AIMS 2014 — Session on Nonlocally Coupled Dynamical Systems

A group-theoretic approach to metastability in networks of interacting SDEs

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Interacting SDEs with noise

Example 1 [B, Fernandez, Gentz, Nonlinearity 2007]

- N particles on a circle $\mathbb{Z}/N\mathbb{Z}$
- Bistable local dynamics
- Ferromagnetic nearest neighbour coupling
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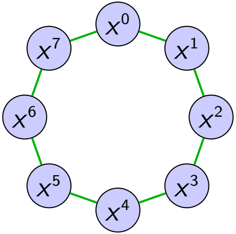

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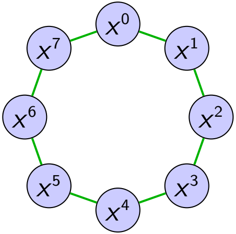

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Gradient system $\qquad \mathrm{d}x_t = -\nabla V(x_t)\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t$

potential $V(x) = \sum_i U(x^i) + \frac{\gamma}{4}\sum_i (x^{i+1} - x^i)^2 \qquad U(\xi) = \frac{1}{4}\xi^4 - \frac{1}{2}\xi^2$

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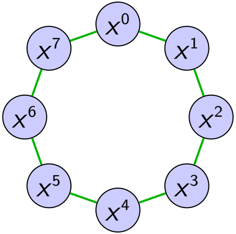

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Example 2 [B, Dutercq, preprint 2013]: Same potential + constraint $\sum_i x^i = 0$

General gradient systems with noise

$$dx_t = -\nabla V(x_t)\,dt + \sqrt{2\varepsilon}\,dW_t$$

$V : \mathbb{R}^N \to \mathbb{R}$: confining potential, class $\mathcal{C}^2$

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- Local minima: $\mathcal{X}_0 = \{x \in \mathcal{X} \colon$ all ev of Hessian $\nabla^2 V(x)$ are $> 0\}$
- Saddles of index 1: $\mathcal{X}_1 = \{x \in \mathcal{X} \colon \nabla^2 V(x)$ has 1 negative ev $\}$

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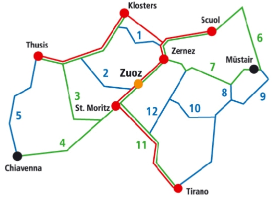
Dynamics $\sim$ markovian jump process on $\mathcal{G} = (\mathcal{X}_0, \mathcal{E})$, $\mathcal{E} \subset \mathcal{X}_1$



Rot Rhätische Bahn
Grün ganzjährig offen
Blau Wintersperre

Nr.	Pass	Land	Passhöhe (m.ü.M.)
1	Flüela	CH	2383
2	Albula	CH	2312
3	Julier	CH	2284
4	Maloja	CH	1815
5	Splügen	I - CH	2115
6	Reschen	A - I	1507
7	Ofen	CH	2149
8	Umbrail	CH - I	2502
9	Stilfserjoch	I	2757
10	Foscagno	I	2291
11	Bernina	CH - I	2323
12	Fla. di Livigno	I	2315

Eyring–Kramers law

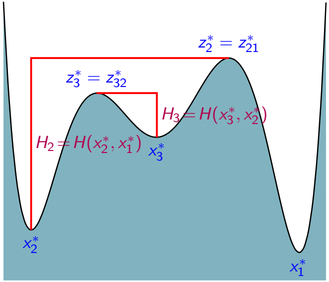
Definition: Communication height

$$H(x_i^*, x_j^*) = \inf_{\gamma: x_i^* \to x_j^*} \sup_t V(\gamma_t) - V(x_i^*)$$
$$= V(z_{ij}^*) - V(x_i^*)$$

Definition: Metastable hierarchy

$x_1^* \prec x_2^* \prec \cdots \prec x_n^* \Leftrightarrow \exists \theta > 0\colon \forall k$

$H_k := H(x_k^*, \{x_1^*, \dots, x_{k-1}^*\})$
$\leqslant \min_{i<k} H(x_i^*, \{x_1^*, \dots, x_{i-1}^*, x_{i+1}^*, \dots, x_k^*\}) - \theta$



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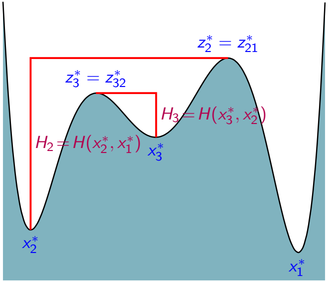

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Theorem: Eyring–Kramers law, [Bovier,Eckhoff,Gayrard,Klein 2004]

τ_k = first-hitting time of nbh of $\{x_1^*, \dots, x_k^*\}$ $\qquad \lambda_k = k^{\mathrm{th}}$ ev of generator

$$\mathbb{E}^{x_k^*}[\tau_{k-1}] = \frac{2\pi}{|\lambda_-(z_k^*)|} \sqrt{\frac{|\det \nabla^2 V(z_k^*)|}{\det \nabla^2 V(x_k^*)}} \, \mathrm{e}^{H_k/\varepsilon} \big[1 + \mathcal{O}_\varepsilon(1)\big] \simeq \lambda_k^{-1}$$

Potential landscape for Example 1

$$V(x) = \sum_{i \in \mathbb{Z}/N\mathbb{Z}} U(x^i) + \frac{\gamma}{4} \sum_{i \in \mathbb{Z}/N\mathbb{Z}} (x^{i+1} - x^i)^2 \qquad U(\xi) = \tfrac{1}{4}\xi^4 - \tfrac{1}{2}\xi^2$$

$\gamma = 0$: $\mathcal{X} = \{-1, 0, 1\}^N$, $\mathcal{X}_0 = \{-1, 1\}^N$, $\mathcal{X}_1 = \{x \in \mathcal{X} : \text{one } x^i = 0\}$

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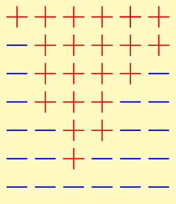
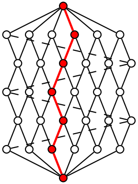
Theorem [BFG, Nonlinearity 2007]

No bifurcation for $0 \leqslant \gamma \leqslant \gamma^*(N)$

where $\gamma^*(N) > \frac{1}{4} \quad \forall N \geqslant 2$





$V_\gamma(z^*_\gamma) = V_0(z^*_0) + \gamma(\#\ \text{interfaces}) + \cdots$

Ising-like dynamics

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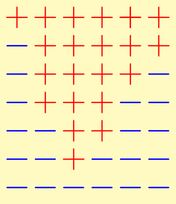

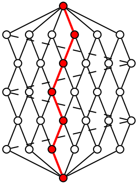

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Ising-like dynamics

Theorem [BFG, Nonlinearity 2007]

$\gamma > \frac{1}{2\sin^2(\pi/N)} \Leftrightarrow \mathcal{X}_0 = \{\pm(1, \dots, 1)\}$, $\mathcal{X}_1 = \{0\} \Leftrightarrow$ Synchronization

Transition to synchronization

Symmetry group

$G = D_N \times \mathbb{Z}_2 = \langle r, s, c \rangle$

$$r(x) = (x^2, x^3, \dots, x^N, x^1)$$
$$s(x) = (x^N, x^{N-1}, \dots, x^1)$$
$$c(x) = -x$$

$\mathcal{X}$ partitionned into

group orbits $O_x = \{gx \colon x \in G\}$

Stabilizer: $G_x = \{g \in G \colon gx = x\}$

$\Rightarrow |O_x||G_x| = |G|$

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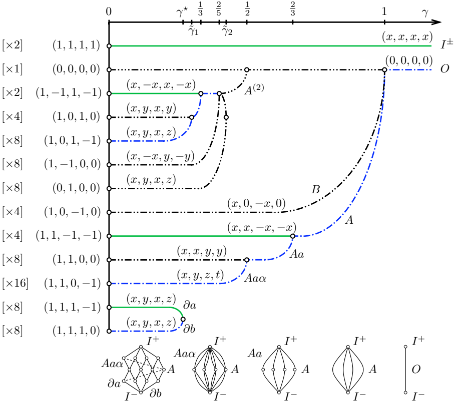

Example: $N = 4$, $|\mathcal{X}_0| = 81$

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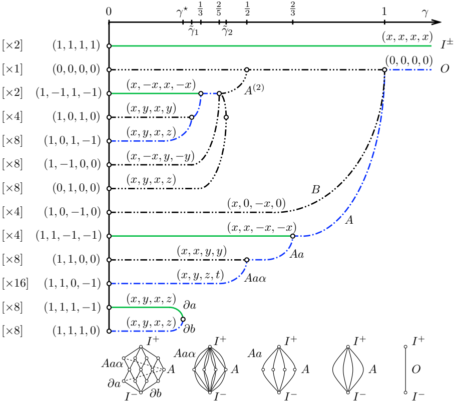

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Problem: no metastable hierarchy $\Rightarrow$ Usual Eyring–Kramers law invalid

Representation theory of finite groups

- Representation: $\pi : G \mapsto \mathrm{GL}(n, \mathbb{C})$ s.t. $\pi(gh) = \pi(g)\pi(h)\ \forall g, h \in G$
- π, π' are equivalent $\Leftrightarrow \exists S \in \mathrm{GL}(n, \mathbb{C})$: $S\pi(g)S^{-1} = \pi'(g)\ \forall g \in G$
- π reducible if equiv to π', $\pi'(g) = \begin{pmatrix} \pi_1(g) & 0 \\ 0 & \pi_2(g) \end{pmatrix}$ i.e. $\pi = \pi_1 \oplus \pi_2$

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Theorem

- A finite group G has finitely many irreducible repr. $\pi^{(0)}, \dots, \pi^{(r-1)}$
- Any repr. π admits a unique (up to order) decomposition

$$\pi = \bigoplus_{p=0}^{r-1} \alpha^{(p)} \pi^{(p)} \qquad \sum_p \alpha^{(p)} \dim(\pi^{(p)}) = \dim(\pi)$$

- Projector on invariant subspace associated with $\pi^{(p)}$

$$P^{(p)} = \frac{\dim(\pi^{(p)})}{|G|} \sum_{g \in G} \overline{\chi^{(p)}(g)} \pi(g) \qquad \chi^{(p)}(g) = \mathrm{Tr}(\pi^{(p)}(g))$$

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Example: Dihedral group $D_N = \langle r, s \rangle, r^N = s^2 = \mathrm{id}, rs = sr^{-1}$
N even: 4 1-dim irred repr: $\pi(r^i s^j) = (\pm 1)^i (\pm 1)^j$, $(\frac{N}{2} - 1)$ 2-dim irred repr

Markovian jump processes with symmetries

Generator L: transition rates $L_{ij} = \frac{c_{ij}}{m_i} \mathrm{e}^{-h_{ij}/\varepsilon}, \quad c_{ij} = c_{ji} \quad \forall i, j \in \mathcal{X}_0$

Assumptions

- Reversibility: $m_i \mathrm{e}^{-V_i/\varepsilon} L_{ij} = m_j \mathrm{e}^{-V_j/\varepsilon} L_{ji} \qquad \forall i, j \in \mathcal{X}_0$
- Symmetry: $\pi(g)L = L\pi(g) \quad \forall g \in G$
  where $\pi(g)_{ab} = 1_{\{g(a)=b\}}$ permutation matrix
- Metastable order on the set of G-orbits $A_1 \prec \cdots \prec A_m$
- No accidental degeneracy:
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Main observation

$$P^{(p)} L = L P^{(p)} \qquad p = 0, \dots, r-1$$

$\Rightarrow$ each subspace $P^{(p)} \mathbb{C}^n$ is invariant for L
The generator restricted to this subspace satisfies an asymmetric Eyring–Kramers law

Modified Eyring–Kramers law

Method: find basis of $P^{(p)}\mathbb{C}^n$, compute matrix elements, apply asym algo

Trivial representation: $\pi^{(0)}(g) = 1 \ \forall g \quad \Rightarrow \quad m$ ev ($m = \#$ orbits)

Theorem [B, Dutercq 2013]

τ_k = first-hitting time of $A_1 \cup A_2 \cup \dots \cup A_k$, μ uniform measure on A_k

$$\mathbb{E}^{\mu}[\tau_{k-1}] = \frac{|G_{a_i} \cap G_{a_j}|}{|G_{a_k}|} \frac{m_{a_k}}{c_{a_i a_j}} \mathrm{e}^{H_k/\varepsilon} \big[1 + \mathcal{O}(\mathrm{e}^{-\theta/\varepsilon})\big] = \frac{1 + \mathcal{O}(\mathrm{e}^{-\theta/\varepsilon})}{\lambda_k^{(0)}}$$

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Other 1-dim representations:

A_i is active for $\pi^{(p)} \Leftrightarrow \pi^{(p)}(h) = 1 \; \forall h \in G_a, a \in A_i$

Similar result for process on set of active orbits, inactive orbits = ☠

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Representations of dim $\geqslant 2$:

Explicit expression for $L^{(p)} = L|_{P^{(p)}\mathbb{C}^n}$ (involving characters $\chi^{(p)} = \operatorname{Tr} \pi^{(p)}$)

Generically, ev are those of blocks $L_{ii}^{(p)}$ or $L_{ii}^{(p)} - L_{ij}^{(p)}(L_{jj}^{(p)})^{-1} L_{ji}^{(p)}$

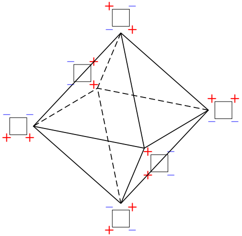
Example 2 for $N = 4$

$V(x) = \sum_i U(x^i) + \frac{\gamma}{4} \sum_i (x^{i+1} - x^i)^2$ restricted to $\{\sum x^i = 0\}$

For $0 \leqslant \gamma < 2/5$: $|\mathcal{X}_0| = 6$

A_1 = orbit of $(1, 1, -1, -1) + \mathcal{O}(\gamma)$

A_2 = orbit of $(1, -1, 1, -1) + \mathcal{O}(\gamma)$



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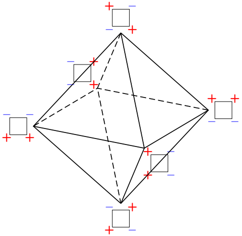

$\pi = 2\pi_{+++} \oplus \pi_{-++} \oplus \pi_{---} \oplus \pi_{1,-}$

$\pi_{\alpha\beta\gamma}(r^i s^j c^k) = \alpha^i \beta^j \gamma^k$

$\pi_{1,-}(r^i s^j c^k) = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}^i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^j (-1)^k$

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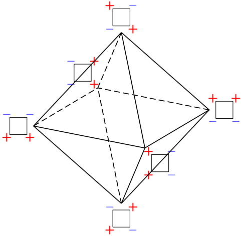
For $0 \leqslant \gamma < 2/5$: $|\mathcal{X}_0| = 6$

A_1 = orbit of $(1, 1, -1, -1) + \mathcal{O}(\gamma)$

A_2 = orbit of $(1, -1, 1, -1) + \mathcal{O}(\gamma)$



$\pi = 2\pi_{+++} \oplus \pi_{-++} \oplus \pi_{---} \oplus \pi_{1,-}$

$\pi_{\alpha\beta\gamma}(r^i s^j c^k) = \alpha^i \beta^j \gamma^k$

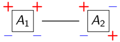
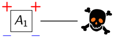
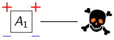
$\pi_{1,-}(r^i s^j c^k) = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}^i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^j (-1)^k$

$L_{a_1 a_2} = \frac{c_{a_1 a_2}}{m_{a_1}} \mathrm{e}^{-h_{a_1 a_2}/\varepsilon}$

$L_{a_2 a_1} = \frac{c_{a_1 a_2}}{m_{a_2}} \mathrm{e}^{-h_{a_2 a_1}/\varepsilon}$

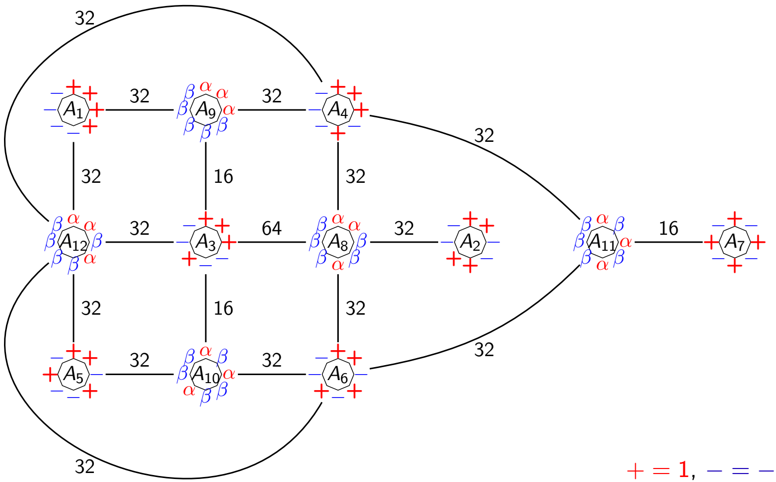
$L_{a_1 a_1'} = \frac{c_{a_1 a_1}}{m_{a_1}} \mathrm{e}^{-h_{a_1 a_1'}/\varepsilon}$

$L_{a_1 a_2} \ll L_{a_1 a_1'} \ll L_{a_2 a_1}$

	A_1	A_2	dim	eigenvalues	graph
π_{+++}	1	1	2	$0, -4L_{a_2 a_1}$	
π_{-++}	1	0	1	$-4L_{a_1 a_1'}$	
π_{---}	0	1	1	$-4L_{a_2 a_1}$	
$\pi_{1,-}$	2	0	2	$(-2L_{a_1 a_1'})^{\times 2}$	
$\lvert A\rvert$	4	2	6	6	

Example 2 for $N = 8$

For small γ: $|\mathcal{X}_0| = 182$, $|\mathcal{X}_1| = 560$



$+ = 1$, $- = -1$

$\alpha = 5/\sqrt{19}$, $\beta = -3/\sqrt{19}$

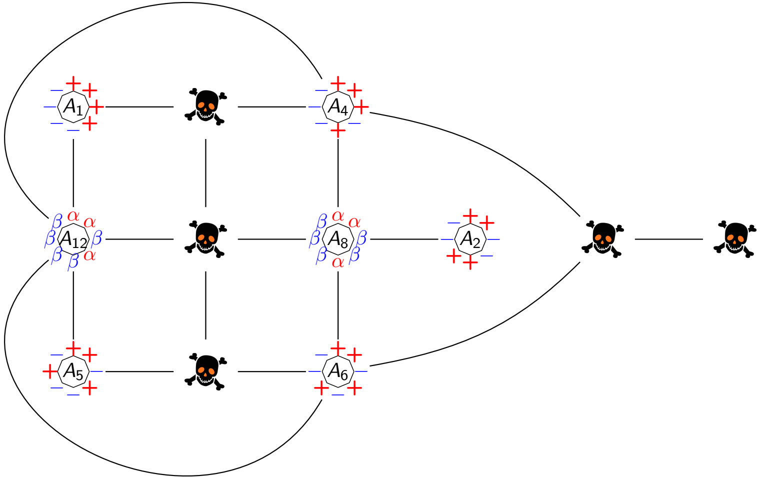
Example 2 for $N = 8$

eigenvalues: $\mathrm{Tr}(P^{(p)}|_{A_i}) = \frac{\dim \pi^{(p)}}{|G_a|} \sum_{h \in G_a} \chi^{(p)}(h) \qquad \forall a \in A_i$

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}	dim
π_{+++}	1	1	1	1	1	1	1	1	1	1	1	1	12
π_{++-}	0	0	1	0	0	0	0	1	1	1	1	1	6
π_{+-+}	0	0	0	0	0	0	0	1	0	0	0	1	2
π_{+--}	0	0	0	1	0	1	0	1	0	0	0	1	4
π_{-++}	1	1	0	1	1	1	0	1	0	0	0	1	7
π_{-+-}	0	0	0	0	0	0	0	1	0	0	0	1	2
π_{--+}	0	0	1	0	0	0	0	1	1	1	1	1	6
π_{---}	0	0	1	1	0	1	1	1	1	1	1	1	9
$\pi_{1,+}$	0	0	2	2	0	2	0	4	2	2	2	4	20
$\pi_{1,-}$	2	0	2	2	2	2	0	4	2	2	2	4	24
$\pi_{2,+}$	2	0	2	2	2	2	0	4	2	2	2	4	24
$\pi_{2,-}$	0	2	2	2	0	2	0	4	2	2	2	4	22
$\pi_{3,+}$	0	0	2	2	0	2	0	4	2	2	2	4	20
$\pi_{3,-}$	2	0	2	2	2	2	0	4	2	2	2	4	24
$\lvert A \rvert$	8	4	16	16	8	16	2	32	16	16	16	32	182

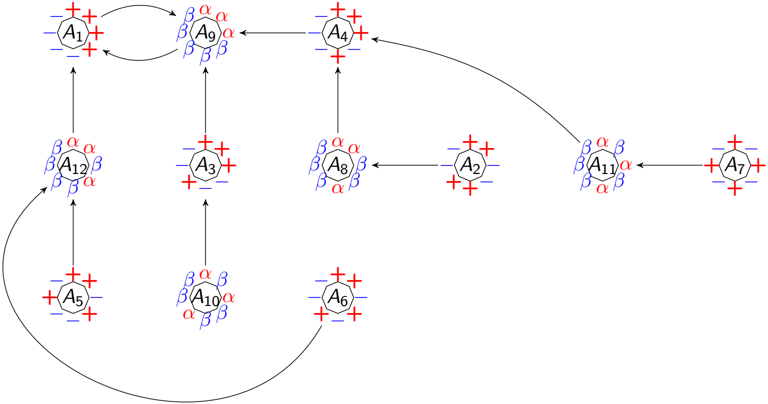
Example 2 for $N = 8$

Irreducible representation π_{-++}



Example 2 for $N = 8$

Irreducible representations of dimension 2: graph of successors



Eigenvalues are those of $L^{(p)}_{22}, \dots, L^{(p)}_{12,12}$ and $L^{(p)}_{11} - L^{(p)}_{19}(L^{(p)}_{99})^{-1}L^{(p)}_{91}$

Concluding remarks

- Also understood: behaviour at bifurcations, $\det \nabla^2(z^*) = 0$
- Also understood: Example 1 in limit $N \to \infty$: Allen–Cahn SPDE
- In progress: from Markovian jump processes to diffusions
- In progress: Example 2 for arbitrary N

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