Identification of Green's Functions Singularities by Cross Correlation of Ambient Noise Signals

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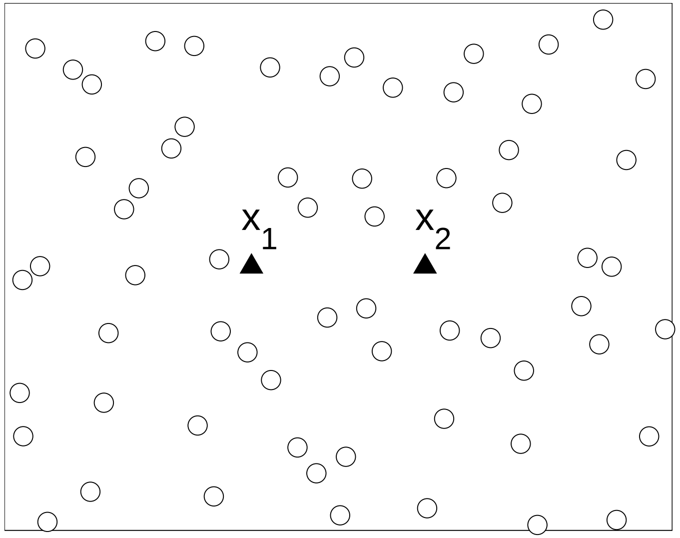
With C. Bardos (Paris 7), G. Papanicolaou (Stanford), and K. Sølna (UC Irvine).

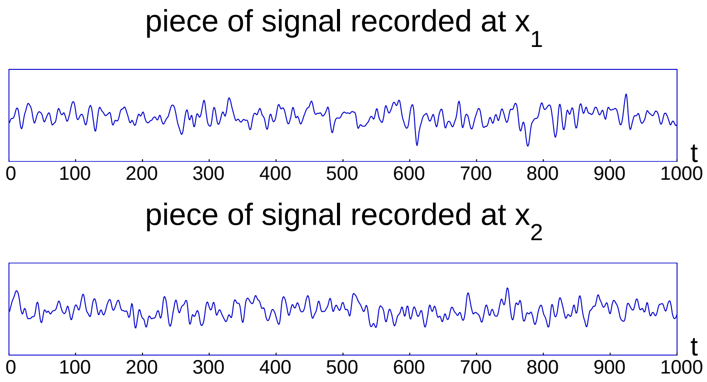
Classical problem in geophysics: Travel time estimation (for background velocity estimation).

- Method 1: Use of earthquake signals.
- Method 2: Use of seismic noise and cross correlation techniques in order to estimate the Green's function.

Travel time estimation by cross correlation

- Ambient noise sources ($\circ$) emit stationary random signals.
- The waves propagate in the (inhomogeneous) medium.
- The signals $u(t, \boldsymbol{x}_1)$ and $u(t, \boldsymbol{x}_2)$ are recorded at two sensors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$.





- What information (about the medium) can possibly be in these signals ?

Travel time estimation by cross correlation

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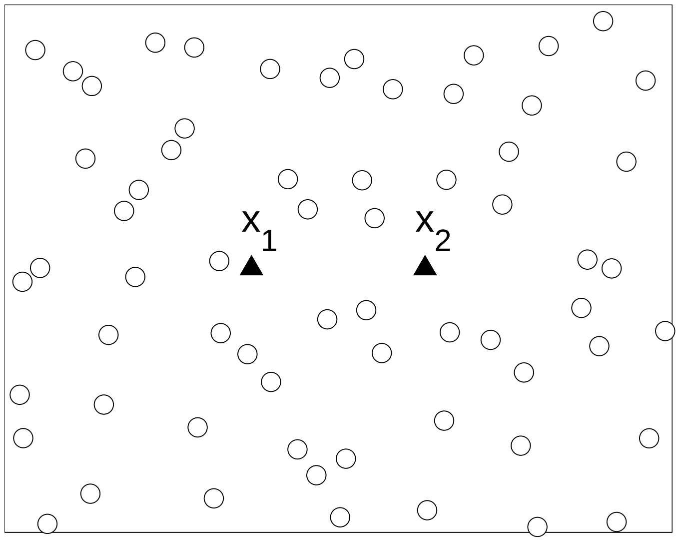
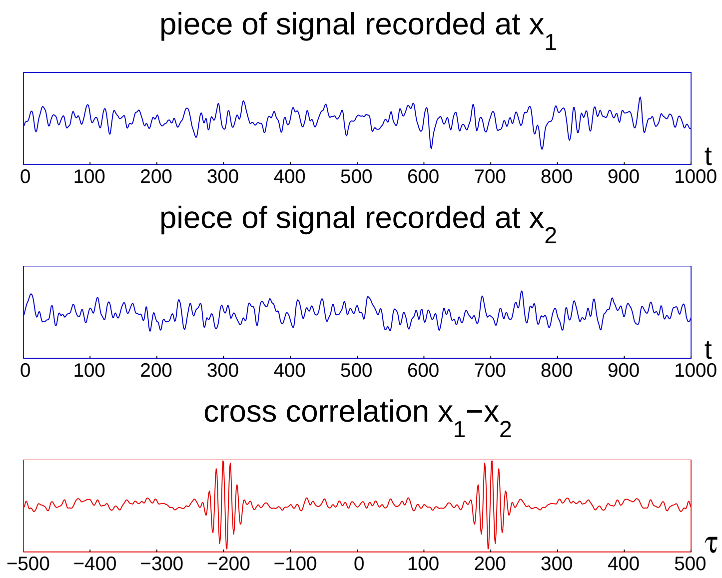


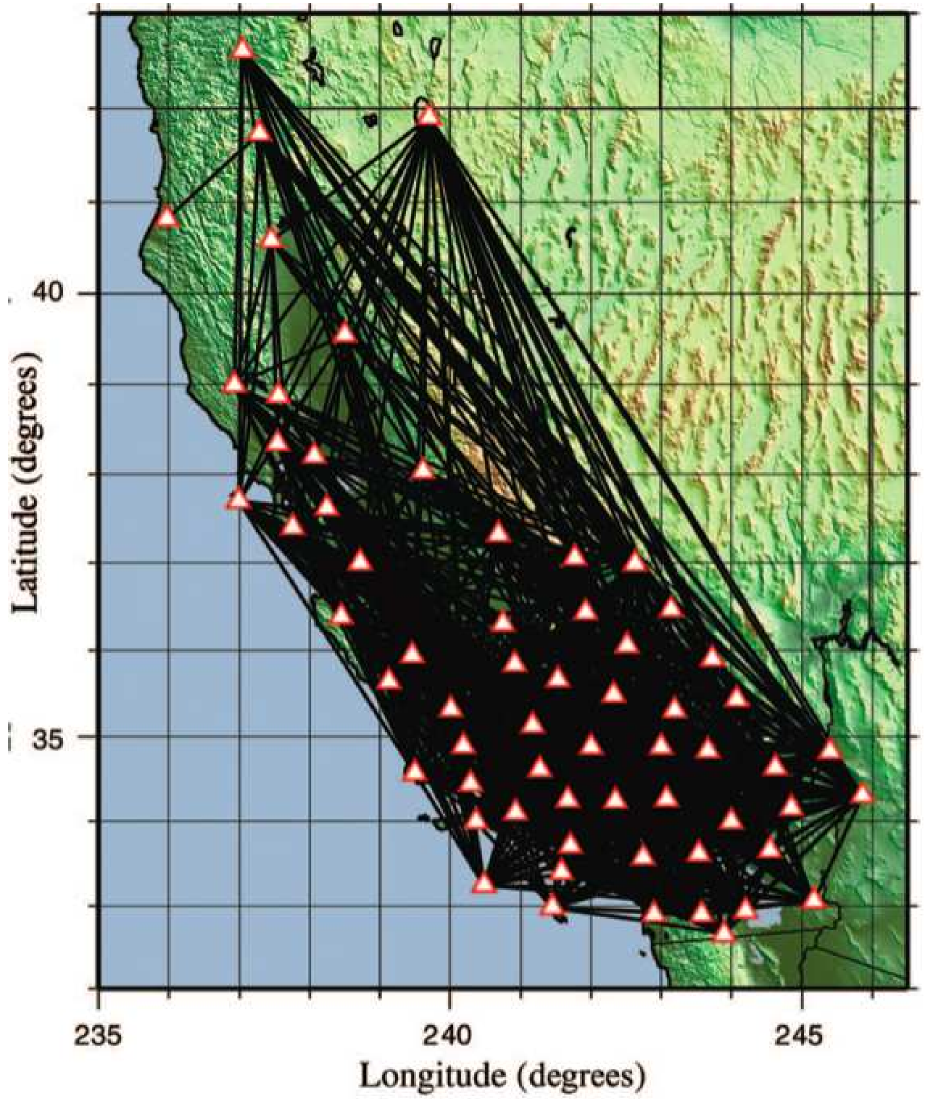


- Compute the empirical cross correlation:

$$C_T(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{1}{T} \int_0^T u(t, \boldsymbol{x}_1) u(t + \tau, \boldsymbol{x}_2) dt$$

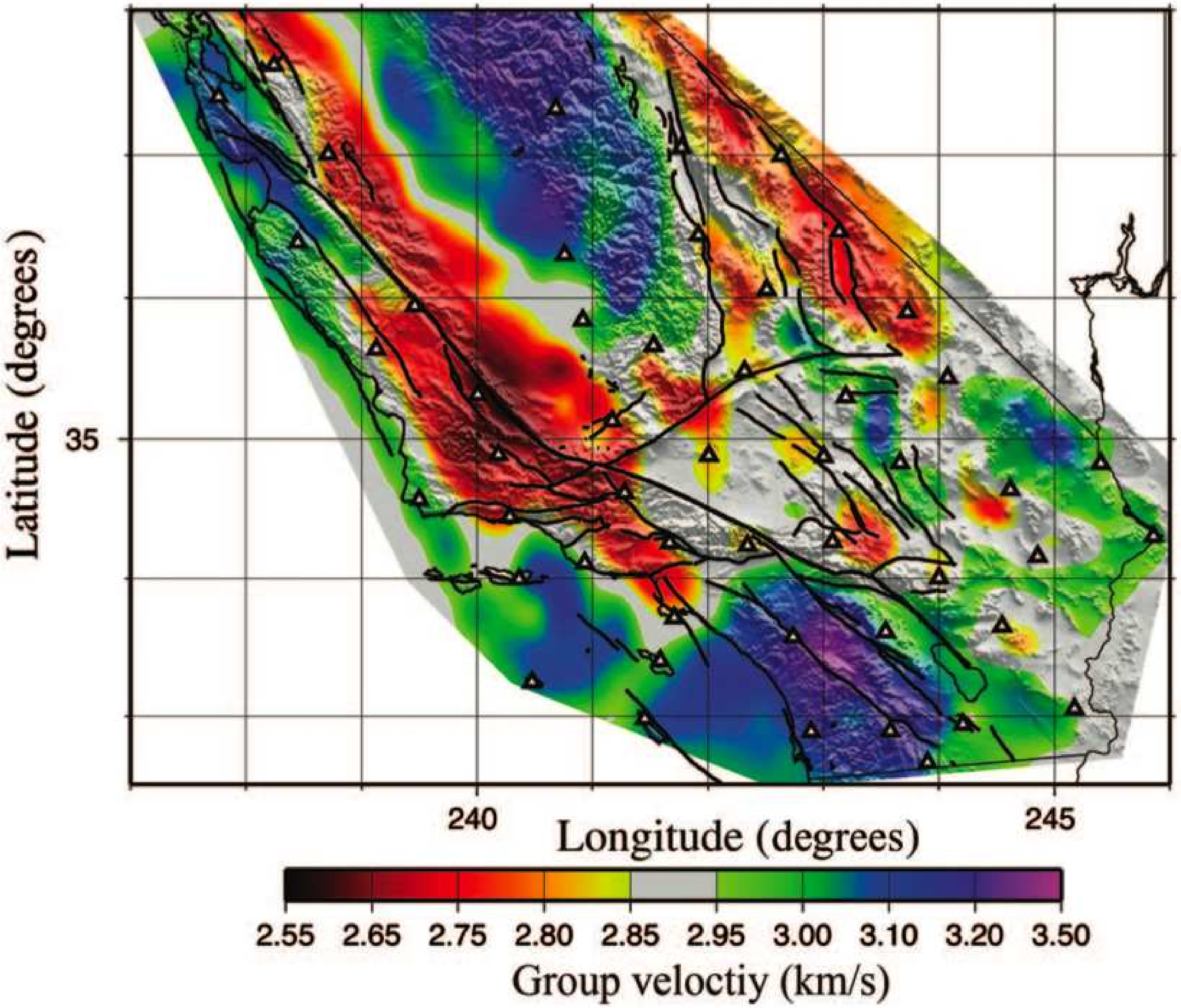
- $C_T(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2)$ is related to the Green's function from $\boldsymbol{x}_1$ to $\boldsymbol{x}_2$!
- The singular component of the Green's function from $\boldsymbol{x}_1$ to $\boldsymbol{x}_2$ gives the travel time from $\boldsymbol{x}_1$ to $\boldsymbol{x}_2$.

Estimations of travel times between pairs of sensors



Surface (Rayleigh) waves [from Shapiro, Campillo, et al, Science 307 (2005), 1615]

Background velocity estimation from travel time estimations



[from Shapiro, Campillo, et al, Science 307 (2005), 1615]

Wave equation

$$\frac{1}{c^2(\boldsymbol{x})}\frac{\partial^2 u}{\partial t^2} - \Delta_{\boldsymbol{x}} u = n(t,\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^d$$

+ radiation condition.

• Sources $n(t,\boldsymbol{x})$: Gaussian process, with mean zero, with covariance

$$\langle n(t_1,\boldsymbol{y}_1)n(t_2,\boldsymbol{y}_2)\rangle = F(t_2-t_1)\Gamma(\boldsymbol{y}_1,\boldsymbol{y}_2)$$

• Stationary in time: $\big(n(t,\boldsymbol{y})\big)_{t,\boldsymbol{y}}$ et $\big(n(t+h,\boldsymbol{y})\big)_{t,\boldsymbol{y}}$ have the same statistical distribution for any $h \Longrightarrow$ the time correlation function F depends only on t_2-t_1.

• The spatial distribution of the sources is characterized by $\Gamma(\boldsymbol{y}_1,\boldsymbol{y}_2)$. To simplify:

$$\Gamma(\boldsymbol{y}_1,\boldsymbol{y}_2) = \Gamma_0(\boldsymbol{y}_1)\delta(\boldsymbol{y}_1-\boldsymbol{y}_2)$$

The function Γ_0 characterizes the spatial support of the sources.

Empirical cross correlation and statistical cross correlation

Empirical cross correlation:

$$C_T(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{1}{T}\int_0^T u(t, \boldsymbol{x}_1) u(t+\tau, \boldsymbol{x}_2) dt$$

with $u(t, \boldsymbol{x}) = \iint G(s, \boldsymbol{x}, \boldsymbol{y}) n(t-s, \boldsymbol{y}) ds d\boldsymbol{y}$ and G = causal time-dependent Green's function.

1. The expectation of C_T (with respect to the distribution of the sources) is independent of the integration time T:

$$\langle C_T(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) \rangle = C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2)$$

where the statistical cross correlation $C^{(1)}$ is given by

$$C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{1}{2\pi}\int d\boldsymbol{y} \int d\omega \overline{\hat{G}}(\omega, \boldsymbol{x}_1, \boldsymbol{y}) \hat{G}(\omega, \boldsymbol{x}_2, \boldsymbol{y}) \Gamma_0(\boldsymbol{y}) \hat{F}(\omega) e^{-i\omega\tau}$$

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2. The empirical cross correlation is a self-averaging quantity:

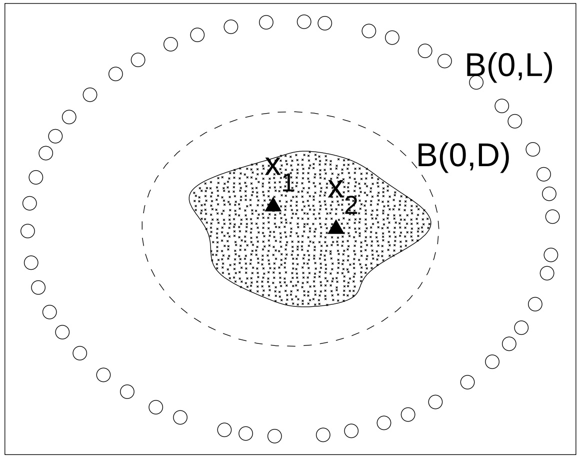
$$C_T(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) \overset{T\to\infty}{\longrightarrow} C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2)$$

in probability.
More precisely, the fluctuations of C_T around its expectation $C^{(1)}$ are of order $T^{-1/2}$.

The ideal situation

Cross correlation with noise sources distributed on a closed surface $\partial B(\mathbf{0}, L)$:

$$C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{1}{2\pi} \int d\omega \int_{\partial B(\mathbf{0},L)} dS(\boldsymbol{y}) \overline{\hat{G}}(\omega, \boldsymbol{x}_1, \boldsymbol{y}) \hat{G}(\omega, \boldsymbol{x}_2, \boldsymbol{y}) \hat{F}(\omega) e^{-i\omega\tau}$$



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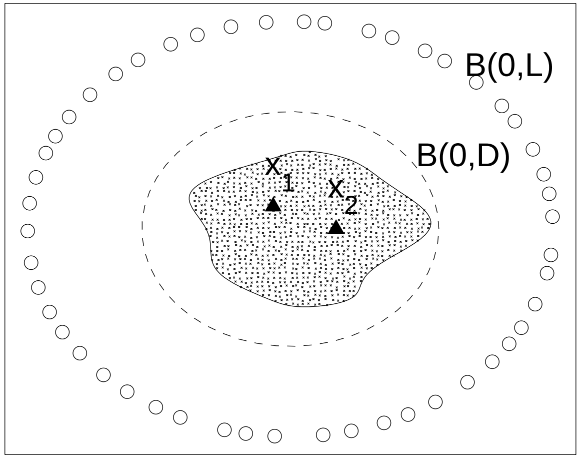

Helmholtz-Kirchhoff theorem:

$$\hat{G}(\omega, \boldsymbol{x}_1, \boldsymbol{x}_2) - \overline{\hat{G}}(\omega, \boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{2i\omega}{c_e} \int_{\partial B(\mathbf{0},L)} dS(\boldsymbol{y}) \overline{\hat{G}}(\omega, \boldsymbol{x}_1, \boldsymbol{y}) \hat{G}(\omega, \boldsymbol{x}_2, \boldsymbol{y})$$

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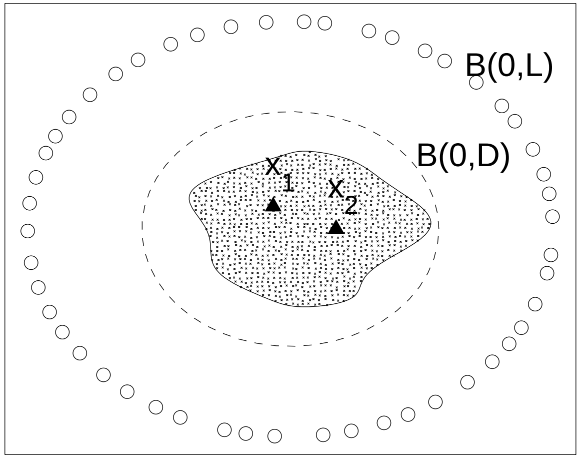

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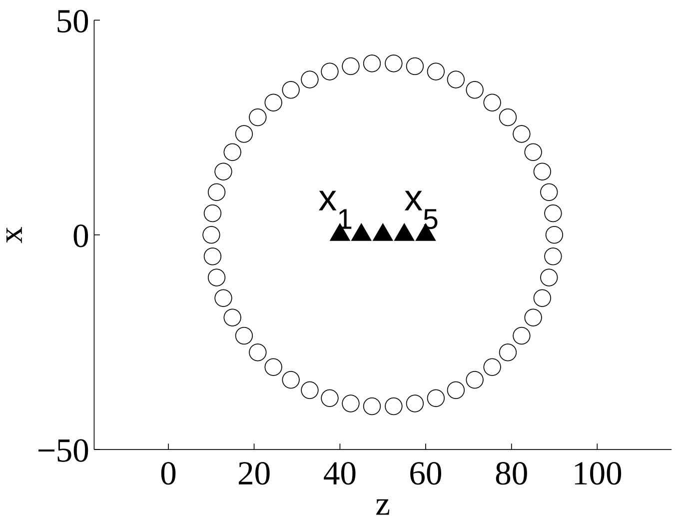
If 1) the medium is homogeneous outside $B(\mathbf{0}, D)$,

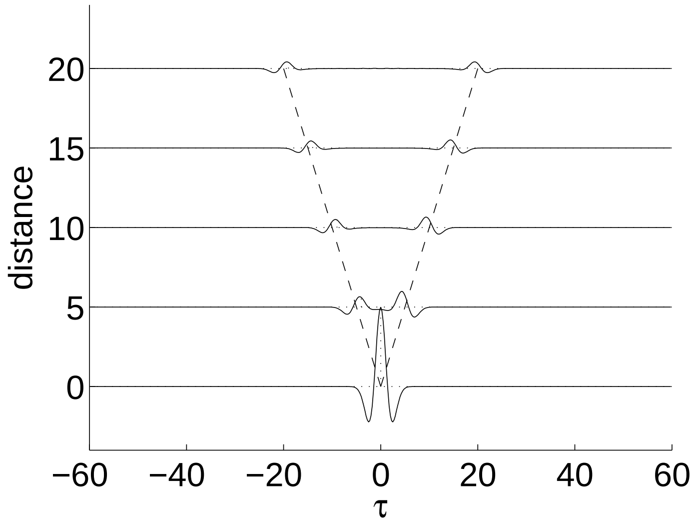
2) the sources are distributed uniformly on the sphere $\partial B(\mathbf{0}, L)$, with $L \gg D$.

Then for any $\boldsymbol{x}_1, \boldsymbol{x}_2 \in B(\mathbf{0}, D)$ we have (up to a multiplicative constant):

$$\frac{\partial}{\partial \tau} C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) = F *_\tau \left[G(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) - G(-\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) \right]$$

Ideal situation:

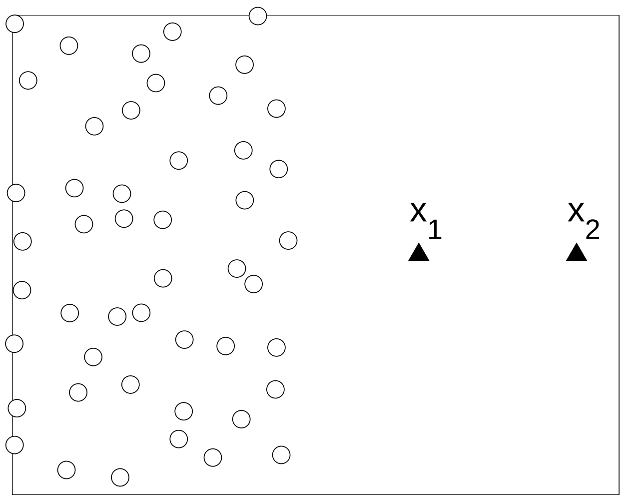


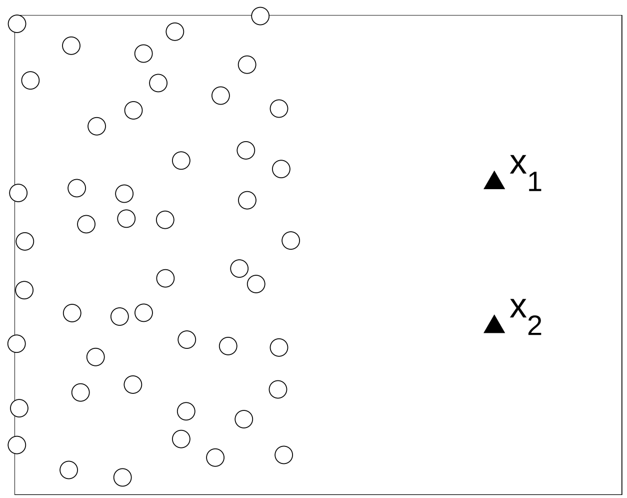


Configuration

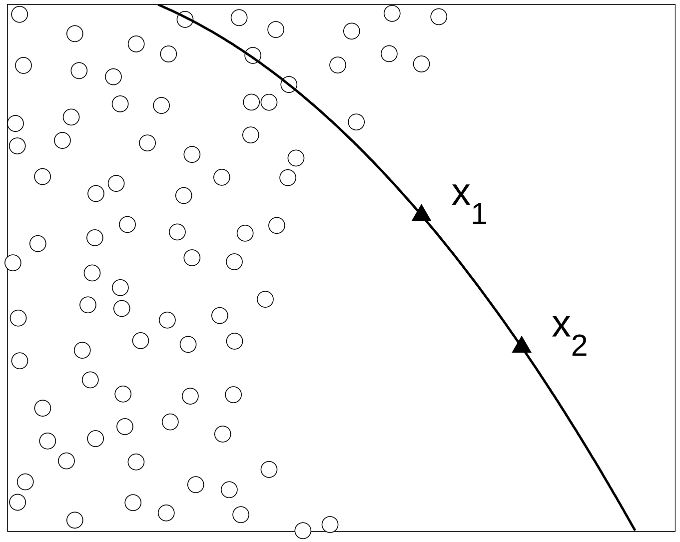
$C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_j)$

What about situations such as:

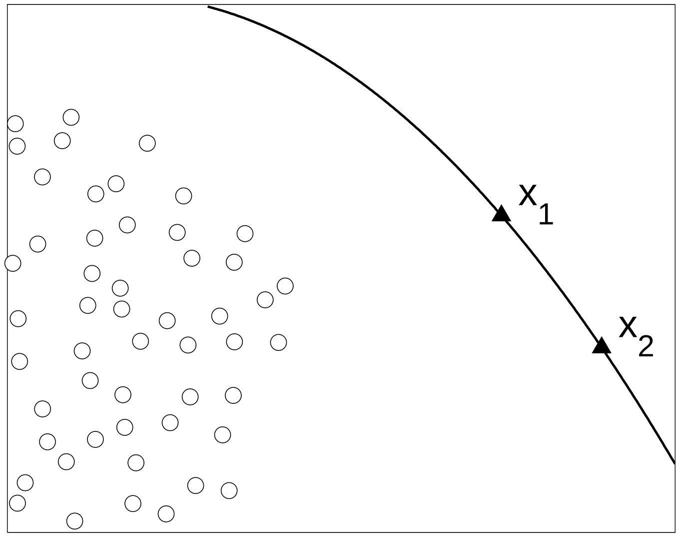




?



Singular component at $\mathcal{T}(\boldsymbol{x}_1, \boldsymbol{x}_2)$



No singular component

High-frequency analysis (geometric optics regime): The cross correlation $C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2)$ has singular components iff the ray joining $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ reaches into the source region (i.e. the support of Γ_0). Then there are one or two singular components at $\tau = \pm\mathcal{T}(\boldsymbol{x}_1, \boldsymbol{x}_2)$ (travel time from $\boldsymbol{x}_1$ to $\boldsymbol{x}_2$).

[More exactly:

the rays $\boldsymbol{y} \to \boldsymbol{x}_1 \to \boldsymbol{x}_2$ contribute to the singular component at $\tau = \mathcal{T}(\boldsymbol{x}_1, \boldsymbol{x}_2)$,

the rays $\boldsymbol{y} \to \boldsymbol{x}_2 \to \boldsymbol{x}_1$ contribute to the singular component at $\tau = -\mathcal{T}(\boldsymbol{x}_1, \boldsymbol{x}_2)$.]

Sketch of proof: WKB approximation + stationary phase analysis

Time correlation function of the sources $F^\varepsilon(t) = F(\frac{t}{\varepsilon}) \Longrightarrow \hat{F}^\varepsilon(\omega) = \varepsilon \hat{F}(\varepsilon\omega)$.

$$C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{1}{2\pi}\int d\boldsymbol{y}\int d\omega\, \overline{\hat{G}}(\omega, \boldsymbol{x}_1, \boldsymbol{y})\hat{G}(\omega, \boldsymbol{x}_2, \boldsymbol{y})\Gamma_0(\boldsymbol{y})\varepsilon\hat{F}(\varepsilon\omega)e^{-i\omega\tau}$$

$$\overset{\omega\to\frac{\omega}{\varepsilon}}{=} \frac{1}{2\pi}\int d\boldsymbol{y}\int d\omega\, \overline{\hat{G}}\Big(\frac{\omega}{\varepsilon}, \boldsymbol{x}_1, \boldsymbol{y}\Big)\hat{G}\Big(\frac{\omega}{\varepsilon}, \boldsymbol{x}_2, \boldsymbol{y}\Big)\Gamma_0(\boldsymbol{y})\hat{F}(\omega)e^{-i\frac{\omega}{\varepsilon}\tau}$$

WKB approximation for $\hat{G}\big(\frac{\omega}{\varepsilon}, \boldsymbol{x}, \boldsymbol{y}\big) \sim a(\boldsymbol{x}, \boldsymbol{y})e^{i\frac{\omega}{\varepsilon}\mathcal{T}(\boldsymbol{x},\boldsymbol{y})}$:

$$C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{1}{2\pi}\int d\boldsymbol{y}\int d\omega\, a(\boldsymbol{x}_1, \boldsymbol{y})a(\boldsymbol{x}_2, \boldsymbol{y})\Gamma_0(\boldsymbol{y})\hat{F}(\omega)e^{i\frac{\omega}{\varepsilon}T(\boldsymbol{y})}$$

with the rapid phase

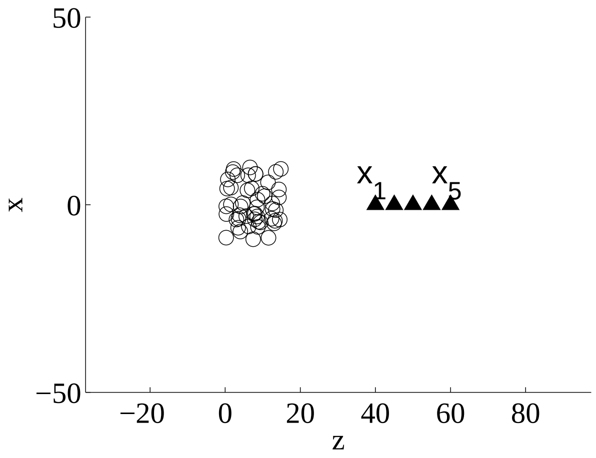
$$\omega T(\boldsymbol{y}) = \omega[\mathcal{T}(\boldsymbol{x}_2, \boldsymbol{y}) - \mathcal{T}(\boldsymbol{x}_1, \boldsymbol{y}) - \tau]$$

Use of the stationary phase theorem. The dominant contribution comes from the stationary points $(\omega, \boldsymbol{y})$ satisfying:

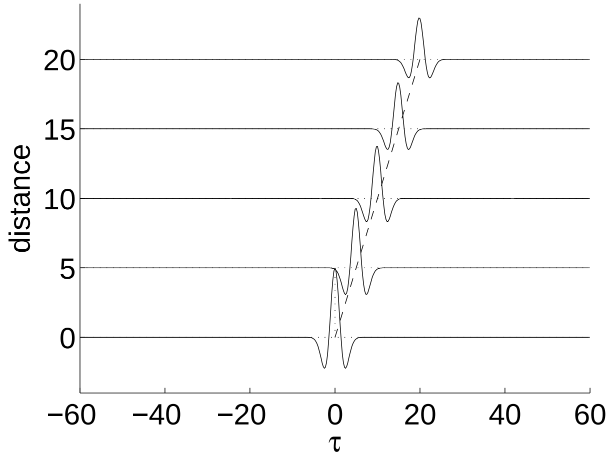
$$\nabla_{\boldsymbol{y}}\Big(\omega T(\boldsymbol{y})\Big) = \boldsymbol{0}, \quad \partial_\omega\Big(\omega T(\boldsymbol{y})\Big) = 0$$

$\hookrightarrow$ two conditions:

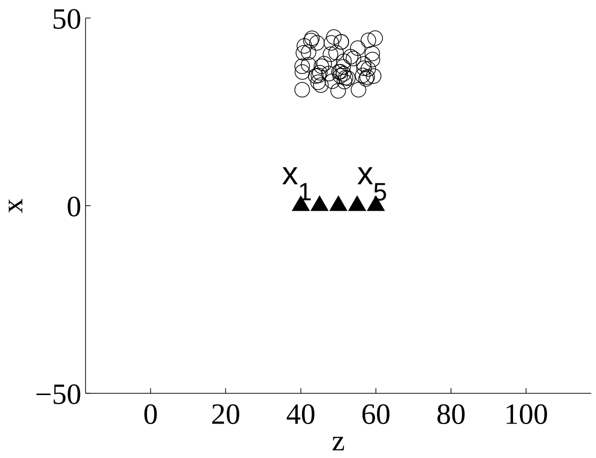
$$\nabla_{\boldsymbol{y}}\mathcal{T}(\boldsymbol{y}, \boldsymbol{x}_2) = \nabla_{\boldsymbol{y}}\mathcal{T}(\boldsymbol{y}, \boldsymbol{x}_1), \quad \mathcal{T}(\boldsymbol{x}_2, \boldsymbol{y}) - \mathcal{T}(\boldsymbol{x}_1, \boldsymbol{y}) = \tau$$



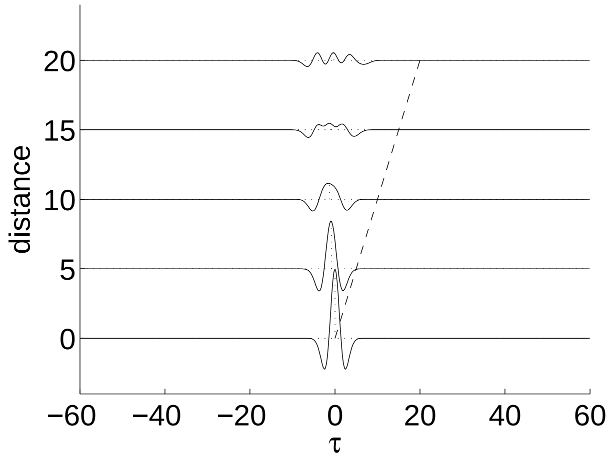
a) Favorable configuration



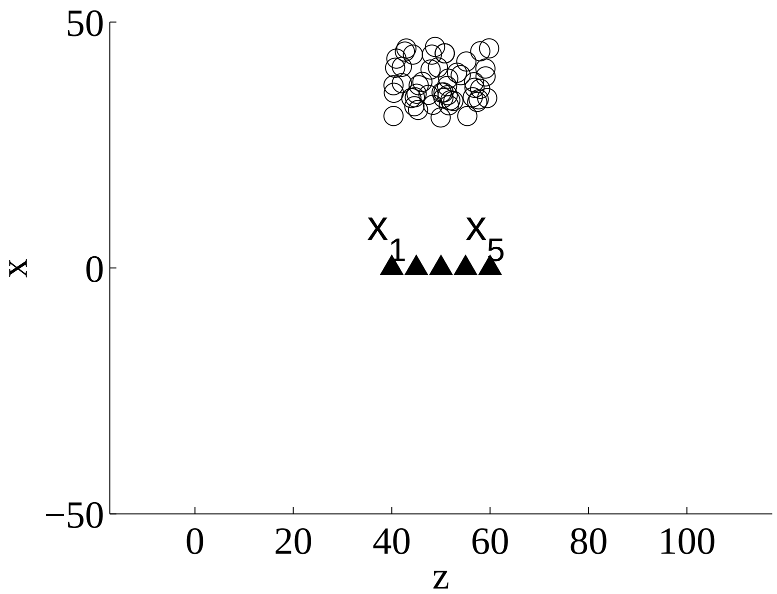
b) $C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_j)$

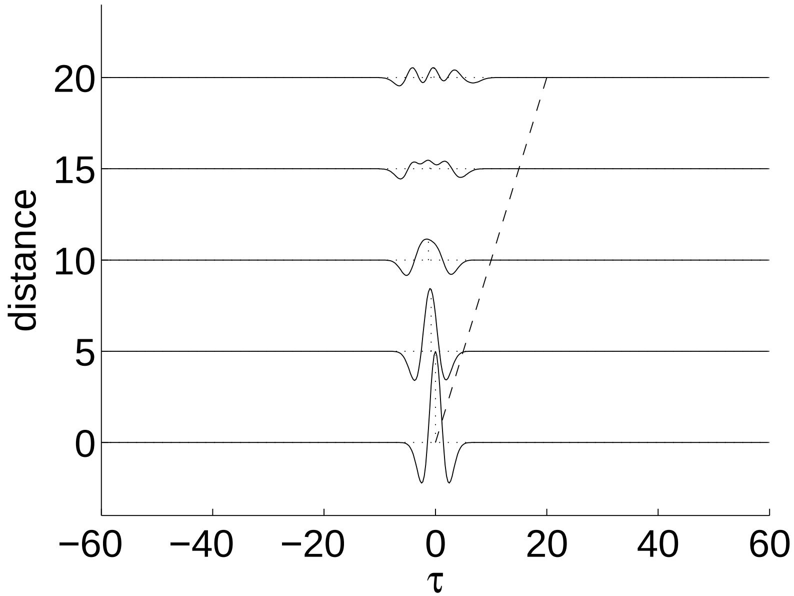


c) Unfavorable configuration

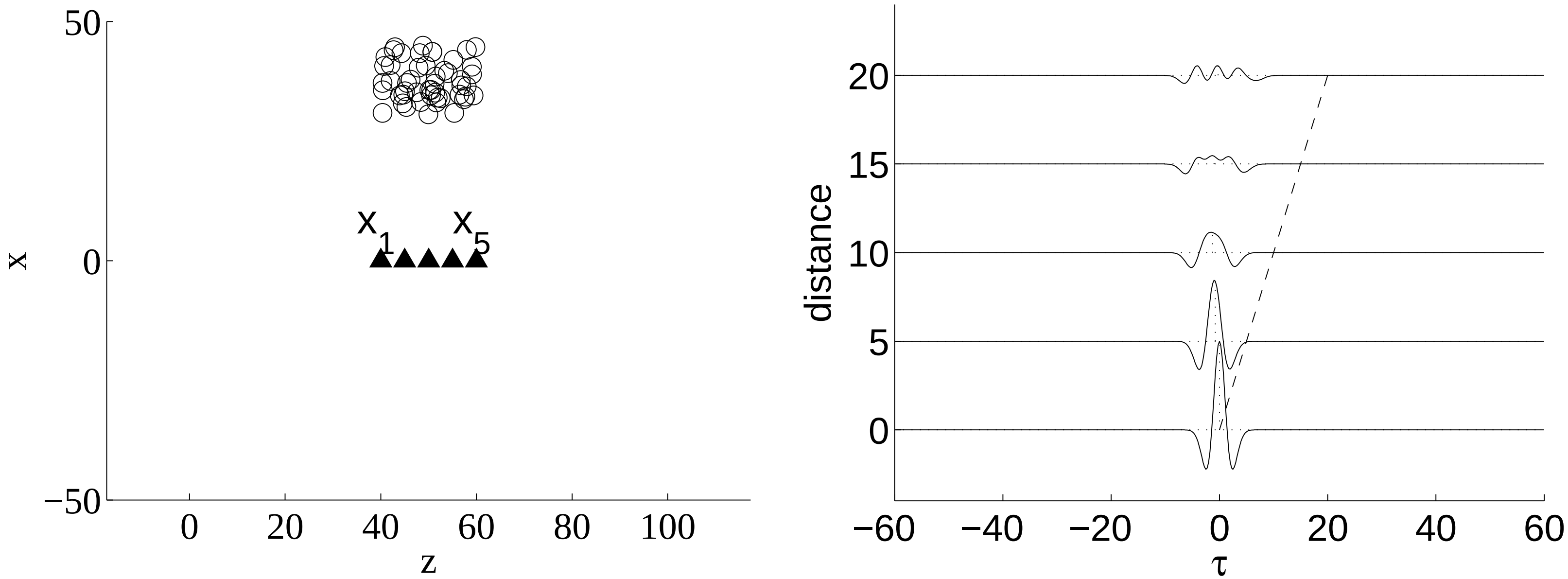


d) $C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_j)$





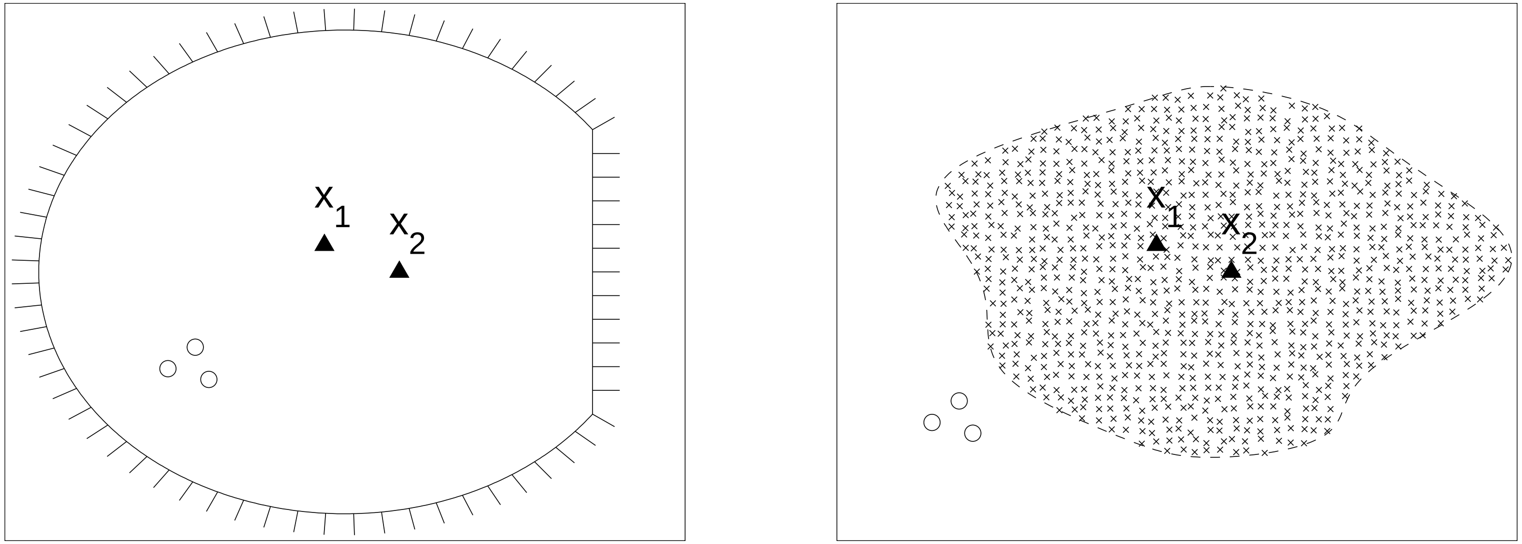
• Here, the cross correlation method does not allow for travel time estimation, because there is not enough “directional diversity”.



• Here, the cross correlation method does not allow for travel time estimation, because there is not enough “directional diversity”.

• Idea (first suggested by M. Campillo [1]): exploit the scattering properties of the medium in order to enhance the “directional diversity”.

[1] M. Campillo and L. Stehly, *Eos Trans. AGU* **88**(52) (2007), Fall Meet. Suppl., Abstract S51D-07.



Configurations in which wave fields have directional diversity:
An ergodic cavity (left figure) and a randomly inhomogeneous medium (right figure).

The wave equation in an ergodic cavity

$$\Big(\frac{1}{T_a}+\frac{\partial}{\partial t}\Big)^2 u - \nabla_{\boldsymbol{x}}\cdot\big[c^2(\boldsymbol{x})\nabla_{\boldsymbol{x}}\big]u = n(t,\boldsymbol{x}), \quad \boldsymbol{x} \text{ in } \Omega, \qquad\qquad u(\boldsymbol{x})=0, \quad \boldsymbol{x}\in\partial\Omega$$

- Ω is a bounded open set $\Omega \subset \mathbb{R}^d$.
- $c(\boldsymbol{x})$ is smooth.
- T_a is the attenuation time.
- $n(t,\boldsymbol{x})$: noisy sources.

It is a zero-mean stationary (in time) Gaussian process with autocorrelation function

$$\langle n(t_1,\boldsymbol{y}_1)n(t_2,\boldsymbol{y}_2)\rangle = F(t_2-t_1)\Gamma\Big(\frac{\boldsymbol{y}_1+\boldsymbol{y}_2}{2},\boldsymbol{y}_2-\boldsymbol{y}_1\Big)$$

The function $\boldsymbol{x}\mapsto\Gamma(\boldsymbol{x},0)$ models the spatial support of the sources.
The function $\boldsymbol{z}\mapsto\Gamma(\boldsymbol{x},\boldsymbol{z})$ is the local spatial autocorrelation function.

Example: $\Gamma(\boldsymbol{x},\boldsymbol{z})=\delta(\boldsymbol{z})$ would mean “sources everywhere, delta-correlated in space”.

- Covariance of the noise:

$$\langle n(t_1, \boldsymbol{y}_1) n(t_2, \boldsymbol{y}_2) \rangle = F^\varepsilon(t_2 - t_1)\Gamma^\varepsilon(\boldsymbol{y}_1, \boldsymbol{y}_2)$$

Time covariance:

$$F^\varepsilon(t_2 - t_1) = F\Big(\frac{t_2 - t_1}{\varepsilon}\Big)$$

Spatial covariance:

$$\Gamma^\varepsilon(\boldsymbol{x}, \boldsymbol{y}) = \Gamma\Big(\frac{\boldsymbol{x}+\boldsymbol{y}}{2}, \frac{\boldsymbol{x}-\boldsymbol{y}}{\varepsilon}\Big)$$

The spatial correlation radius of the noise sources is of the same order as the decoherence time (ε)
$\hookrightarrow$ time and space noise correlations contribute to the Green's function estimation at the same order of magnitude.

- The covariance operator $\Theta^\varepsilon : L^2(\Omega) \to L^2(\Omega)$ defined by

$$\Theta^\varepsilon \psi(\boldsymbol{x}) = \int \Gamma^\varepsilon(\boldsymbol{x},\boldsymbol{y})\psi(\boldsymbol{y})dy$$

is a zero-order pseudodifferential operator with symbol $\hat{\Gamma}(\boldsymbol{x},\boldsymbol{\xi})$

$$\Theta^\varepsilon = \mathrm{Op}^\varepsilon\left[\hat{\Gamma}(\boldsymbol{x},\boldsymbol{\xi})\right],$$

where the Fourier transform $\hat{\Gamma}(\boldsymbol{x},\boldsymbol{\xi})$ of the function $\boldsymbol{z} \mapsto \Gamma(\boldsymbol{x},\boldsymbol{z})$ is

$$\hat{\Gamma}(\boldsymbol{x},\boldsymbol{\xi}) = \int \Gamma(\boldsymbol{x},\boldsymbol{z}) e^{-i\boldsymbol{\xi}\cdot\boldsymbol{z}} d\boldsymbol{z},$$

and we have used the Weyl quantization Op^ε defined by

$$\mathrm{Op}^\varepsilon\left[\hat{\Gamma}(\boldsymbol{x},\boldsymbol{\xi})\right]\psi(\boldsymbol{x}) = \frac{1}{(2\pi)^d}\iint \hat{\Gamma}\Big(\frac{\boldsymbol{x}+\boldsymbol{y}}{2},\boldsymbol{\xi}\Big) e^{\frac{i}{\varepsilon}\boldsymbol{\xi}\cdot(\boldsymbol{x}-\boldsymbol{y})}\psi(\boldsymbol{y})d\boldsymbol{y}d\boldsymbol{\xi}$$

It is possible to reconstruct the singular components of the Green's function in the ergodic case, up to a smoothing operator that depends on Γ^ε and F^ε.
There are two ingredients in the proof:
1) Approximation of full wave propagation by classical ray dynamics (Egorov theorem): the singular (high-frequency) components propagate along the rays $(\boldsymbol{X}_t, \boldsymbol{\xi}_t)$ of geometric optics (Hamiltonian flow $h(\boldsymbol{x}, \boldsymbol{\xi}) = c(\boldsymbol{x})|\boldsymbol{\xi}|$) defined by

$$\begin{aligned} \frac{d\boldsymbol{X}_t}{dt} &= c(\boldsymbol{X}_t)\frac{\boldsymbol{\xi}_t}{|\boldsymbol{\xi}_t|}, \qquad \boldsymbol{X}_0(\boldsymbol{x}, \boldsymbol{\xi}) = \boldsymbol{x}, \\ \frac{d\boldsymbol{\xi}_t}{dt} &= -\nabla c(\boldsymbol{X}_t)|\boldsymbol{\xi}_t|, \qquad \boldsymbol{\xi}_0(\boldsymbol{x}, \boldsymbol{\xi}) = \boldsymbol{\xi}, \end{aligned}$$

and with specular reflection at the boundary $\partial\Omega$.
2) Ergodicity of the ray dynamics in the cavity Ω: starting from almost any point $\boldsymbol{x}$ and almost any direction $\boldsymbol{\xi}$, the ray $(\boldsymbol{X}_t, \boldsymbol{\xi}_t)$ visits all the energy surface.
For any $f \in L^\infty(S^*(\Omega))$ and for $(\boldsymbol{x}, \boldsymbol{\xi})$ in a subset of full measure of $S^*(\Omega)$,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f\big(\boldsymbol{X}_s(\boldsymbol{x}, \boldsymbol{\xi}), \boldsymbol{\xi}_s(\boldsymbol{x}, \boldsymbol{\xi})\big)ds = \bar{f} := \frac{1}{\mu(S^*(\Omega))}\int_{S^*(\Omega)} f(\boldsymbol{m})d\mu(\boldsymbol{m}).$$

where $S^*(\Omega)$ is the cotangent spherical bundle (energy surface)

$$S^*(\Omega) = \{(\boldsymbol{x}, \boldsymbol{\xi}) \in T^*\Omega\,,\ c(\boldsymbol{x})|\boldsymbol{\xi}| = 1\}$$

Main result [1]: If $c \in W^{4,\infty}(\Omega)$, $\hat{\Gamma}$ is smooth, bounded, and integrable, then $\partial_\tau C^{(1)}(\tau, \boldsymbol{x}, \boldsymbol{y})$ is the kernel of the operator

$$e^{-\frac{\tau}{T_a}} \mathcal{K}_\Gamma^\varepsilon \mathcal{F}_F^\varepsilon [G(\tau) - G(-\tau)] + R^\varepsilon(\tau) + R_{T_a}(\tau) \,,$$

for any $\tau > 0$, where

- $G(\tau)$ is the Green's function operator with kernel $G(\tau, \boldsymbol{x}, \boldsymbol{y})$,
- $\mathcal{F}_F^\varepsilon$ is the convolution operator in τ (due to the time correlations of the sources):

$$\mathcal{F}_F^\varepsilon G(\tau) = \int F^\varepsilon(s) G(\tau - s) ds$$

- $\mathcal{K}_\Gamma^\varepsilon$ is the smoothing operator (due to the spatial correlations of the sources):

$$\mathcal{K}_\Gamma^\varepsilon = \mathrm{Op}^\varepsilon \left[\hat{k}_\Gamma \big(c(\boldsymbol{x}) \boldsymbol{\xi} \big) \right], \qquad \hat{k}_\Gamma(\tilde{\boldsymbol{\xi}}) = \frac{\int_\Omega d\boldsymbol{z} c(\boldsymbol{z})^{-d} \int_{\partial B(\mathbf{0},1)} dS(\boldsymbol{\eta}) \hat{\Gamma}\big(\boldsymbol{z}, |\tilde{\boldsymbol{\xi}}| \frac{\boldsymbol{\eta}}{c(\boldsymbol{z})}\big)}{\int_\Omega d\boldsymbol{z} c(\boldsymbol{z})^{-d} \int_{\partial B(\mathbf{0},1)} dS(\boldsymbol{\eta})}$$

[1] C. Bardos, J. Garnier, and G. Papanicolaou, *Inverse Problems* **24**, 015011 (2008).

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- $G(\tau)$ is the Green's function operator with kernel $G(\tau, \boldsymbol{x}, \boldsymbol{y})$,
- $\mathcal{F}_F^\varepsilon$ is the convolution operator in τ (due to the time correlations of the sources):

$$\mathcal{F}_F^\varepsilon G(\tau) = \int F^\varepsilon(s) G(\tau - s) ds$$

- $\mathcal{K}_\Gamma^\varepsilon$ is the smoothing operator (due to the spatial correlations of the sources):

$$\mathcal{K}_\Gamma^\varepsilon = \mathrm{Op}^\varepsilon \left[\hat{k}_\Gamma\big(c(\boldsymbol{x})\boldsymbol{\xi}\big) \right], \qquad \hat{k}_\Gamma(\tilde{\boldsymbol{\xi}}) = \frac{\int_\Omega d\boldsymbol{z} c(\boldsymbol{z})^{-d} \int_{\partial B(\mathbf{0},1)} dS(\boldsymbol{\eta}) \hat{\Gamma}\big(\boldsymbol{z}, |\tilde{\boldsymbol{\xi}}| \frac{\boldsymbol{\eta}}{c(\boldsymbol{z})}\big)}{\int_\Omega d\boldsymbol{z} c(\boldsymbol{z})^{-d} \int_{\partial B(\mathbf{0},1)} dS(\boldsymbol{\eta})}$$

- the remainder $R^\varepsilon(\tau)$ is determined by the error in the semiclassical approximation and it is small if ε is small (Egorov theorem).
- the remainder $R_{T_a}(\tau)$ is determined by the rate of convergence of the ergodic theorem for the function $\hat{\Gamma}$ of the classical Hamiltonian flow.

If T_{erg} is the characteristic convergence time of $\frac{1}{t}\int_0^t \hat{\Gamma}(\mathbf{X}_s, \boldsymbol{\xi}_s) ds$ to its ergodic limit $\hat{k}_\Gamma(c(\boldsymbol{x})\boldsymbol{\xi})$, then $R_{T_a}(\tau)$ is small if $T_a \gg T_{\mathrm{erg}}$.

[1] C. Bardos, J. Garnier, and G. Papanicolaou, *Inverse Problems* **24**, 015011 (2008).

The wave equation in a randomly scattering medium: the radiative transport regime

$$\frac{1}{c^\varepsilon(\boldsymbol{x})^2}\frac{\partial^2 u^\varepsilon}{\partial t^2} - \Delta u^\varepsilon = n^\varepsilon(t,\boldsymbol{x}), \quad \boldsymbol{x}\in\mathbb{R}^3$$

- The velocity in the medium $c^\varepsilon(\boldsymbol{x})$ is of the form

$$\frac{1}{c^\varepsilon(\boldsymbol{x})^2} = \frac{1}{c_0^2(\boldsymbol{x})}\left[1+\nu^\varepsilon(\boldsymbol{x})\right]$$

The slowly varying background velocity $c_0(\boldsymbol{x})$ is smooth and bounded.
The rapid random fluctuations of the medium are described by the process $\nu^\varepsilon(\boldsymbol{x})$ with mean zero and covariance

$$\mathbb{E}\big[\nu^\varepsilon(\boldsymbol{x}_1)\nu^\varepsilon(\boldsymbol{x}_2)\big] = \varepsilon R\Big(\frac{\boldsymbol{x}_1+\boldsymbol{x}_2}{2},\frac{\boldsymbol{x}_2-\boldsymbol{x}_1}{\varepsilon}\Big)$$

(locally stationary random medium with small fluctuations with amplitude $\sim\varepsilon^{1/2}$ and correlation length $\sim\varepsilon$).

- The noise sources $n^\varepsilon(t,\boldsymbol{x})$ have mean zero and covariance

$$\Big\langle n^\varepsilon(t_1,\boldsymbol{y}_1)n^\varepsilon(t_2,\boldsymbol{y}_2)\Big\rangle = F\Big(\frac{t_2-t_1}{\varepsilon}\Big)\Gamma\Big(\frac{\boldsymbol{y}_1+\boldsymbol{y}_2}{2},\frac{\boldsymbol{y}_2-\boldsymbol{y}_1}{\varepsilon}\Big)$$

- Consider the *Radiative Transport Equation* [1]:

$$\frac{\partial W}{\partial t} + \nabla_{\boldsymbol{\kappa}}\omega \cdot \nabla_{\boldsymbol{x}} W - \nabla_{\boldsymbol{x}}\omega \cdot \nabla_{\boldsymbol{\kappa}} W = \int \sigma(\boldsymbol{x}, \boldsymbol{\kappa}, \boldsymbol{\kappa}')\big(W(t, \boldsymbol{x}, \boldsymbol{\kappa}') - W(t, \boldsymbol{x}, \boldsymbol{\kappa})\big) d\boldsymbol{\kappa}'$$

with the initial condition

$$W(t = 0, \boldsymbol{x}, \boldsymbol{\kappa}) = \frac{1}{2}\hat{\Gamma}(\boldsymbol{x}, \boldsymbol{\kappa}) c_0^2(\boldsymbol{x})$$

the dispersion relation

$$\omega(\boldsymbol{x}, \boldsymbol{\kappa}) = c_0(\boldsymbol{x})|\boldsymbol{\kappa}|$$

and the scattering cross section

$$\sigma(\boldsymbol{x}, \boldsymbol{\kappa}, \boldsymbol{\kappa}') = \frac{\pi c_0^2(\boldsymbol{x})|\boldsymbol{\kappa}|^2}{2(2\pi)^3} \hat{R}(\boldsymbol{x}, \boldsymbol{\kappa} - \boldsymbol{\kappa}')\delta\big(c_0(\boldsymbol{x})|\boldsymbol{\kappa}| - c_0(\boldsymbol{x})|\boldsymbol{\kappa}'|\big)$$

$\hookrightarrow$ Amplitude factor (energy density illuminating $\boldsymbol{x}$):

$$A(\boldsymbol{x}, \boldsymbol{\kappa}) = \int_0^{\infty} W(t, \boldsymbol{x}, \boldsymbol{\kappa}) dt$$

[1] L. Ryzhik, J. Keller, and G. Papanicolaou, *Wave Motion* **24**, 327 (1996).

• Diffusive regime.

Diffusion-approximation: When the distance L between the sources and the observation area around $\boldsymbol{x}$ is much larger than the transport mean free path (determined by $\hat{R}$), then the amplitude factor $A(\boldsymbol{x}, \boldsymbol{\kappa})$ becomes independent of the direction $\boldsymbol{\kappa}/|\boldsymbol{\kappa}|$:

$$A(\boldsymbol{x}, \boldsymbol{\kappa}) = \mathcal{A}(\boldsymbol{x}, |\boldsymbol{\kappa}|)$$

We then have [1]

$$\frac{1}{\varepsilon}\frac{\partial}{\partial \tau} C^{(1)}\big(\varepsilon\tau, \boldsymbol{x} - \frac{\varepsilon \boldsymbol{y}}{2}, \boldsymbol{x} + \frac{\varepsilon \boldsymbol{y}}{2}\big)$$
$$\stackrel{\varepsilon\to 0}{\longrightarrow} \iint F(\tau-\tau')K_{\boldsymbol{x}}(\boldsymbol{y}-\boldsymbol{y}')\big[G_{c_0(\boldsymbol{x})}(\tau', \boldsymbol{y}') - G_{c_0(\boldsymbol{x})}(-\tau', \boldsymbol{y}')\big] d\boldsymbol{y}' d\tau'$$

where

$$G_{c_0}(\tau, \boldsymbol{y}) = \frac{1}{4\pi|\boldsymbol{y}|}\delta\Big(\tau - \frac{|\boldsymbol{y}|}{c_0}\Big)$$

and

$$K_{\boldsymbol{x}}(\boldsymbol{y}) = \frac{1}{2\pi^2}\int_0^\infty \mathcal{A}(\boldsymbol{x}, k)\mathrm{sinc}(k|\boldsymbol{y}|)k^2 dk$$

↪ smoothing due to the time and space correlations of the sources.

Note: estimation of the Green's function only between two nearby points.

[1] J. Garnier and K. Sølna, to appear in *Commun. Math. Sciences*

The wave equation in a randomly scattering medium: overview

• Trade-off between
- directional diversity enhancement due to scattering
- fluctuations of the cross correlations with respect to the distribution of the random medium

• Example: three-dimensional scattering medium in the diffusion approximation regime [1]
$\rightarrow$ good trade-off when the distance from the sources to the observation points is larger than the mean free path and when the distance between the observation points is smaller than the mean free path.

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• Example: randomly layered media [2]
$\rightarrow$ large fluctuations, but poor directional diversity enhancement.

[2] J. Garnier and K. Sølna, to appear in *SIAM J. Imaging Sciences*

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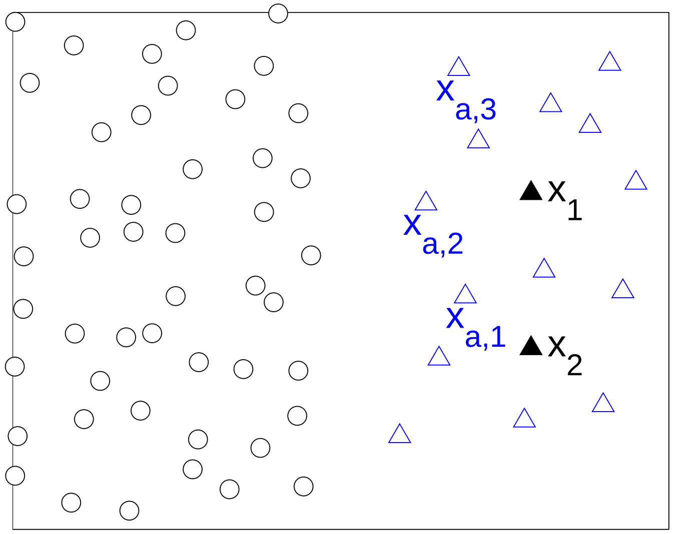
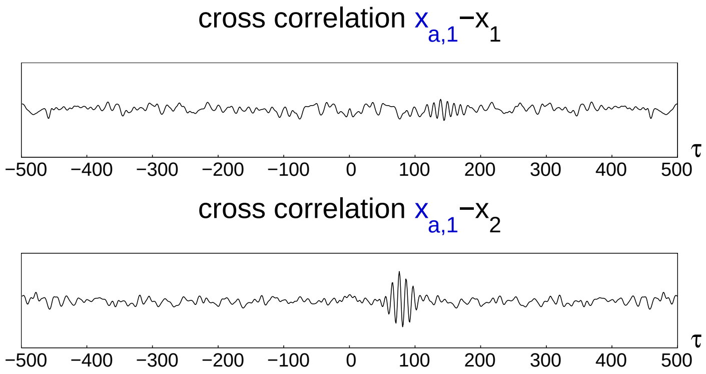
• Example: randomly layered media [2]
→ large fluctuations, but poor directional diversity enhancement.

• Example: three-dimensional weakly scattering medium in the single-scattering regime [3]
→ the scattered waves have small amplitudes compared to the primary energy flux.
→ it is possible to enhance the stability by using fourth-order cross correlations.

[3] J. Garnier and G. Papanicolaou, *SIAM J. Imaging Sciences* **2**, 396 (2009).

Fourth-order cross correlations for travel time estimation



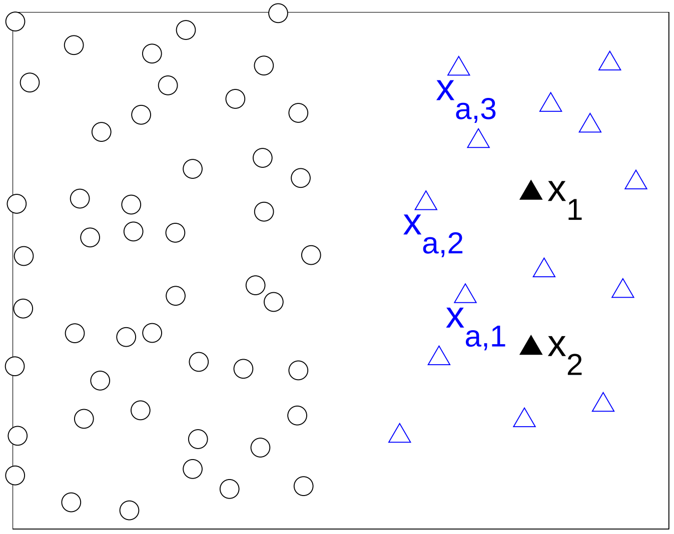


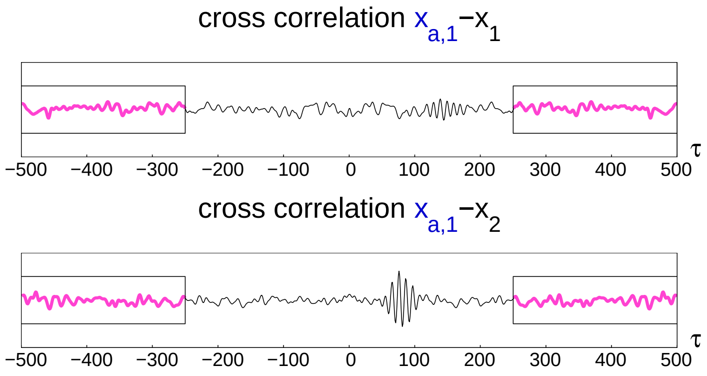
Use of auxiliary sensors $\boldsymbol{x}_{a,j}$, $j = 1, \ldots, N$. Algorithm:

1) for each j, compute the cross correlations $C_T(\tau, \boldsymbol{x}_{a,j}, \boldsymbol{x}_1)$ and $C_T(\tau, \boldsymbol{x}_{a,j}, \boldsymbol{x}_2)$:

$$C_T(\tau, \boldsymbol{x}_{a,j}, \boldsymbol{x}_l) = \frac{1}{T} \int_0^T u(t, \boldsymbol{x}_{a,j}) u(t + \tau, \boldsymbol{x}_l) dt \,, \quad l = 1, 2$$

Fourth-order cross correlations for travel time estimation





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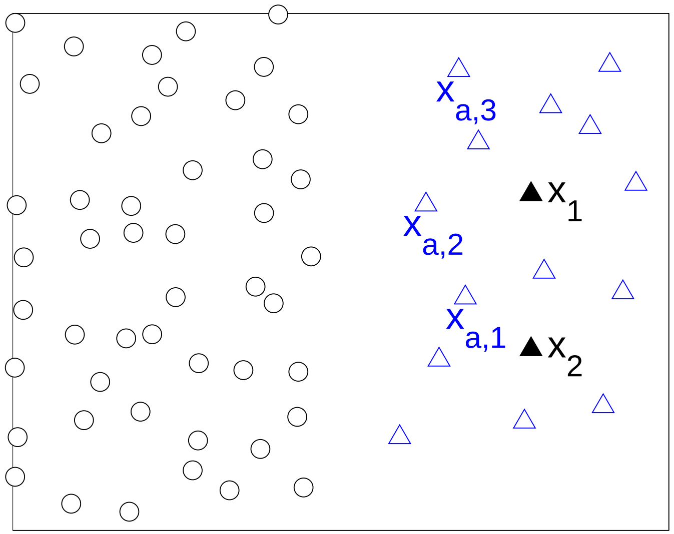
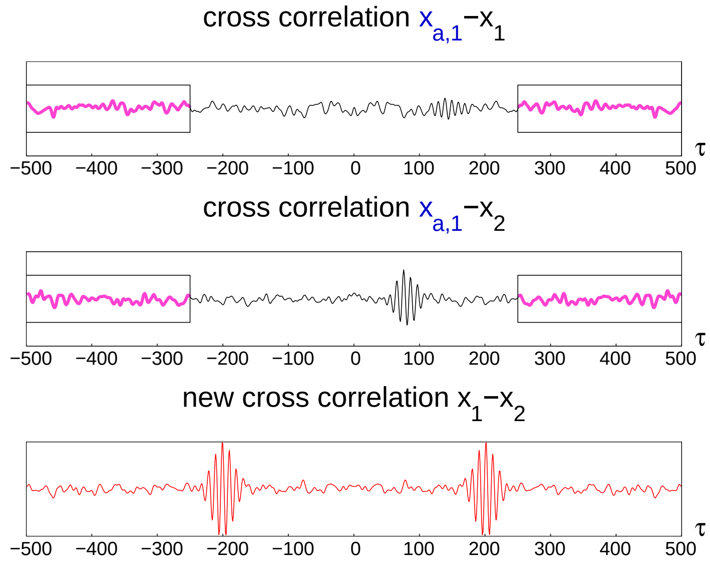
2) consider the tails of $C_T(\tau, \boldsymbol{x}_{a,j}, \boldsymbol{x}_1)$ and $C_T(\tau, \boldsymbol{x}_{a,j}, \boldsymbol{x}_2)$:

$$C_{T,\mathrm{coda}}(\tau, \boldsymbol{x}_{a,j}, \boldsymbol{x}_l) = C_T(\tau, \boldsymbol{x}_{a,j}, \boldsymbol{x}_l) \big[\mathbf{1}_{(-T_{c2}, -T_{c1})}(\tau) + \mathbf{1}_{(T_{c1}, T_{c2})}(\tau) \big], \quad l = 1, 2$$

Fourth-order cross correlations for travel time estimation





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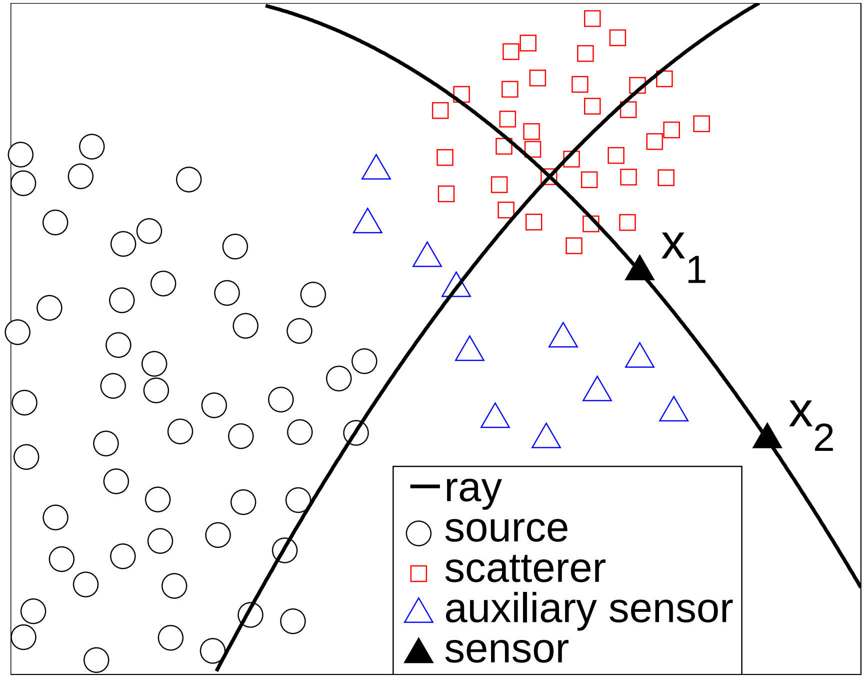
3) compute the cross correlations between the tails and sum over j:

$$C_T^{(3)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) = \sum_{j=1}^{N} \int_{-\infty}^{\infty} C_{T,\mathrm{coda}}(\tau', \boldsymbol{x}_{a,j}, \boldsymbol{x}_1) C_{T,\mathrm{coda}}(\tau' + \tau, \boldsymbol{x}_{a,j}, \boldsymbol{x}_2) d\tau'$$



Using asymptotic analysis [1]: $C^{(3)}$ has singular components if

1) there are scatterers along the ray joining $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$.

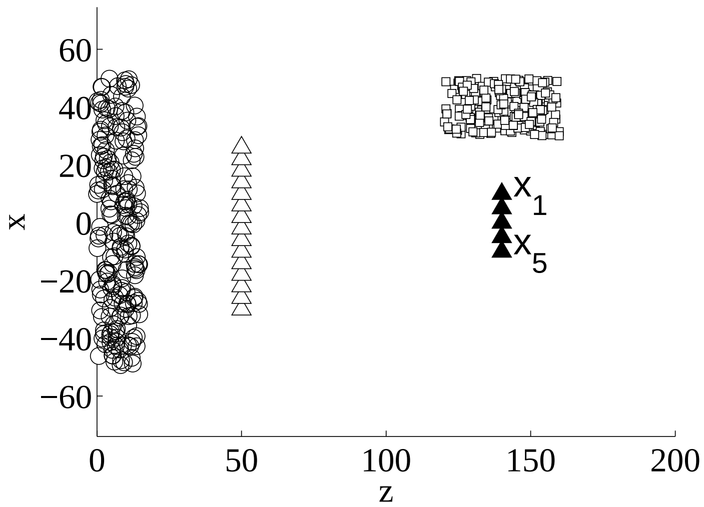
2) there are auxiliary sensors along rays joining sources and scatterers.

These singular components are at $\tau = \pm\mathcal{T}(\boldsymbol{x}_1, \boldsymbol{x}_2)$.

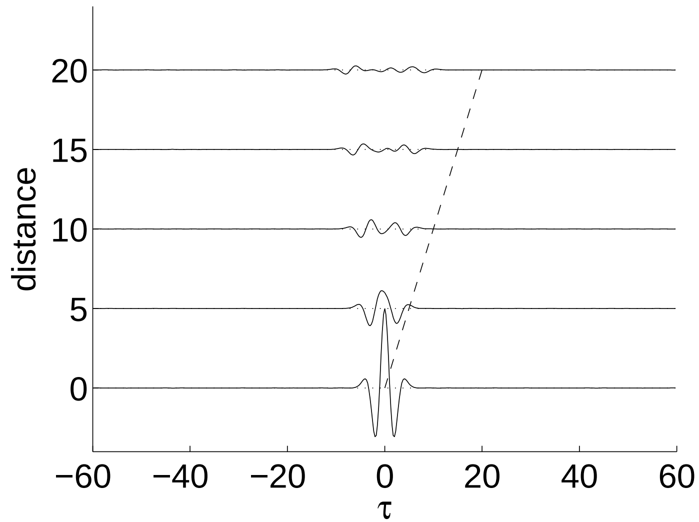
It is not required that the ray joining $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ reaches into the source region !

If the scattering region covers the region of interest or surrounds it, then $C^{(3)}$ has singular components at $\tau = \pm\mathcal{T}(\boldsymbol{x}_1, \boldsymbol{x}_2)$!

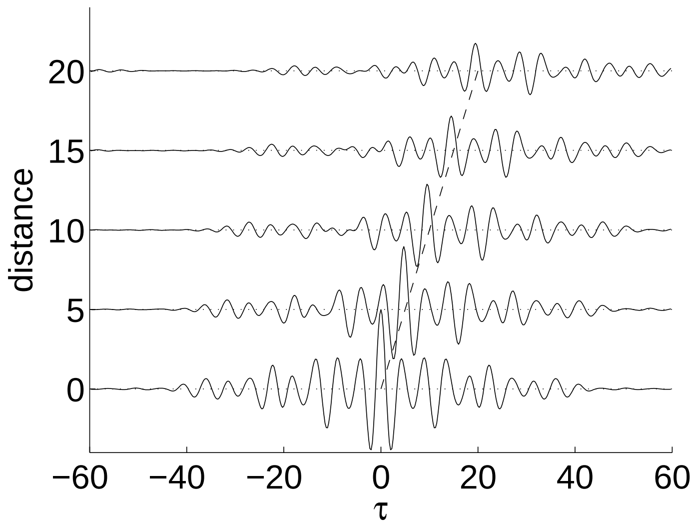
[1] J. Garnier and G. Papanicolaou, *SIAM J. Imaging Sciences* **2**, 396 (2009).



Configuration



$C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_j)$



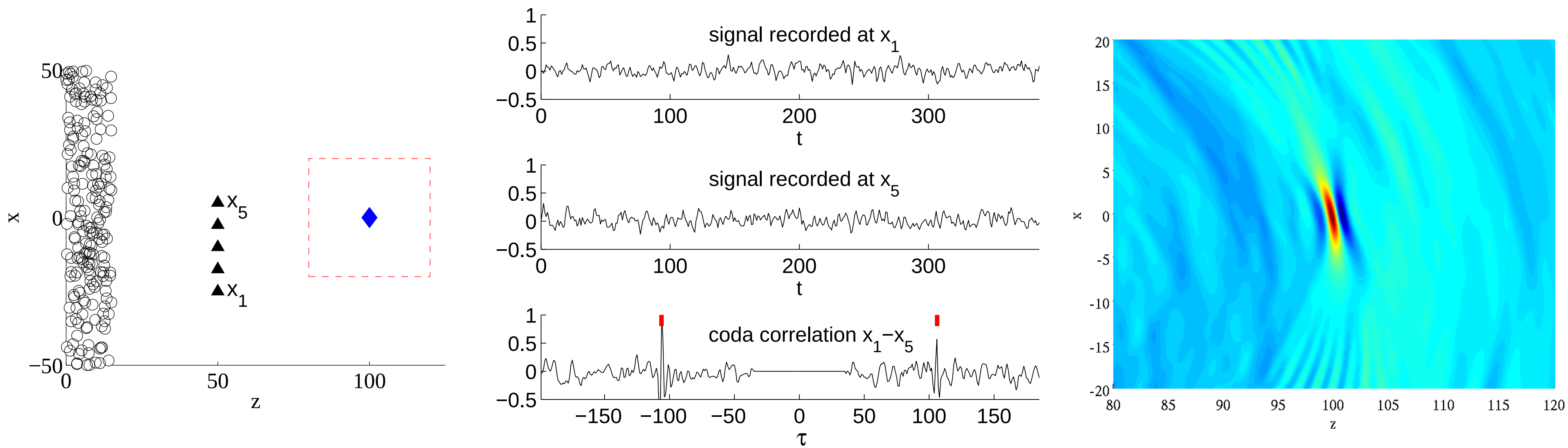
$C^{(3)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_j)$

Reflector imaging with ambient noise signals [1]

- Ambient noise sources ($\circ$) emit stationary random signals.
- The signals $(u(t,\boldsymbol{x}_j))_{j=1,\dots,n}$ are recorded by the sensors $(\boldsymbol{x}_j)_{j=1,\dots,n}$ ($\blacktriangle$).
- The cross correlation matrix is computed:

$$C_{\boldsymbol{x}_i,\boldsymbol{x}_j}(t) = \frac{1}{T}\int_0^T u(s,\boldsymbol{x}_i)u(s+t,\boldsymbol{x}_j)ds$$

- Reflector imaging ($\blacklozenge$) by migration of the cross correlation matrix.



- There is information in the noise (in its correlation structure) !

[1] J. Garnier and G. Papanicolaou, *SIAM J. Imaging Sciences* **2**, 396 (2009).

Conclusion

• Travel time estimation is possible using cross correlation of ambient noise signals. Also: source localization and imaging of reflectors.

• It is possible to exploit the scattering properties of the medium for travel time estimation. The use of special fourth-order cross correlation is helpful.

• It is possible to exploit the scattering properties of the medium for imaging by cross correlating and migrating the coda cross correlations (migration with $C^{(1)}$ and/or $C^{(3)}$).

• Main applications in geophysics (global, regional, and local scales: volcano monitoring, oil reservoir monitoring). Also in microwave imaging.