SIAM Conference on Uncertainty Quantification

Theory and Applications of Random Poincaré Maps

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Joint works with Manon Baudel (Orléans),
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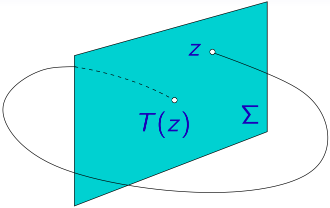
Deterministic Poincaré maps

ODE $\dot{z} = f(z) \quad z \in \mathbb{R}^n$

Flow: $z_t = \varphi_t(z_0)$

$\Sigma \subset \mathbb{R}^n$: $(n-1)$-dimensional manifold

Poincaré map (or first-return map):
$T : \Sigma \to \Sigma$



$T(z) = \varphi_\tau(z)$ where $\tau = \inf\{t > 0 \colon \varphi_\tau(z) \in \Sigma\}$

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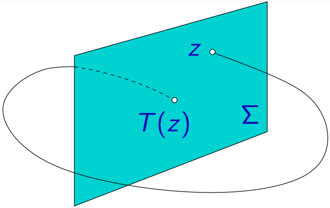

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Benefits:

1. Dimension reduction: T is an $(n-1)$-dimensional map
2. Stability of periodic orbits: no neutral direction
3. Bifurcations of periodic orbits easier to study (period doubling, Hopf, . . .)

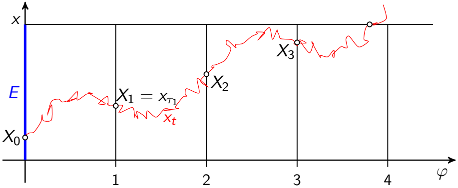
Question: how about SDEs $\quad \mathrm{d}z_t = f(z_t)\,\mathrm{d}t + \sigma g(z_t)\,\mathrm{d}W_t$?

Random Poincaré maps

In appropriate coordinates

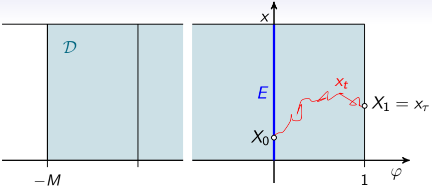
$$\begin{aligned} d\varphi_t &= f(\varphi_t, x_t)\,dt + \sigma F(\varphi_t, x_t)\,dW_t \qquad \varphi \in \mathbb{R} \quad (\text{or } \mathbb{R}/\mathbb{Z}) \\ dx_t &= g(\varphi_t, x_t)\,dt + \sigma G(\varphi_t, x_t)\,dW_t \qquad x \in E \subset \Sigma \end{aligned}$$

- all functions periodic in φ (say period 1)
- $f \geqslant c > 0$ and σ small $\Rightarrow$ φ_t likely to increase
- process may be killed when x leaves E
  (one may want to consider process conditionned on staying in E)



$X_0, X_1, \dots$ form (substochastic) Markov chain

Harmonic measure



- τ: first-exit time of $z_t = (\varphi_t, x_t)$ from $\mathcal{D} = (-M, 1) \times E$
- $A \subset \partial\mathcal{D}$: $\mu_z(A) = \mathbb{P}^z\{z_\tau \in A\}$ harmonic measure (wrt generator $\mathcal{L}$)
- [Ben Arous, Kusuoka, Stroock '84]: under hypoellipticity cond, μ_z admits (smooth) density $h(z, y)$ wrt arclength on $\partial\mathcal{D}$

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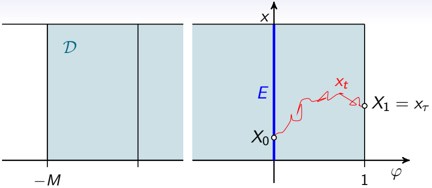

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- [Ben Arous, Kusuoka, Stroock '84]: under hypoellipticity cond, μ_z admits (smooth) density $h(z, y)$ wrt arclength on $\partial\mathcal{D}$
- Remark: $\mathcal{L}_z h(z, y) = 0$ (kernel is harmonic)
- For $B \subset E$ Borel set

$$\mathbb{P}^{X_0}\{X_1 \in B\} = K(X_0, B) := \int_B K(X_0, \mathrm{d}y)$$

where $K(x, \mathrm{d}y) = h((0, x), (1, y))\,\mathrm{d}y =: k(x, y)\,\mathrm{d}y$

Fredholm theory

Consider integral operator K acting

- on L^∞ via $f \mapsto (Kf)(x) = \int_E k(x,y) f(y) \, \mathrm{d}y = \mathbb{E}^x[f(X_1)]$
- on L^1 via $m \mapsto (mK)(y) = \int_E m(x) k(x,y) \, \mathrm{d}x = \mathbb{P}^\mu\{X_1 \in \mathrm{d}y\}$

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Thm [Fredholm 1903]:
If $k \in L^2$, then K has eigenvalues λ_n of finite multiplicity
Right/left eigenfunctions: $Kh_n = \lambda_n h_n$, $h_n^* K = \lambda_n h_n^*$, form complete ON basis

Thm [Perron, Frobenius, Jentzsch 1912, Krein–Rutman '50, Birkhoff '57]:
Principal eigenvalue λ_0 is real, simple, $|\lambda_n| < \lambda_0 \ \forall n \geqslant 1$, $h_0, h_0^* > 0$

Spectral decomp: $k\ (x,y) = \lambda_0 h_0(x) h_0^*(y) + \lambda_1 h_1(x) h_1^*(y) + \dots$

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$$\Rightarrow \mathbb{P}^x\{X_n \in \mathrm{d}y | X_n \in E\} = \pi_0(\mathrm{d}x) + \mathcal{O}((|\lambda_1|/\lambda_0)^n)$$

where $\pi_0 = h_0^* / \int_E h_0^*$ is quasistationary distribution (QSD)
[Yaglom '47, Bartlett '57, Vere-Jones '62, ...]

How to estimate the principal eigenvalue

▷ "Trivial" bounds: $\forall A \subset E$ with $\text{Lebesgue}(A) > 0$,

$$\inf_{x \in A} K(x, A) \leqslant \lambda_0 \leqslant \sup_{x \in E} K(x, E)$$

Proof: $x^* = \operatorname{argmax} h_0 \Rightarrow \lambda_0 = \int_E k(x^*, y) \frac{h_0(y)}{h_0(x^*)} \, dy \leqslant K(x^*, E)$

$$\lambda_0 \int_A h_0^*(y) \, dy = \int_E h_0^*(x) K(x, A) \, dx \geqslant \inf_{x \in A} K(x, A) \int_A h_0^*(y) \, dy$$

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- Donsker–Varadhan-type bound:

$$\lambda_0 \leqslant 1 - \frac{1}{\sup_{x \in E} \mathbb{E}^x[\tau_\Delta]} \qquad \text{where} \quad \tau_\Delta = \inf\{n > 0 \colon X_n \notin E\}$$

- Bounds using Laplace transforms (see below)
  Also provide estimates on $h_0(x)$

How to estimate the spectral gap

Various approaches: coupling, Poincaré/log-Sobolev inequalities, Lyapunov functions [Meyn & Tweedie], Laplace transform, ...

Thm [Garett Birkhoff '57] Under uniform positivity condition

$$s(x)\nu(A) \leqslant K(x,A) \leqslant Ls(x)\nu(A) \qquad \forall x \in E, \forall A \subset E$$

one has $|\lambda_1|/\lambda_0 \leqslant 1 - L^{-2}$

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Localised version: [B & Gentz, SIAM J Math Analysis (2014)]
Assume $\exists\, A \subset E$ and $m : A \to \mathbb{R}_+^*$ such that

$$m(y) \leqslant k(x,y) \leqslant Lm(y) \qquad \forall x, y \in A \tag{1}$$

Then

$$|\lambda_1| \leqslant L - 1 + \mathcal{O}\Big(\sup_{x\in E} K(x, E \setminus A)\Big) + \mathcal{O}\Big(\sup_{x\in A} [1 - K(x,E)]\Big)$$

To prove the restricted positivity condition (1):

- Show that $|Y_n - X_n|$ likely to decrease exp for $X_0, Y_0 \in A$
- Use Harnack inequalities once $|Y_n - X_n| = \mathcal{O}(\sigma^2)$

Application 1: Stochastic FitzHugh–Nagumo eq.

$$\mathrm{d}x_t = \frac{1}{\varepsilon}[x_t - x_t^3 + y_t]\,\mathrm{d}t + \frac{\sigma_1}{\sqrt{\varepsilon}}\,\mathrm{d}W_t^{(1)} \qquad \text{neuron membrane potential}$$

$$\mathrm{d}y_t = [a - x_t - by_t]\,\mathrm{d}t + \sigma_2\,\mathrm{d}W_t^{(2)} \qquad \text{open ion channels}$$

- $b = 0$ for simplicity in this talk, bifurcation parameter $\delta := \frac{3a^2-1}{2}$
- $W_t^{(1)}, W_t^{(2)}$: independent Wiener processes
- $0 < \sigma_1, \sigma_2 \ll 1$, $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$

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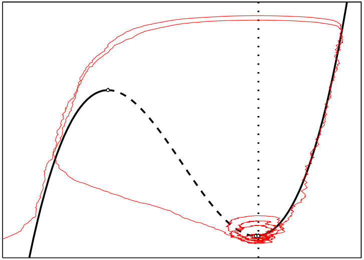
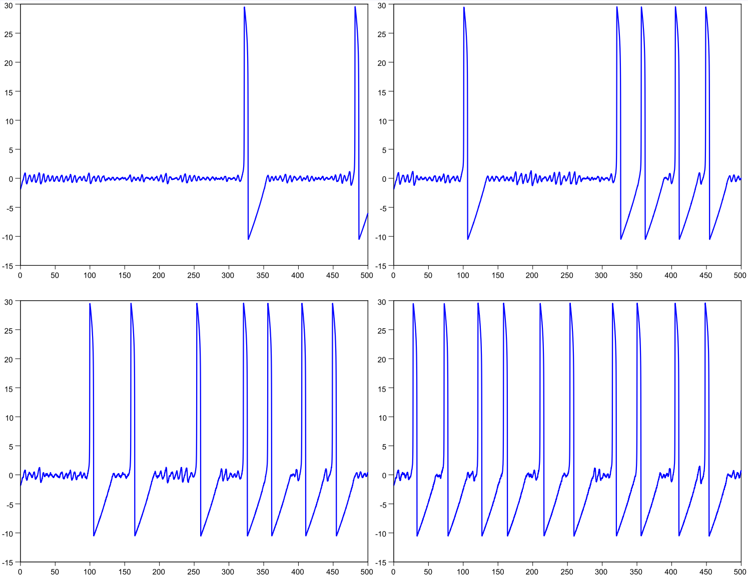
$\varepsilon = 0.1$
$\delta = 0.02$
$\sigma_1 = \sigma_2 = 0.03$

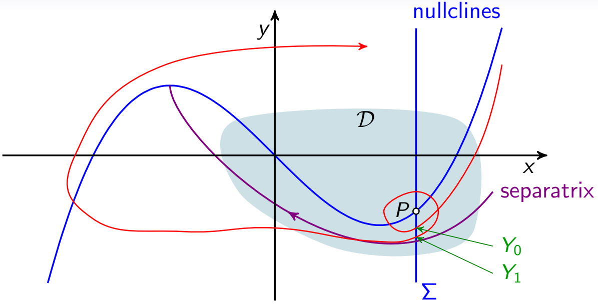


Mixed-mode oscillations (MMOs)



Time series $t \mapsto -x_t$ for $\varepsilon = 0.01,\ \delta = 3 \cdot 10^{-3},\ \sigma = 1.46 \cdot 10^{-4}, \dots, 3.65 \cdot 10^{-4}$

Random Poincaré map



$Y_0, Y_1, \dots$ substochastic Markov chain describing process killed on $\partial\mathcal{D}$
Number of small oscillations: $N = \inf\{n \geqslant 1 \colon Y_n \notin \Sigma\}$

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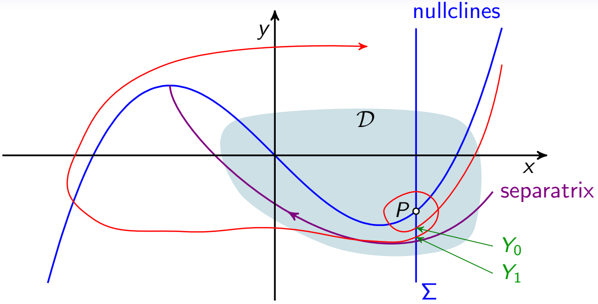

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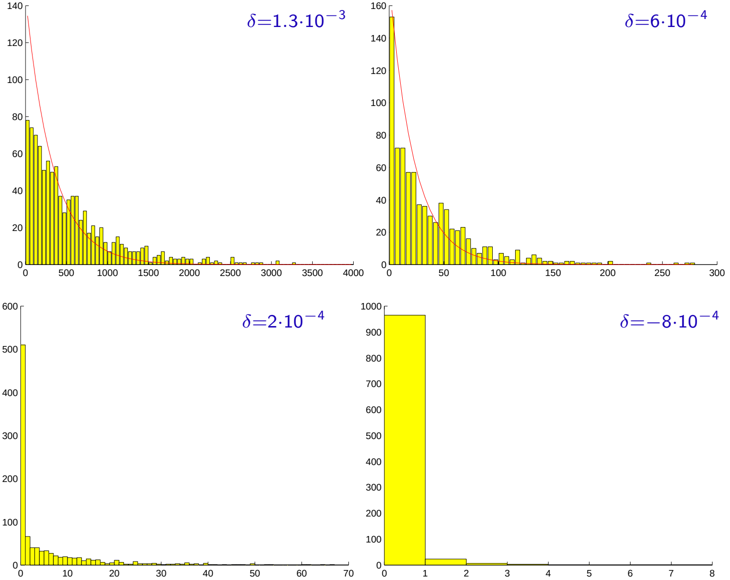
Theorem 1 [B & Landon, Nonlinearity 2012]

N is asymptotically geometric: $\lim_{n\to\infty} \mathbb{P}\{N = n+1 | N > n\} = 1 - \lambda_0$
where $\lambda_0 \in \mathbb{R}_+$: principal eigenvalue of the chain, $\lambda_0 < 1$ if $\sigma > 0$

Proof: follows from existence of spectral gap □

Histograms of distribution of N (1000 spikes)

$\sigma = \varepsilon = 10^{-4}$



Weak-noise regime

Theorem 2 [B & Landon, Nonlinearity 2012]

Assume ε and $\delta/\sqrt{\varepsilon}$ sufficiently small
There exists $\kappa > 0$ s.t. for $\sigma^2 \leqslant (\varepsilon^{1/4}\delta)^2/\log(\sqrt{\varepsilon}/\delta)$

- Principal eigenvalue:

$$1 - \lambda_0 \leqslant \exp\left\{-\kappa\frac{(\varepsilon^{1/4}\delta)^2}{\sigma^2}\right\}$$

- Expected number of small oscillations:

$$\mathbb{E}^{\mu_0}[N] \geqslant C(\mu_0)\exp\left\{\kappa\frac{(\varepsilon^{1/4}\delta)^2}{\sigma^2}\right\}$$

where $C(\mu_0)$ = probability of starting on Σ above separatrix

Proof: Construct $A \subset \Sigma$ such that $K(x, A)$ exponentially close to 1 for all $x \in A$, use trivial lower bound □

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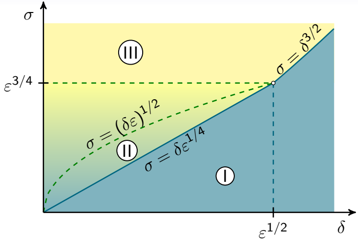
- Linear approximation around separatrix provides good description for dynamics with stronger noise

Summary: Parameter regimes

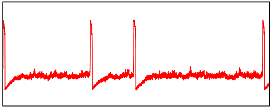


$\sigma_1 = \sigma_2$:
$\mathbb{P}\{N = 1\} \simeq \Phi\Big(-\frac{(\pi\varepsilon)^{1/4}(\delta-\sigma^2/\varepsilon)}{\sigma}\Big)$

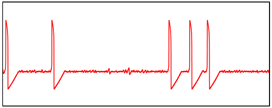
see also

[Muratov & Vanden Eijnden '08]

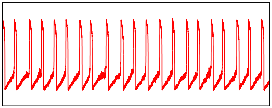
Regime I: rare isolated spikes
Theorem 2 applies ($\delta \ll \varepsilon^{1/2}$)
Interspike interval $\simeq$ exponential



Regime II: clusters of spikes
$\#$ interspike osc asympt geometric
$\sigma = (\delta\varepsilon)^{1/2}$: geom(1/2)



Regime III: repeated spikes
$\mathbb{P}\{N = 1\} \simeq 1$
Interspike interval $\simeq$ constant



Application 2: Exit through unstable periodic orbit

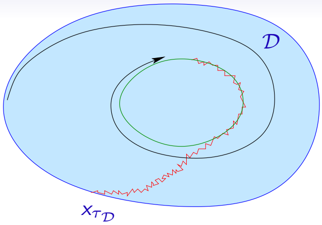
Planar SDE

$dx_t = f(x_t)\,dt + \sigma g(x_t)\,dW_t$

$\mathcal{D} \subset \mathbb{R}^2$: int of unstable periodic orbit

First-exit time: $\tau_{\mathcal{D}} = \inf\{t > 0 \colon x_t \notin \mathcal{D}\}$

Law of first-exit location $x_{\tau_{\mathcal{D}}} \in \partial\mathcal{D}$?



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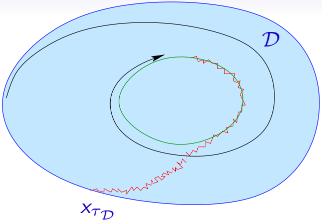

Large-deviation principle with rate function

$$I(\gamma) = \frac{1}{2}\int_0^T (\dot{\gamma}_t - f(\gamma_t))^T D(\gamma_t)^{-1}(\dot{\gamma}_t - f(\gamma_t))\,dt \qquad D = gg^T$$

Quasipotential:
$V(y) = \inf\{I(\gamma) \colon \gamma : \text{stable orbit} \to y \in \partial\mathcal{D} \text{ in arbitrary time}\}$

Theorem [Wentzell, Freidlin '69]: If V reaches its min at a unique $y^* \in \partial\mathcal{D}$, then $x_{\tau_{\mathcal{D}}}$ concentrates in y^* as $\sigma \to 0$

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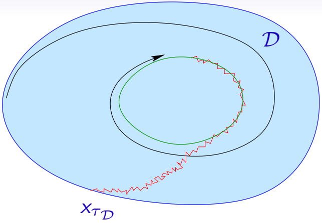

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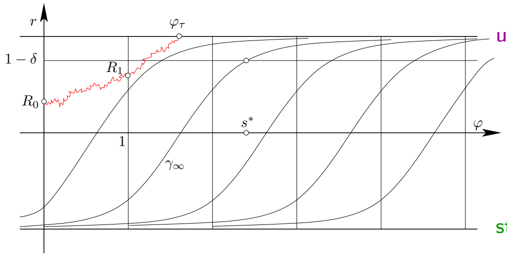
Theorem [Wentzell, Freidlin '69]: If V reaches its min at a unique $y^* \in \partial\mathcal{D}$, then $x_{\tau_{\mathcal{D}}}$ concentrates in y^* as $\sigma \to 0$

Problem: V is constant on $\partial\mathcal{D}$!

However, generically there is still an optimal exit path γ_∞

Random Poincaré maps



Consequences of spectral decomp $k(x,y) = \sum_{k\geqslant 0} \lambda_k h_k(x) h_k^*(y)$ assuming spectral gap $|\lambda_1|/\lambda_0 < 1$:

- $\mathbb{P}^{R_0}\{R_n \in A\} = \lambda_0^n h_0(R_0) \int_A h_0^*(y)\,\mathrm{d}y \big[1 + \mathcal{O}((|\lambda_1|/\lambda_0)^n)\big]$
- If $t = n + s$,

$$\mathbb{P}^{R_0}\{\varphi_\tau \in \mathrm{d}t\} = \lambda_0^n h_0(R_0) \int h_0^*(y) \mathbb{P}^y\{\varphi_\tau \in \mathrm{d}s\}\,\mathrm{d}y \big[1 + \mathcal{O}((|\lambda_1|/\lambda_0)^n)\big]$$

Periodically modulated exponential distribution: $f(t+1) \simeq \lambda_0 f(t)$

Main result: cycling

Theorem [B & Gentz, SIAM J Math Analysis (2014), B (2014)]

$$\lim_{m\to\infty}\lim_{\sigma\to0}\operatorname{Law}\bigl(\theta(\varphi_\tau)-\log(\sigma^{-1})-\lambda T Y_m^\sigma\bigr)=\operatorname{Law}\Bigl(\frac{Z}{2}-\frac{\log 2}{2}\Bigr)$$

- T period of unstable orbit, λ Lyapunov exponent
- θ explicit parametrization of ∂D, $\theta' > 0$, $\theta(\varphi+1)=\theta(\varphi)+\lambda T$
- $Y_m^\sigma \in \mathbb{N}$ asymptotically geometric r.v. with success prob e^{-I_m/σ^2}
  $I_m = I(\gamma_\infty)+\mathcal{O}(e^{-2m\lambda T})$; principal eigenvalue $\lambda_0 \simeq 1-e^{-I(\gamma_\infty)/\sigma^2}$
- Z follows a stardard Gumbel law: $\mathbb{P}\{Z\leqslant t\}=e^{-e^{-t}}$

In other words, $\theta(\varphi_\tau)\sim\lambda T Y_\infty^\sigma+\log(\sigma^{-1})+\frac{Z}{2}-\frac{\log 2}{2}$

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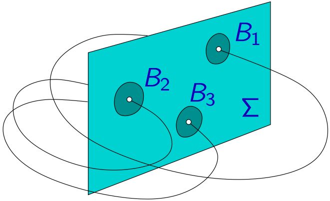
- Y_∞^σ is the number of the tracked translate of γ_∞
- Cycling: Exit distribution is shifted by $\log(\sigma^{-1})$ (no convergence!)
- $\frac{Z}{2}$: random distribution around a given translate

More

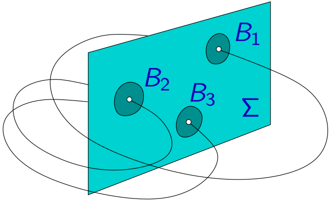
Work in progress: spectral theory

- Stable periodic orbits in $x_1, \dots, x_N$
- Eigenvalue equation
  $(Kh)(x) = \mathrm{e}^{-u}\, h(x)$
- B_i small ball around x_i, $B = \bigcup_{i=1}^N B_i$
- Harnack ineq. $\Rightarrow$ $h(x) \simeq h_i$ in B_i



Work in progress: spectral theory

- Stable periodic orbits in $x_1, \dots, x_N$
- Eigenvalue equation $(Kh)(x) = \mathrm{e}^{-u}\, h(x)$
- B_i small ball around x_i, $B = \bigcup_{i=1}^N B_i$
- Harnack ineq. $\Rightarrow$ $h(x) \simeq h_i$ in B_i



Feynman–Kac-type formula: If $|\mathrm{e}^{-u}| > \sup_{x\in\Sigma\setminus B} \mathbb{P}^x\{X_1 \notin \Sigma \setminus B\}$

$$h(x) = \mathbb{E}^x[\mathrm{e}^{u\tau_B}\, h(X_{\tau_B})] \simeq \sum_{j=1}^N h_j \left[1 + u\mathbb{P}^x\{X_{\tau_B} \in B_j\}\right]$$

Theorem [Baudel & B, in progress (2016)]
For appropriate (metastable) ordering of B_j, $\mathcal{M}_j = B_1 \cup \dots \cup B_j$
K has N eigenvalues exponentially close to 1

$$\lambda_k \simeq 1 - \mathbb{P}^{B_k}\{X_{\tau_B} \in \mathcal{M}_{k-1}\}$$

Concluding remarks

- Random Poincaré maps are useful to understand irreversible SDEs with periodic orbits
- The spectral decomposition allows to prove sharp results
- Noise can induce spikes that may have non-Poisson interval statistics
- Cycling of exit distribution trough unstable periodic orbit

References

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- N. B. and Barbara Gentz, *On the noise-induced passage through an unstable periodic orbit II: General case*, SIAM J Math Anal **46**, 310–352 (2014)
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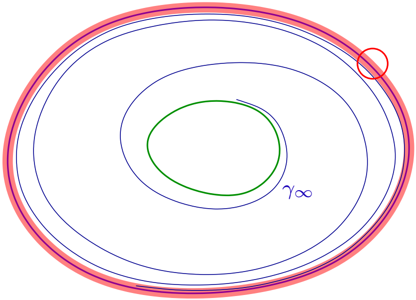
Heuristics for cycling

$\theta(\varphi)$: parametrisation in which effective normal diffusion is constant

$\text{dist}(\gamma_\infty, \text{unst orbit}) \simeq e^{-\lambda T \theta(\varphi)}$

Escape when

$e^{-\lambda T \theta(\varphi)} = \sigma \;\Rightarrow\; \theta(\varphi) = \dfrac{|\log \sigma|}{\lambda T}$



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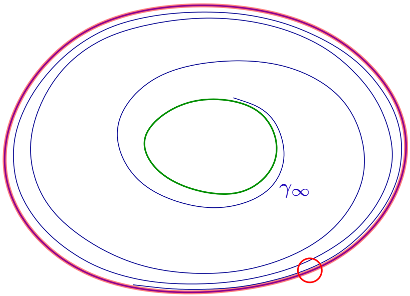

Influence of noise intensity: cycling

$\lambda = 1,\ T = 4,\ V = 1$

$1 \geqslant \sigma \geqslant 0.0001$

(area under curve not normalized)

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