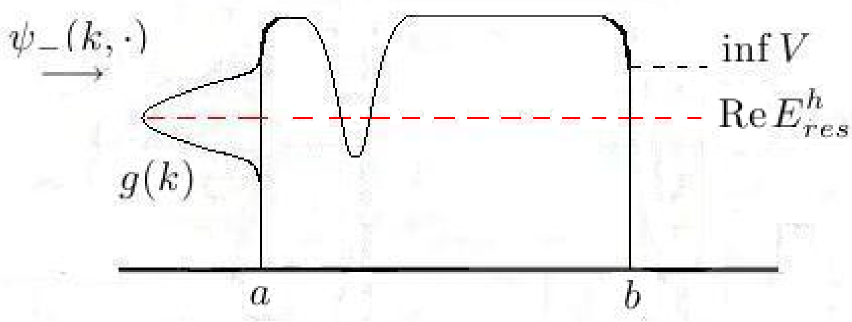

Transport problem

$$\begin{cases} H^h = -h^2\Delta + V - W^h + V^h_{NL} \\ -\Delta^D_{(a,b)} V^h_{NL} = \rho^h \\ \rho^h(x) = \int \frac{dk}{2\pi h} g(k)\ |\psi_-(k,x)|^2 \end{cases},$$

non-equilibrium condition

$$\text{supp } g = \left\{k > 0,\ k^2 \sim \text{Re } E^h_{res}\right\}$$

$h \to 0$, *stationary case (Nier, Bonaillie-Noël, Patel 08,09)*

"Non-linear phenomena are governed by a finite number of resonant states"

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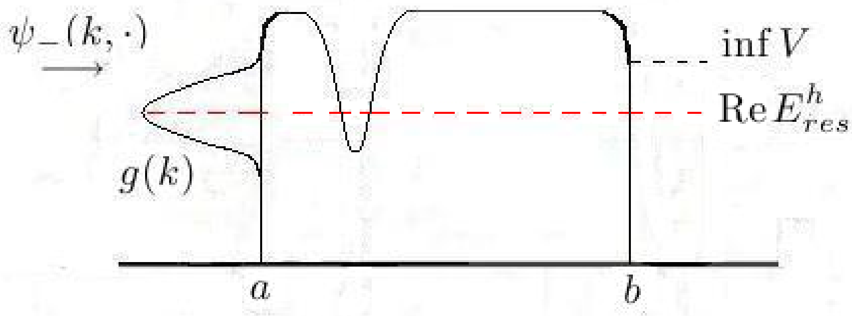

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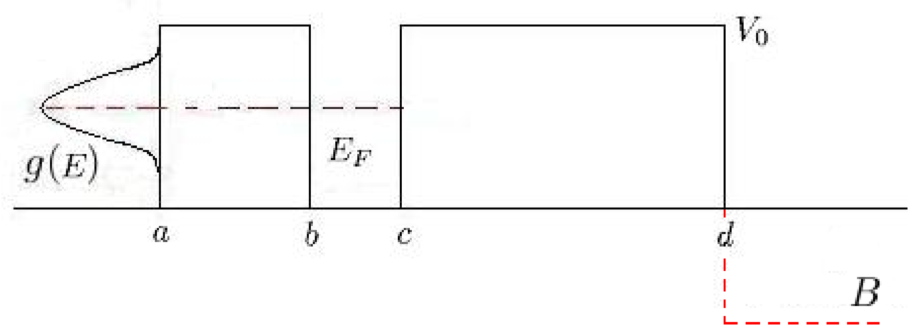
Our guess: in the non-stationary case V^h_{NL} is adiabatic perturbation

Motivations

Rappel: G. Jona-Lasinio,C.Presilla,J.Sjöstrand (95), C.Presilla,J.Sjöstrand (96,97)

$$\begin{cases} (i\partial_t + \Delta - \mathcal{V})\,\phi(\cdot, t, E) = 0 \\ \mathcal{V} = 1_{(a,b)\cup(c,d)} V_0 - B 1_{(d,-\infty)} + W(s,\cdot) \end{cases}$$



$$W(s,x) = \frac{8\pi}{a_B} s(\phi) 1_{(a,d)} \begin{cases} \frac{(x-a)(d-c)}{(b-a)+(d-c)} & x \in (a,b) \\ \frac{(b-a)(d-c)}{(b-a)+(d-c)} & x \in (b,c) \\ \frac{(b-a)(d-x)}{(b-a)+(d-c)} & x \in (c,d) \end{cases}$$

$$\boxed{s(\phi) = \int dE\, g(E) \int_{\frac{a+b}{2}}^{\frac{c+d}{2}} dx\ |\phi(x,t,E)|^2}$$

(charge sheet density)

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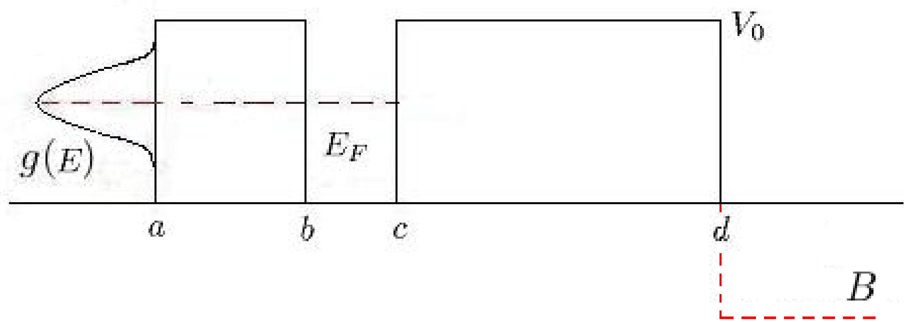

approximations

$|\partial_t s| << 1$ (*adiabatic evolution*)

$(b-a) + (d-c) \to \infty$ (*WKB*)

$$\Rightarrow \begin{cases} \partial_t s(t) = -2\Gamma(s(t))\,[s(t) - f(s(t))] \\ f = 2\pi\, g(E_R(s))\,\Gamma^{-1}(s)\,V_0^2 \left|\left\langle \mu, \psi_{q,E_{res}} \right\rangle\right|^2 \end{cases}$$

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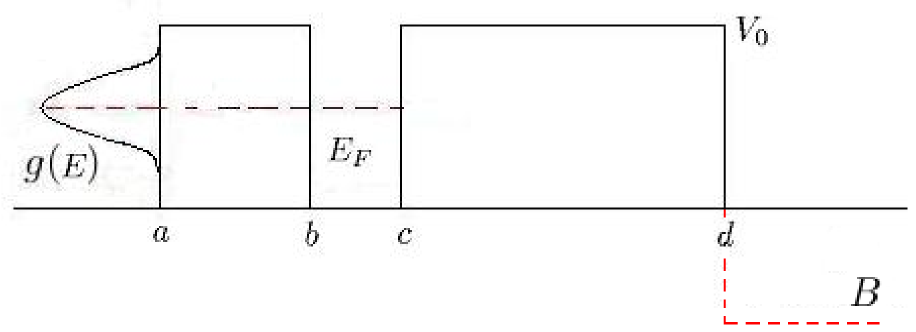

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open questions

- Adiabatic (non-linear) evolution of resonances: $\psi_{q,E_{res}(s(0))}(t) \sim z(t)\,\psi_{q,E_{res}(s(t))}$
- Rôle of the geometry in the deduction of the reduced model

Adiabatic evolution of resonances: the linear case

$$H^h(t) = -h^2\Delta + V(\varepsilon t) - W^h(\varepsilon t)$$

...differents strategies are possibles

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...*differents strategies are possibles*

i) (G. Perelman 2000) evolution of quasi-resonant states

$$\psi_{q,E_{res}(0)}(t) = \mu(t)\,\psi_{q,E_{res}(t)} + R(t); \qquad R \sim \mathcal{O}(\varepsilon) \quad \text{pour: } t \lesssim \frac{1}{|\mathrm{Im}\,E_{res}|} + \varepsilon^{-\nu},\ \nu \in (0,1)$$

· many time scales

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· many time scales

ii) Complex deformations (Main problem: lack of estimates uniform in time for the semigroup)
(A. Joye, 07 $\longrightarrow$ adiabatic theorem for dynamical systems without uniform time estimates)

· analyticity assumptions

A modified model

For $\theta_0 \in \mathbb{C}$: $\quad H^h_{\theta_0} = -h^2 \Delta_{\theta_0} + V - W^h, \qquad W^h = \underbrace{W^h_1}_{L^\infty} + \underbrace{W^h_2}_{\mathcal{M}_b}, \qquad supp W^h_{i=1,2} \subset (a,b)$

$$\begin{cases} D(\Delta_{\theta_0}) = H^2(\mathbb{R} \setminus \{a,b\}), & \left[\begin{array}{l} e^{-\frac{\theta_0}{2}} u(b^+) = u(b^-);\ e^{-\frac{3}{2}\theta_0} u'(b^+) = u'(b^-) \\ e^{-\frac{\theta_0}{2}} u(a^-) = u(a^+);\ e^{-\frac{3}{2}\theta_0} u'(a^-) = u'(a^+) \end{array} \right. \\ \Delta_{\theta_0} u = \partial_x^2 u, \quad \mathbb{R} \setminus \{a,b\} \end{cases}$$

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Assumptions

$[A_1]$ $\qquad c 1_{[a,b]} \leq V\,, \quad \|V\|_{L^\infty} \leq \frac{1}{c}, \quad \|W^h_1\|_{L^\infty} \leq \frac{1}{c}\,, \qquad \|W^h_2\|_{\mathcal{M}_b} \leq \frac{1}{c} h$

$[A_2]$ $\qquad H^h_D = -h^2 \Delta^D_{(a,b)} + V - W^h$

$$\sigma\left(H^h_D\right) \cap (0, \inf V) = \left\{\lambda^h_j\right\}_{j \leq l}\,, \qquad d\left(\left\{\lambda^h_j\right\}, \sigma\left(H^h_D\right) \setminus \left\{\lambda^h_j\right\}\right) > c$$

$$\lim_{h\to 0} \lambda^h_j = \lambda^0 \in (c, \inf V - c)\ ; \quad \left|\lambda^h_j - \lambda^0\right| < \frac{h}{c}$$

A modified model

the artificial interface conditions

- introduce small perturbations controlled by $|\theta_0|$

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Lemma Let

$$W_{\theta_0} : W_{\theta_0}(x,y) = \int \frac{dk}{2\pi h} \psi_-(k,x) e^{-i\frac{k}{h}y}, \qquad \left(\Delta_{\theta_0} + k^2\right) \psi_-(k,x) = 0$$

We have: $\quad \Delta_{\theta_0} W_{\theta_0} = W_{\theta_0} \Delta\,,$

$$W_{\theta_0} = Id + \mathcal{O}(\theta_0) \Longrightarrow \boxed{e^{it\Delta_{\theta_0}} = e^{it\Delta} + \mathcal{O}(\theta_0)}$$

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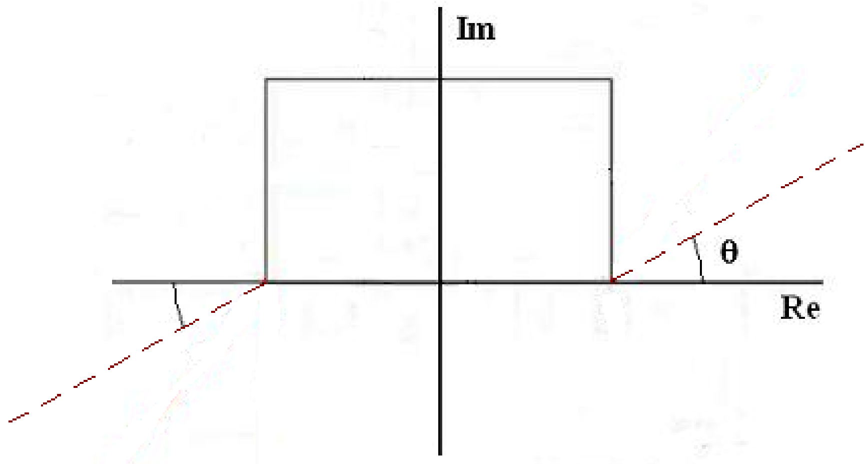
Complex deformations method
(Aguilar, Balsev, Combes, Simon)



Deformed operator $$H^h_{\theta_0}(\theta) = -h^2 e^{-2\theta\, 1_{\mathbb{R}\setminus(a,b)}} \Delta_{\theta_0+\theta} + V - W^h$$

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A Krein-like resolvent's formula

$$\left(H_{\theta_0}^h(\theta)-z\right)^{-1}-\left(H_{ND}^h(\theta)-z\right)^{-1}=\sum_{i,j=1}^{4} M_{ij}^{-1}(z,\theta,1_{(a,b)}u_{i,z})\left\langle\overline{\gamma_j\, u_{j,z}},\cdot\right\rangle_{L^2(\mathbb{R})}\gamma_i\, u_{i,z}$$

$$H_{ND}^h(\theta)=-h^2e^{-2\theta}\Delta_{\mathbb{R}\backslash(a,b)}^N\oplus\left[-h^2\Delta_{(a,b)}^D+V-W^h\right]$$

$$\left(-h^2e^{-2\theta\,1_{\mathbb{R}\backslash(a,b)}}\partial_x^2+V-W^h-z\right)u_{i,z}=0,\qquad u_{i,z}\in H^2\left(\mathbb{R}\backslash\left\{a,b\right\}\right)$$

for $\mathrm{Re}\,z<0$ $\Rightarrow$ $\boxed{M_{ij}(z,\theta,1_{(a,b)}u_{i,z})=M_{ij}(z,1_{(a,b)}u_{i,z})}$

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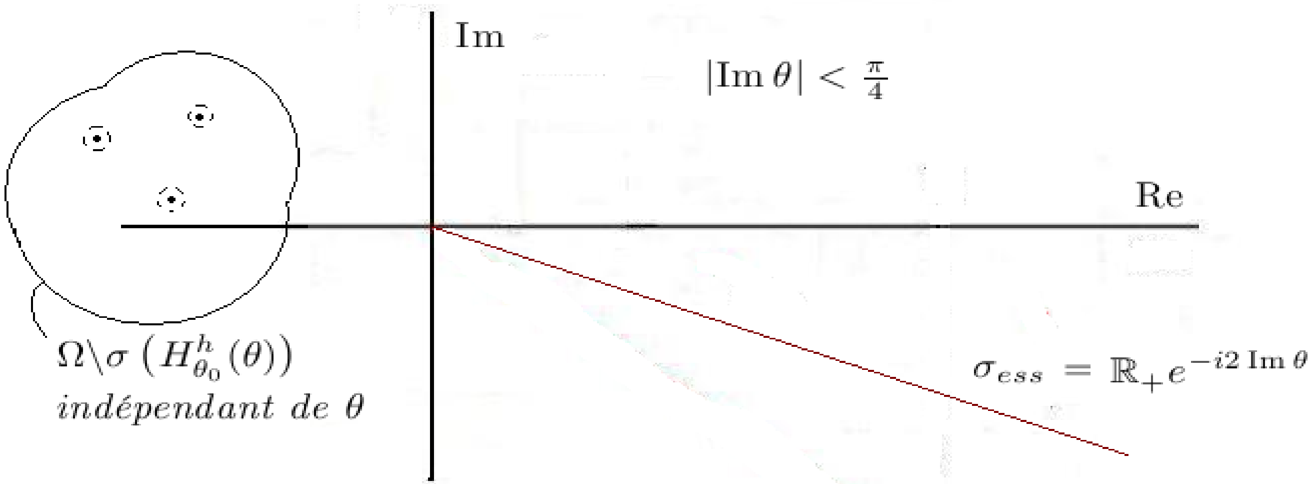

For: $\Omega \subset \{\mathrm{Re}\, z < 0\}$, $z \in \Omega\backslash\sigma\left(H^h_{\theta_0}(\theta)\right)$

$$\theta \to \left(H^h_{\theta_0}(\theta) - z\right)^{-1}$$ *is* holomorphic *in the strip:* $|\mathrm{Im}\,\theta| < \dfrac{\pi}{4}$

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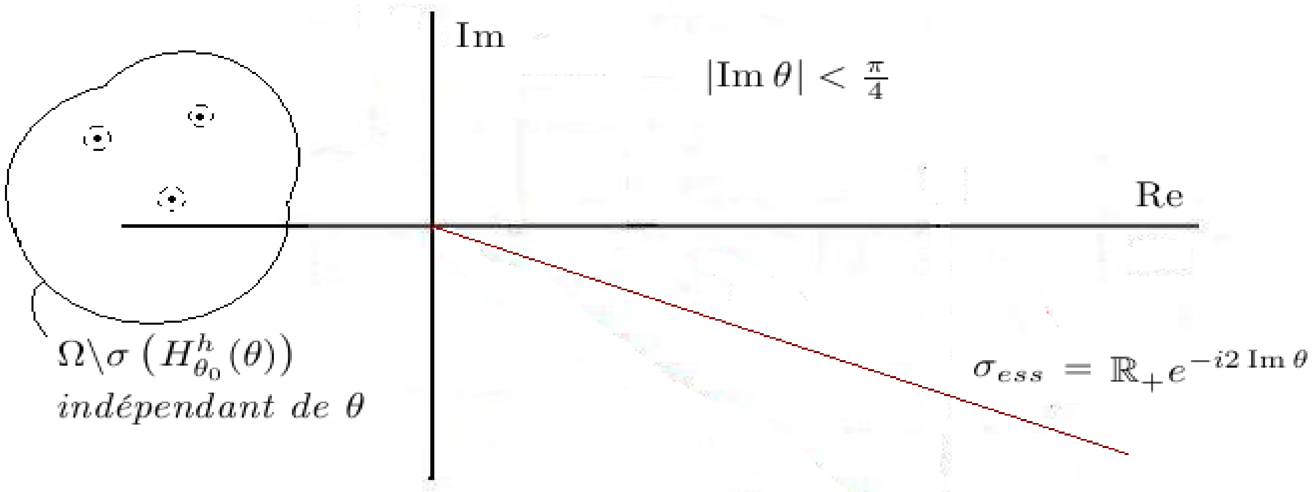

Proposition *(Faraj, M., Nier 2010).* For $\mathrm{Im}\,\theta > 0$, the resonances of $H^h_{\theta_0}$ coincide with the eigenvalues of $H^h_{\theta_0}(\theta)$ in $\{\arg z \in (-2\,\mathrm{Im}\,\theta, 0)\}$

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Lemma Let $\theta = \theta_0 = i\tau$, $\tau > 0$. Then: $iH^h_{\theta_0}(\theta_0)$ is maximal accretive.

Proof. $-i\mathbb{R}_+ \in res\left(H^h_{\theta_0}(\theta_0)\right)$ and

$$\mathrm{Re}\left\langle u, iH^h_{\theta_0}(\theta_0)u\right\rangle_{L^2} = h^2 \sin 2\tau \int_{\mathbb{R}\setminus(a,b)} \left|u'\right|^2 > 0\,, \qquad u \in D\left(H^h_{\theta_0}(\theta_0)\right)$$

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...*work plan:*

1. Localisation of resonances (controlling the error with θ_0)

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2. Adiabatic evolution

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2. Adiabatic evolution

3. A model of quantum transport

Localizing resonances

in: $\omega_{ch} = \left\{ z : d\left(z, \left\{\lambda_j^h\right\}\right) < ch \right\}$

$\left(H_{\theta_0}^h(\theta) - z\right) u = 0$

$u \in L^2(\mathbb{R}), \quad \mathrm{Im}\,\theta > |\arg z|$

$$\longleftrightarrow \quad \begin{cases} \left(-h^2\partial_x^2 + V - W^h - z\right) u = 0, & x \in (a,b) \\ \left(h\partial_x + iz^{\frac{1}{2}}e^{-\theta_0}\right) u(a^+) = 0 \\ \left(h\partial_x - iz^{\frac{1}{2}}e^{-\theta_0}\right) u(b^-) = 0 \end{cases}$$

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Proposition *(Faraj, M., Nier, 2010).* Under the assumptions $[A_1]$, $[A_2]$, $|\theta_0| < \frac{c^2 h}{8}$: There exist l solutions $\left\{z_j^h(\theta_0)\right\}_{j \le l}$ to the problem: $\left(H_{\theta_0}^h(\theta) - z\right) u = 0$ in $\omega_{\frac{ch}{2}}$:

$$\cdot \left|z_j^h(\theta_0) - \lambda_j^h\right| = \mathcal{O}\left(h^{-3} e^{-\frac{2S_0}{h}}\right), \qquad S_0 = d_{Ag}\left(\mathrm{supp} W^h, \{a, b\}, \lambda^0\right)$$

Moreover, if: i) $e^{-\frac{S_0}{4h}} < |\theta_0| < \frac{c^2 h}{8}$; ii) $\lim_{h \to 0} h^3 e^{\frac{2S_0}{h}} \left|\lambda_j^h - \lambda^0\right| = +\infty \Rightarrow$

$$\cdot \left|z_j^h(\theta_0) - z_j^h(0)\right| = \mathcal{O}\left(\theta_0 h^{-3} e^{-\frac{2S_0}{h}}\right)$$

Adiabatic theory

Assumption $[A_3]$:

i) $\theta = \theta_0 = ih^{N_0}\,,\ N_0 \geq 4$;

ii) $V(t), W^h(t), \lambda^0(t) \in \mathcal{C}^K(0,T)\,,\ K \geq 2$:

$$W_1^h \in \sum_{n \leq N} w_n \left(\frac{x - x_n}{h}, t \right)\,, \quad W_2^h = \sum_{m \leq N} h\alpha_m(t)\delta\left(x - y_m\right)$$

iii) $[A_1]\,,[A_2]$ *unif* $\diagup$ *t*.

Adiabatic theory

an evolution problem: $u_s \in L^2(\mathbb{R})$, $\quad \varepsilon = e^{-\frac{\tau}{h}}$, $\tau > 0$

$$\begin{cases} i\varepsilon\partial_t u_t = H^h_{\theta_0}(\theta_0, t)u_t \\ u_s = P_0(s)u_s \,, \qquad P_0(t) = \textit{spectral projector over } l.c. \left\{ z^h_j(\theta_0, t) \right\}_{j \le l} \end{cases}$$

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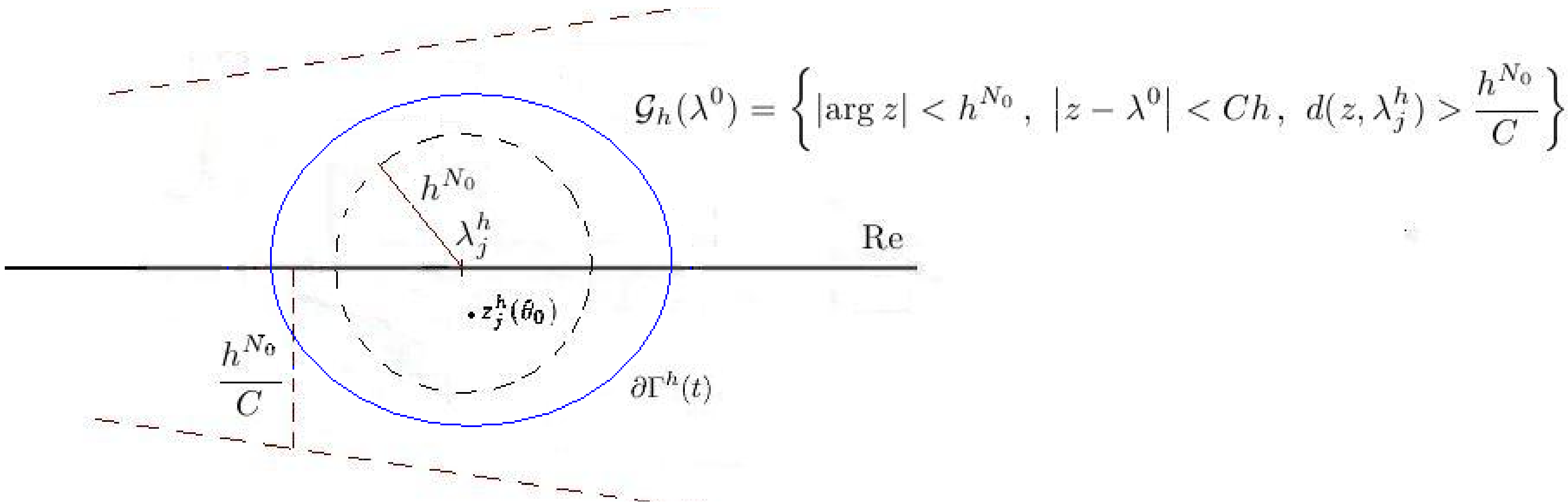

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Theorem *(Faraj, M., Nier, 2010).* In the assumptions $[A_3]$; let $\delta \in (0,1)$, $\phi_0(s,t)$ be the parallel transport of $P_0(t)$ and v_t such that

$$\begin{cases} i\varepsilon\partial_t v_t = \phi_0(s,t)P_0(t)H^h_{\theta_0}(\theta_0, t)P_0(t)\phi_0(t,s)v_t \\ v_s = u_s \end{cases} ; \quad \begin{cases} \partial_t\phi_0 = [\partial_t P_0(s), P_0(t)]\,\phi_0 \\ \phi_0(t = s, s) = Id \end{cases}$$

Then,

$$\max_{t\in[sT]} \|u_t - \phi_0(t,s)v_t\|_{L^2(\mathbb{R})} \le C_{a,b,c,\delta,T}\varepsilon^{1-\delta} \|u_s\|_{L^2(\mathbb{R})}$$

Évolution adiabatique d'une couche de charge

Hypothéses

$$V_0 > 0, \quad c \in (a,b)\,,$$

$$H^h_{\theta_0}(t) = -h^2\Delta_{\theta_0} + 1_{(a,b)}V_0 + h\alpha(t)\delta_c\,,$$

$$\alpha \in C^2\left(0,T;\left(-2V_0^{\frac{1}{2}},0\right)\right)$$

$$\text{supp } g(k) = \left\{k>0,\ \left|k^2-\lambda_0\right| < \frac{h}{d}\right\}$$

$$\rho_0^h = \int \frac{dk}{2\pi h} g(k)\ |\psi_-(k,\cdot,\alpha_0)\rangle\,\langle\psi_-(k,\cdot,\alpha_0)|\,,$$

$$\left(H^h_{\theta_0}(0) - k^2\right)\psi_-(k,\cdot,\alpha_0)$$

$$A_{\theta_0}(t) = Tr\left[\chi\,\rho_t^h\right]\,,$$

$$\text{supp } \chi = (c-2\eta, c+2\eta)$$

Adiabatic evolution of a charge's density sheet

Assumption $[A_4]$:

· $h \in (0, h_0)$, h_0 small; $\theta_0 = ih^{N_0}$, $N_0 > 2$.

· $\alpha \in C^\infty\left(0, T; \left(-2V_0^{\frac{1}{2}}, 0\right)\right)$, $|\alpha_t - \alpha_s| \leq d_0\, h$, such that: $\left\{\partial_t^j \alpha(t)\right\}_{j \leq J} \neq 0$.

· $\lambda_t = V_0 - \frac{\alpha_t^2}{4}$, and $d_0 > 0$ such that $\forall t$:

$$\lambda_t^{\frac{1}{2}} \in \text{supp}\ g = \left\{k > 0,\ \left|k^2 - \lambda_0\right| < 2\frac{h}{d_0}\right\}.$$

· $g(E^{\frac{1}{2}})$ is holomorphic in a complex neighbourhood of λ_0, of size $\frac{h}{d_0}$.

· $\varepsilon = e^{-\frac{|\alpha_0|}{h} d(c, \{a, b\})}$

Adiabatic evolution of a charge's density sheet

Theorem *(Faraj, M., Nier, 2010).* In the assumption $[A_4]$, we have

1. There exists an unique resonance of $H^h_{\theta_0,\alpha(t)}$, $E(t)$, with: $\mathrm{Re}\, E^{\frac{1}{2}}(t) \in (0, V_0)$. With the notation: $E(t) = E_R(t) - i\Gamma_t$,

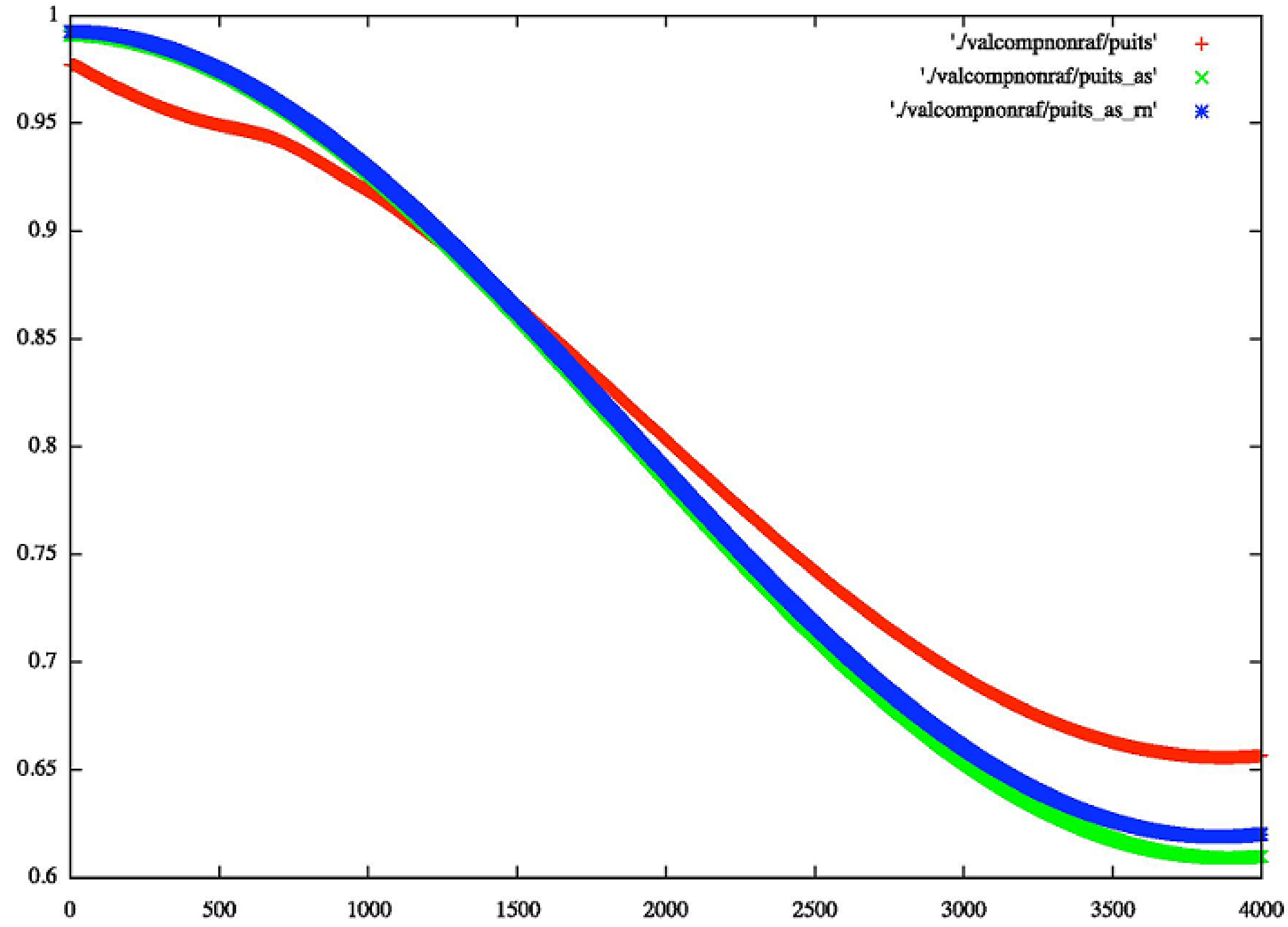
$$E_R(t) = \lambda_t + \mathcal{O}\left(e^{-\frac{|\alpha_t|}{h} d(c,\{a,b\})}\right) \;;\quad \Gamma_t = \mathcal{O}\left(e^{-\frac{|\alpha_t|}{h} d(c,\{a,b\})}\right).$$

The corresponding resonant state is $G(t)$: $\left(H^h_{\theta_0,0} - E(t)\right) G = \delta_c\,,\quad$ in $L^2(a,b)$

2. For $d(c,\{a,b\}) = c - a$, $A_{\theta_0}(t) = a(t) + \mathcal{J}(t) + \mathcal{O}\left(|\theta_0|\right)$

$$\begin{cases} \partial_t a(t) = \left(-2\frac{\Gamma_t}{\varepsilon}\right)\left(a(t) - \left|\frac{\alpha_t}{\alpha_0}\right|^3 g\left(\lambda_t^{\frac{1}{2}}\right)\right) \\ a(0) = g\left(\lambda_0^{\frac{1}{2}}\right) \end{cases}, \quad \mathcal{J}(t) = \left|1 - \left|\frac{\alpha_t}{\alpha_0}\right|^{\frac{3}{2}}\right|^2 g\left(\lambda_t^{\frac{1}{2}}\right) = \mathcal{O}\left(h^2\right)$$

For $d(c,\{a,b\}) = b - c$, there exists $\beta > 0$ such that: $A_{\theta_0}(t) = \mathcal{O}\left(e^{-\frac{\beta}{h}}\right)$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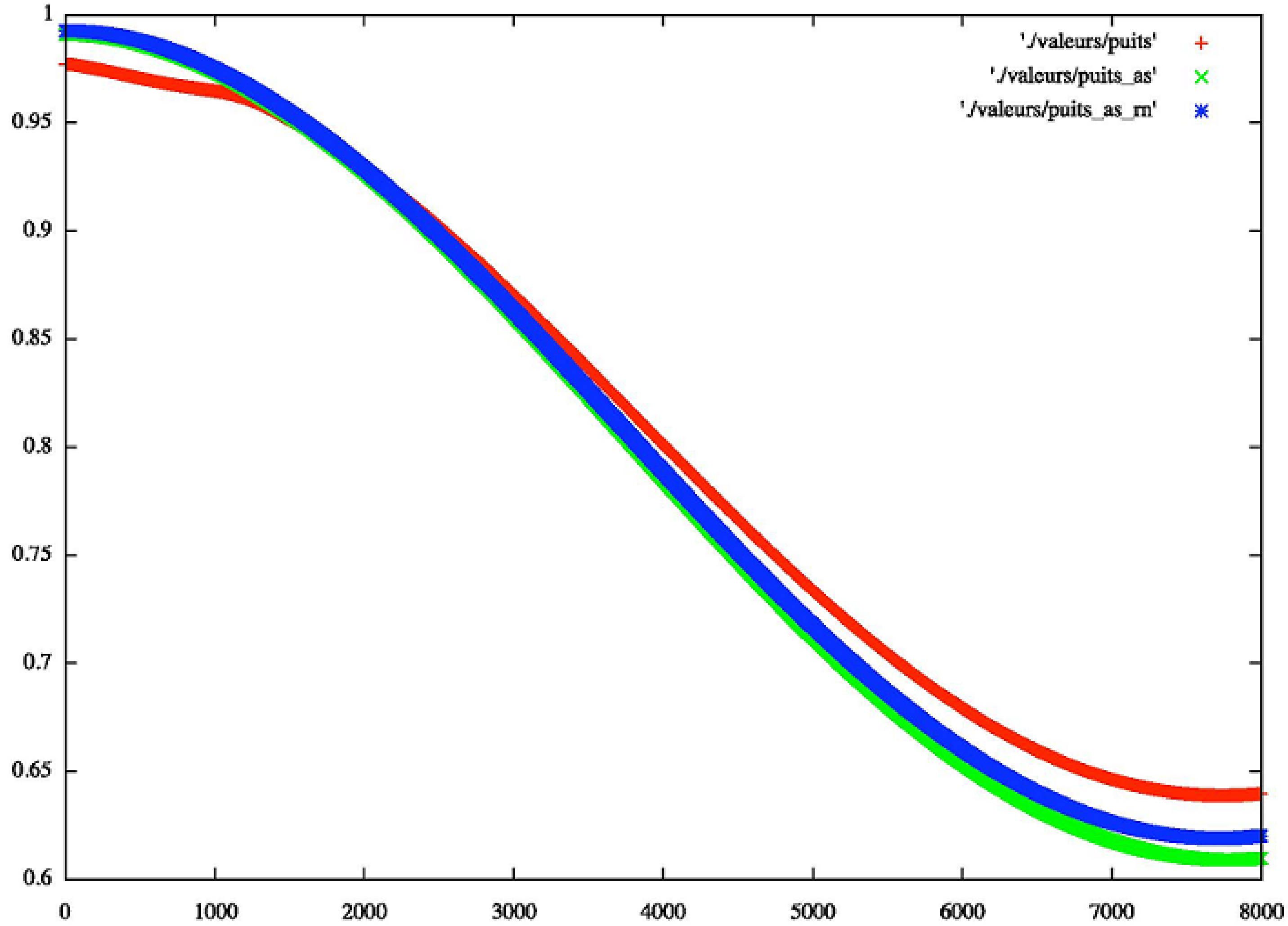

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