

Stochastic analysis seminar – Wuhan, Great Bay University
& Huazhong University of Science and Technology

(Online talk)

Stochastic resonance in stochastic PDEs

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Project
PERISTOCH

Stochastic resonance in an SDE

$$\begin{aligned} dx_t &= \underbrace{\left[-x_t^3 + x_t + A \cos(\varepsilon t)\right]}_{= -\frac{\partial}{\partial x} \left[\frac{1}{4}x^4 - \frac{1}{2}x^2 - Ax \cos(\varepsilon t)\right]} dt + \sigma dW_t \end{aligned}$$

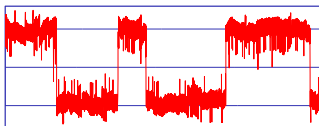
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Stochastic resonance in an SDE

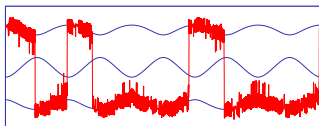
$$\begin{aligned} dx_t &= \underbrace{\left[-x_t^3 + x_t + A \cos(\varepsilon t) \right]}_{=} dt + \sigma dW_t \\ &= -\frac{\partial}{\partial x} \left[\frac{1}{4} x^4 - \frac{1}{2} x^2 - A x \cos(\varepsilon t) \right] \Big|_{x_t} \end{aligned}$$

- ▷ Ice Ages: deterministically bistable climate [Croll, Milankovitch]
- ▷ random perturbations due to weather [Benzi-Sutera-Vulpiani, Nicolis-Nicolis]

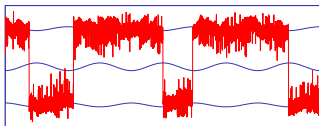
Sample paths $\{x_t\}_t$ for $\varepsilon = 0.001$:



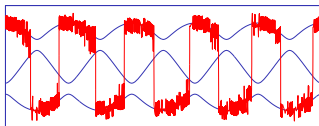
$A = 0, \sigma = 0.3$



$A = 0.24, \sigma = 0.2$



$A = 0.1, \sigma = 0.27$



$A = 0.35, \sigma = 0.2$

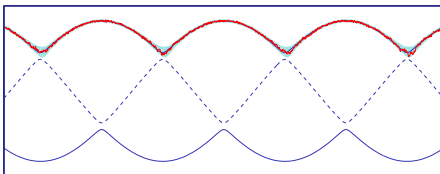
Descriptions of stochastic resonance

- ▷ Fokker–Planck equation: [Caroli, Caroli, Roulet & Saint-James '81]
- ▷ Two-state Markov chain: [Eckmann & Thomas '82], [Imkeller & Pavlyukevich '02], [Herrmann & Imkeller '02]
- ▷ Signal-to-noise ratio: [Gammaitoni, Menichella-Saetta & ... '89], [Fox '89], [Jung & Hänggi '89], [McNamara & Wiesenfeld '89]
- ▷ Slow forcing: [Jung & Hänggi '91], [Talkner '99], [Talkner & Łuczka '04]
- ▷ Large deviations: [Freidlin '00, Freidlin '01]
- ▷ Residence-time distributions: [Zhou, Moss & Jung '90], [Choi, Fox & Jung '98], ...
- ▷ Overview articles:
[Moss, Pierson & O'Gorman '94], [Wiesenfeld & Moss '95], [McNamara & Wiesenfeld '95], [Wiesenfeld & Jaramillo '98], [Gammaitoni, Hänggi, Jung & Marchesoni '98], [Hänggi '02], [Wellens, Shatokhin & Buchleitner '04], ...
- ▷ Monograph: [Herrmann, Imkeller, Pavlyukevich & Peithmann '14]

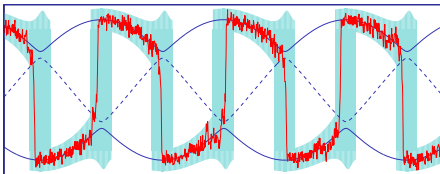
The synchronisation regime

$A_c = \frac{2}{3\sqrt{3}}$, $A = A_c - \delta$, $0 < \delta \ll 1$. Critical noise intensity: $\sigma_c = \max\{\delta, \varepsilon\}^{3/4}$

$\sigma \ll \sigma_c$:
transitions unlikely



$\sigma \gg \sigma_c$:
synchronisation



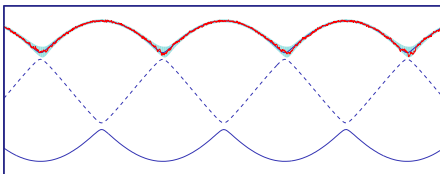
Theorem [B & Gentz, Annals App. Proba 2002]

- ▷ Away from (avoided) bifurcations, sample paths concentrated in σ -neighbourhood of deterministic stable periodic solutions
- ▷ $\sigma \ll \sigma_c$: transition probability per period $\leq e^{-\sigma_c^2/\sigma^2}$
- ▷ $\sigma \gg \sigma_c$: transition probability per period $\geq 1 - e^{-c\sigma^{4/3}/(\varepsilon|\log \sigma|)}$

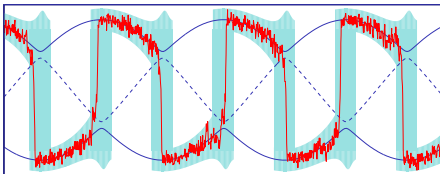
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(Stochastic) Allen–Cahn equation on \mathbb{T}^2

$$du(t, x) = [\nu(\varepsilon t)\Delta u(t, x) + u(t, x) - u(t, x)^3] dt + \sigma dW(t, x)$$

(Online: <https://youtu.be/yXOEAxZHNCQ>)

Stochastic resonance in stochastic PDEs

$$d\phi(t, x) = \left[\Delta\phi(t, x) + \phi(t, x) - \phi(t, x)^3 + \underbrace{A\cos(\varepsilon t)}_{h(\varepsilon t)} \right] dt + \sigma dW(t, x)$$

Stochastic resonance in bistable SPDEs on \mathbb{T}^1

$$d\phi(t, x) = [\Delta\phi(t, x) + f(\varepsilon t, \phi(t, x))] dt + \sigma dW(t, x)$$

- ▷ $\phi = \phi(t, x) \in \mathbb{R}$, $\varepsilon t \in [0, T]$ or f is T -periodic, $x \in \mathbb{T} = \mathbb{R}/L\mathbb{Z}$, $L > 0$
- ▷ $\phi \mapsto f(s, \phi)$ bistable, \mathcal{C}^2 , confining, e.g. $f(s, \phi) = \phi - \phi^3 + A \cos(s)$
- ▷ $dW(t, x)$ space-time white noise on $\mathbb{R}_+ \times \mathbb{T}$
- ▷ $0 < \varepsilon, \sigma \ll 1$
- ▷ δ measures closeness to bifurcation (e.g. $A_c - A$)

Theorem [B & Nader, Stoch. & PDEs: Analysis & Comput., 2023]

- ▷ Away from bifurcations, solutions are concentrated around deterministic solutions in Sobolev H^s -norm for any $s < \frac{1}{2}$
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Proof ideas, 1D SDE below threshold

On slow time scale $\varepsilon t \rightarrow t$:

$$dx_t = \frac{1}{\varepsilon} f(t, x_t) dt + \frac{\sigma}{\sqrt{\varepsilon}} dW_t$$

$f(t, x^*(t)) = 0$, $\partial_x f(t, x^*(t)) < 0$, det. slow solution $\bar{x}(t) = x^*(t) + \mathcal{O}(\varepsilon)$

Write $x_t = \bar{x}(t) + \xi_t$ and Taylor-expand:

$$d\xi_t = \frac{1}{\varepsilon} \left[\bar{a}(t)\xi_t + \underbrace{b(t, \xi_t)}_{=\mathcal{O}(\xi_t^2)} \right] dt + \frac{\sigma}{\sqrt{\varepsilon}} dW_t$$

where $\bar{a}(t) = \partial_x f(t, \bar{x}(t)) = \partial_x f(t, x^*(t)) + \mathcal{O}(\varepsilon) < 0$

Variations of constants (Duhamel formula), if $\xi_0 = 0$:

$$\xi_t = \underbrace{\frac{\sigma}{\sqrt{\varepsilon}} \int_0^t e^{\bar{a}(t,s)/\varepsilon} dW_s}_{\xi_t^0: \text{sol of linearised system}} + \underbrace{\frac{1}{\varepsilon} \int_0^t e^{\bar{a}(t,s)/\varepsilon} b(s, \xi_s) ds}_{\text{treat as a perturbation}}$$

where $\bar{\alpha}(t, s) = \int_s^t \bar{a}(u) du$

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Proof ideas, 1D SDE below threshold

Properties of OU-like process $\xi_t^0 = \frac{\sigma}{\sqrt{\varepsilon}} \int_0^t e^{\bar{\alpha}(t,s)/\varepsilon} dW_s$:

- ▷ Gaussian process, $\mathbb{E}[\xi_t^0] = 0$, $\text{Var}(\xi_t^0) = \frac{\sigma^2}{\varepsilon} \int_0^t e^{2\bar{\alpha}(t,s)/\varepsilon} ds$
- ▷ Confidence interval: $\mathbb{P}\{|\xi_t^0| > \frac{h}{\sigma} \sqrt{\text{Var}(\xi_t^0)}\} = \mathcal{O}(e^{-h^2/2\sigma^2})$
- ▷ $\sigma^{-2} \text{Var}(\xi_t^0)$ satisfies ODE $\varepsilon \dot{v} = 2\bar{a}(t)v + 1$

Lemma [B & Gentz, PTRF 2002]

$\bar{v}(t)$ solution of ODE bounded away from 0: $\bar{v}(t) = \frac{1}{-2\bar{a}(t)} + \mathcal{O}(\varepsilon)$

$$\mathbb{P}\left\{\sup_{0 \leq s \leq t} \frac{|\xi_s^0|}{\sqrt{\bar{v}(s)}} > h\right\} = C_0(t, \varepsilon) e^{-h^2/2\sigma^2}$$

where $C_0(t, \varepsilon) = \sqrt{\frac{2}{\pi} \frac{1}{\varepsilon} \left| \int_0^t \bar{a}(s) ds \right|} \frac{h}{\sigma} [1 + \mathcal{O}(\varepsilon + \frac{t}{\varepsilon} e^{-h^2/\sigma^2})]$

Proof based on Doob's submartingale inequality and partition of $[0, t]$

Proof ideas, 1D SDE below threshold

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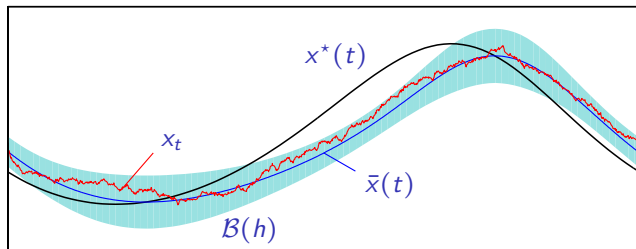
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Proof ideas, 1D SDE below threshold

Nonlinear equation: $d\xi_t = \frac{1}{\varepsilon} [\bar{a}(t)\xi_t + b(t, \xi_t)] dt + \frac{\sigma}{\sqrt{\varepsilon}} dW_t$

Confidence strip: $\mathcal{B}(h) = \{|\xi| \leq h\sqrt{\bar{v}(t)} \forall t\} = \{|x - \bar{x}(t)| \leq h\sqrt{\bar{v}(t)} \forall t\}$



Theorem B & Gentz, PTRF 2002

$$C(t, \varepsilon) e^{-\kappa_- h^2 / 2\sigma^2} \leq \mathbb{P}\{\text{leaving } \mathcal{B}(h) \text{ before time } t\} \leq C(t, \varepsilon) e^{-\kappa_+ h^2 / 2\sigma^2}$$

where $\kappa_{\pm} = 1 \mp \mathcal{O}(h)$ and $C(t, \varepsilon) = C_0(t, \varepsilon)[1 + \mathcal{O}(h)]$ (requires $h \leq h_0$)

Avoided transcritical bifurcation

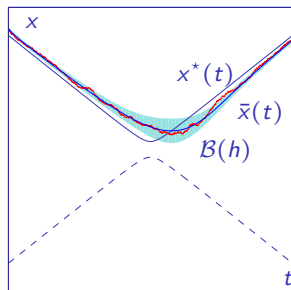
$$dx_t = \frac{1}{\varepsilon} [t^2 + \delta - x_t^2 + \dots] dt + \frac{\sigma}{\sqrt{\varepsilon}} dW_t$$

Equil. curve: $x^*(t) \simeq \sqrt{t^2 + \delta}$

Slow sol.: $\bar{x}(t) = x^*(t) + \mathcal{O}(\min\{\frac{\varepsilon}{|t|}, \frac{\varepsilon}{\sqrt{\delta+\varepsilon}}\})$

$$\bar{a}(t) = \partial_x f(t, \bar{x}(t)) \asymp \begin{cases} -|t| & |t| \geq \sqrt{\delta + \varepsilon} \\ -\sqrt{\delta + \varepsilon} & |t| \leq \sqrt{\delta + \varepsilon} \end{cases}$$

Confidence strip $\mathcal{B}(h)$: width $\asymp h/\sqrt{|\bar{a}(t)|}$



Theorem [B & Gentz, AAP 2002]

$$\mathbb{P}\{\text{leaving } \mathcal{B}(h) \text{ before time } t\} \leq C(t, \varepsilon) e^{-\kappa h^2 / 2\sigma^2}$$

where $\kappa = 1 - \mathcal{O}(\sup_{s \leq t} h|\bar{a}(s)|^{-3/2}) - \mathcal{O}(\varepsilon) \Rightarrow$ requires $h < h_0 \inf_{s \leq t} |\bar{a}(s)|^{3/2}$

▷ $\sigma < \sigma_c = \max\{\delta, \varepsilon\}^{3/4}$: result applies $\forall t$, $\mathbb{P}\{\text{trans}\} = \mathcal{O}(e^{-\kappa\sigma_c^2/\sigma^2})$

▷ $\sigma > \sigma_c = \max\{\delta, \varepsilon\}^{3/4}$: result applies up to $t \asymp -\sigma^2/3$

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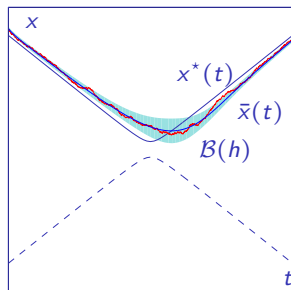
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Above threshold

What happens for $\sigma > \sigma_c$ and $t > -\sigma^{2/3}$?

General principle: partition $t_0 = s_0 < s_1 < s_2 < \dots < s_n = t$ of $[t_0, t]$

Lemma Let $P_k = \mathbb{P}\{\text{making no transition during } (s_{k-1}, s_k]\}$. Then

$$\mathbb{P}\{\text{making no transition during } [t_0, t]\} \leq \prod_{k=1}^n P_k$$

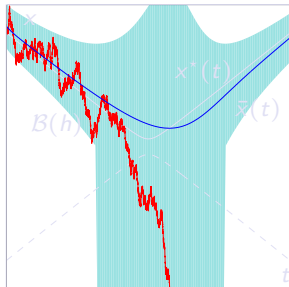
Choose partition s.t. each $P_k \leq q < 1 \Rightarrow \mathbb{P}\{\text{no transition}\} \leq e^{-n \log q}$

Define partition such that

$$\int_{s_{k-1}}^{s_k} |\bar{a}(s)| ds = c\varepsilon |\log \sigma| \Rightarrow P_k \leq \frac{2}{3}$$

Thm [B & Gentz, AAP 2002]

Transition probability $\geq 1 - e^{-\kappa \sigma^{4/3} / (\varepsilon |\log \sigma|)}$



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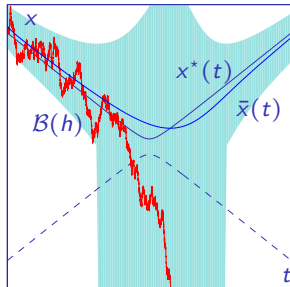
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SPDE on \mathbb{T}^1 : stable case

$$d\phi(t, x) = \frac{1}{\varepsilon} [\Delta\phi(t, x) + f(t, \phi(t, x))] dt + \frac{\sigma}{\sqrt{\varepsilon}} dW(t, x)$$

- ▷ $f(t, \phi^*(t)) = 0$ for all $t \in I = [0, T]$
- ▷ $a(t) = \partial_\phi f(t, \phi^*(t)) \leq -a_- < 0$ for all $t \in I$

In deterministic case $\sigma = 0$: \exists particular solution $\bar{\phi}(t, x)$ such that

$$\|\bar{\phi}(t, \cdot) - \phi^*(t)\mathbf{e}_0\|_{H^1} \leq C\varepsilon \quad \forall t \in I$$

Theorem [B & Nader 2023]

Fix $s < \frac{1}{2}$, and let $\mathcal{B}(h) = \{(t, \phi) : t \in I, \|\phi - \bar{\phi}(t, \cdot)\|_{H^s} < h\}$

For any $\nu > 0$

$$\mathbb{P}\{\text{leaving } \mathcal{B}(h) \text{ before time } t\} \leq C(t, \varepsilon, s) \exp\left\{-\kappa \frac{h^2}{\sigma^2} \left[1 - \mathcal{O}\left(\frac{h}{\varepsilon^\nu}\right)\right]\right\}$$

holds for some $\kappa > 0$, $h = \mathcal{O}(\varepsilon^\nu)$ and $C(t, \varepsilon, s) = \mathcal{O}(t/\varepsilon)$.

SPDE on \mathbb{T}^1 : stable case

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Ideas of proof

- ▷ $\phi(x) = \sum_{k \in \mathbb{Z}} \phi_k e_k(x) \Rightarrow \|\phi\|_{H^s}^2 = \sum_{k \in \mathbb{Z}} \langle k \rangle^{2s} \phi_k^2, \quad \langle k \rangle = \sqrt{1 + k^2}$
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Choose $h_k^2 \sim h^2 \langle k \rangle^{-2+2s+\eta}$, $\eta > 0$

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▷ Schauder estimate: $\beta \in H^r$, $0 < r < \frac{1}{2} \Rightarrow$

$$\|e^{t\Delta} \beta\|_{H^q} \leq M(q, r) t^{-(q-r)/2} \|\beta\|_{H^r} \quad \forall q < r+2$$

Consequence: $\psi = \psi^0 + \psi^1$ where nonlinear term satisfies

$$\|\psi^1\|_{H^q} \leq M' \varepsilon^{(q-r)/2-1} \sup_t \|b(t, \psi(y, \cdot))\|_{H^r}$$

SPDE near a bifurcation point

$$d\phi = \frac{1}{\varepsilon} [\Delta\phi + g(t) - \phi^2 - b(t, \phi)] dt + \frac{\sigma}{\sqrt{\varepsilon}} dW(t, x)$$

with $g(t) = \delta + t^2 + \mathcal{O}(t^3)$ and $b = \mathcal{O}(\phi^3 + t\phi^2 + t^2\phi)$

- ▷ Decompose $\phi(t, x) = \phi_0(t)e_0(x) + \phi_\perp(t, x)$ where e_0 constant fct
- ▷ ϕ_\perp satisfies similar concentration result as ϕ in stable case
- ▷ ϕ_0 satisfies similar equation as in 1D, with error term of order $\|\phi_\perp\|_{H^s}^2$

Thm 1: Transverse component

$$\mathbb{P}\{\tau_{B_1}(h_\perp) < t \wedge \tau_{B_0}(h)\} \leq C(t, \varepsilon, s) \exp\left\{-\kappa \frac{h^2}{\sigma^2} \left[1 - \mathcal{O}\left(\frac{h}{\varepsilon^\nu}\right)\right]\right\}$$

Thm 2: Mean

$$\mathbb{P}\{\tau_{B_0}(h) < t \wedge \tau_{B_1}(h_1)\} \leq C(t, \varepsilon) e^{-\kappa h^2/2\sigma^2} \quad \kappa = 1 - \mathcal{O}\left(\sup_s h |\bar{a}(s)|^{3/2}\right)$$

Thm 3: Escape

$$\mathbb{P}\{\phi_0(t_1) > -d \forall t \in [-\sigma^{2/3}, t \wedge \tau_{B_1}(h_1)]\} \leq \frac{3}{2} e^{-\hat{\alpha}(t, -\sigma^{2/3})/[\varepsilon \log(\sigma^{-1})]}$$

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$$d\phi(t, x) = \frac{1}{\varepsilon} \left[\Delta \phi(t, x) + \sum_{j=1}^n A_j(t) \phi(t, x)^j \right] dt + \frac{\sigma}{\sqrt{\varepsilon}} dW(t, x) \quad x \in \mathbb{T}^2$$

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- ▷ Use Besov–Hölder spaces $\mathcal{B}_{2, \infty}^\alpha$, $\alpha < 0$, instead of Sobolev spaces H^s :

$$\|\phi\|_{\mathcal{B}_{2, \infty}^\alpha} = \sup_{q \geq 0} 2^{q\alpha} \|\delta_q \phi\|_{L^2} \quad \delta_q \phi = \sum_{2^{q-1} \leq |k| < 2^q} \phi_k e_k$$

SPDE on the 2d torus

Theorem [B & Nader 2024]

For $\alpha < 0$, $m \in \mathbb{N}$,

$$\mathbb{P}\left\{\sup_{0 \leq t \leq T} \|\cdot \psi(t, \cdot)^m\|_{\mathcal{B}_{2,\infty}^\alpha} > h^m\right\} \leq C_m(T, \varepsilon, \alpha) e^{-\kappa_m(\alpha) h^2 / \sigma^2}$$

where

$$\kappa_m(\alpha) \geq c_0 \frac{\alpha^2}{m^7} \quad C_m(T, \varepsilon, \alpha) \leq c_1 \frac{T}{\varepsilon} \frac{m^{3/2} e^m m^m}{|\alpha|}$$

▷ Binomial formula

$$\cdot \psi^m \cdot = H_m(\psi; C_N) = \sum_{|n|=m} \frac{m!}{n!} \prod_{q \geq 0} H_{n_q}(\delta_q \psi; c_q) \quad c_q = \mathcal{O}(1)$$

▷ Doob submartingale inequality for $\sup_{t \in I_\ell} \|\delta_{q_0}(\prod_{q \geq 0} H_{n_q}(\delta_q \hat{\psi}; c_q))\|_{L^2}^2$

where $\hat{\psi}$ martingale approximating ψ on intervals I_ℓ depending on q_0

▷ Upgrade to bound for $\sup_{t \in I_\ell} \|\delta_{q_0}(\prod_{q \geq 0} H_{n_q}(\delta_q \psi; c_q))\|_{L^2}^2$

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Concentration estimates

Theorem [B & Nader 2024]

Let $\phi_1 = \phi - \phi^* - \psi$. Then $\forall \gamma < 2, \forall \nu < 1 - \frac{\gamma}{2}, \forall h < h_0 \varepsilon^\nu$

$$\mathbb{P}\left\{ \sup_{t \in [0, T]} \|\phi_1(t)\|_{C^{\gamma-1}} > M \varepsilon^{-\nu} h (h + \varepsilon) \right\} \leq C(T, \varepsilon) e^{-\kappa h^2 / \sigma^2}$$

- ▷ Use $\|\phi^\ell : \psi^m\|_{\mathcal{B}_{2, \infty}^{(2\ell+1)\alpha}} \leq \|\phi\|_{\mathcal{B}_{2, \infty}^\ell} \|\psi^m\|_{\mathcal{B}_{2, \infty}^\alpha}$ to bnd nonlin term in $d\phi_1$
- ▷ Use Schauder estimate and $\mathcal{B}_{2, \infty}^\gamma \hookrightarrow \mathcal{B}_{\infty, \infty}^{\gamma-1} = C^{\gamma-1}$

Example: Dynamic pitchfork bifurcation

$$d\phi(t, x) = \frac{1}{\varepsilon} [\Delta \phi(t, x) + a(t)\phi(t, x) - : \phi(t, x)^3 :] dt + \frac{\sigma}{\sqrt{\varepsilon}} dW(t, x)$$

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- ▷ Fractional Brownian motion (fBm) of Hurst parameter $H \in (0, 1)$:
Centred Gaussian process with covariance

$$\mathbb{E}[W_t^H W_s^H] = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H})$$

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Let $dx_t = \frac{1}{\varepsilon} a(t)x_t dt + \frac{\sigma}{\varepsilon^H} dW_t^H$, with $a(t) \leq a_0 < 0$ and $H \in (0, 1)$. Then

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Open questions

- ▷ Case $x \in \mathbb{T}^3$? Regularity structures or similar needed ...

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Slides available at <https://www.idpoisson.fr/berglund/Wuhan24.pdf>

