Oscillations multimodales dans les équations différentielles stochastiques

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Oscillations in natural systems

Belousov-Zhabotinsky reaction [Hudson 79]

Stellate cells [Dickson 00]

Mean temperature based on ice core measurements [Johnson et al 01]
Oscillations in natural systems

Belousov-Zhabotinsky reaction [Hudson 79]  
Stellate cells [Dickson 00]

- **Deterministic models** reproducing these oscillations exist and have been abundantly studied.
  - They often involve singular perturbation theory.

- We want to understand the effect of **noise** on oscillatory patterns.
  - Noise may also induce oscillations not present in deterministic case.
Example: Van der Pol oscillator

\[ x'' + \varepsilon^{-1/2}(x^2 - 1)x' + x = 0 \]

\[ \dot{x} = y + x - \frac{1}{3}x^3 \quad \text{and} \quad \varepsilon \dot{x} = y + x - \frac{1}{3}x^3 \]

\[ \dot{y} = -\varepsilon x \quad \text{and} \quad \dot{y} = -x \]
Example: Van der Pol oscillator

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\begin{align*}
\dot{x} &= y + x - \frac{1}{3}x^3 \\
\dot{y} &= -\varepsilon x
\end{align*}
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\begin{align*}
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\dot{y} &= -x
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\]

\[
\begin{align*}
\varepsilon \to 0 \\
\downarrow
\end{align*}
\]

\[
\begin{align*}
\dot{x} &= y + x - \frac{1}{3}x^3 \\
\dot{y} &= 0
\end{align*}
\]

\[
\begin{align*}
\dot{x} &= y + x - \frac{1}{3}x^3 \\
\dot{y} &= -(x - \frac{1}{3}x^3)
\end{align*}
\]

\[
\begin{align*}
\varepsilon \to 0 \\
\downarrow
\end{align*}
\]

\[
\begin{align*}
\dot{x} &= \frac{x}{1 - x^2}
\end{align*}
\]
Example: Van der Pol oscillator

\[ x'' + \varepsilon^{-1/2}(x^2 - 1)x' + x = 0 \]

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\begin{align*}
\dot{x} &= y + x - \frac{1}{3}x^3 & \quad \varepsilon \rightarrow 0 \\
\dot{y} &= -\varepsilon x & \quad \varepsilon \rightarrow 0 \\
\end{align*}
\]

\[
\begin{align*}
\dot{x} &= y + x - \frac{1}{3}x^3 & \quad t \rightarrow \varepsilon t \\
\dot{y} &= -x & \quad \varepsilon \rightarrow 0 \\
\end{align*}
\]

\[
\begin{align*}
\dot{x} &= y + x - \frac{1}{3}x^3 & \quad \varepsilon \rightarrow 0 \\
\dot{y} &= 0 & \quad \varepsilon \rightarrow 0 \\
\end{align*}
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\begin{align*}
\dot{x} &= y + x - \frac{1}{3}x^3 & \quad \varepsilon \rightarrow 0 \\
\dot{y} &= -x & \quad \varepsilon \rightarrow 0 \\
\Rightarrow \dot{x} &= \frac{x}{1 - x^2} \\
\end{align*}
\]
Example: Van der Pol oscillator

\[ x'' + \varepsilon^{-1/2}(x^2 - 1)x' + x = 0 \]

\[ \dot{x} = y + x - \frac{1}{3}x^3 \]
\[ \dot{y} = -\varepsilon x \]

Relaxation oscillations
Effect of noise on the Van der Pol oscillator

\[ dx_t = \left[ y_t + x_t - \frac{x_t^3}{3} \right] dt + \sigma \, dW_t \]

\[ dy_t = -\varepsilon x_t \, dt \]
Effect of noise on the Van der Pol oscillator

\[
\begin{align*}
\, dx_t &= \left[ y_t + x_t - \frac{x_t^3}{3} \right] \, dt + \sigma \, dW_t \\
\, dy_t &= -\varepsilon x_t \, dt
\end{align*}
\]

Theorem [B & Gentz 2006]

- \( \sigma < \sqrt{\varepsilon} \): Cycles comparable to deterministic ones with probability \( 1 - \mathcal{O}(e^{-\varepsilon/\sigma^2}) \)
- \( \sigma > \sqrt{\varepsilon} \): Cycles are smaller, by \( \mathcal{O}(\sigma^4/3) \), than deterministic cycles, with probability \( 1 - \mathcal{O}(e^{-\sigma^2/\varepsilon|\log \sigma|}) \)
Single neuron communicates by generating action potential

- **Excitable**: small change in parameters yields spike generation
- May display **Mixed-Mode Oscillations (MMOs) and Relaxation Oscillations**
Conductance-based models for membrane potential

Hodgkin–Huxley model (1952)

\[
\begin{align*}
C \frac{\text{d}v}{\text{d}t} &= - \sum_i \bar{g}_i \varphi_i^\alpha \chi_i^\beta (v - v_i^*) \\
\tau_{\varphi, i}(v) \frac{\text{d}\varphi_i}{\text{d}t} &= - (\varphi_i - \varphi_i^*(v)) \\
\tau_{\chi, i}(v) \frac{\text{d}\chi_i}{\text{d}t} &= - (\chi_i - \chi_i^*(v))
\end{align*}
\]

\(\varphi_i^*(v), \chi_i^*(v)\) sigmoïdal functions, e.g. \(\tanh(av + b)\)

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Conductance-based models for membrane potential

Hodgkin–Huxley model (1952)

\[ C\dot{v} = -\sum_i \bar{g}_i \varphi_i^\alpha \chi_i^\beta (v - v_i^*) \] voltage

\[ \tau_{\varphi,i}(v) \dot{\varphi}_i = -(\varphi_i - \varphi_i^*(v)) \] activation

\[ \tau_{\chi,i}(v) \dot{\chi}_i = -(\chi_i - \chi_i^*(v)) \] inactivation

\( \triangleright i \in \{\text{Na}^+, \text{K}^+, \ldots\} \) describes different types of ion channels

\( \triangleright \varphi_i^*(v), \chi_i^*(v) \) sigmoïdal functions, e.g. \( \tanh(av + b) \)

For \( C/\bar{g}_i \ll \tau_{x,i} \): slow–fast systems of the form

\[ \varepsilon \dot{v} = f(v, w) \]

\[ \dot{w}_i = g_i(v, w) \]
Conductance-based models for membrane potential

Fitzhugh–Nagumo model (1962)

\[ \varepsilon \dot{x} = x - x^3 + y \]
\[ \dot{y} = \alpha - \beta x - \gamma y \]
**Conductance-based models for membrane potential**

**Fitzhugh–Nagumo model (1962)**

\[
\begin{align*}
\varepsilon \dot{x} &= x - x^3 + y \\
\dot{y} &= \alpha - \beta x - \gamma y \\
&= \frac{1}{\sqrt{3}} + \delta - x
\end{align*}
\]

**The canard (french duck) phenomenon**


\[
\begin{align*}
\varepsilon &= 0.05 \\
\alpha &= \frac{1}{\sqrt{3}} + \delta \\
\beta &= 1 \\
\gamma &= 0 \\
\delta_1 &= -0.003 \\
\delta_2 &= -0.003765458 \\
\delta_3 &= -0.003765459 \\
\delta_4 &= -0.005
\end{align*}
\]
Conductance-based models for membrane potential

Fitzhugh–Nagumo model (1962)

\[\begin{align*}
\varepsilon \dot{x} &= x - x^3 + y \\
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\end{align*}\]
The canard (french duck) phenomenon

Normal form near fold point

\[
\begin{align*}
\varepsilon \dot{x} &= y - x^2 \\
\dot{y} &= \delta - x
\end{align*}
\] ( + higher-order terms)
Folded node singularity

Normal form [Benoît, Lobry '82, Szmolyan, Wechselberger '01]:

\[ \begin{align*}
\epsilon \dot{x} &= y - x^2 \\
\dot{y} &= -(\mu + 1)x - z \\
\dot{z} &= \frac{\mu}{2}
\end{align*} \] (\(+\) higher-order terms)
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Folded node singularity

**Theorem** [Benoït, Lobry '82, Szmolyan, Wechselberger '01]:
For $2k + 1 < \mu^{-1} < 2k + 3$, the system admits $k$ canard solutions
The $j^{th}$ canard makes $(2j + 1)/2$ oscillations

**Mixed-mode oscillations** (MMOs)

Picture: Mathieu Desroches
Effect of noise

\[
\begin{align*}
    dx_t &= \frac{1}{\varepsilon}(y_t - x_t^2) \, dt + \frac{\sigma}{\sqrt{\varepsilon}} \, dW_t^{(1)} \\
    dy_t &= \left[ -(\mu + 1)x_t - z_t \right] \, dt + \sigma \, dW_t^{(2)} \\
    dz_t &= \frac{\mu}{2} \, dt
\end{align*}
\]

- Noise smears out small amplitude oscillations
- Early transitions modify the mixed-mode pattern
**Covariance tubes**

Linearized stochastic equation around a canard \((x^\text{det}_t, y^\text{det}_t, z^\text{det}_t)\)

\[
d\zeta_t = A(t)\zeta_t \, dt + \sigma \, dW_t \quad A(t) = \begin{pmatrix} -2x^\text{det}_t & 1 \\ -1 & 0 \end{pmatrix}
\]

\[
\zeta_t = U(t)\zeta_0 + \sigma \int_0^t U(t, s) \, dW_s \quad (U(t, s) : \text{principal solution of } \dot{U} = AU)
\]

Gaussian process with covariance matrix

\[
\text{Cov}(\zeta_t) = \sigma^2 V(t) \quad V(t) = U(t)V(0)U(t)^{-1} + \int_0^t U(t, s)U(t, s)^T \, ds
\]
Covariance tubes

Linearized stochastic equation around a canard \((x_t^{\det}, y_t^{\det}, z_t^{\det})\)

\[
d\zeta_t = A(t)\zeta_t \, dt + \sigma \, dW_t \quad A(t) = \begin{pmatrix} -2x_t^{\det} & 1 \\ -(1+\mu) & 0 \end{pmatrix}
\]

\[
\zeta_t = U(t)\zeta_0 + \sigma \int_0^t U(t,s) \, dW_s \quad (U(t,s) : \text{principal solution of } \dot{U} = AU)
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Covariance tube:

\[
\mathcal{B}(h) = \left\{ \langle (x,y) - (x_t^{\det}, y_t^{\det}), V(t)^{-1}[(x,y) - (x_t^{\det}, y_t^{\det})] \rangle < h^2 \right\}
\]

Theorem [B, Gentz, Kuehn 2010]

Probability of leaving covariance tube before time \(t\) (with \(z_t \leq 0\)):

\[
\mathbb{P}\{\tau_{\mathcal{B}(h)} < t\} \leq C(t) e^{-\kappa h^2/2\sigma^2}
\]
Covariance tubes

Theorem [B, Gentz, Kuehn 2010]
Probability of leaving covariance tube before time $t$ (with $z_t \leq 0$):

$$\mathbb{P}\{\tau_B(h) < t\} \leq C(t) e^{-\kappa h^2/2\sigma^2}$$

Sketch of proof:
- (Sub)martingale: $\{M_t\}_{t \geq 0}$, $\mathbb{E}\{M_t | M_s\} = (\geq)M_s$ for $t \geq s \geq 0$
- Doob’s submartingale inequality: $\mathbb{P}\left\{\sup_{0 \leq t \leq T} M_t \geq L\right\} \leq \frac{1}{L} \mathbb{E}[M_T]$
Covariance tubes

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- Linear equation: $\zeta_t = \sigma \int_0^t U(t,s) \, dW_s$ is no martingale
  but can be approximated by martingale on small time intervals

- $\exp\{\gamma \langle \zeta_t, V(t)^{-1}\zeta_t \rangle\}$ approximated by submartingale

- Doob’s inequality yields bound on probability of leaving $B(h)$ during small time intervals. Then sum over all time intervals
Covariance tubes

Theorem [B, Gentz, Kuehn 2010]
Probability of leaving covariance tube before time $t$ (with $z_t \leq 0$):

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\mathbb{P}\left\{ \tau_{B(h)} < t \right\} \leq C(t) e^{-\kappa h^2/2\sigma^2}
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- Linear equation: $\zeta_t = \sigma \int_0^t U(t,s) \, dW_s$ is no martingale
  but can be approximated by martingale on small time intervals
- $\exp\{\gamma \langle \zeta_t, V(t)^{-1} \zeta_t \rangle\}$ approximated by submartingale
- Doob’s inequality yields bound on probability of leaving $B(h)$ during small
  time intervals. Then sum over all time intervals
- Nonlinear equation: $d\zeta_t = A(t)\zeta_t \, dt + b(\zeta_t, t) \, dt + \sigma \, dW_t$

$$
\zeta_t = \sigma \int_0^t U(t,s) \, dW_s + \int_0^t U(t,s)b(\zeta_s, s) \, ds
$$

Second integral can be treated as small perturbation for $t \leq \tau_{B(h)}$
Small-amplitude oscillations and noise

One shows that for $z = 0$

- The distance between the $k^{th}$ and $(k+1)^{st}$ canard has order $e^{-(2k+1)^2\mu}$
- The section of $\mathcal{B}(h)$ is close to circular with radius $\mu^{-1/4}h$
Small-amplitude oscillations and noise

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Sketch of proof:
- Dynamic diagonalization of equation linearized around central (“weak”) canard
- $V(t) = \sigma^{-2} \text{Cov}(\zeta_i)$ satisfies fast-slow equation

$$\mu \frac{dV}{dz} = A(z)V + VA(z)^T + 1$$

which can be studied by singular perturbation theory.

Note: Hopf bifurcation at $z = 0$!
Small-amplitude oscillations and noise

One shows that for \( z = 0 \)

\[ \nabla \text{The distance between the } k^{\text{th}} \text{ and } k + 1^{\text{st}} \text{ canard has order } e^{- (2k+1)^2 \mu} \]

\[ \nabla \text{The section of } B(h) \text{ is close to circular with radius } \mu^{-1/4} h \]

Corollary

Let

\[ \sigma_k(\mu) = \mu^{1/4} e^{- (2k+1)^2 \mu} \]

Canards with \( \frac{2k+1}{4} \) oscillations become indistinguishable from noisy fluctuations for \( \sigma > \sigma_k(\mu) \)
Small-amplitude oscillations and noise

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**Corollary**

Let

$$\sigma_k(\mu) = \mu^{1/4} e^{-(2k+1)^2\mu}$$

Canards with $\frac{2k+1}{4}$ oscillations become indistinguishable from noisy fluctuations for $\sigma > \sigma_k(\mu)$
Early transitions

Let $\mathcal{D}$ be neighbourhood of size $\sqrt{z}$ of a canard for $z > 0$ (unstable).

Theorem [B, Gentz, Kuehn 2010]

\[ \exists \kappa, C, \gamma_1, \gamma_2 > 0 \text{ such that for } \sigma |\log \sigma|^{\gamma_1} \leq \mu^{3/4} \text{ probability of leaving } \mathcal{D} \text{ after } z_t = z \text{ satisfies} \]

\[ \mathbb{P}\{z_{\tau_{\mathcal{D}}} > z\} \leq C |\log \sigma|^{\gamma_2} e^{-\kappa(z^2 - \mu)/(\mu |\log \sigma|)} \]

Small for $z \gg \sqrt{\mu |\log \sigma|/\kappa}$
Early transitions

Let $D$ be neighbourhood of size $\sqrt{z}$ of a canard for $z > 0$ (unstable)

Theorem [B, Gentz, Kuehn 2010]

$\exists \kappa, C, \gamma_1, \gamma_2 > 0$ such that for $\sigma |\log \sigma|^{\gamma_1} \leq \mu^{3/4}$ probability of leaving $D$ after $z_t = z$ satisfies

$$
\mathbb{P}\{z_{\tau_D} > z\} \leq C |\log \sigma|^{\gamma_2} e^{-\kappa(z^2-\mu)/(\mu|\log \sigma|)}
$$

Small for $z \gg \sqrt{\mu|\log \sigma|/\kappa}$

Sketch of proof :

- Escape from neighbourhood of size $\sigma |\log \sigma|/\sqrt{z}$ : compare with linearized equation on small time intervals + Markov property
- Escape from annulus $\sigma |\log \sigma|/\sqrt{z} \leq ||\zeta|| \leq \sqrt{z}$ : use polar coordinates and averaging
- To combine the two regimes : use Laplace transforms
Early transitions

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**Theorem** [B, Gentz, Kuehn 2010]

$\exists \kappa, C, \gamma_1, \gamma_2 > 0$ such that for $\sigma|\log \sigma|^{\gamma_1} \leq \mu^{3/4}$ probability of leaving $\mathcal{D}$ after $z_t = z$ satisfies

$$\mathbb{P}\{z_{\tau_D} > z\} \leq C|\log \sigma|^{\gamma_2} e^{-\kappa(z^2 - \mu)/(\mu|\log \sigma|)}$$

Small for $z \gg \sqrt{\mu|\log \sigma|/\kappa}$
Further work

▷ Better understanding of distribution of noise-induced transitions
▷ Effect on mixed-mode pattern in conjunction with global return mechanism
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▷ Effect on mixed-mode pattern in conjunction with global return mechanism
Noise-induced MMOs  [D. Landon, PhD thesis, in progress]

FitzHugh–Nagumo, normal form near bifurcation point:

\[
\begin{align*}
    \frac{dx_t}{dt} &= (y_t - x_t^2) \, dt + \sigma \, dW_t \\
    \frac{dy_t}{dt} &= \varepsilon(\delta - x_t) \, dt
\end{align*}
\]

\( \delta > \sqrt{\varepsilon} \): equilibrium \((\delta, \delta^2)\) is a node, effectively 1D problem

- \( \sigma \ll \delta^{3/2} \): rare spikes, approx. exponential interspike times
- \( \sigma \gg \delta^{3/2} \): repeated spikes

\( \delta < \sqrt{\varepsilon} \): equilibrium \((\delta, \delta^2)\) is a focus. Two-dimensional problem
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Noise-induced MMOs [D. Landon, PhD thesis, in progress]

Conjectured bifurcation diagram [Muratov and Vanden Eijnden (2007)]:

\[\sigma = \delta^{3/2}\]

\[\sigma = (\delta \varepsilon)^{1/2}\]

\[\sigma = \delta \varepsilon^{1/4}\]
Noise-induced MMOs [D. Landon, PhD thesis, in progress]

Conjectured bifurcation diagram [Muratov and Vanden Eijnden (2007)]:

\[
\begin{align*}
&\sigma = \delta^{3/2} \\
&\varepsilon^{3/4} = (\delta \varepsilon)^{1/2} \\
&\varepsilon^{1/2} = \delta \varepsilon^{1/4}
\end{align*}
\]

Work in progress:

▷ Prove bifurcation diagram is correct
▷ Characterize interspike time statistics and spike train statistics
▷ Characterize distribution of mixed-mode patterns
Noise-induced MMOs

[D. Landon, PhD thesis, in progress]

Definition of random number of SAOs $N$:

$N$ = survival time of substochastic Markov chain

Theorem (2011):

- $\lim_{n \to \infty} P\{N = n + 1 | N > n\} = 1 - \lambda_0$, $\lambda_0$ = principal ev
- Weak noise: $\sigma_1^2 + \sigma_2^2 \leq (\varepsilon^{1/4}\delta)^2 \Rightarrow 1 - \lambda_0 \leq e^{-\kappa(\varepsilon^{1/4}\delta)^2/(\sigma_1^2 + \sigma_2^2)}$
- Increasing noise:

$$1 - \lambda_0 \simeq \Phi\left( -\frac{(\pi\varepsilon)^{1/4}(\delta - \sigma_1^2/\varepsilon)}{\sqrt{\sigma_1^2 + \sigma_2^2}} \right)$$
References

