Online probability seminar, Seoul (Online)

Renormalisation and metastable dynamics of the two-dimensional Allen–Cahn equation

Nils Berglund

Institut Denis Poisson, University of Orléans, France

INSTITUT ORLÉANS · TOURS Mathématiques & Physique Théorique DENIS POISSON

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Project PERIS-TOCH

(Stochastic) Allen–Cahn equation on $\mathbb{T}^2$

$$\mathrm{d}\phi(t,x) = \left[\nu(\varepsilon t)\Delta\phi(t,x) + \phi(t,x) - \phi(t,x)^3\right]\mathrm{d}t + \sigma\,\mathrm{d}W(t,x)$$

(Online: https://youtu.be/yX0EAxZHNCQ)

Eyring–Kramers law for 1D SPDEs: heuristics

$$\partial_t u(t,x) = \Delta u(t,x) + f(u(t,x)) \quad + \sqrt{2\varepsilon}\,\xi(t,x) \qquad (f(u) = u - u^3)$$

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Initial condition: u_{in} near $u_- \equiv -1$ with eigenvalues $\nu_k = (\frac{\beta k\pi}{L})^2 + 2$

Target: $u_+ \equiv 1$, $\tau_+ = \inf\{t > 0 \colon \|u_t - u_+\|_{L^\infty} < \rho\}$

Transition state: ($\beta = 1$ for Neumann b.c., $\beta = 2$ for periodic b.c.)

$$u_{\text{ts}}(x) = \begin{cases} u_0(x) \equiv 0 & \text{if } L \leqslant \beta\pi \quad \text{with ev } \lambda_k = (\frac{\beta k\pi}{L})^2 - 1 \\ u_1(x)\ \beta\text{-kink stationary sol.} & \text{if } L > \beta\pi \quad \text{with ev } \lambda'_k \end{cases}$$

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[Faris & Jona-Lasinio 82]: large-deviation principle

$\Rightarrow$ Arrhenius law: $\mathbb{E}^{u_{\text{in}}}[\tau_+] \simeq e^{(V[u_{\text{ts}}]-V[u_-])/\varepsilon}$

[Maier & Stein 01]: formal computation; for Neumann b.c.

$$\Rightarrow \mathbb{E}^{u_{\text{in}}}[\tau_+] \simeq 2\pi\sqrt{\frac{1}{|\lambda_0|\nu_0}\prod_{k=1}^\infty \frac{\lambda_k}{\nu_k}}\, e^{(V[u_{\text{ts}}]-V[u_-])/\varepsilon}$$

The two-dimensional case

- ▷ Large-deviation principle: [Hairer & Weber, 2015]
- ▷ Naive computation of prefactor fails:

$$\log \prod_{k\in(\mathbb{N}^2)^*} \frac{1-\left(\frac{L}{|k|\pi}\right)^2}{1+2\left(\frac{L}{|k|\pi}\right)^2} \simeq \sum_{k\in(\mathbb{N}^2)^*} \log\left(1-\frac{3L^2}{|k|^2\pi^2}\right)$$

$$\simeq -\sum_{k\in(\mathbb{N}^2)^*} \frac{3L^2}{|k|^2\pi^2} \simeq -\frac{3L^2}{\pi^2}\int_1^\infty \frac{r\,\mathrm{d}r}{r^2} = -\infty$$

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- In fact, the equation needs to be renormalised

Theorem: [Da Prato & Debussche 2003]
Let ξ^δ be a mollification on scale δ of white noise. Then

$$\partial_t u = \Delta u + \left[1 + 3\varepsilon C(\delta)\right]u - u^3 + \sqrt{2\varepsilon}\xi^\delta$$

with $C(\delta) \simeq \log(\delta^{-1})$ admits local solution converging as $\delta \to 0$
(Global version: [Mourrat & Weber 2015])
[Mourrat & Weber 2014]: Renormalised eq = scaling limit of Ising–Kac model

Renormalisation

Problem: Stoch. convolution $w_t(x) = \int_0^t \mathrm{e}^{\Delta(t-s)} \xi(s,x) \, \mathrm{d}s$ is distribution

- δ-mollification should be equivalent to Galerkin approx. $|k| \leqslant N = \delta^{-1}$:

$$w_N(x,t) = \sum_{|k| \leqslant N} a_k(t) \frac{1}{L} \mathrm{e}^{\mathrm{i}\,\Omega k \cdot x} \qquad a_k = \int_0^t \mathrm{e}^{-\mu_k(t-s)} \, \mathrm{d}W_s^{(k)}$$

$$\mu_k = (\Omega |k|)^2 \qquad \Omega = \beta\pi/L$$

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- $\lim_{t\to\infty} \int_0^t e^{(\Delta-1)(t-s)} \xi_N(s,x)\, ds = \phi_N$ is a Gaussian free field, s.t.

$$L^2 C_N := L^2 \mathbb{E}\phi_N^2 = \mathbb{E}\|\phi_N\|_{L^2}^2 = \sum_{|k|\leqslant N} \frac{1}{2(\mu_k+1)} = \frac{\mathrm{Tr}(P_N[-\Delta+\mathbb{1}]^{-1})}{2} \simeq \log(N)$$

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- Wick powers

$$:\phi_N^2: = \phi_N^2 - C_N$$
$$:\phi_N^3: = \phi_N^3 - 3C_N\phi_N$$
$$:\phi_N^4: = \phi_N^4 - 6C_N\phi_N^2 + 3C_N^2$$

have zero mean and uniformly bounded variance (when integrated)

Nelson's estimate

Lemma: For X random variable in n^{th} inhomogeneous Wiener chaos

$$\mathbb{E}[X^{2p}]^{\frac{1}{2p}} \leqslant C_n(2p-1)^{\frac{n}{2}}\mathbb{E}[X^2]^{\frac{1}{2}}$$

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$$\sup_N \mathbb{E}\Big[\Big(\int_{\mathbb{T}^2} :\phi_N^n(x): \mathrm{d}x\Big)^{2p}\Big] < \infty$$

$$\forall M > N: \ \mathbb{E}\Big[\Big(\int_{\mathbb{T}^2} :\phi_M^n(x): \mathrm{d}x - \int_{\mathbb{T}^2} :\phi_N^n(x): \mathrm{d}x\Big)^{2p}\Big]^{\frac{1}{p}} \leqslant C'_n (2p-1) \frac{(\log N)^{n-2}}{N}$$

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Corollary:

$$\mathbb{E}\Big[\exp\Big\{-\frac{\varepsilon}{4}\int_{\mathbb{T}^2} :\phi_N^4(x):\mathrm{d}x\Big\}\Big] \leqslant 1 + \mathcal{O}(\varepsilon)$$

- Integrand is bounded below by term of order $-C_N^2$
- Use Markov's ineq to bound tails of integral and integrate by parts

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Useful as $\operatorname{cap}(A,B) \leqslant \sqrt{\frac{|\lambda_0|\varepsilon}{2\pi}} \prod_{0<|k|\leqslant N} \sqrt{\frac{2\pi\varepsilon}{\lambda_k}}\mathbb{E}\Big[\exp\Big\{-\frac{\varepsilon}{4}\int_{\mathbb{T}^2} :u_N^4(x): \mathrm{d}x\Big\}\Big]$

Computation of the prefactor

- Consider for simplicity $L < \beta\pi \Rightarrow$ transition state in 0
- Galerkin-truncated renormalised potential

$$V_N = \frac{1}{2}\int_{\mathbb{T}^2} \left[\|\nabla u_N(x)\|^2 - u_N(x)^2 \right] \mathrm{d}x + \frac{1}{4}\int_{\mathbb{T}^2} :u_N(x)^4: \mathrm{d}x$$

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- Using Nelson estimate: $\operatorname{cap}(A,B) \simeq \sqrt{\frac{|\lambda_0|\varepsilon}{2\pi}}\prod_{0<|k|\leqslant N}\sqrt{\frac{2\pi\varepsilon}{\lambda_k}}$
- Symmetry argument:

$$\int_{B^c} h_{A,B}(z)\,\mathrm{e}^{-V_N(z)/\varepsilon}\,\mathrm{d}z = \frac{1}{2}\int \mathrm{e}^{-V_N(z)/\varepsilon}\,\mathrm{d}z = \frac{1}{2}\mathcal{Z}_N(\varepsilon)$$

- $\mathcal{Z}_N(\varepsilon) \simeq 2\prod_{|k|\leqslant N}\sqrt{\frac{2\pi\varepsilon}{\nu_k}}\,\mathrm{e}^{-V_N(L,0)/\varepsilon}$ where $-V_N(L,0) = \frac{1}{4}L^2 + \frac{3}{2}L^2 C_N\varepsilon$
- Prefactor proportional to (since $\nu_k = \lambda_k + 3$)

$$\prod_{0<|k|\leqslant N}\frac{\lambda_k}{\lambda_k+3}\mathrm{e}^{3/\lambda_k} \qquad \text{converges since} \qquad \log\left[\frac{\lambda_k}{\lambda_k+3}\mathrm{e}^{3/\lambda_k}\right] = \mathcal{O}\left(\frac{1}{|k|^4}\right)$$

Main result in dimension 2

Theorem: [B, Di Gesù, Weber, 2017]

For $L < 2\pi$, appropriate $A \ni u_-$, $B \ni u_+$, $\exists \mu_N$ probability measures on ∂A:

$$\limsup_{N\to\infty} \mathbb{E}^{\mu_N}\big[\tau_B\big] \leqslant \frac{2\pi}{|\lambda_0|}\sqrt{\prod_{k\in\mathbb{Z}^2} \frac{|\lambda_k|}{\nu_k}\, \mathrm{e}^{\frac{\nu_k-\lambda_k}{|\lambda_k|}}}\, \mathrm{e}^{(V[u_{\mathrm{ts}}]-V[u_-])/\varepsilon}\big[1 + c_+\sqrt{\varepsilon}\,\big]$$

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- Inverse of prefactor involves Carleman–Fredholm determinant:

  $$\det{}_2(\mathbb{1} + \mathrm{T}) = \det(\mathbb{1} + \mathrm{T})e^{-\operatorname{Tr}\mathrm{T}}$$

  with $T = 3(-\Delta_\perp - 1)^{-1}$

  $\det_2$ defined whenever T is only Hilbert–Schmidt (true for $d \leqslant 3$)
- [Tsatsoulis & Weber 2018]: Same result for $\mathbb{E}^{u_0}\big[\tau_B\big]$

References

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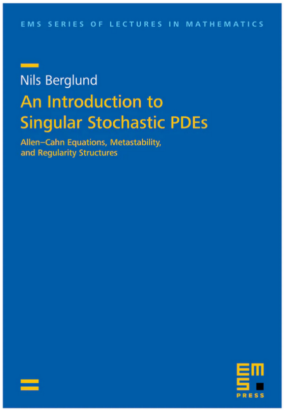

Thanks for your attention!

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