Théorie des valeurs extrêmes et problème de sortie stochastique

Séminaire tournant de probabilités et statistiques, Tours, 1/6/15

1. Théorie des valeurs extrêmes

$X_1, X_2, \dots$ v.a.r. i.i.d.

$S_n = \sum_{i=1}^{n} X_i$

thms limites: $\lim_{n\to\infty} \frac{S_n - b_n}{a_n} \overset{\text{loi}}{=} Y$

ex: TCL : $X_1 \in L^2$, $a_n \sim n^{1/2}$, $Y = N(0,1)$

Cauchy: $\frac{S_n}{n} \overset{\text{loi}}{=} X_1$

$\frac{S_n - b_n}{a_n} \overset{\text{loi}}{=} Y \Rightarrow Y$ stable

(ex: $\mathbb{E}(e^{itY}) = e^{-|t|^\alpha}$)

$F(t) = \mathbb{P}\{X_1 \leqslant t\}$

$M_n = \max\{X_1, \dots, X_n\}$

$\Rightarrow \mathbb{P}\{M_n \leqslant t\} = F(t)^n$

$\mathbb{P}\{Y \leqslant t\} = \Phi(t)$

Déf: $F \in D(\Phi) \Leftrightarrow \lim_{n\to\infty} \frac{M_n - b_n}{a_n} = Y$

$\Leftrightarrow \lim_{n\to\infty} F(a_n t + b_n)^n = \Phi(t) \quad \forall t$

Rem: 1) $X_1 \in D(\Phi) \Rightarrow X_1 \in D(\Phi(a\cdot + b))$ $\forall a > 0$, $\forall b \in \mathbb{R}$

2) $X_1 \in D(\Phi) \Rightarrow \exists a > 0, b \in \mathbb{R}$: $\Phi(ax+b)^2 = \Phi(x) \quad \forall x$

Thm: [Fréchet, Fisher & Tippett, Gnedenko '43]

$F(t) \neq 1_{\{t \geqslant c\}}$

$F \in D(\Phi) \Rightarrow \Phi \in \{\Phi_\alpha, \Psi_\alpha, \Lambda\}$ $(\alpha > 0)$

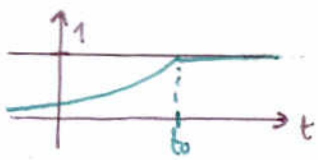
$$\begin{cases} \Phi_\alpha(t) = e^{-t^{-\alpha}} 1_{\{t>0\}} & \text{Fréchet} \\ \Psi_\alpha(t) = e^{-(-t)^\alpha} 1_{\{t\leqslant 0\}} + 1_{\{t>0\}} & \text{Weibull inv} \\ \Lambda(t) = e^{-e^{-t}} & \text{Gumbel} \end{cases}$$

lemme: $R(t) := 1 - F(t) = \mathbb{P}\{X_1 > t\}$

$F \in D(\Phi) \Leftrightarrow \exists a_n, b_n : \lim_{n\to\infty} n R(a_n t + b_n) = -\log \Phi(t) \quad \forall t : \Phi(t) > 0$

Gumbel: choix possible $\begin{cases} b_n = \inf\{t : F(t) > 1 - \frac{1}{n}\} \\ a_n = \inf\{t : F(t + b_n) > 1 - \frac{1}{ne}\} \end{cases}$

Ex: $N \in D(\Lambda)$



Thm: $t_0 := \inf\{t : F(t) = 1\} \in \mathbb{R} \cup \{\infty\}$

$F \in D(\Lambda) \Leftrightarrow \exists A(z), \lim_{z\to t_0-} A(z) = 0$

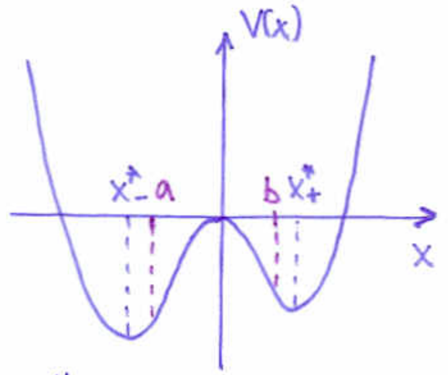
t.q. $\lim_{z\to t_0-} \underbrace{\frac{R(z[1+A(z)t])}{R(z)}}_{= \mathbb{P}\{X_1 > z[1+A(z)t] \mid X_1 > z\}} = -\log \Lambda(t) = e^{-t} \quad \forall t$

durée de vie résiduelle

$z = b_n$, $A(b_n) = \frac{a_n}{b_n} \quad \forall n$

2. Problème de sortie stochastique

a) dim 1



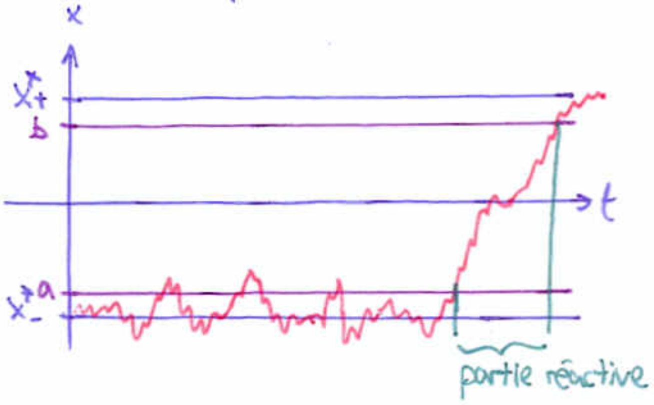
$$dx_t = -V'(x_t)\,dt + \sigma\, dW_t$$

$$\lambda = -V''(0) > 0$$

$$\tau_x = \inf\{t>0 : x_t = x\}$$

Connu: $\mathbb{E}^a[\tau_b] = C(\varepsilon)\, e^{2[V(0)-V(x_-^*)]/\sigma^2}$

$$\tau_b / \mathbb{E}^a[\tau_b] \xrightarrow[\sigma\to 0]{\text{loi}} \mathcal{E}xp(1)$$



Thm 1: [Cérou, Guyader, Lelièvre, Malrieu 2013]

$a < x_0 < 0$

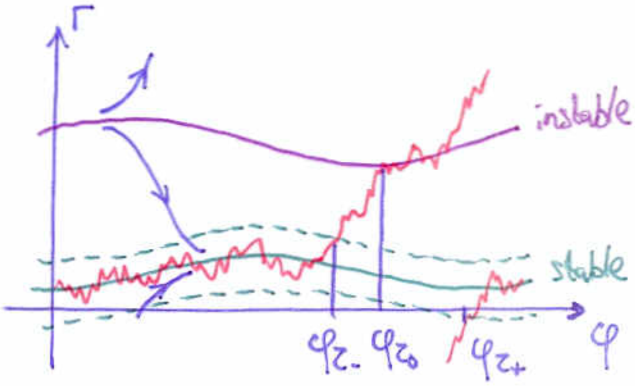
$$\lim_{\sigma\to 0} \text{Loi}(\lambda\tau_b - 2|\log\sigma| \mid \tau_b < \tau_a) = \text{Loi}(\underbrace{Z}_{\text{Gumbel}} + \underbrace{T(x_0,b)}_{\text{déterministe}})$$

preuve: h-transf. de Doob

b) dim 2 cond "polaires"

$$\begin{cases} dr_t = f_r(r_t,\varphi_t)\,dt + \sigma g_r(r_t,\varphi_t)\,dW_t \\ d\varphi_t = f_\varphi(r_t,\varphi_t)\,dt + \sigma g_\varphi(r_t,\varphi_t)\,dW_t \end{cases}$$

Syst. dét: $(\sigma=0)$ $\begin{cases} \dot r = f_r(r,\varphi) \\ \dot\varphi = f_\varphi(r,\varphi) \end{cases}$ a - 1 orbite stable
- 1 orbite instable



Thm 2: [B&Gentz, SIAM J Math Anal 2014]

Sous une cond. de non-dég:

$$\lim_{m\to\infty}\left(\lim_{\sigma\to 0} \text{Loi}\big(\theta(\varphi_{\tau_0}) - |\log\sigma| - \lambda T\, Y_m^\sigma\big)\right) = \text{Loi}\left(\frac{Z}{2} - \frac{\log 2}{2}\right)$$

où * Y_m^σ : v.a. $\mathbb{N}$ asympt. géom : $\lim_{n\to\infty} \mathbb{P}\{Y_m^\sigma = n+1 \mid Y_m^\sigma \geq n\} = e^{-I_m/\sigma^2}$

$I_m = I_\infty + O(e^{-2m\lambda T})$

* $\theta'(\varphi) > 0$, $\theta(\varphi+1) = \theta(\varphi) + \lambda T$ fct explicite
* λT : exp. de Lyapunov x période de l'orbite

Rem: $\theta(\varphi_{\tau_+}) - \theta(\varphi_{\tau_-}) - 2|\log\sigma| \xrightarrow[\sigma\to 0]{\text{loi}} Z + \text{const}$

3. Lien entre les deux

[Bakhtin, Stoch Dyn 2014]

$$dx_t = \lambda x_t\,dt + \sigma dW_t \quad \Rightarrow \quad x_t = e^{\lambda t}\tilde{x}_t$$

$\lambda > 0,\ x_0 < 0$

$$\tilde{x}_t = x_0 + \sigma\int_0^t e^{-\lambda s}\,dW_s \stackrel{\text{loi}}{=} x_0 + \sigma\sqrt{\frac{1-e^{-2\lambda t}}{2\lambda}}\,N$$

Principe de réflection: $\mathbb{P}\{\tau_0 < t\} = 2\,\mathbb{P}\{\tilde{x}_t > 0\}$

$$\mathbb{P}\{\tau_0 < t \mid \tau_0 < \infty\} = \frac{\mathbb{P}\{\tau_0 < t\}}{\mathbb{P}\{\tau_0 < \infty\}} = \frac{2\,\mathbb{P}\{\tilde{x}_t > 0\}}{2\,\mathbb{P}\{\tilde{x}_\infty > 0\}} = \mathbb{P}\{\tilde{x}_t > 0 \mid \tilde{x}_\infty > 0\}$$

$$\mathbb{P}\{\tau_0 < t + \tfrac{1}{\lambda}|\log\sigma| \mid \tau_0 < \infty\} = \mathbb{P}\{\tilde{x}_{t+\frac{1}{\lambda}|\log\sigma|} > 0 \mid \tilde{x}_\infty > 0\}$$

durée de vie résiduelle

$$= \mathbb{P}\Big\{N > \frac{|x_0|}{\sigma}\sqrt{\frac{2\lambda}{1-\sigma^2 e^{-2\lambda t}}} \;\Big|\; N > \frac{|x_0|}{\sigma}\sqrt{2\lambda}\Big\}$$

$$\xrightarrow[\sigma\to 0]{} \exp\big\{-x_0^2\lambda e^{-2\lambda t}\big\} = \mathbb{P}\Big\{\frac{Z}{2} + \frac{\log(x_0^2\lambda)}{2\lambda} < t\Big\}$$

Rem: se sert de $-\log\Lambda(e^{-x}) = \Lambda(x)$

$$\boxed{\text{Thm 3: [Bakhtin, Day]} \quad \lim_{\sigma\to 0} \mathrm{Loi}\big(\lambda\tau_0 - |\log\sigma| \mid \tau_0 < \tau_a\big) = \mathrm{Loi}\Big(\frac{Z}{2} + \frac{\log(x_0^2\lambda)}{2\lambda}\Big)}$$

$\{\tau_0 < \tau_a\}$ et $\{\tau_0 < \infty\}$ sont asympt équivalents

$$\boxed{\begin{array}{l}\text{Thm 4: [Day]} \quad x_0 = 0,\ a < 0 < b \qquad \Theta := -\log|N| \\ \displaystyle\lim_{\sigma\to 0} \mathrm{Loi}\big(\lambda\tau_b - |\log\sigma| \mid \tau_b < \tau_a\big) = \mathrm{Loi}\Big(\Theta + \log\frac{(2b^2\lambda)}{2}\Big)\end{array}}$$

basé sur $\quad b = x_{\tau_b} = \sigma e^{\lambda\tau_b}\displaystyle\int_0^{\tau_b} e^{-\lambda s}\,dW_s \cong \sigma\frac{1}{\sqrt{2\lambda}}e^{\lambda\tau_b}N$

Thm 3 & Thm 4 $\Rightarrow$ Thm 1 $\qquad$ car $\frac{1}{2}Z + \Theta \stackrel{\text{loi}}{=} Z + \frac{1}{2}\log(2)$