Non-Hermitian operators in QM
&
$\mathcal{PT}$-symmetry

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Hors de ligne

(Outline)

1. QM with non-Hermitian operators

   (just some conceptual remarks)

2. $\mathcal{PT}$-symmetry

   (what is known and my point of view)

3. physical $\mathcal{PT}$-symmetric model in QM

   (non-self-adjoint Robin boundary conditions)

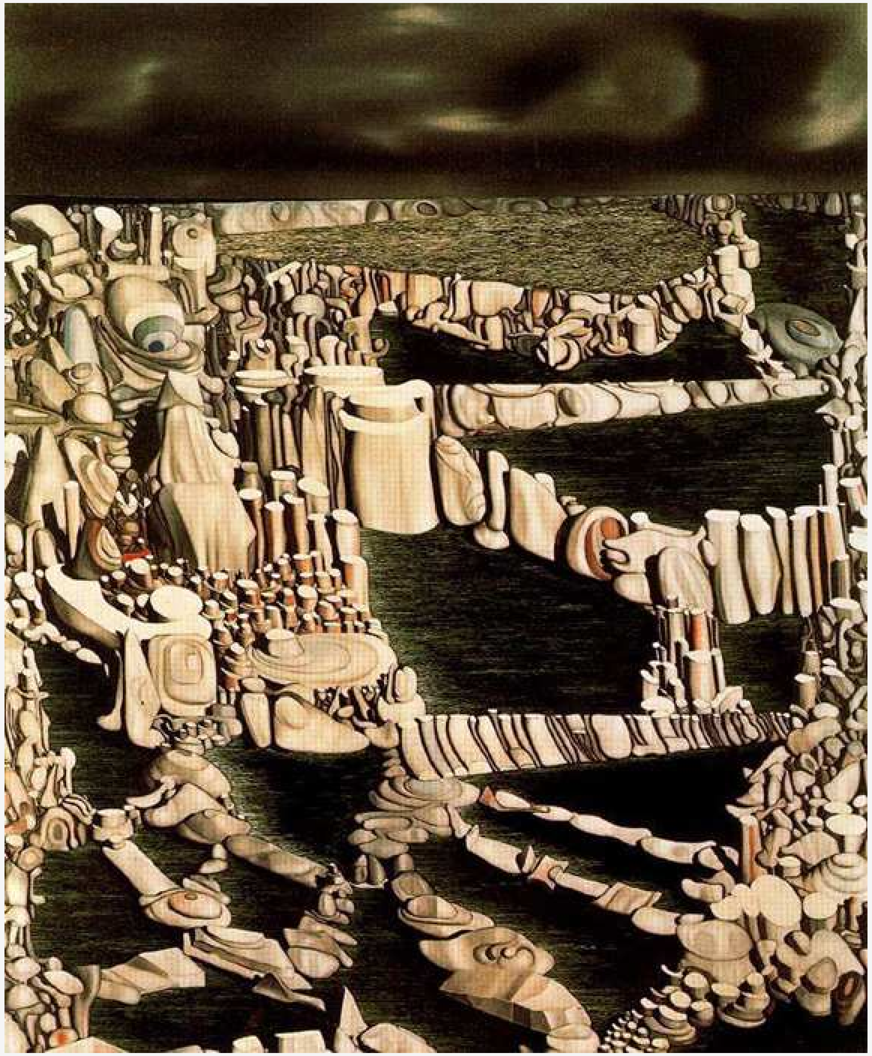
4. Conclusions

¿ QM with non-Hermitian operators ?

$$\mathbb{C}$$

$$\mathbb{R}$$

$$H^* = H$$



Imaginary Numbers by Yves Tanguy, 1954

(Museo Thyssen-Bornemisza, Madrid)

$$\mathbb{I}$$

$$H^{\mathcal{PT}} = H$$

Insignificant non-Hermiticity

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Theorem (spectral theorem).
Let $H = H^*$. Then

$$f(H) = \int_{\sigma(H)} f(\lambda)\; dE_H(\lambda)$$

for any complex-valued continuous function f.

Technical non-Hermiticity

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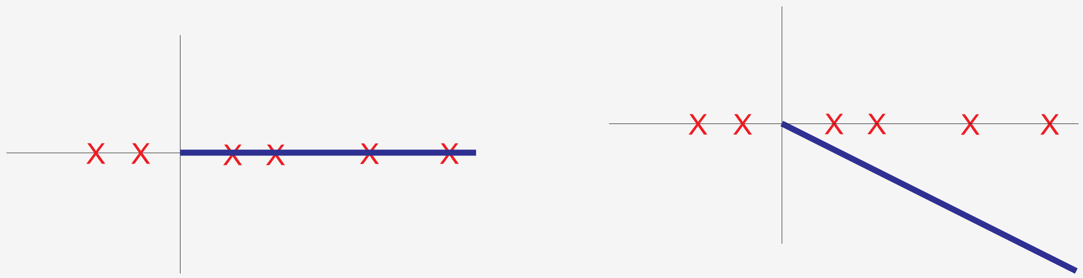
Example 1. complex scaling $H_\theta := S_\theta(-\Delta + V)S_\theta^{-1}, \quad (S_\theta \psi)(x) := e^{\theta/2}\,\psi(e^\theta x)$

$\theta = 0$ $\quad\quad$ $\Im\theta > 0$



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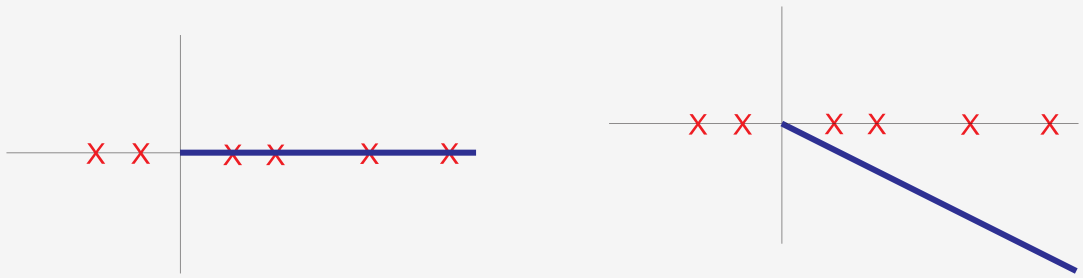

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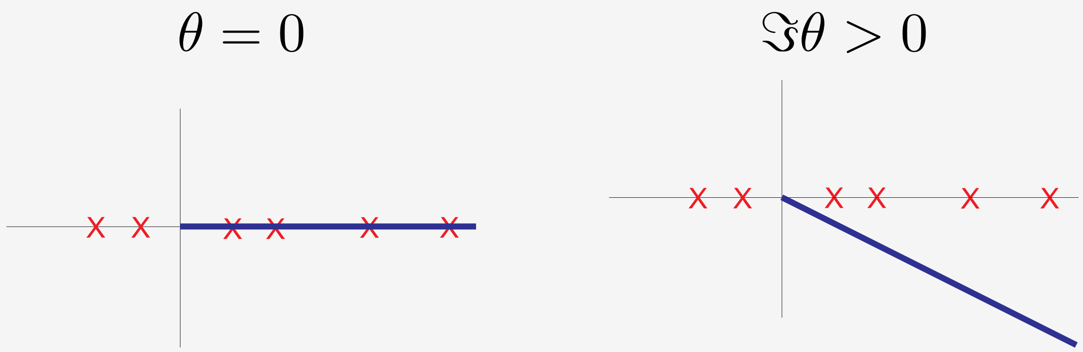

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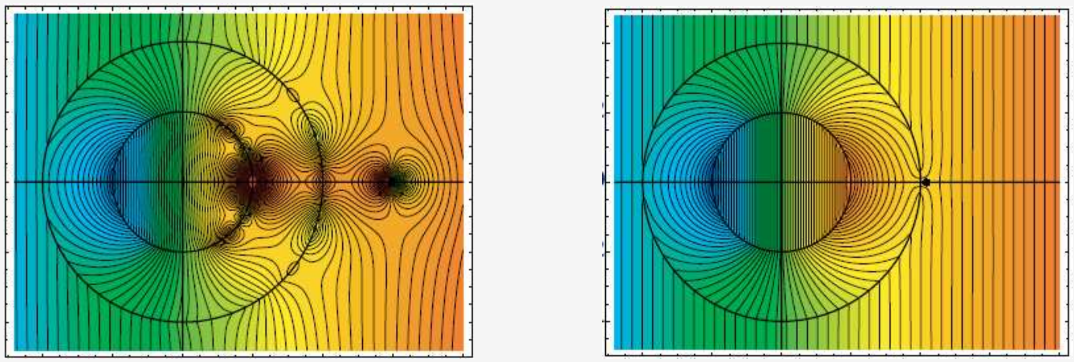
Example 3. cloaking effects in metamaterials $H_\eta := -\nabla\cdot a_\eta\nabla, \;\; a_\eta(x) := \begin{cases} +1, & x \in \Omega_+ \\ -1 + i\eta, & x \in \Omega_- \end{cases}$

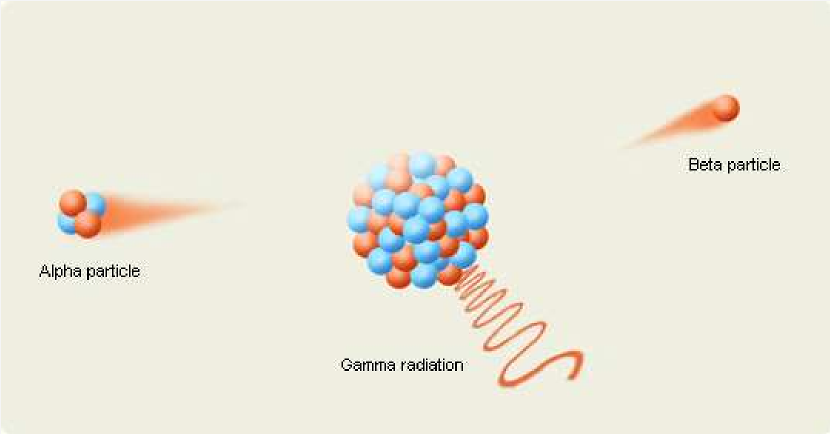


[Pendry 2004], [Milton, Nicorovici 2006], [Bouchitté, Schweizer 2009], . . .

Approximate non-Hermiticity

open systems

Example 1. radioactive decay



Example 2. dissipative Schrödinger operators in semiconductor physics

Baro, Behrndt, Kaiser, Neidhardt, Rehberg, . . .

Example 3. repeated interaction quantum systems

Bruneau, Joye, Merkli, Pillet, . . .

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Unitary groups on *a Hilbert space* are generated by self-adjoint operators.

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¡ no !

Theorem (Stone's theorem).

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¿ yes ?

by changing the Hilbert space

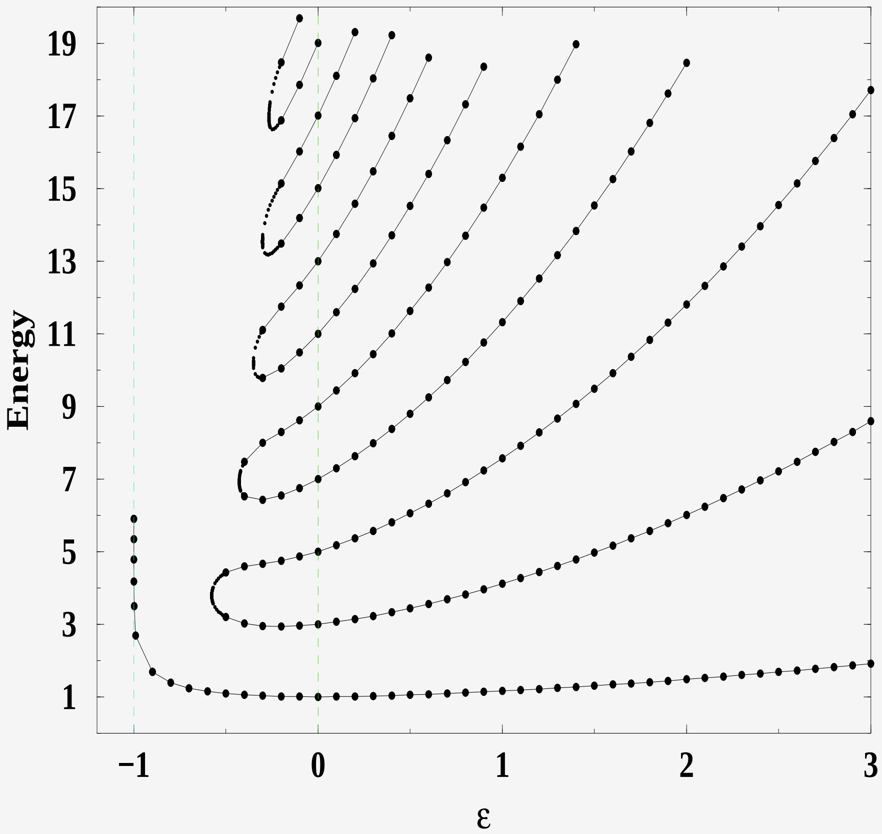
...

Non-Hermitian Hamiltonians with real spectra

$$-\Delta + V \quad \text{in} \quad L^2(\mathbb{R})$$

$V(x) = x^2 + ix^3$ [Caliceti, Graffi, Maioli 1980]

$V(x) = x^2\,(ix)^\varepsilon$



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$$V(x) = \begin{cases} i\,\mathrm{sgn}(x) & \text{if} \quad x \in (-L, L) \\ \infty & \text{elsewhere} \end{cases}$$

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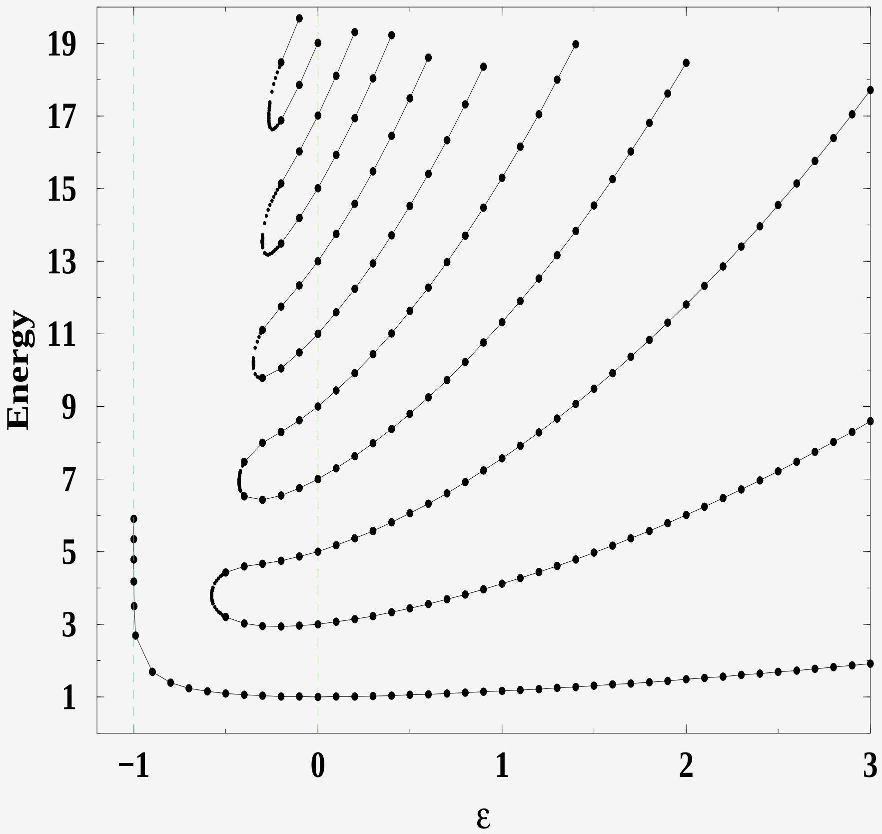

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¿ What is behind the reality of the spectrum ?

$\mathcal{PT}$ - symmetry

$$[H, \mathcal{PT}] = 0$$

$$(\mathcal{P}\psi)(x) := \psi(-x)$$

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Here we assume that H *has purely discrete spectrum.*

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Let $H_0 := -\Delta + V_0$ be self-adjoint, with purely discrete and simple spectrum.
Let V be bounded and $\mathcal{PT}$-symmetric. Define $H_\varepsilon := H_0 + \varepsilon V$.

$\implies$ $\sigma(H_\varepsilon)$ is dicrete and simple $\stackrel{*}{\implies}$ $\sigma(H_\varepsilon) \cap J \subset \mathbb{R}$ for every bounded J and small ε

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Moreover, let the eigenstates of H_ε form a Riesz basis. $H\psi_n = E_n\psi_n$, $H^*\phi_n = E_n\phi_n$

$\implies$ $H^* = \Theta H \Theta^{-1}$ where $\Theta := \sum_n \phi_n \langle \phi_n, \cdot \rangle$ is self-adjoint, bounded and positive

$\implies$ H is Hermitian in $\big(L^2, \langle \cdot, \Theta\, \cdot \rangle\big)$, *i.e.* $\Theta^{1/2} H \Theta^{-1/2}$ is Hermitian in $\big(L^2, \langle \cdot, \cdot \rangle\big)$

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Albeverio-Fei-Kurasov, Bender-Boettcher, Caliceti-Graffi-Sjöstrand, Boulton-Levitin-Marletta, Kretschmer-Szymanowski, Fring, Langer-Tretter, Mostafazadeh, Scholtz-Geyer-Hahne, Znojil, . . .

Mathematical frameworks

to understand $\boxed{\mathcal{PT}\, H\, \mathcal{PT} = H}$ in a more general setting than:

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Remark. In general (in ∞-dimensional spaces), all the classes of operators are unrelated.
[Siegl 2008]

¿ Physical relevance ?

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suggestions:

- nuclear physics [Scholtz, Geyer, Hahne 1992]
- optics [Klaiman, Günther, Moiseyev 2008], [Schomerus 2010], [West, Kottos, Prosen 2010]
- solid state physics [Bendix, Fleischmann, Kottos, Shapiro 2009]
- superconductivity [Rubinstein, Sternberg, Ma 2007]
- electromagnetism [Ruschhaupt, Delgado, Muga 2005], [Mostafazadeh 2009]

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¡ but !

"So far, there have been no experiments that prove clearly and definitively that quantum systems defined by non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians do exist in nature."

[Bender 2007]

The simplest $\mathcal{PT}$-symmetric model

[D.K., Bíla, Znojil 2006]

$$\mathcal{H} := L^2(0,\pi), \quad H_\alpha \psi := -\psi'', \quad D(H_\alpha) := \left\{ \psi \in W^{2,2}(0,\pi) \,\middle|\, \begin{array}{l} \psi'(0) + i\alpha\psi(0) = 0 \\ \psi'(\pi) + i\alpha\psi(\pi) = 0 \end{array} \right\}$$

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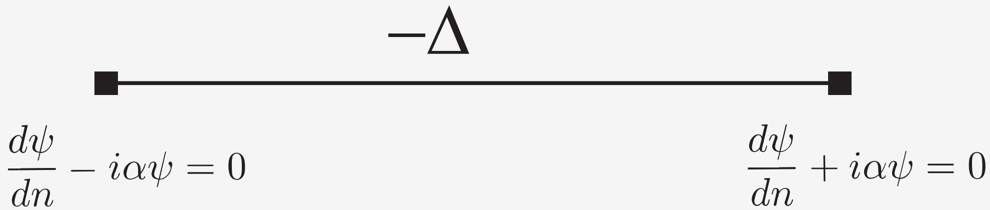

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$$\boxed{H_\alpha^* = H_{-\alpha} = \mathcal{T} H_\alpha \mathcal{T}} \qquad (\mathcal{T}\text{-self-adjointness})$$

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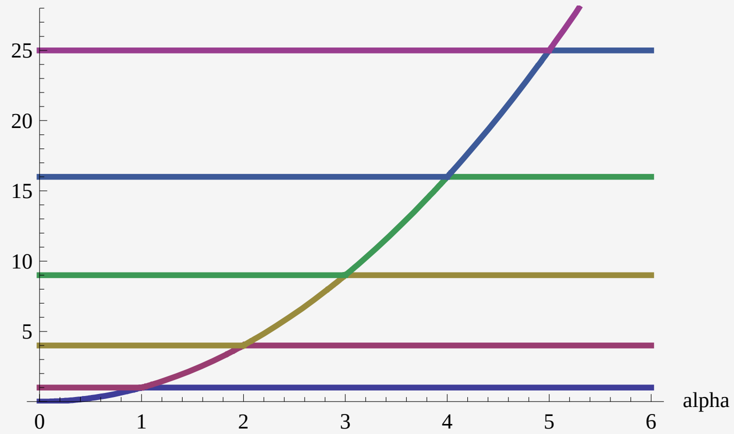

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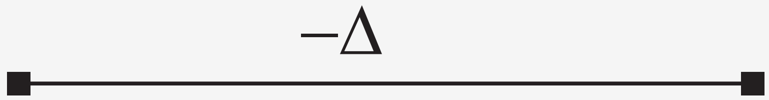

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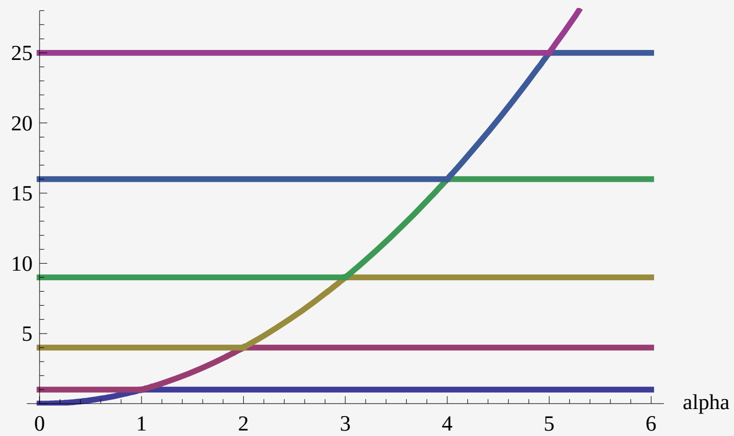

Corollary. The spectrum of H_α is $\begin{cases} \text{always real,} \\ \text{simple if} \quad \alpha \notin \mathbb{Z} \setminus \{0\}. \end{cases}$

The metric operator

[D.K. 2007], [D.K., Siegl, Železný 2010]

Theorem 3. Let $\alpha \in (-1, 1)$.

Then H_α is similar to a self-adjoint operator $\tilde{H}_\alpha := \Theta_\alpha^{1/2} \, H_\alpha \, \Theta_\alpha^{-1/2}$ with

$$\Theta_\alpha := I + K_\alpha$$

$$K_\alpha(x, x') := \alpha \, e^{i\alpha(x-x')} \left[\tan(\pi\alpha/2) + i \, \cos(\pi\alpha/2) \, \mathrm{sgn}(x - x') \right]$$

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[D.K. 2007], [D.K., Siegl, Železný 2010]

Theorem 3. Let $\alpha \in (-1, 1)$.
Then H_α is similar to a self-adjoint operator $\tilde{H}_\alpha := \Theta_\alpha^{1/2} H_\alpha \Theta_\alpha^{-1/2}$ with

$$\Theta_\alpha := I + K_\alpha$$

$$K_\alpha(x, x') := \alpha\, e^{i\alpha(x-x')} \left[\tan(\pi\alpha/2) + i\, \cos(\pi\alpha/2)\, \mathrm{sgn}(x - x') \right]$$

Remark. As $\alpha \to 0$, $\tilde{H}_\alpha = T_\alpha + \mathcal{O}(\alpha^3)$ with

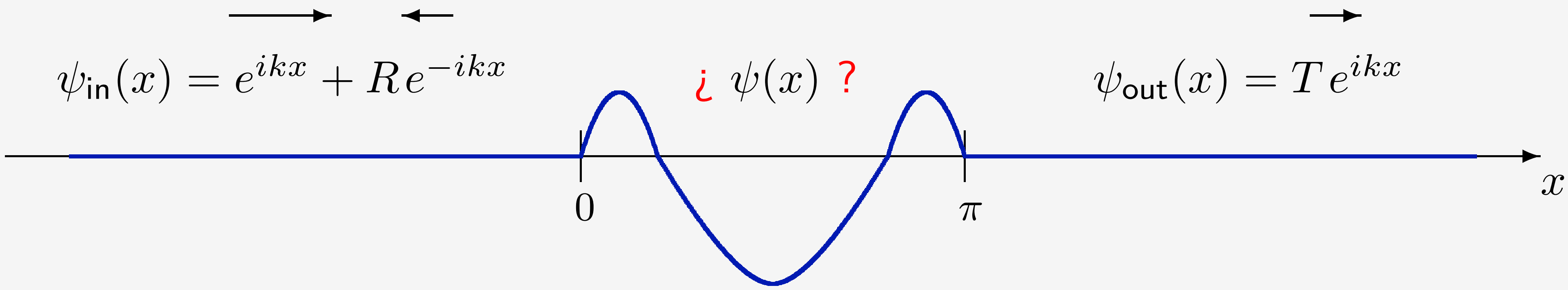
$$(T_\alpha\psi)(x) := -\psi''(x) - \alpha^2\, \psi(x) + \tfrac{1}{4}\, \alpha^2\, [\psi(0) + \psi(\pi)]$$

$$D(T_\alpha) := \left\{ \psi \in W^{2,2}(0, \pi) \,\middle|\, \psi'(0) = -\psi'(\pi) = \tfrac{1}{4}\, \alpha^2 \int_0^\pi \psi(x)\, dx \right\}$$

The physical realisation

[Hernandez-Coronado, D.K., Siegl 2010]

scattering by a compactly supported *even* potential V: $-\psi'' + V\psi = k^2\psi$ $\quad k > 0$



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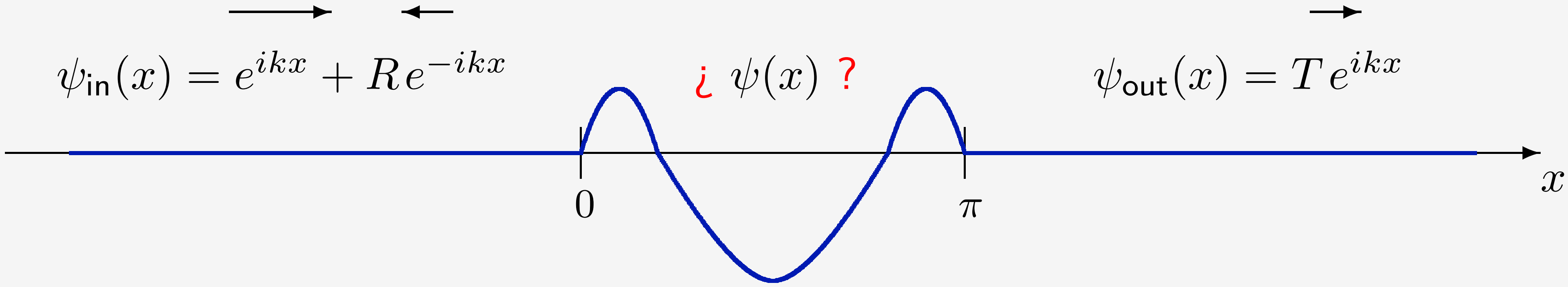

perfect transmission $\Longrightarrow$ $\begin{cases} -\psi'' + V\psi = k^2\psi & \text{in} \quad (0,\pi) \\ \psi' - ik\psi = 0 & \text{at} \quad 0,\pi \end{cases}$

(*i.e.* $R = 0$)

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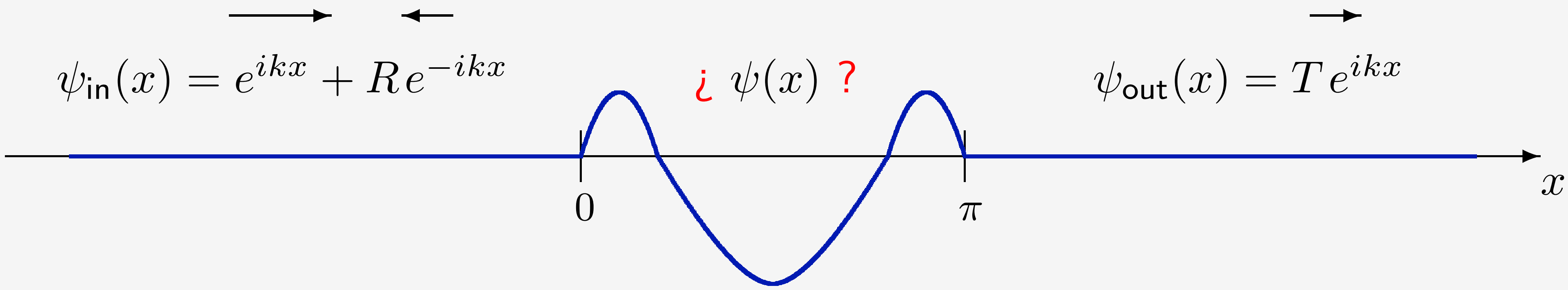

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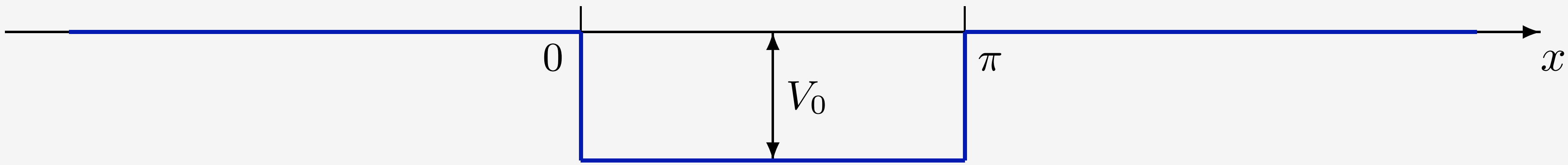
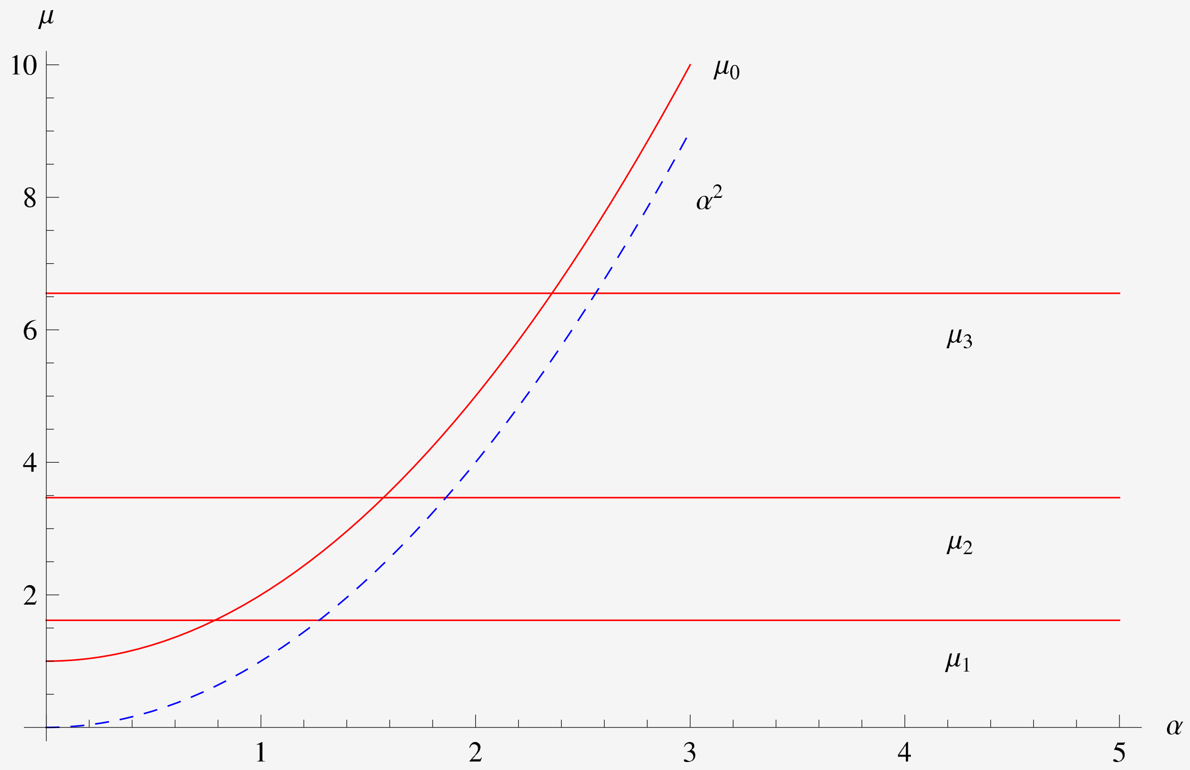
solutions given by a non-self-adjoint $\mathcal{PT}$-symmetric spectral problem:

$$\begin{cases} -\psi'' + V\psi = \mu(\alpha)\psi & \text{in} \quad (0,\pi) \\ \psi' + i\alpha\psi = 0 & \text{at} \quad 0,\pi \\ \mu(\alpha) = \alpha^2 \end{cases}$$

Square-well potential



$$\mu_n(\alpha) = \begin{cases} \alpha^2 - V_0 & \text{if} \quad n = 0 \\ n^2 - V_0 & \text{if} \quad n \geq 1 \end{cases}$$



perfect transmission energies: $\{n^2 - V_0\}_{n=1}^{\infty}$

¿ Significance of complex spectra ?

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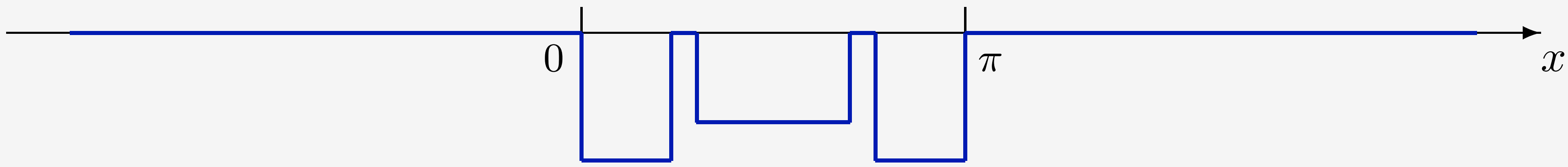
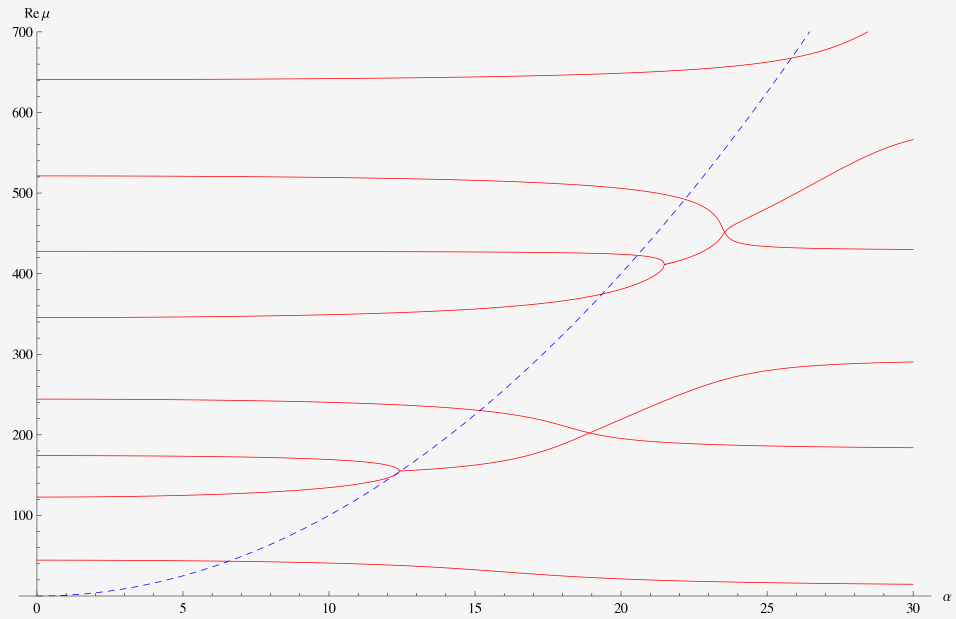
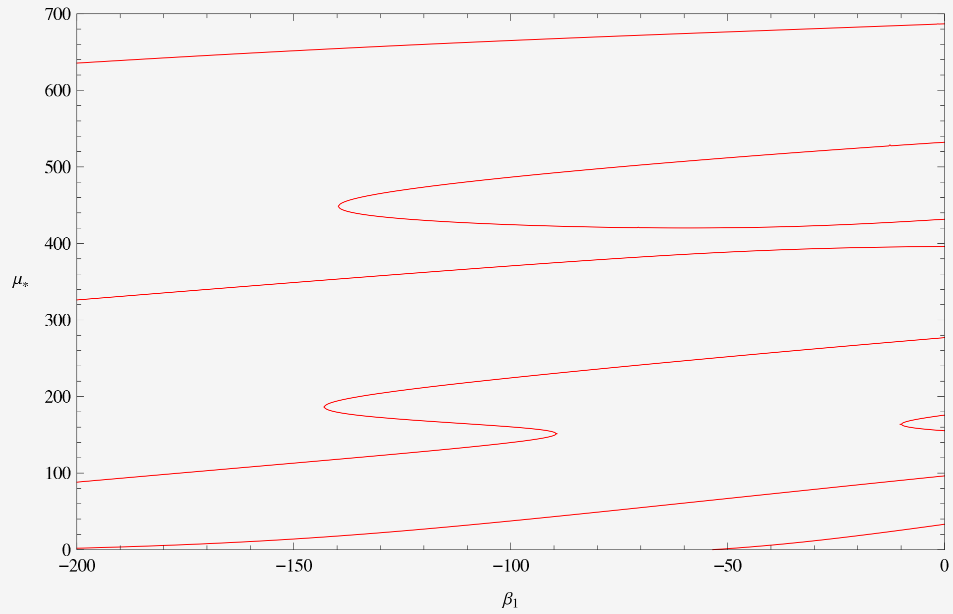
¡ loss of perfect transmission energies !





dispersion relations $\mu(\alpha) = \alpha^2$



perfect-transmission energies

The inverse problem

scattering data $\longrightarrow$ spectrum

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scattering data $\longrightarrow$ spectrum

initial PTE problem

$$\begin{cases} -\psi'' + V\psi = \mu(\alpha)\psi & \text{in} \quad (0,\pi) \\ \psi' + i\alpha\psi = 0 & \text{at} \quad 0,\pi \\ \mu(\alpha) = \alpha^2 & \end{cases}$$

shifted scatterer

$$\begin{cases} -\psi'' + (V + V_0)\psi = \mu_0(\alpha)\psi & \text{in} \quad (0,\pi) \\ \psi' + i\alpha\psi = 0 & \text{at} \quad 0,\pi \\ \mu(\alpha) = \alpha^2 & \end{cases}$$

$$\Longleftrightarrow$$

modified initial problem

$$\begin{cases} -\psi'' + V\psi = \mu(\alpha)\psi & \text{in} \quad (0,\pi) \\ \psi' + i\alpha\psi = 0 & \text{at} \quad 0,\pi \\ \mu(\alpha) = \alpha^2 - V_0 & \end{cases}$$

Consequently:

$$V_0 \mapsto \text{PTEs}(V_0) \quad \Longrightarrow \quad \alpha \mapsto \mu(\alpha)$$

(provided that $V_0 \mapsto \text{PTEs}(V_0)$ are invertible)

Conclusions

Ad $\mathcal{PT}$-symmetry :

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Generalisations:

→ higher-dimensional models with both the point and continuous spectra

→ curvature-induced effects

¡ many open problems ! (⇐ spectral theory of non-self-adjoint operators is "in its infancy")

ESF exploratory workshop on

Mathematical aspects of the physics with non-self-adjoint operators

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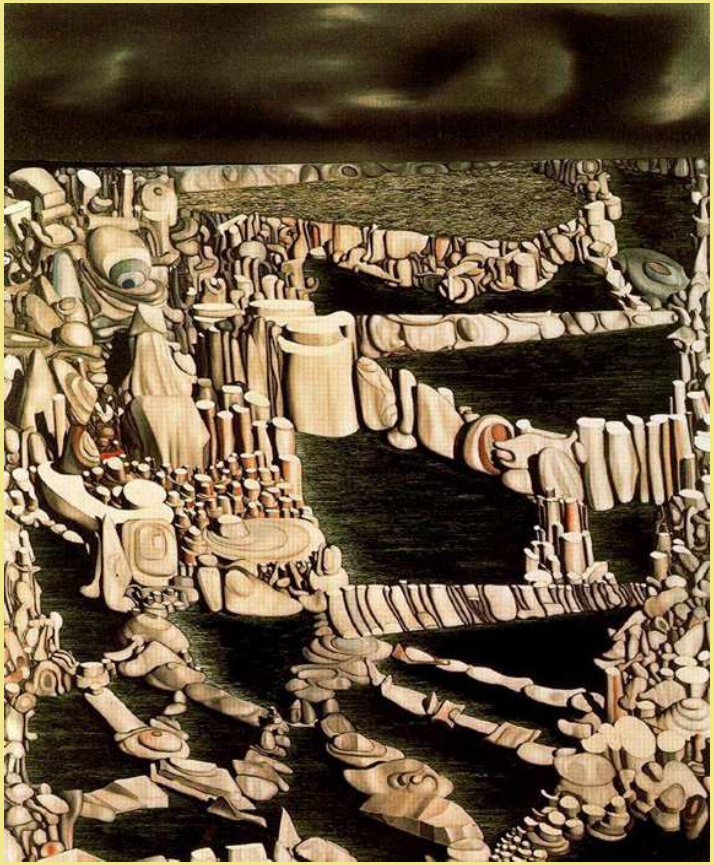

Imaginary numbers 1954 by Y. Tanguy

Studying non-self-adjoint operators is like being a vet rather than a doctor: one has to acquire a much wider range of knowledge, and to accept that one cannot expect to have as high a rate of success when confronted with particular cases.

E. B. Davies 2007

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My $\mathcal{PT}$-symmetric life

http://gemma.ujf.cas.cz/~david/

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