Geometric singular perturbation theory for stochastic differential equations

Nils Berglund
Centre de Physique Théorique - CNRS
Marseille Luminy
http://berglund.univ-tln.fr

Joint work with Barbara Gentz, WIAS, Berlin

ZIF, Bielefeld, June 2005

Slow–fast systems: heuristics

In fast time s:

$$\begin{aligned} x' &= f(x,y) && x \in \mathbb{R}^n, \text{ fast variable} \\ y' &= \varepsilon g(x,y) && y \in \mathbb{R}^m, \text{ slow variable} \end{aligned}$$

- Perturbation of $x' = f(x,\lambda)$, with slowly moving parameter λ
- Simplest case: $x^\star(\lambda)$ asympt. stable equilibrium point

In slow time $t = \varepsilon s$:

$$\begin{aligned} \varepsilon\dot{x} &= f(x,y) && x \in \mathbb{R}^n, \text{ fast variable} \\ \dot{y} &= g(x,y) && y \in \mathbb{R}^m, \text{ slow variable} \end{aligned}$$

- Slow manifold: $f(x^\star(y),y) = 0$ (for all y in some domain)
- Reduced equation:

$$\dot{y} = g(x^\star(y),y)$$

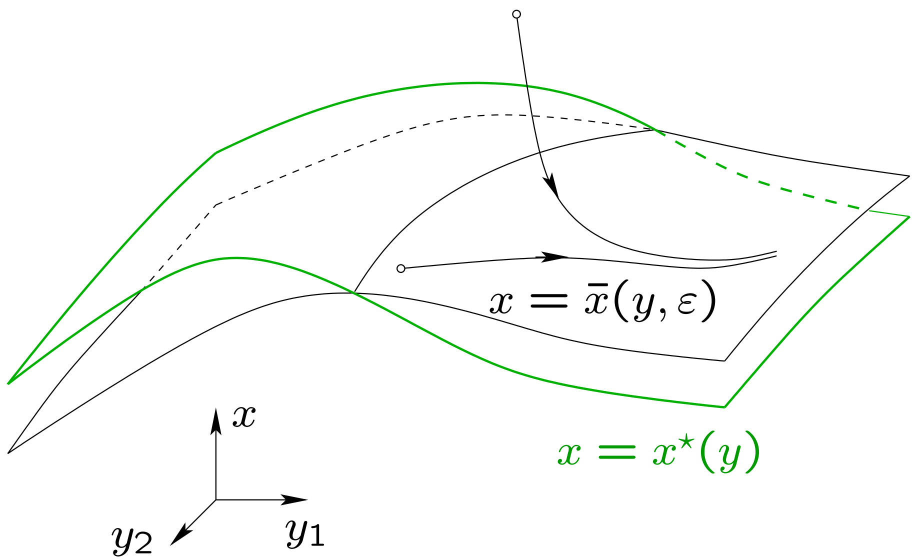
Geometric singular perturbation theory

$$\varepsilon \dot{x} = f(x,y) \qquad x \in \mathbb{R}^n, \text{ fast variable}$$
$$\dot{y} = g(x,y) \qquad y \in \mathbb{R}^m, \text{ slow variable}$$

- Slow manifold: $f(x^\star(y), y) = 0$ (for all y in some domain)
- Stability: Eigenvalues of $\partial_x f(x^\star(y), y)$ have negative real parts

Theorem [Tihonov '52, Fenichel '79]
$\exists$ *adiabatic manifold* $x = \bar{x}(y, \varepsilon)$
s.t.

- $\bar{x}(y,\varepsilon)$ is invariant
- $\bar{x}(y,\varepsilon)$ attracts nearby solutions
- $\bar{x}(y,\varepsilon) = x^\star(y) + \mathcal{O}(\varepsilon)$



Stochastic perturbation: one-dimensional case

$$\mathrm{d}x_t = \frac{1}{\varepsilon} f(x_t, t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

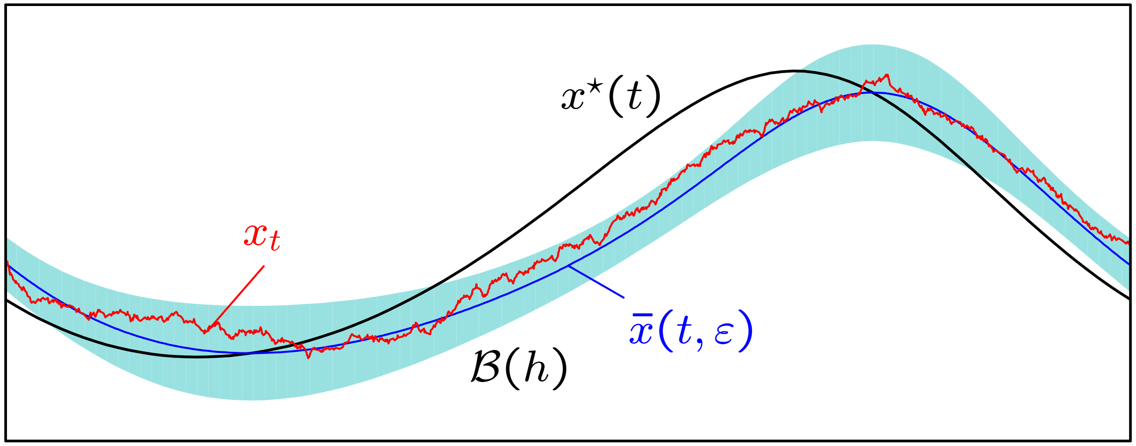
Slow–fast system with $y_t = t$

Stable equil. branch: $f(x^\star(t), t) = 0$, $a^\star(t) = \partial_x f(x^\star(t), t) \leqslant -a_0$

Adiabatic solution: $\bar{x}(t, \varepsilon) = x^\star(t) + \mathcal{O}(\varepsilon)$

$\bar{a}(t, \varepsilon) = \partial_x f(\bar{x}(t, \varepsilon), t) = a^\star(t) + \mathcal{O}(\varepsilon)$

$\mathcal{B}(h)$: strip of width $\simeq h/|\bar{a}(t, \varepsilon)|$ around $\bar{x}(t, \varepsilon)$.



Stochastic perturbation: one-dimensional case

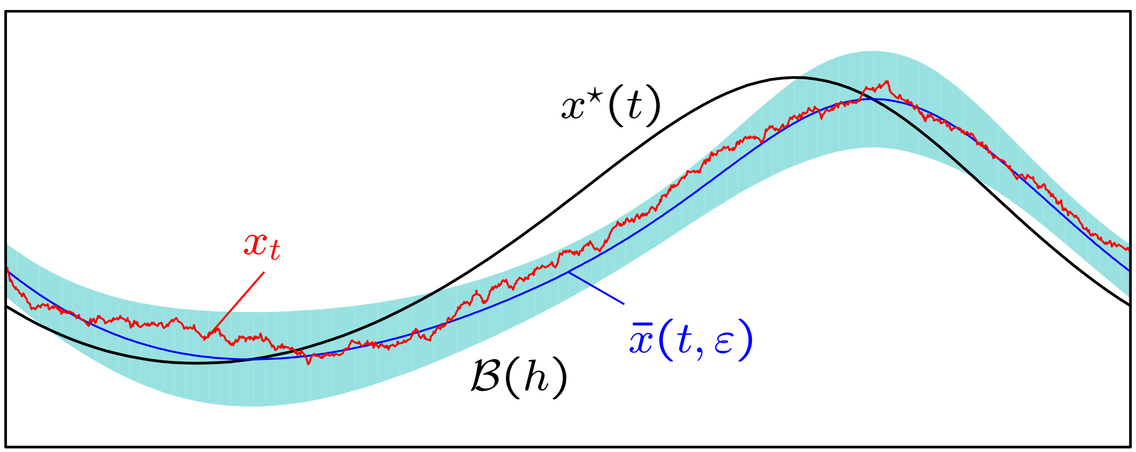
$$\mathrm{d}x_t = \frac{1}{\varepsilon} f(x_t, t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

Theorem: [B. & G., PTRF 2002]

$$C(t,\varepsilon)\mathrm{e}^{-\kappa_- h^2/2\sigma^2} \leqslant \mathbb{P}\Big\{\text{leaving } \mathcal{B}(h) \text{ before time } t\Big\} \leqslant C(t,\varepsilon)\mathrm{e}^{-\kappa_+ h^2/2\sigma^2}$$

$$\kappa_\pm = 1 \mp \mathcal{O}(h)$$

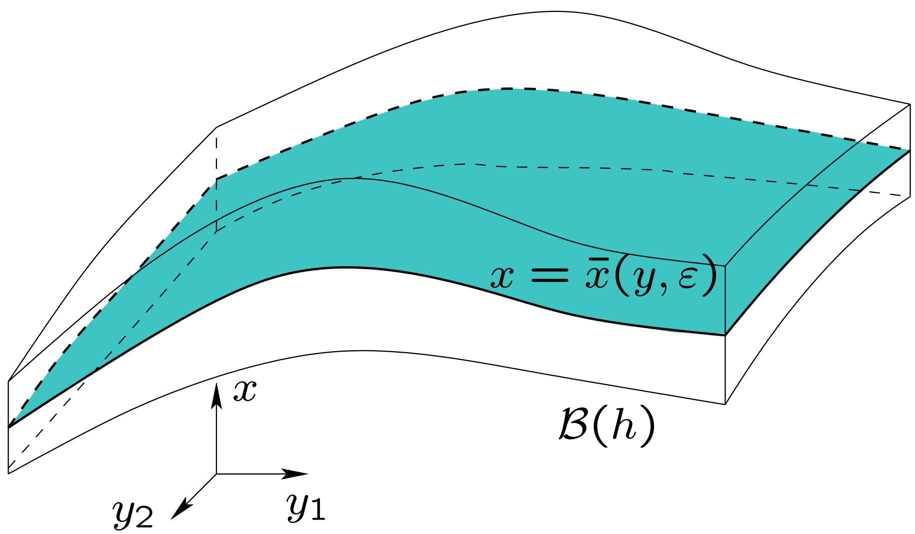
$$C(t,\varepsilon) = \sqrt{\frac{2}{\pi}\frac{1}{\varepsilon}}\left|\int_0^t \bar{a}(s,\varepsilon)\,\mathrm{d}s\right|\frac{h}{\sigma}\Big[1 + \text{error}\Big]$$



Stochastic perturbation: n-dimensional case

$$\begin{cases} \mathrm{d}x_t = \dfrac{1}{\varepsilon} f(x_t, y_t)\,\mathrm{d}t + \dfrac{\sigma}{\sqrt{\varepsilon}} F(x_t, y_t)\,\mathrm{d}W_t & \text{(fast variables } \in \mathbb{R}^n\text{)} \\ \mathrm{d}y_t = \quad g(x_t, y_t)\,\mathrm{d}t + \sigma'\,G(x_t, y_t)\,\mathrm{d}W_t & \text{(slow variables } \in \mathbb{R}^m\text{)} \end{cases}$$

Stable slow manifold: $f(x^\star(y), y) = 0$, $A(y) = \partial_x f(x^\star(y), y)$ stable



$$\mathcal{B}(h) := \Big\{ (x, y) \colon \Big\langle \big[x - \bar{x}(y, \varepsilon)\big], X^\star(y)^{-1} \big[x - \bar{x}(y, \varepsilon)\big] \Big\rangle < h^2 \Big\}$$

$X^\star(y)$ solution of $A(y)X^\star + X^\star A(y)^\mathsf{T} + F(x^\star, y)F(x^\star, y)^\mathsf{T} = 0$

Stochastic perturbation: n-dimensional case

$$\begin{cases} \mathrm{d}x_t = \dfrac{1}{\varepsilon} f(x_t, y_t)\,\mathrm{d}t + \dfrac{\sigma}{\sqrt{\varepsilon}} F(x_t, y_t)\,\mathrm{d}W_t & \text{(fast variables } \in \mathbb{R}^n) \\ \mathrm{d}y_t = \quad g(x_t, y_t)\,\mathrm{d}t + \sigma'\, G(x_t, y_t)\,\mathrm{d}W_t & \text{(slow variables } \in \mathbb{R}^m) \end{cases}$$

Theorem [B. & G., JDE 2003]

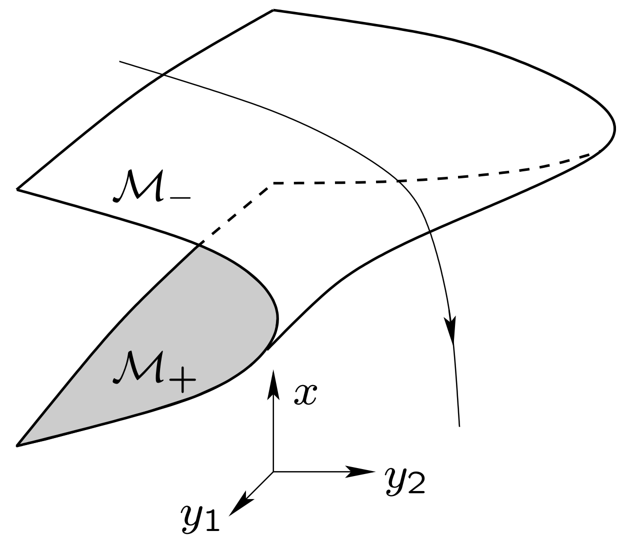
- $\mathbb{P}\Big\{\text{leaving } \mathcal{B}(h) \text{ before time } t\Big\} \simeq C(t,\varepsilon)\,\mathrm{e}^{-\kappa h^2/2\sigma^2}$
  $\kappa = 1 - \mathcal{O}(h) - \mathcal{O}(\varepsilon)$.
- Projection on $\bar{x}(y,\varepsilon)$:

$$\mathrm{d}y_t^0 = g(\bar{x}(y_t^0,\varepsilon), y_t^0)\,\mathrm{d}t + \sigma' G(\bar{x}(y_t^0,\varepsilon), y_t^0)\,\mathrm{d}W_t$$

  y_t^0 approximates y_t to order $\sigma\sqrt{\varepsilon}$ up to Lyapunov time of $\dot{y} = g(\bar{x}(y,\varepsilon)y)$.

Bifurcations

$x^\star(y)$ slow manifold for $y \in \mathcal{D}_0$
$A(y) = \partial_x f(x^\star(y), y)$
Some ev of $A(y)$ cross imaginary axis as y approaches $\partial\mathcal{D}_0$

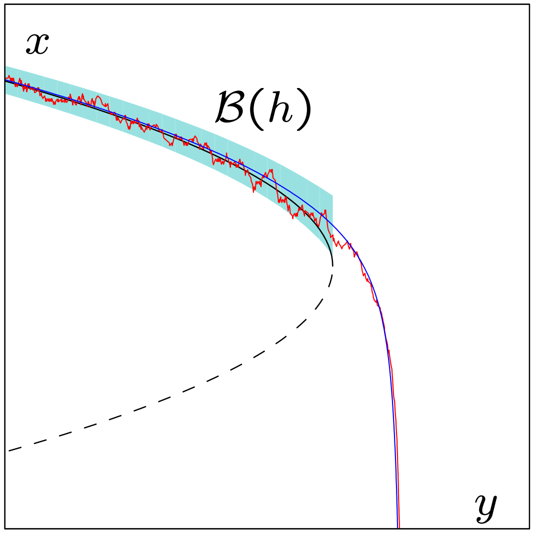


Theorem [B. & G., JDE 2003]
System can be approximated by projection on centre manifold.

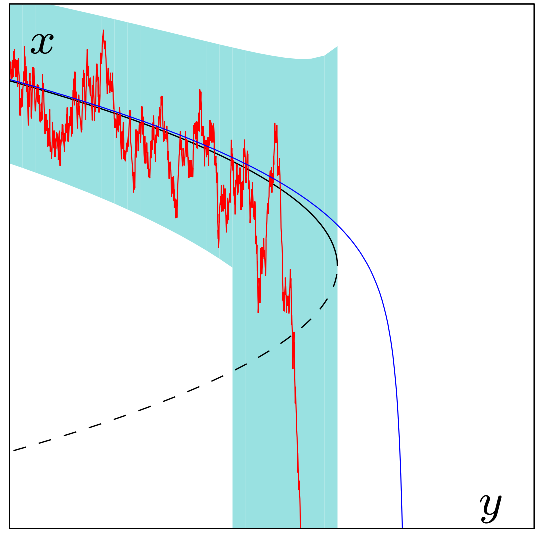
- Saddle–node bifurcation: transitions through unstable manifold, relaxation oscillations, hysteresis
- (Avoided) transcritical bifurcation: stochastic resonance
- Pitchfork bifurcation: decrease of bifurcation delay

Saddle–node bifurcation   e.g. $f(x,y) = -y - x^2$

$\sigma \ll \sqrt{\varepsilon}$



$\sigma \gg \sqrt{\varepsilon}$



Deterministic case $\sigma = 0$: Solutions stay at distance $\varepsilon^{1/3}$ above bifurcation point until time $\varepsilon^{2/3}$ after bifurcation.

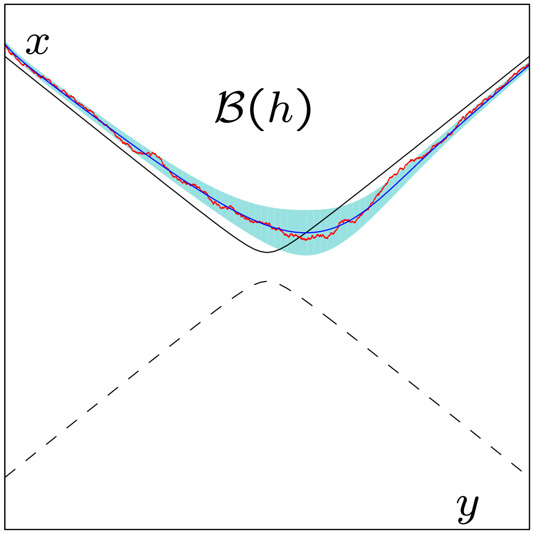
Theorem: [B. & G., Nonlinearity 2002]

1. If $\sigma \ll \sqrt{\varepsilon}$: Paths likely to stay in $\mathcal{B}(h)$ until time $\varepsilon^{2/3}$ after bifurcation, maximal spreading $\sigma\varepsilon^{-1/6}$.
2. If $\sigma \gg \sqrt{\varepsilon}$: Paths likely to escape at time $\sigma^{4/3}$ *before* bifurcation.

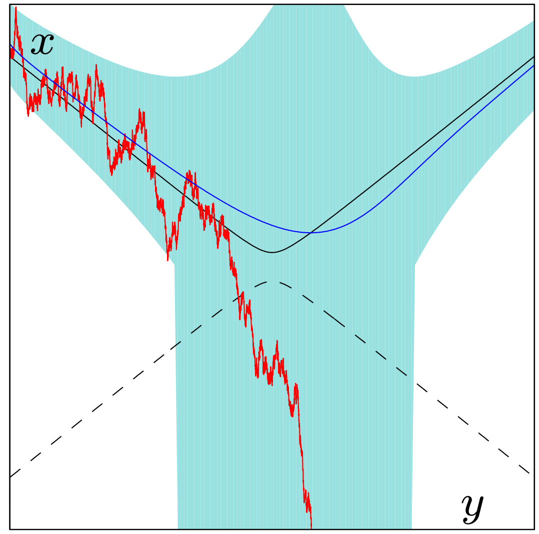
Avoided transcritical bifurcation

e.g. $f(x,y) = y^2 + \delta - x^2$

$\sigma \ll (\delta \vee \varepsilon)^{3/4}$



$\sigma \gg (\delta \vee \varepsilon)^{3/4}$



Minimal distance between branches $= \delta^{1/2}$
Det. case $\sigma = 0$: Solutions stay $(\delta \vee \varepsilon)^{1/2}$ above bif. point

Theorem: [B. & G., Ann. App. Probab. 2002]

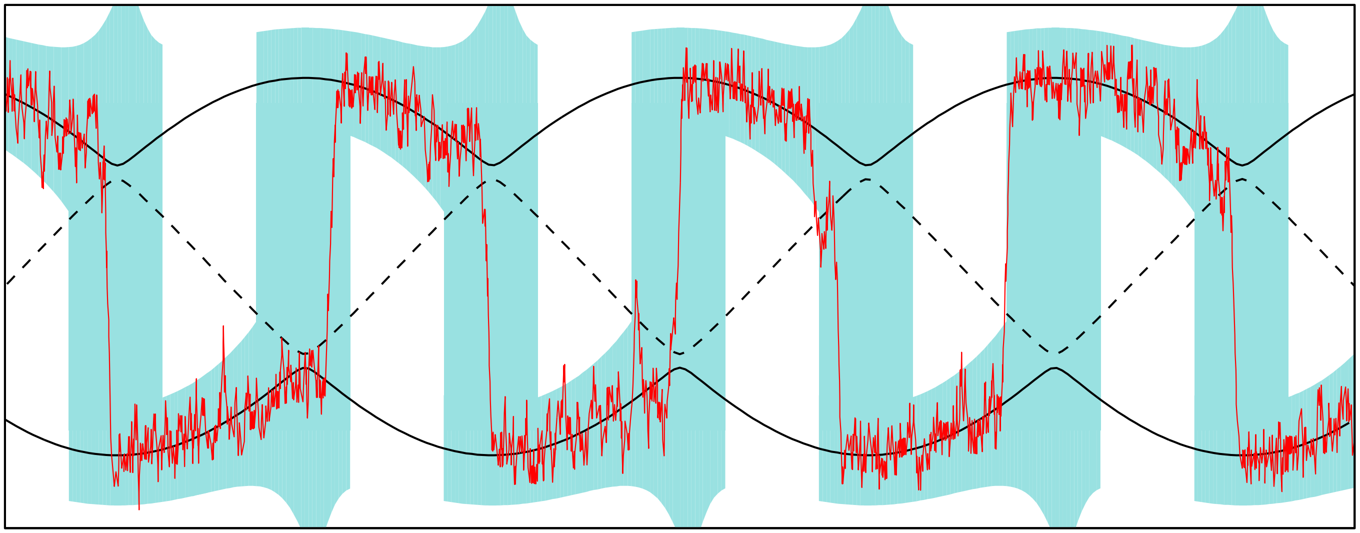
1. If $\sigma \ll (\delta \vee \varepsilon)^{3/4}$: Paths likely to stay in $\mathcal{B}(h)$, maximal spreading $\sigma(\delta \vee \varepsilon)^{-1/4}$.
2. If $\sigma \gg (\delta \vee \varepsilon)^{3/4}$: Paths likely to escape at time $\sigma^{2/3}$ *before* avoided bifurcation.

Stochastic resonance

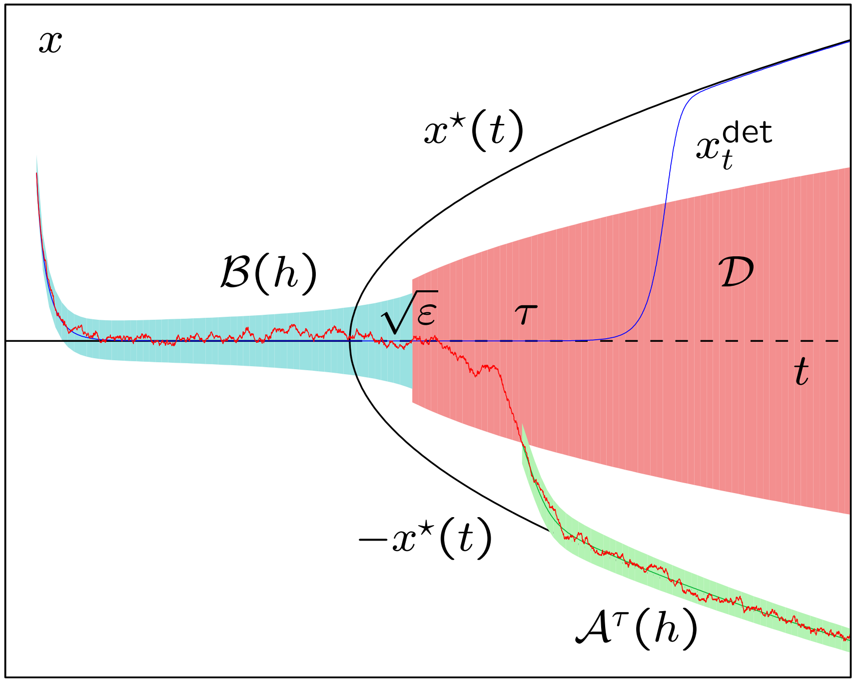
$$\mathrm{d}x_t = \underbrace{\Big[x_t - x_t^3 + A\cos\varepsilon t\Big]}_{= -\frac{\partial}{\partial x}V(x_t,t)} \mathrm{d}t + \sigma\,\mathrm{d}W_t$$

Potential: $$V(x,t) = \frac{1}{4}x^4 - \frac{1}{2}x^2 - Ax\cos\varepsilon t.$$

$\sigma \gg (\delta \vee \varepsilon)^{3/4}$, $\delta = A - A_c$, $A_c = 2/3\sqrt{3}$: synchronisation



Pitchfork bifurcation e.g. $f(x,y) = yx - x^3$

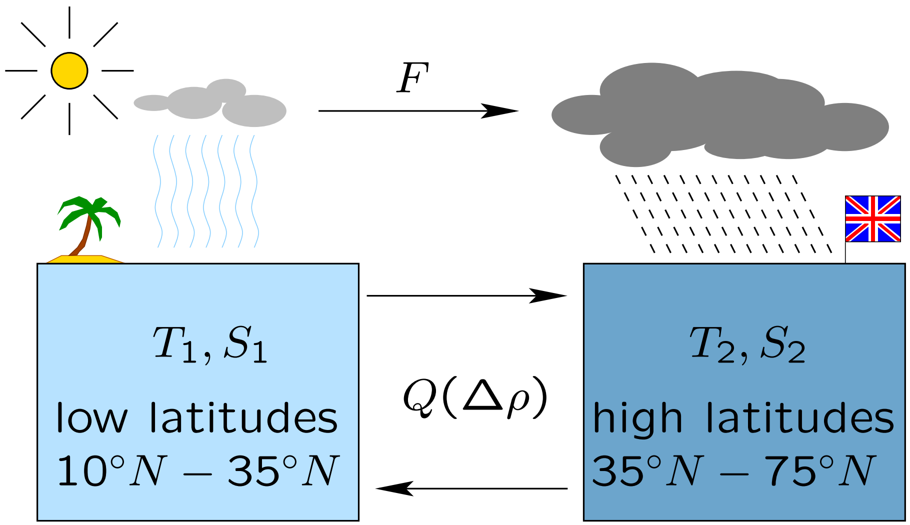


Theorem [B. & G., PTRF 2002]

- Paths concentrated in $\mathcal{B}(h)$ up to time $\sqrt{\varepsilon}$
  Typical spreading $\sigma\varepsilon^{-1/4}$
- Paths likely to leave $\mathcal{D}$ at time $\sqrt{\varepsilon|\log\sigma|}$
- Paths likely to stay in $\mathcal{A}^\tau(h)$ after leaving $\mathcal{D}$

Application. North-Atlantic thermohaline circulation: Stommel's Box Model ('61)

- T_i: temperatures
- S_i: salinities
- F: freshwater flux
- $Q(\Delta\rho)$: mass exchange
- $\Delta\rho = \alpha_S \Delta S - \alpha_T \Delta T$
- $\Delta T = T_1 - T_2$
- $\Delta S = S_1 - S_2$



$$\frac{\mathrm{d}}{\mathrm{d}s}\Delta T = -\frac{1}{\tau_r}(\Delta T - \theta) - Q(\Delta\rho)\Delta T$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\Delta S = \frac{S_0}{H}F - Q(\Delta\rho)\Delta S$$

Model for Q (Cessi): $Q(\Delta\rho) = \frac{1}{\tau_d} + \frac{q}{V}\Delta\rho^2$.

Slow–fast system

Separation of time scales: $\tau_r \ll \tau_d$
Scaling: $x = \Delta T/\theta$, $y = \Delta S \alpha_S/(\alpha_T \theta)$, $s = \tau_d t$, ...

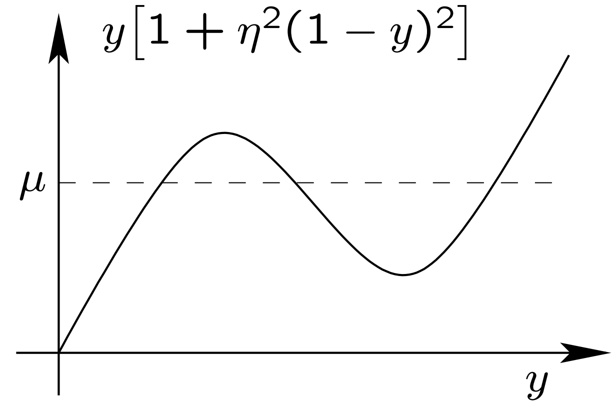
$$\begin{aligned} \varepsilon \dot{x} &= -(x-1) - \varepsilon x \Big[1 + \eta^2 (x-y)^2\Big] \\ \dot{y} &= \mu - y\Big[1 + \eta^2 (x-y)^2\Big] \end{aligned}$$

$\varepsilon = \tau_r/\tau_d \ll 1$

Slow manifold: $x = 1 + \mathcal{O}(\varepsilon) \Rightarrow \varepsilon \dot{x} = 0$.

Reduced equation on slow manifold:

$$\dot{y} = \mu - y\Big[1 + \eta^2 (1-y)^2 + \mathcal{O}(\varepsilon)\Big]$$



One or two stable equilibria,
depending on μ (and η).

Time-dependent freshwater flux

$$\mathrm{d}x_t = \frac{1}{\varepsilon_0}\Big[-(x_t - 1) - \varepsilon_0 x_t Q(x_t - y_t)\Big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon_0}}\,\mathrm{d}W_t^0$$

$$\mathrm{d}y_t = \Big[z_t - y_t Q(x_t - y_t)\Big]\,\mathrm{d}t + \sigma_1\,\mathrm{d}W_t^1$$

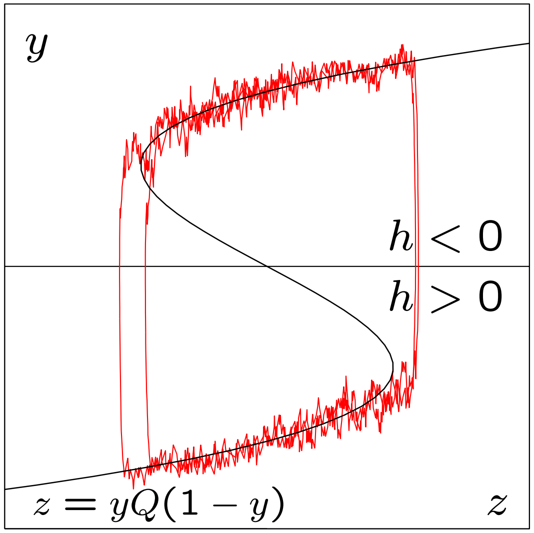
$$\mathrm{d}z_t = \varepsilon h(x_t, y_t, z_t)\,\mathrm{d}t + \sqrt{\varepsilon}\sigma_2\,\mathrm{d}W_t^2$$

Reduced equation, $t \mapsto \varepsilon t$:

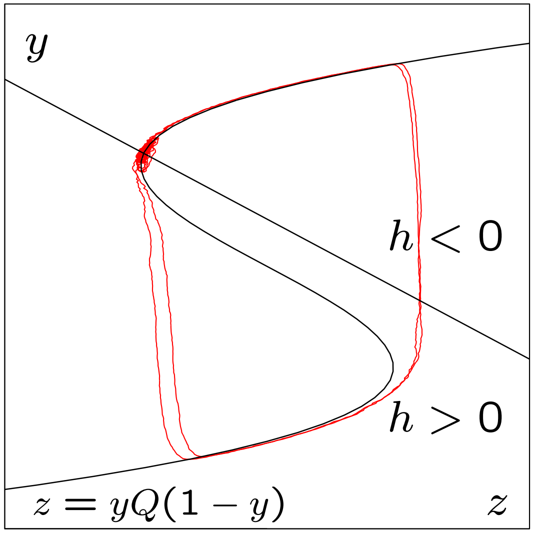
$$\mathrm{d}y_t = \frac{1}{\varepsilon}\Big[z_t - y_t Q(1 - y_t)\Big]\,\mathrm{d}t + \frac{\sigma_1}{\sqrt{\varepsilon}}\,\mathrm{d}W_t^1$$

$$\mathrm{d}z_t = h(1, y_t, z_t)\,\mathrm{d}t + \sigma_2\,\mathrm{d}W_t^2$$

Relaxation oscillations



Excitability



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Nils Berglund
Barbara Gentz

Noise-Induced Phenomena in Slow–Fast Dynamical Systems

A Sample-Paths Approach

May 26, 2005

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