KAM for NLS with harmonic potential

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(Joint work with Benoît Grébert)

Introduction

The equation : We consider the nonlinear Schrödinger equation with harmonic potential

$$\begin{cases} i\partial_t u + \partial_x^2 u - x^2 u = |u|^2 u, & (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(0,x) = u_0(x), \end{cases}$$

Physical interest : Model for Bose-Einstein condensates.

Litterature : R. Fukuizumi, K. Yajima - G. Zang, R. Carles, ...

Motivation : The equation is globally well-posed in the energy space.

Let $p > 1$. Behaviour of

$$\|u(t)\|_{\mathcal{H}^p(\mathbb{R})}$$

when $t \to \infty$?

Introduction

Difficulty : Spectral structure of $-\partial_x^2 + x^2$: the eigenvalues are $\lambda_j = 2j - 1$, $j \geq 1$ and are completely resonant in the sense that there exist many $k \in \mathbb{N}^\infty$ of finite length so that $k \cdot \lambda = \sum_{j \geq 1} k_j \lambda_j = 0$.
Therefore, we consider

$$\begin{cases} i\partial_t u + \partial_x^2 u - x^2 u + \varepsilon V(x) u = \varepsilon |u|^2 u, & (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(0,x) = u_0(x). \end{cases} \qquad (NLS)$$

where $\varepsilon \ll 1$ and $V \in \mathcal{S}(\mathbb{R}, \mathbb{R})$.

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where $\varepsilon \ll 1$ and $V \in \mathcal{S}(\mathbb{R}, \mathbb{R})$.

Aim :
Construction of quasi-periodic in time solutions to (NLS) for typical V. In particular, the $\mathcal{H}^p$ norm of these solutions will be bounded.

Quasi-periodicity :
$f : \mathbb{R} \longrightarrow \mathbb{C}$, $t \mapsto f(t)$ is quasi-periodic if there exist $n \geq 1$, a periodic function $U : \mathbb{T}^n \longrightarrow \mathbb{C}$ and $(\omega_1, \dots, \omega_n) \in \mathbb{R}^n$ so that for all $t \in \mathbb{R}$, $f(t) = U(\omega_1 t, \dots, \omega_n t)$.

Introduction

Denote by $A = -\partial_x^2 + x^2 - \varepsilon V(x)$.

- There exists an Hilbertian basis of $L^2(\mathbb{R})$ of eigenfunctions $(\varphi_j)_{j\geq 1}$ of A

$$A\varphi_j = \lambda_j \varphi_j, \quad \text{with} \quad \lambda_j \sim 2j-1 \quad \text{and} \quad \varphi_j \sim h_j,$$

  where $(h_j)_{j\geq 1}$ are the Hermite functions.

- For $p \geq 0$, we define the Sobolev spaces

$$\mathcal{H}^p = \mathcal{H}^p(\mathbb{R}) = \big\{ u \in \mathcal{S}'(\mathbb{R}) \ : \ A^{p/2} u \in L^2(\mathbb{R}) \big\}.$$

- Let $u = \sum_{j\geq 1} w_j \varphi_j \in \mathcal{H}^p$, then

$$\|u\|^2_{\mathcal{H}^p} \sim \sum_{j\geq 1} j^p |w_j|^2.$$

The result on the nonlinear equation

Our main result concerning the nonlinear Schrödinger equation (NLS) is the following

Theorem (B. Grébert - LT)

Let $n \geq 1$ be an integer. Then there exist a large class of $V \in \mathcal{S}(\mathbb{R})$ and $\varepsilon_0 > 0$ such that for each $\varepsilon < \varepsilon_0$ the solution of (NLS) with initial datum

$$u_0(x) = \sum_{j=1}^{n} I_j^{1/2} e^{i\theta_j} \varphi_j(x), \qquad (IC)$$

with $(I_1, \cdots, I_n) \subset (0,1]^n$ and $(\theta_1, \ldots, \theta_n) \in \mathbb{T}^n$, is quasi-periodic.

- When θ covers $\mathbb{T}^n$, the set of solutions of (NLS) with initial condition (IC) covers a n dimensional torus which is invariant by (NLS).
- Our result also applies to any non linearity $\pm|u|^{2m}u$, with $m \geq 1$.
- The set $\{1, \cdots, n\}$ can be replaced by any finite set of $\mathbb{N}$ of cardinality n.

The more precise result

Let $n \geq 1$ and $\Pi = [-1,1]^n$. There exist $(f_k)_{1\leq k\leq n} \in \mathcal{S}(\mathbb{R})$ such that if we set

$$V(x,\xi) = \sum_{j=1}^{n} \xi_k f_k(x),$$

with $\xi = (\xi_1, \ldots, \xi_n) \in \Pi$ we have

Theorem (B. Grébert - LT)

Let $n \geq 1$ be an integer. Then there exists a Cantor set $\widetilde{\Pi} \subset \Pi$ of full measure and $\varepsilon_0 > 0$ so that for each $\varepsilon < \varepsilon_0$ and $\xi \in \widetilde{\Pi}$ the solution of (NLS) with initial datum

$$u_0(x) = \sum_{j=1}^{n} I_j^{1/2} e^{i\theta_j} \varphi_j(\xi, x), \qquad (IC)$$

with $(I_1, \cdots, I_n) \subset (0,1]^n$ and $(\theta_1, \ldots, \theta_n) \in \mathbb{T}^n$, is quasi-periodic.

The solution u reads

$$u(t,x) = \sum_{j=1}^{n} \left(I_j + y_j(t)\right)^{\frac{1}{2}} e^{i\theta_j(t)} \varphi_j(\xi, x) + \sum_{j\geq 1} z_j(t) \varphi_{j+n}(\xi, x).$$

Some previous results

- S.B. Kuksin '93 and J. Pöschel '96 :
  Case $\lambda_j \sim cj^d$ with $d > 1$ or with smoothing nonlinearity.
- S.B. Kuksin & J. Pöschel '96 :
  NLS on $[0, \pi]$ without external parameter ξ.
- H. Eliasson & S.B. Kuksin '08 :
  NLS on $\mathbb{T}^d$ with $V \star u$ perturbation.
- B. Grébert, R. Imekraz & E. Paturel '08 :
  Normal forms technics.

Key ingredient

Use of dispersive properties of the Hermite functions

$$\forall\, r > 2, \;\; \exists\, \beta(r) > 0, \;\; \|h_j\|_{L^r(\mathbb{R})} \leq C_r j^{-\beta(r)} \|h_j\|_{L^2(\mathbb{R})}.$$

(K. Yajima-G. Zhang '01)

The symplectic structure

We consider the (complex) Hilbert space ℓ^2_p defined by the norm

$$\|w\|_p^2 = \sum_{j\geq 1} |w_j|^2 j^p.$$

We define the symplectic phase space $\mathcal{P}^p$ as

$$\mathcal{P}^p = \mathbb{T}^n \times \mathbb{R}^n \times \ell^2_p \times \ell^2_p \ni (\theta, y, z, \bar{z}),$$

equipped with the canonic symplectic structure

$$\sum_{j=1}^{n} \mathrm{d}\theta_j \wedge \mathrm{d}y_j \; + \; i \sum_{j\geq 1} \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j.$$

The Hamiltonian formulation

Let $p \geq 2$ and $n \geq 1$. Fix $(I_1, \dots, I_n) \in]0,1]^n$ and write

$$\begin{cases} u(x) = \sum_{j=1}^n (y_j + I_j)^{\frac{1}{2}} e^{i\theta_j} \varphi_j(\xi, x) + \sum_{j\geq 1} z_j \varphi_{j+n}(\xi, x), \\ \bar{u}(x) = \sum_{j=1}^n (y_j + I_j)^{\frac{1}{2}} e^{-i\theta_j} \varphi_j(\xi, x) + \sum_{j\geq 1} \bar{z}_j \varphi_{j+n}(\xi, x), \end{cases}$$

where $(\theta, y, z, \bar{z}) \in \mathcal{P}^p = \mathbb{T}^n \times \mathbb{R}^n \times \ell_p^2 \times \ell_p^2$ are regarded as variables. In this setting equation (NLS) reads as the Hamilton equations associated to the Hamiltonian function $H = N + P$ where

$$N = \sum_{j=1}^n \lambda_j(\xi) y_j + \sum_{j\geq 1} \Lambda_j(\xi) z_j \bar{z}_j,$$

$\Lambda_j(\xi) = \lambda_{j+n}(\xi)$ and

$$P(\theta, y, z, \bar{z}) = \frac{\varepsilon}{2} \int_{\mathbb{R}} \Big| \sum_{j=1}^n (y_j + I_j)^{\frac{1}{2}} e^{i\theta_j} \varphi_j(\xi, x) + \sum_{j\geq 1} z_j \varphi_{j+n}(\xi, x),$$

$$\sum_{j=1}^n (y_j + I_j)^{\frac{1}{2}} e^{-i\theta_j} \varphi_j(\xi, x) + \sum_{j\geq 1} \bar{z}_j \varphi_{j+n}(\xi, x) \Big|^4 dx.$$

The Hamiltonian formulation

In other words, we obtain the following system, which is equivalent to (NLS)

$$\begin{cases} \dot{\theta}_j = \frac{\partial H}{\partial y_j}, \quad \dot{y}_j = -\frac{\partial H}{\partial \theta_j}, \quad 1 \leq j \leq n \\ \dot{z}_j = i\frac{\partial H}{\partial \overline{z}_j}, \qquad \dot{\overline{z}}_j = -i\frac{\partial H}{\partial z_j}, \quad j \geq 1 \\ (\theta_j(0), y_j(0), z_j(0), \overline{z}_j(0)) = (\theta_j^0, y_j^0, z_j^0, \overline{z}_j^0), \end{cases}$$

where the initial conditions are chosen so that

$$u_0(x) = \sum_{j=1}^{n} (I_j + y_j^0)^{\frac{1}{2}} e^{i\theta_j^0} \varphi_j(\varepsilon, x) + \sum_{j \geq 1} z_j^0 \varphi_{j+n}(\xi, x).$$

The general KAM strategy

Consider a smooth function $F = F(\theta, y, z, \overline{z})$, and denote by X_F^t the flow of the equation

$$\begin{cases} \dot{\theta}_j = \frac{\partial F}{\partial y_j}, \quad \dot{y}_j = -\frac{\partial F}{\partial \theta_j}, \quad 1 \leq j \leq n \\ \dot{z}_j = i\frac{\partial F}{\partial \overline{z}_j}, \qquad \dot{\overline{z}}_j = -i\frac{\partial F}{\partial z_j}, \quad j \geq 1 \\ (\theta_j(0), y_j(0), z_j(0), \overline{z}_j(0)) = (\theta_j^0, y_j^0, z_j^0, \overline{z}_j^0). \end{cases}$$

If F is small enough, X_F^1 is well defined and we have

(i) The application X_F^1 preserves the symplectic structure.

(ii) For any smooth G we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(G \circ X_F^t) = \{ G, F \} \circ X_F^t.$$

Idea of the KAM iteration

Find F so that $H \circ X_F^1$ is in a better form than $H = N + P$.

The general KAM strategy

Write the expansion

$$P = \sum_{m,q,\overline{q}} \sum_{k\in\mathbb{Z}^n} P_{kmq\overline{q}}\, e^{ik\cdot\theta} y^m z^q \overline{z}^{\overline{q}},$$

We then consider the second order Taylor approximation of P which is

$$R = \sum_{2|m|+|q+\overline{q}|\leq 2} \sum_{k\in\mathbb{Z}^n} R_{kmq\overline{q}}\, e^{ik\cdot\theta} y^m z^q \overline{z}^{\overline{q}},$$

where $R_{kmq\overline{q}} = P_{kmq\overline{q}}$
Thanks to the Taylor formula we can write

$$\begin{aligned} H \circ X_F^1 &= N \circ X_F^1 + R \circ X_F^1 + (P - R) \circ X_F^1 \\ &= N + \{ N, F \} + \int_0^1 (1-t)\{ \{ N, F \}, F \} \circ X_F^t \, dt + \\ &\quad + R + \int_0^1 \{ R, F \} \circ X_F^t \, dt + (P - R) \circ X_F^1. \end{aligned}$$

Assume that we can find F and $\widehat{N}$ which has the same form as N and which satisfy the so-called homological equation

$$\{N, F\} + R = \widehat{N}.$$

The general KAM strategy

Once the homological equation is solved : We define the new normal form by $N_+ = N + \widehat{N}$, the frequencies of which are given by

$$\lambda^+(\xi) = \lambda(\xi) + \widehat{\lambda}(\xi) \text{ and } \Lambda^+(\xi) = \Lambda(\xi) + \widehat{\Lambda}(\xi),$$

where

$$\widehat{\lambda}_j(\xi) = \frac{\partial \widehat{N}}{\partial y_j}(0,0,0,0,\xi) \text{ and } \widehat{\Lambda}_j(\xi) = \frac{\partial^2 \widehat{N}}{\partial z_j \partial \overline{z}_j}(0,0,0,0,\xi).$$

We define the new perturbation term P_+ by

$$P_+ = (P - R) \circ X_F^1 + \int_0^1 \{ R(t), F \} \circ X_F^t \, dt,$$

where $R(t) = (1-t)\widehat{N} + tR$ in such a way that

$$H \circ X_F^1 = N_+ + P_+ .$$

Convergence : If $P = O(\varepsilon)$ and $F = O(\varepsilon)$. Then $R = O(\varepsilon)$ and the quadratic part of P_+ is $O(\varepsilon^2)$.

The general KAM strategy

At the end we obtain a symplectic transformation Φ (near the origin) so that $H^\star = H \circ \Phi = N^\star + P^\star$, where

$$N^\star = \sum_{j=1}^{n} \lambda_j^\star(\xi) y_j + \sum_{j \geq 1} \Lambda_j^\star(\xi) z_j \bar{z}_j,$$

and $P^\star$ has no quadratic part in $z, \bar{z}$ and no linear part in y. Then the new coordinates $(y', \theta', z', \bar{z}') = \Phi^{-1}(y, \theta, z, \bar{z})$ satisfy

$$\begin{cases} \dot{\theta}'_j = \frac{\partial H^\star}{\partial y'_j}, \quad \dot{y}'_j = -\frac{\partial H^\star}{\partial \theta'_j}, \quad 1 \leq j \leq n \\ \dot{z}'_j = i\frac{\partial H^\star}{\partial \bar{z}'_j}, \quad \dot{\bar{z}}'_j = -i\frac{\partial H^\star}{\partial z'_j}, \quad j \geq 1. \end{cases} \qquad (NH)$$

In particular, the solution to (NH) with initial condition $(\theta'_j(0), y'_j(0), z'_j(0), \bar{z}'_j(0)) = (\theta'^0_j, 0, 0, 0)$ reads

$$(\theta'_j(t), y'_j(t), z'_j(t), \bar{z}'_j(t)) = (t\lambda_j^\star + \theta'^0_j, 0, 0, 0).$$

Hence we have constructed a quasi-periodic solution to (NLS).

The homological equation

Aim : Solve

$$\{N, F\} + R = \widehat{N},$$

with

$$N = \sum_{j=1}^{n} \lambda_j(\xi) y_j + \sum_{j\geq 1} \Lambda_j(\xi) z_j \bar{z}_j \ .$$

We look for a solution F of the form

$$F = \sum_{2|m|+|q+\overline{q}|\leq 2} \ \sum_{k\in\mathbb{Z}^n} F_{kmq\overline{q}} \, e^{ik\cdot\theta} y^m z^q \overline{z}^{\overline{q}}.$$

A direct computation gives

$$iF_{kmq\overline{q}} = \begin{cases} \dfrac{R_{kmq\overline{q}}}{k\cdot\lambda(\xi) + (q-\overline{q})\cdot\Lambda(\xi)}, & \text{if} \quad |k| + |q-\overline{q}| \neq 0, \\ 0, & \text{otherwise}, \end{cases}$$

$$\widehat{N} = [R] = \sum_{|m|+|q|=1} R_{0mqq} y^m z^q \overline{z}^q.$$

Control of the frequencies

We show that we can find $(f_k)_{1\leq k\leq n}$ such that

Small divisors control : There exist a subset $\Pi_\alpha \subset \Pi$ with $\text{Meas}(\Pi \backslash \Pi_\alpha) \longrightarrow 0$ when $\alpha \longrightarrow 0$ and $\tau > 1$, such that for all $\xi \in \Pi_\alpha$

$$\big| k\cdot\lambda(\xi) + l\cdot\Lambda(\xi)\big| \geq \alpha\frac{\langle l\rangle}{1+|k|^\tau}, \quad (k,l)\in\mathcal{Z},$$

where $\mathcal{Z} := \{(k,l)\in\mathbb{Z}^n\times\mathbb{Z}^\infty,\ (k,l)\neq 0,\ |l|\leq 2\}$.

To perform the KAM method, we now have to check this condition persists after each iteration. This will be the case if the perturbation $\widehat{\Lambda}_j$ satisfies $|\widehat{\Lambda}_j| \leq C\varepsilon j^{-\beta}$ for some $\beta > 0$.

Control of the frequencies

We have

$$\widehat{\Lambda}_j(\xi) = \frac{\partial^2 \widehat{N}}{\partial z_j \partial \overline{z}_j}(0,0,0,0,\xi) = \frac{\partial^2 P}{\partial z_j \partial \overline{z}_j}(0,0,0,0,\xi).$$

In our case, we have $P = \frac{\varepsilon}{2}\int_{\mathbb{R}} |u|^4$, therefore $\frac{\partial^2 P}{\partial z_j \partial \overline{z}_j} = 2\varepsilon \int_{\mathbb{R}} \varphi_{j+n}^2 |u|^2$.

Now by the dispersive estimate $\|\varphi_j\|_{L^\infty(\mathbb{R})} \leq C j^{-1/12}$ we get

$$\left|\frac{\partial^2 P}{\partial z_j \partial \overline{z}_j}\right| \leq \varepsilon \|\varphi_{j+n}\|^2_{L^\infty(\mathbb{R})} \|u\|^2_{L^2(\mathbb{R})} \leq C\varepsilon j^{-1/6} \|u\|^2_{L^2(\mathbb{R})}.$$

Reducibility of the linear equation

We consider the linear equation

$$\begin{cases} i\partial_t u + \partial_x^2 u - x^2 u + \epsilon V(t\omega, x)u = 0, & (t,x) \in \mathbb{R}\times\mathbb{R}, \\ u(0,x) = u_0(x), \end{cases} \qquad (LS)$$

where the potential $V : \ \mathbb{T}^n \times \mathbb{R} \ni (\theta, x) \mapsto \mathbb{R}$ satisfies

- V is analytic in θ.
- V is $\mathcal{C}^\infty$ in x, with bounded derivatives.
- V satisfies $|V(\theta,x)| \leq C(1+x^2)^{-\delta}$ for some $\delta > 0$.

Reducibility of the linear equation

$$\begin{cases} i\partial_t u + \partial_x^2 u - x^2 u + \epsilon V(t\omega, x)u = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(0,x) = u_0(x). \end{cases} \qquad (LS)$$

Theorem (B. Grébert - LT)

There exists ϵ_0 *such that for all* $0 \leq \epsilon < \epsilon_0$ *there exists* $\Lambda_\varepsilon \subset [0, 2\pi)^n$ *such that* $|[0, 2\pi)^n \backslash \Lambda_\varepsilon| \to 0$ *as* $\epsilon \to 0$*, and such that for all* $\omega \in \Lambda_\varepsilon$*, the linear Schrödinger equation* (LS) *reduces, in* $L^2(\mathbb{R})$*, to a linear equation with constant coefficients.*

Reducibility of the linear equation

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Theorem (B. Grébert - LT)

There exists ϵ_0 such that for all $0 \le \epsilon < \epsilon_0$ there exists $\Lambda_\varepsilon \subset [0, 2\pi)^n$ such that $|[0,2\pi)^n \backslash \Lambda_\varepsilon| \to 0$ as $\epsilon \to 0$, and such that for all $\omega \in \Lambda_\varepsilon$, the linear Schrödinger equation (LS) reduces, in $L^2(\mathbb{R})$, to a linear equation with constant coefficients.

Corollary

Let $\omega \in \Lambda_\varepsilon$, then any solution u of (LS) is almost-periodic in time and we have the bounds

$$(1 - \varepsilon C)\|u_0\|_{\mathcal{H}^p} \le \|u(t)\|_{\mathcal{H}^p} \le (1 + \varepsilon C)\|u_0\|_{\mathcal{H}^p}, \quad \forall\, t \in \mathbb{R},$$

for some $C = C(p, \omega)$.

Reducibility of the linear equation

Some previous results

- D. Bambusi & S. Graffi '01 ; J. Lui & X. Yuan '10 :
  Case x^{β}, $\beta > 2$.
- W.-M. Wang '08 :
  Case x^2, for some particular V.
- H. Eliasson & S.B. Kuksin '08 :
  NLS on $\mathbb{T}^d$.
- W.-M. Wang '08, J.-M. Delort '10, D. Fang & Q. Zhang '10 :
  NLS on $\mathbb{T}^d$: Bounds $\|u(t)\|_{H^p} \lesssim \big(\ln t\big)^{cp}$ if V is analytic.
- J.-M. Delort '10 :
  Existence of solutions so that $\|u(t)\|_{\mathcal{H}^p} \gtrsim t^{p/2}$ if V is allowed to be a pseudo-differential operator.

Hamiltonian formulation

Equation (LS) reads as a non autonomous Hamiltonian system

$$\begin{cases} \dot{z}_j = -i(2j-1)z_j - i\varepsilon \frac{\partial}{\partial \bar{z}_j} \widetilde{Q}(t,z,\bar{z}), & j \geq 1 \\ \dot{\bar{z}}_j = i(2j-1)\bar{z}_j + i\varepsilon \frac{\partial}{\partial z_j} \widetilde{Q}(t,z,\bar{z}), & j \geq 1 \end{cases}$$

where

$$\widetilde{Q}(t,z,\bar{z}) = \int_{\mathbb{R}} V(\omega t, x)\Big(\sum_{j\geq 1} z_j h_j(x)\Big)\Big(\sum_{j\geq 1} \bar{z}_j h_j(x)\Big)\mathrm{d}x.$$

We re-interpret this system as an autonomous Hamiltonian system in an extended phase space

$$\begin{cases} \dot{z}_j = -i(2j-1)z_j - i\varepsilon \frac{\partial}{\partial \bar{z}_j} Q(\theta,z,\bar{z}) & j \geq 1 \\ \dot{\bar{z}}_j = i(2j-1)\bar{z}_j + i\varepsilon \frac{\partial}{\partial z_j} Q(\theta,z,\bar{z}) & j \geq 1 \\ \dot{\theta}_j = \omega_j & j = 1, \cdots, n \\ \dot{y}_j = -\varepsilon \frac{\partial}{\partial \theta_j} Q(\theta,z,\bar{z}) & j = 1, \cdots, n \end{cases}$$

where

$$Q(\theta,z,\bar{z}) = \int_{\mathbb{R}} V(\theta, x)\Big(\sum_{j\geq 1} z_j h_j(x)\Big)\Big(\sum_{j\geq 1} \bar{z}_j h_j(x)\Big)\mathrm{d}x.$$

Linear dynamics

Here the external parameters are directly the frequencies $\omega = (\omega_j)_{1 \leq j \leq n} \in [0, 2\pi)^n =: \Pi$ and the normal frequencies $\Omega_j = 2j - 1$ are constant.

Using the KAM scheme, we are able to show the existence of a set of parameters $\Pi_\varepsilon \subset \Pi$ with $|\Pi \backslash \Pi_\varepsilon| \to 0$ when $\varepsilon \to 0$ and a coordinate transformation $\Phi : \Pi_\varepsilon \times \mathcal{P}^0 \longrightarrow \mathcal{P}^0$, such that $H \circ \Phi = N^\star$, where $N^\star$ takes the form

$$N^\star(\omega) = \sum_{j=1}^{n} \omega_j y_j + \sum_{j \geq 1} \Omega_j^\star z_j \bar{z}_j.$$

In the new coordinates, $(y', \theta', z', \bar{z}') = \Phi^{-1}(y, \theta, z, \bar{z})$, the dynamic is linear with y' invariant :

$$\begin{cases} \dot{z}_j' = i\Omega_j^\star z_j' & j \geq 1 \\ \dot{\bar{z}}_j' = -i\Omega_j^\star \bar{z}_j' & j \geq 1 \\ \dot{\theta}_j' = \omega_j & j = 1, \cdots, n \\ \dot{y}_j' = 0 & j = 1, \cdots, n. \end{cases}$$