Remarks on effective dynamics in Quantum Hall systems and the Chalker Coddington model

Joachim Asch

Centre de Physique Théorique, Marseille

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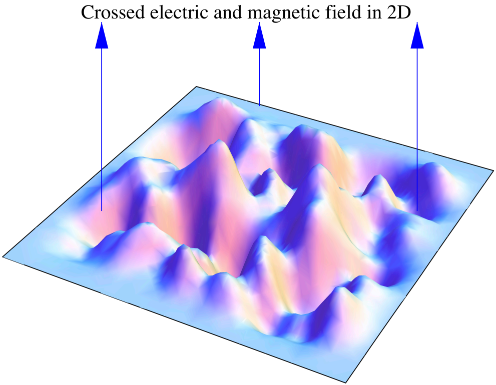
Motivation

Understand the dynamics of a 2d electron gas in crossed magnetic and electric fields. In particular the "localization–delocalization" question of the Quantum Hall effect.



Here: No electron-electron interactions, zero temperature, planar geometry, constant magnetic field
$\rightarrow$ selfadjoint one particle Hamiltonian:

$$H = \frac{1}{2}\left(D - \frac{1}{2}q^{\perp}\right)^2 + V(\ell q) \qquad \text{on } \mathbb{D}(H).$$

$$D := -i\nabla, V \in C^{\infty}(\mathbb{R}^2, \mathbb{R}), \quad \ell := \sqrt{\frac{\hbar}{|eB|}}.$$

$$v := D - \frac{q^{\perp}}{2} \qquad \text{velocity,}$$

$$\frac{1}{2}v^2 =: H^{La} \qquad \text{kinetic energy.}$$

$V = 0$: v rotates around center c.

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Known (Kruskal, Gardner & cie.) :
the kinetic energy (magnetic moment) is an adiabatic invariant of the corresponding classical system for ℓ small; the variation in time in small to any order in ℓ for times up any order in $1/\ell$.

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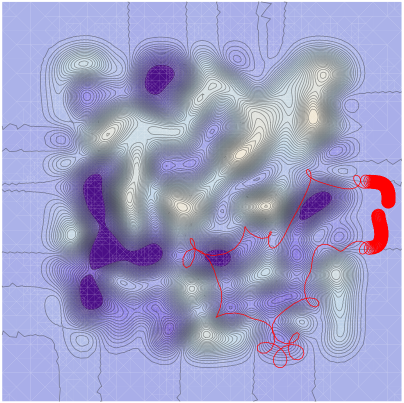
Trajectory in Gaussian random field



Classical Intuition

Precision of the Quantum Hall effect
$\longrightarrow$ Question: Is it possible to go to the limit of infinite times and to construct an exact invariant in the quantum case ?

V quadratic polynomial

Guiding center coordinates:

$$c := -D^{\perp} + \frac{q}{2}. \qquad q = c + v^{\perp};$$

$$[v_1, v_2] = i \ , \quad [c_2, c_1] = i \ , \quad [c_i, v_j] = 0.$$

Degree 1 case: $E \in \mathbb{R}^2, \quad V(q) := -\langle E, q \rangle$. There exists an effective Hamiltonian H_{eff} with $[H_{eff}, H^{La}] = 0$ and a constant of motion H_{inv} with $[H_{inv}, H] = 0$.

Proposition

Define $\mathcal{U} := e^{i\langle E, v \rangle}$.

$H_{eff} := \mathcal{U}(H^{La} + V)\mathcal{U}^{-1} = H^{La} - \langle E, c \rangle - \frac{1}{2}E^2 = \mathbf{D}H - \frac{1}{2}E^2,$

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$$H = \frac{1}{2}\omega\left(\begin{pmatrix} c \\ v^\perp \end{pmatrix}, \mathbb{H}\begin{pmatrix} c \\ v^\perp \end{pmatrix}\right)$$

with ω the symplectic structure and

$$\mathbb{H} := \begin{pmatrix} \sigma V'' & \sigma V'' \\ -\sigma V'' & -\sigma - \sigma V'' \end{pmatrix}, \quad \sigma := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Proposition

If $1 + trV'' > 0$ and $(1 + trV'')^2 - 4\det V'' \geq 0$ then there exists a metaplectic transformation $\mathcal{U}$ and an $\omega > 0$ such that

$$H_{eff} := \mathcal{U} H \mathcal{U}^{-1} = \frac{1}{2}\omega\left(\begin{pmatrix} c \\ v^\perp \end{pmatrix}, \begin{pmatrix} \mathbb{H}_c & 0 \\ 0 & -\omega\sigma \end{pmatrix}\begin{pmatrix} c \\ v^\perp \end{pmatrix}\right);$$

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Exemple: Straight parabolic channel

Proposition

Let $V(q) = \frac{a}{2} q_1^2$. Let $\omega := \sqrt{1+a}$ be real then there exists a unitary metaplectic transformation $\mathcal{U}$ such that

$$H_{eff} := \mathcal{U} H \mathcal{U}^{-1} = \frac{1}{2}\left(\frac{a}{\omega^2} c_1^2 + \omega v^2\right);$$

$$H_{inv} := \mathcal{U}^{-1} H^{La} \mathcal{U} = \frac{1}{2}\left(\frac{a^2}{\omega^3} c_1^2 + 2\frac{a}{\omega} c_1 v_1 + \omega v_1^2 + \frac{1}{\omega} v_2^2\right)$$

is a constant of motion.

“General” potentials

Consider the class of potentials:

$$\mathcal{G} := \{V : \mathbb{R}^2 \to \mathbb{R}; V = g * \mu, \int_{\mathbb{R}^2} d|\mu| < \infty\}.$$

where

$$g(q) := e^{-\frac{q^2}{2}} \qquad (q \in \mathbb{R}^2)$$

and μ a real valued finite measure μ.

Theorem

For $V \in \mathcal{G}$ small enough there exists a unitary operator $\mathcal{U}$ such that for

$$H_{eff} := \mathcal{U}\left(H^{La} + V\right)\mathcal{U}^{-1}, \qquad H_{inv} := \mathcal{U}^{-1} H^{La} \mathcal{U} :$$

$$\left[H_{eff}, H^{La}\right] = 0, \qquad H_{inv} = e^{iHt} H_{inv} e^{-iHt} \quad (t \in \mathbb{R}).$$

Proof

1. Construct $\mathcal{U}$ with an abstract result on invariant subspace perturbation theory. (Fast converging iterative algorithm à la Bellissard). Needs decay

$$\|P_n^{La} V P_m^{La}\| \leq \frac{c}{\langle n-m \rangle};$$

2. Applicability to the the class of potentials.

Open Problems

1: Decide if an extension of the class of potentials in view of the usual i.i.d distributed models is possible or not.
2: Understand the classical status of the problem.

Remark on the effective Hamiltonian

For $V \in \mathcal{G}$, $H = H^{La} + \varepsilon V$, H_{eff} is given in first order by

$$\mathbf{D}H = \frac{1}{2\pi} \int_0^{2\pi} e^{iH^{La}t} H e^{-iH^{La}t} \, dt = Op^w(< H >)$$

a pseudodifferential operator whose Weyl-Symbol is the average of the symbol of H over the Landau orbits:

$$< H > (q,p) := \underbrace{\frac{1}{2}\left(p - \frac{q^\perp}{2}\right)^2}_{=:h^{La}} + < V > (q,p),$$

$$< V > (q,p) := \frac{1}{2\pi} \int_0^{2\pi} V\left(c + \sqrt{2h^{La}}(\cos t, \sin t)\right) dt$$

with $c := -p^\perp + \frac{q}{2}$.

Remark

$$< V > = < V > (c_1, c_2, h^{La}) \text{ and } \{c_2, c_1\} = \{q_1, p_1\}$$

so for fixed h^{La} the effective $< H >$ generates a Hamiltonian flow on the so called "non commutative" c_2, c_1 plane of guiding center motion.

Remark on one Landau Level

Let P be the projection on one Landau Level and choose $\{P, P^{\perp} := 1 - P\}$ as complete family of projections, then it follows

Theorem

For $V \in \mathbb{B}\left(L^2(\mathbb{R}^2)\right)$ small enough there exists a unitary operator $\mathcal{U}$ such that for

$$H_{eff} := \mathcal{U}\left(H^{La} + V\right)\mathcal{U}^{-1}, \qquad P_{inv} := \mathcal{U}^{-1} P \mathcal{U} :$$

$$[H_{eff}, P] = 0, \qquad P_{inv} = e^{iHt} P_{inv} e^{-iHt} \quad (t \in \mathbb{R}).$$

The restriction of H_{eff} to *Ran* P is determined to first order by

$$Op^w\left(\frac{1}{2\pi}\int_0^{2\pi} V\left(c + (\cos t, \sin t)\right) dt\right)$$

if V is a function.

The Chalker Coddington Model

First step: use the effective Hamiltonian in one Landau band

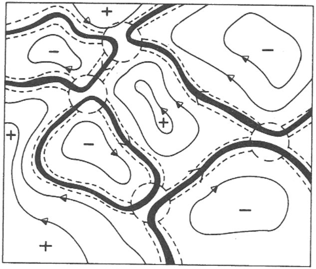


Figure 1. Sketch of a typical potential, $V(x, y)$. Full curves represent equipotentials and arrows give direction of guiding centre motion; + and − denote maxima and minima. Heavy curves indicate contours at potential V_0. Portions of these contours are enclosed in strips and circles (broken lines) which correspond to links and nodes of the network model.

The Chalker Coddington Model

Second step: focus on tunneling at saddle points

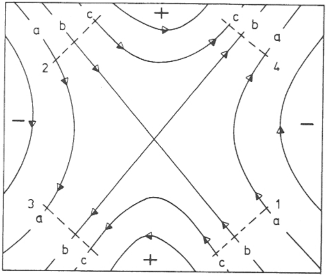


Figure 2. Sketch of a saddle point in the potential, $V(x, y)$. Full curves represent contours and arrows give direction of guiding centre motion. Amplitudes Z_1, Z_2, Z_3 and Z_4 (see

The Chalker Coddington Model

and describe tunneling events by scattering matrices (cf.: [CdVP])

$$S(q) := \begin{pmatrix} q_1 q_2 & 0 \\ 0 & q_1 \overline{q_2} \end{pmatrix} \begin{pmatrix} t & -r \\ r & t \end{pmatrix} \begin{pmatrix} q_3 & 0 \\ 0 & \overline{q_3} \end{pmatrix} \in U(2)$$

where

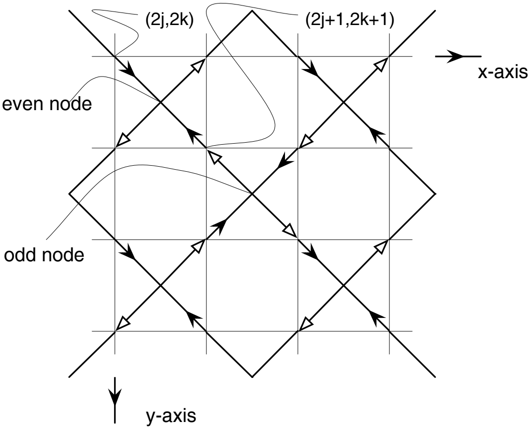
$$r, t \in [0,1], \text{ such that}, r^2 + t^2 = 1,$$

and $q = (q_1, q_2, q_3) \in \mathbb{T}^3$.

The 3 random phases per node account for the randomness of the potential, parameters t, r are deterministic.

The Chalker Coddington Model

Third step: consider a network of tunneling events



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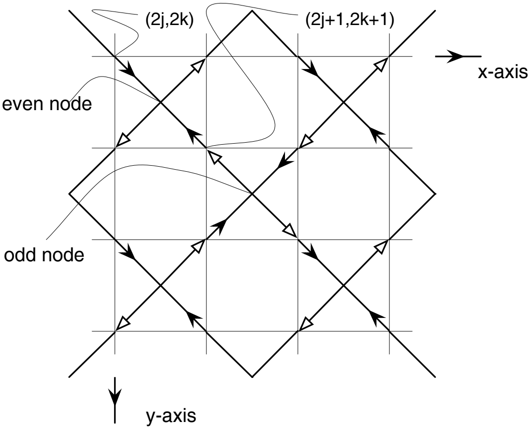

with dynamics described by the family of random unitary (time one) operators (valleys: $|t|$: tunnel; $|r|$: stay)

$$\widehat{U}(p) : l^2(\mathbb{Z}^2) \to l^2(\mathbb{Z}^2)$$

defined by its matrix elements $\widehat{U}_{\mu;\nu} := 0$ except for the blocks describing scattering at the "even" and "odd" nodes.

$$\begin{pmatrix} \widehat{U}(p)_{(2j+1,2k);(2j,2k)} & \widehat{U}(p)_{(2j+1,2k);(2j+1,2k+1)} \\ \widehat{U}(p)_{(2j,2k+1);(2j,2k)} & \widehat{U}(p)_{(2j,2k+1);(2j+1,2k+1)} \end{pmatrix}$$

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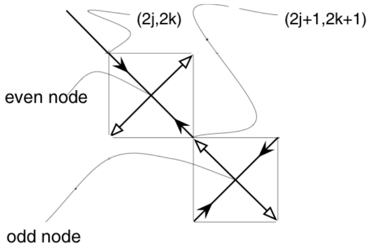
Probability space

We suppose that the random phases are uniformly and independently distributed. The family is ergodic for translations shifting the blocks



The probability space is $\widehat{\Omega}$, with

$$p \in \widehat{\Omega}, \quad p(2j,2k) =: (\underbrace{p_1, p_2, p_3}_{p_e(2j,2k)}, \underbrace{p_4, p_5, p_6}_{p_o(2j+1,2k+1)}).$$

The measure $\otimes_{(2\mathbb{Z})^2} d^6 l$

Purpose of the model

Study the localization length as a method to understand the delocalization phenomenon.
CC's numerical result:

-the localization length is finite for $t \neq r$

-the localization length diverges as $t/r \to 1$ as

$$\left| \frac{1}{\ln\left|\frac{t}{r}\right|} \right|^{\alpha}$$

with critical exponent $\alpha = 2.5 \pm 0.5$.

CC's method: finite size scaling based on the restriction of the model to a cylinder of perimeter M.
Our aim here: proof finiteness of the localization length for the restriction of the model to a cylinder of perimeter M, get bounds.

Exponents, construction

–Reduce the model by a unitary diagonal transformation to a model with two phases per node sitting in the outgoing links $\rightarrow$ new $\widehat{U} = D\mathbb{S}$ with D random diagonal, $\mathbb{S}$ deterministic;
–restrict to a cylinder of perimeter $2M$ $\rightarrow$ new random unitary U_M on $l^2(\mathbb{Z} \times \mathbb{Z}_{2m})$;
–for U_M write the spectral- as a transfer matrix problem;
–define the Lyaponov exponents for the cocycle defined by the transfer matrices;
–define the localization length as the inverse of the smallest positive Lyapunov exponent;

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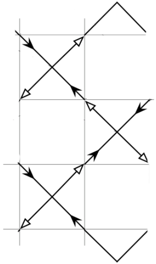
–restrict to a cylinder of perimeter $2M \rightarrow$ new random unitary U_M on $l^2(\mathbb{Z} \times \mathbb{Z}_{2m})$;
–for U_M write the spectral- as a transfer matrix problem;
–define the Lyaponov exponents for the cocycle defined by the transfer matrices;
–define the localization length as the inverse of the smallest positive Lyapunov exponent;

Exponents, construction



$$\sum_{\nu \in \mathbb{Z}\times\mathbb{Z}_{2M}} U_{\mu\nu}\psi_\nu = z\psi_\mu \qquad \forall \mu \in \mathbb{Z}\times\mathbb{Z}_{2M}$$

$$\Longleftrightarrow$$

$$\begin{pmatrix} \psi_{2j+1,2k} \\ \psi_{2j+1,2k+1} \end{pmatrix} = T_{eo}(z,p) \begin{pmatrix} \psi_{2j,2k} \\ \psi_{2j,2k+1} \end{pmatrix},$$

$$\begin{pmatrix} \psi_{2j+2,2k+1} \\ \psi_{2j+2,2k+2} \end{pmatrix} = T_{oe}(z,p) \begin{pmatrix} \psi_{2j+1,2k+1} \\ \psi_{2j+1,2k+2} \end{pmatrix}.$$

Exponents, construction

$$T \in \underbrace{\left\{ B \in \mathbb{M}_{2,2}(\mathbb{C}); B^* J B = J, \quad J := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}}_{=: U(1,1) = \text{ Lorentzgroup}}$$

$$T_{eo} = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{t} \begin{pmatrix} z^{-1} & -r \\ -r & z \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix},$$

$$T_{oe} = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \frac{1}{r} \begin{pmatrix} z & -t \\ t & -z^{-1} \end{pmatrix} \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}.$$

Exponents, construction

$$D(q) := \begin{pmatrix} q_1 & & \\ & \ddots & \\ & & q_{2M} \end{pmatrix},$$

$$M_1(z) := \frac{1}{t}\begin{pmatrix} z^{-1} & -r & & & \\ -r & z & & & \\ & & \ddots & & \\ & & & z^{-1} & -r \\ & & & -r & z \end{pmatrix},$$

$$M_2(z) := \frac{1}{r}\begin{pmatrix} -z^{-1} & & & & & & t \\ & z & -t & & & & \\ & t & -z^{-1} & & & & \\ & & & \ddots & & & \\ & & & & z & -t & \\ & & & & t & -z^{-1} & \\ -t & & & & & & z \end{pmatrix}.$$

Exponents, construction

Define for a fixed $z \neq 0$

$$\begin{aligned} A_z : \Omega &\rightarrow U_M(1,1) \\ A_z(p) &:= D(p_l)\, M_2(z) D(p_m)\, M_1(z) D(p_r)\,. \end{aligned}$$

Then A generates the cocycle Φ over the ergodic dynamical system

$$(\Omega, \mathcal{F}, \mathbb{P}, (\Theta^n)_{n \in \mathbb{Z}})$$

defined by $\Phi_z : \mathbb{Z} \times \Omega \rightarrow U_M(1,1)$

$$\Phi_z(n,p) := \begin{cases} A(\Theta^{n-1}p) \ldots A(p) & n > 0 \\ \mathbb{I} & n = 0 \\ A^{-1}(\Theta^n p) \ldots A^{-1}(\Theta^{-1}p) & n < 0 \end{cases}$$

with Ω and Θ defined according to the two slice.

Exponents, construction

$$U(p)\psi = z\psi \Longleftrightarrow \Psi_{2N} = \Phi_z(N, f(p))\Psi_0.$$

Oseledets for $\Phi \Longrightarrow$
for $z \neq 0$ there exists an invariant subset of full measure of $p \in \Omega$ such that the limits

$$\lim_{n \to \pm\infty} (\Phi_z^*(n,p)\Phi_z(n,p))^{1/2|n|} =: \Psi_z(p)$$

exist.
Denote by $\gamma_k(p,z) \quad k \in \{1, \ldots, 2M\}$ the eigenvalues of $\Psi_z(p)$ arranged in decreasing order. By ergodicity
$\gamma_k(p,z) = \gamma_k(z) \quad p \ a.e.$

$$\lambda_k(z) := \log \gamma_k(z).$$

Exponents, construction

Scaling $\Longrightarrow$

$$\partial_z \lambda_i = 0.$$

Lorentz symmetry $\Longrightarrow$

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_M \geq 0 \geq -\lambda_M \geq \ldots \geq -\lambda_1.$$

Definition

The localization length $\xi_M \in [0, \infty]$ *is defined as*

$$\xi_M := \frac{1}{\lambda_M}.$$

Exponents: Results

–use "Bougerol - Lacroix" theory to show that the Lyapunov spectrum is simple thus

$$\xi_M < \infty;$$

–proof a Thouless formula and compute the mean exponent

$$\frac{1}{M}\sum_{i}^{M}\lambda_i;$$

–use the mean exponent to get a lower bound on the localization length;
–use "Sh'nol" theory to show that the spectrum of U_M is pure point with exponentially decaying eigenfunctions.

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Exponents: simplicity

Theorem

For $rt \neq 0, z \in \mathbb{T}$ it holds

$$\lambda_1 > \lambda_2 > \ldots > \lambda_M > 0.$$

Proof. Follow the strategy of Bougerol Lacroix.

$G :=$ the smallest subgroup of $U(M,M)$ generated by $\{A(p)\}$.

Show that

$$G \cong U(M,M) \cong Sp(M,\mathbb{C}),$$

in particular that G is connected. Then show irreducibility and contraction properties by hand.

Exponents: simplicity

Remark

We have proved by hand that if the transfer matrices generate the complex symplectic group $Sp(M, \mathbb{C})$ then the results of Bougerol Lacroix apply, i.e.: the Lyapunov spectrum is simple. The results in Bougerol Lacroix are stated for real groups only. While it is remarked in their introduction that these results should hold in the complex case, this seems not to be obvious to specialists in the field.

Problem to be clarified !

Exponents: Thouless formula

Theorem

It holds:

$$\frac{1}{M}\sum_{i=1}^{M}\lambda_i = \frac{1}{2}\log\frac{1}{rt} \geq \frac{1}{2}\log 2$$

Proof. Follow Craig Simon to proof Thouless formula for the unitaries a hand, use that the density of states is flat because of the uniformly distributed phases.

Exponents: Estimate

We estimate the cocycle to derive an upper bound on the largest Lyapunov exponent, which is uniform in the quasienergy and the width M of the strip.

Proposition

1. $rt \neq 0$. *For the generator of the cocycle it holds*

$$\|A(p)\| \leq \frac{1}{rt}(1+r)(1+t);$$

2. *it follows:* $2\lambda_1 \leq \log\left(\frac{1}{rt}\right) + \log\left((1+r)(1+t)\right)$.
3. *There exists a* $c > 0$ *such that for* $M \in \mathbb{N}$ *it holds:*

$$dist(r, \{0,1\}) < e^{-cM} \implies$$
$$\xi_M = \frac{1}{\lambda_M} \leq \frac{2}{\log\left(\frac{1}{rt}\right) - (M-1)\log\left((1+r)(1+t)\right)}.$$

Localization

Theorem

Let $M \in \mathbb{N}$, $rt \neq 0$. Then

1. *the almost sure : spectrum Σ, continuous spectrum Σ_c and pure point spectrum Σ_{pp} of $U(p)$ satisfy*

$$\Sigma = \Sigma_{pp} = \mathbb{T} \qquad \textit{and} \qquad \Sigma_c = \emptyset;$$

2. *the eigenfunctions decay exponentially, almost surely.*

Proof. Follow the strategy known for discrete Schrödinger operators : polynomial boundedness of generalized eigenfunctions, positivity of the Lyapunov exponent and spectral averaging. Use the work of Bourget, Hamza, Howland, Joye, Stolz.

Credits

Collaboration with : Olivier Bourget, CédricMéresse, Alain Joye

This material in

J. Asch, C. Meresse, A constant of quantum motion in two dimensions in crossed magnetic and electric fields J. Phys. A: Math. Theor. 43 (2010) 474002.

J. Asch, O. Bourget, A. Joye, Localization Properties of the Chalker-Coddington Model Annales Henri Poincare, 11 (2010), 1341–1373

Abstract result

$H^{La} = \sum_{n \in \mathbb{N}_0} (n + 1/2) P_n^{La}$ with infinite dimensional projections P_n^{La}.

Consider H with a finitely many or infinitely many bands with non decreasing gap width.

Definition

A selfadjoint operator H is of class $\mathcal{C}_{\mathfrak{g}}$ for a $\mathfrak{g} > 0$ if for a complete family of orthogonal, mutually disjoint projections $\{P_n\}_{n \in I \subseteq \mathbb{N}_0}$ which commute with H, it holds for

$$\sigma_n := spect\left(P_n H P_n \restriction_{RanP_n}\right) :$$

$$min\sigma_{n+1} - \max \sigma_n \geq \mathfrak{g}.$$

Figure: Typical spectrum of H

Define

$$\mathbf{D}V := \sum_{n\in I} P_n V P_n, \qquad \mathbf{O}V := V - \mathbf{D}V.$$

$$\langle a\rangle := \max(1,|a|), \qquad \|A\|_l := \sup_{n,m\in I} \langle n-m\rangle^l \|P_n A P_m\|$$

Then prove

Theorem (Invariant subspace perturbation theory)

Let $H \in \mathcal{C}_{\mathfrak{g}}$ and V be a bounded selfadjoint operator such that

$$\|V\|_1 \leq \frac{\mathfrak{g}}{8}.$$

Then there exists a unitary $\mathcal{U}$ such that $\mathcal{U}^{-1}D(H) \subset D(H)$ with the property that for

$$H_{eff} := \mathcal{U}(H+V)\mathcal{U}^{-1}, \qquad D(H_{eff}) = D(H)$$

it holds for all n:

$$[H_{eff}, P_n] = 0.$$

Main ingredient of the proof:
The uniform gap condition allows for the existence of a bounded solution of the commutator equation:

Lemma (Γ)

Let $H \in \mathcal{C}_{\mathfrak{g}}$ *and* V *a bounded selfadjoint operator such that* $\mathbf{D}V = 0$ *and such that* $\|V\|_1 < \infty$*. Then there exists a bounded antiselfadjoint* W *and* $c > 0$ *such that*

$$[H, W] = V, \qquad \mathbf{D}W = 0$$

such that

$$\|W\| \leq \frac{c}{\mathfrak{g}} \|V\|_1,$$

$$\|W\|_2 \leq \frac{c}{\mathfrak{g}} \|V\|_1.$$

Proof of Theorem (Inv. subspace...)

Define $H_0 := H + V = \mathbf{D}H_0 + \mathbf{O}H_0$. As $\|\mathbf{O}H_0\|_1 = \|OV\|_1 < \frac{\mathfrak{g}}{8}$, $\mathbf{D}H_0 \in \mathcal{G}_{1/4}$. By lemma Γ there exists a bounded solution W_0 of

$$[\mathbf{D}H_0, W_0] = \mathbf{O}H_0, \qquad \mathbf{D}W_0 = 0$$

Define the unitary $\mathcal{U}_0 := e^{W_0}$ .Then

$$e^{W_0}H_0e^{-W_0} - \mathbf{D}H_0 = \mathcal{O}(\|\mathbf{O}H_0\|^2).$$

Suppose that for $s \in \mathbb{N}$ diagonalization has been done up to $H_s, \mathcal{U}_{s-1}$ such that $\|\mathbf{D}H_s - H\| \leq \mathfrak{g}/4$, $\|\mathbf{O}H_s\|_1 < \infty$. To go to step $s+1$, use again lemma Γ to solve

$$[\mathbf{D}H_s, W_s] = \mathbf{O}H_s, \qquad \mathbf{D}W_s = 0$$

for a bounded W_s and define

$$\mathcal{U}_s := e^{W_s}\mathcal{U}_{s-1}, \qquad H_{s+1} := e^{W_s}H_se^{-W_s}.$$

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$$\mathcal{U}_s := e^{W_s}\mathcal{U}_{s-1}, \qquad H_{s+1} := e^{W_s}H_se^{-W_s}.$$

Prove

$$\|[W_s, \mathbf{O}H_s]\|_1 \leq 2c\|W_s\|_2\|\mathbf{O}H_s\|_1.$$

and deduce

$$\|\mathbf{O}H_s\|_1 \leq \lambda^{2^s}$$

with $\lambda < 1$ proportional to $\|V\|_1$. ∎

Class G

It is a nontrivial problem to determine a sufficiently large class of potentials V satisfying the condition

$$\|V\|_1 < \infty.$$

Proposition

Let $|\mu|(\mathbb{R}^2) < \infty$, $V = e^{-\#^2/2} * \mu$ and P_n^{La} the eigenprojections of H^{La} it holds :

$$\|V\|_1 = \sup_{n,m\in\mathbb{N}_0} \langle n-m\rangle^l \|P_n^{La} V P_m^{La}\| < \infty.$$

Proof

The ingredients of the proof are:
1-an explicit calculation of all the transition coefficients of $e^{-q^2/2}$ using an observation of W.M. Wang;
2-a probabilistic estimate of the combinatorial quantities;
3-translation invariance. It is crucial that the variance of the gaussian equals 1.

Lemma

For $g(q) = \exp\left(-\frac{q^2}{2}\right), \quad (q \in \mathbb{R}^2)$ *and the radial Landau eigenfunctions*

$$\psi_{n,l}(r,\Theta) := (-1)^n \sqrt{\frac{n!}{2^l(n+l)!}}\, r^l e^{i\Theta l} L_n^l\left(\frac{r^2}{2}\right) \frac{e^{-\frac{r^2}{4}}}{\sqrt{2\pi}}$$

it holds for $n, m \in \mathbb{N}_0, l \geq -(n \wedge m)$

$$|\langle \psi_{n,l}, g\psi_{m,l}\rangle| = \frac{1}{2^{l+m+n+1}} \frac{(l+m+n)!}{\sqrt{(l+m)!(l+n)!n!m!}} \,.$$

Proof.

$|\langle \psi_{n,l}, g\psi_{m,l} \rangle| =$

$$\frac{1}{2^l}\sqrt{\frac{n!m!}{(l+n)!(l+m)!}}\int_0^\infty e^{-r^2} r^{2l} L_n^l L_m^l \left(\frac{r^2}{2}\right) rdr.$$

For $l \geq 0$. Use that the family $n \mapsto L_n^l(x)$ is orthogonal in $L^2(\mathbb{R}_+, d\nu_l)$, $d\nu_l := x^l e^{-x} dx$ and the identity:

$$L_n^l L_m^l \left(\frac{x}{2}\right) = \sum_{s \geq 0} B_s^{n,m,l} L_s^l(x).$$

As $L_0^l \equiv 1 \, \forall l$, one has

$$\int_0^\infty L_n^l L_m^l \left(\frac{x}{2}\right) d\nu_l(x) = \sum_{s \geq 0} B_s^{n,m,l} \int_0^\infty L_s^l L_0^l(x) d\nu_l(x)$$

$$= B_0^{n,m,l} \Gamma(l+1) = \frac{1}{2^{m+n}} \frac{(l+m+n)!}{l!\, m!\, n!} l!$$

Lemma

For a $c > 0, n, m \in \mathbb{N}_0$, *it holds*

$$\frac{(m+n)!}{2^{m+n} n! m!} \leq \frac{c}{\langle m-n \rangle}$$

$$\frac{(l+m+n)!}{2^{l+m+n+1}\sqrt{(l+m)!(l+n)!n!m!}} \leq \frac{c}{\langle m-n \rangle} \qquad (l \geq m \wedge n).$$