Metastability in systems of coupled multistable SDEs

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Interacting SDEs with noise

Example 1 [B, Fernandez, Gentz, Nonlinearity 2007]

- N particles on a circle $\mathbb{Z}/N\mathbb{Z}$
- Bistable local dynamics
- Ferromagnetic nearest neighbour coupling
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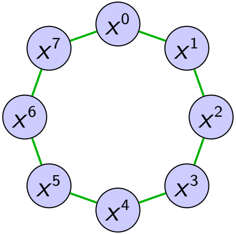

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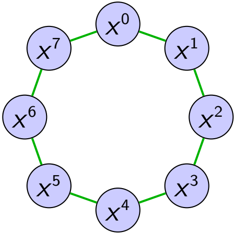

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Gradient system $\qquad dx_t = -\nabla V(x_t)\, dt + \sqrt{2\varepsilon}\, dW_t$

potential $V(x) = \sum_i U(x^i) + \frac{\gamma}{4}\sum_i (x^{i+1} - x^i)^2 \qquad U(\xi) = \frac{1}{4}\xi^4 - \frac{1}{2}\xi^2$

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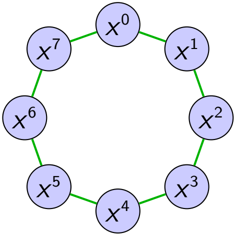

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Example 2 [B, Dutercq, JoTP 2015]: Same potential + constraint $\sum_i x^i = 0$

General gradient systems with noise

$$dx_t = -\nabla V(x_t)\,dt + \sqrt{2\varepsilon}\,dW_t$$

$V : \mathbb{R}^N \to \mathbb{R}$: confining potential, class $\mathcal{C}^2$

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- Stationary points: $\mathcal{X} = \{x : \nabla V(x) = 0\}$
- Local minima: $\mathcal{X}_0 = \{x \in \mathcal{X} :$ all ev of Hessian $\nabla^2 V(x)$ are $> 0\}$
- Saddles of index 1: $\mathcal{X}_1 = \{x \in \mathcal{X} : \nabla^2 V(x)$ has 1 negative ev $\}$

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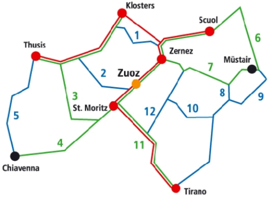
Dynamics $\sim$ markovian jump process on $\mathcal{G} = (\mathcal{X}_0, \mathcal{E})$, $\mathcal{E} \subset \mathcal{X}_1$



Rot Rhätische Bahn
Grün ganzjährig offen
Blau Wintersperre

Nr.	Pass	Land	Passhöhe (m.ü.M.)
1	Flüela	CH	2383
2	Albula	CH	2312
3	Julier	CH	2284
4	Maloja	CH	1815
5	Splügen	I - CH	2115
6	Reschen	A - I	1507
7	Ofen	CH	2149
8	Umbrail	CH - I	2502
9	Stilfserjoch	I	2757
10	Foscagno	I	2291
11	Bernina	CH - I	2323
12	Fla. di Livigno	I	2315

Wentzell–Freidlin theory

$$dx_t = f(x_t)\,dt + \sqrt{2\varepsilon}\,dW_t$$

Large-deviation rate function: $I_{[0,T]}(\varphi) = \dfrac{1}{2}\displaystyle\int_0^T \|\dot\varphi(t) - f(\varphi(t))\|^2\,dt$

Noise-induxed exit from domain $\mathcal{D}$ containing unique attractor $x^\star$

- Mean exit time:
  $\lim\limits_{\varepsilon\to 0} 2\varepsilon \log \mathbb{E}^{x_0}[\tau] = \inf\limits_{z\in\partial\mathcal{D}} \overline{V}(x^*, z) \qquad \overline{V}(x^*, z) = \inf\limits_{T>0}\, \inf\limits_{\varphi: x^*\to z} I_{[0,T]}(\varphi)$
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Case of multiple attractors: $\displaystyle V^{(k)} = \min_{g\ W\text{-graph},|W|=k} \sum_{(\alpha\to\beta)\in g} \overline{V}(\alpha,\beta)$

Small eigenval. λ_k of generator satisfy $\displaystyle -\lim_{\varepsilon\to0} \varepsilon \log(|\lambda_k|) = V^{(k)} - V^{(k+1)}$

Efficient computation of λ_k: [Cameron and Vanden–Eijnden 2014]

Eyring–Kramers law for $\mathrm{d}x_t = -\nabla V(x_t)\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t$

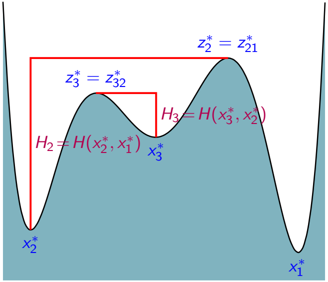
Definition: Communication height

$$H(x_i^*, x_j^*) = \inf_{\gamma: x_i^* \to x_j^*} \sup_t V(\gamma_t) - V(x_i^*)$$
$$= V(z_{ij}^*) - V(x_i^*)$$

Definition: Metastable hierarchy

$x_1^* \prec x_2^* \prec \dots \prec x_n^* \Leftrightarrow \exists \theta > 0\colon \forall k$

$$H_k := H(x_k^*, \{x_1^*, \dots, x_{k-1}^*\})$$
$$\leqslant \min_{i<k} H(x_i^*, \{x_1^*, \dots, x_{i-1}^*, x_{i+1}^*, \dots, x_k^*\}) - \theta$$



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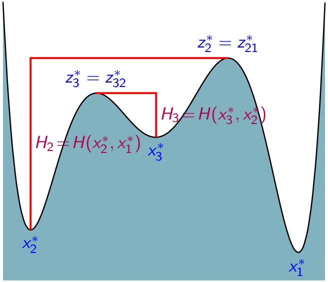

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Theorem: Eyring–Kramers law [Bovier,Eckhoff,Gayrard,Klein 2004]

τ_k = first-hitting time of nbh of $\{x_1^*, \dots, x_k^*\}$ $\quad$ $\lambda_k = k^{\text{th}}$ ev of generator

$$\mathbb{E}^{x_k^*}[\tau_{k-1}] = \frac{2\pi}{|\lambda_-(z_k^*)|} \sqrt{\frac{|\det \nabla^2 V(z_k^*)|}{\det \nabla^2 V(x_k^*)}}\, \mathrm{e}^{H_k/\varepsilon} \bigl[1 + \mathcal{O}_\varepsilon(1)\bigr] \simeq |\lambda_k|^{-1}$$

Potential landscape for Example 1

$$V(x) = \sum_{i \in \mathbb{Z}/N\mathbb{Z}} U(x^i) + \frac{\gamma}{4} \sum_{i \in \mathbb{Z}/N\mathbb{Z}} (x^{i+1} - x^i)^2 \qquad U(\xi) = \tfrac{1}{4}\xi^4 - \tfrac{1}{2}\xi^2$$

$\gamma = 0$: $\mathcal{X} = \{-1, 0, 1\}^N$, $\mathcal{X}_0 = \{-1, 1\}^N$, $\mathcal{X}_1 = \{x \in \mathcal{X} : \text{one } x^i = 0\}$

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Theorem [BFG, Nonlinearity 2007]

No bifurcation for $0 \leqslant \gamma \leqslant \gamma^*(N)$

where $\gamma^*(N) > \frac{1}{4} \quad \forall N \geqslant 2$

+	+	+	+	+	+
−	+	+	+	+	+
−	+	+	+	+	−
−	+	+	+	−	−
−	−	+	+	−	−
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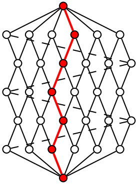

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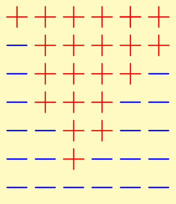

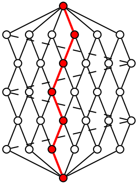

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Ising-like dynamics

Theorem [BFG, Nonlinearity 2007]

$\gamma > \dfrac{1}{2\sin^2(\pi/N)} \Leftrightarrow \mathcal{X}_0 = \{\pm(1, \dots, 1)\}$, $\mathcal{X}_1 = \{0\} \Leftrightarrow$ Synchronization

Transition to synchronization

Symmetry group

$G = D_N \times \mathbb{Z}_2 = \langle r, s, c \rangle$

$$
\begin{aligned}
r(x) &= (x^2, x^3, \dots, x^N, x^1) \\
s(x) &= (x^N, x^{N-1}, \dots, x^1) \\
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\end{aligned}
$$

$\mathcal{X}$ partitionned into

group orbits $O_x = \{gx \colon g \in G\}$

Stabilizer: $G_x = \{g \in G \colon gx = x\}$

$\Rightarrow |O_x||G_x| = |G|$

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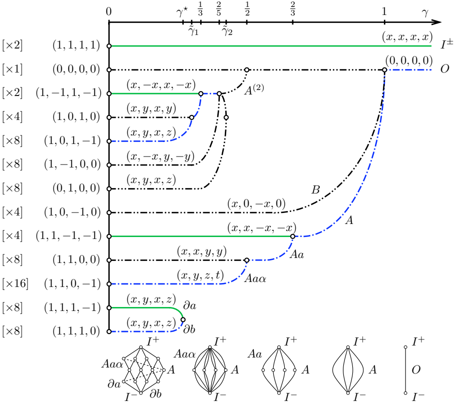

Example: $N = 4$, $|\mathcal{X}| = 3^4$ for $\gamma = 0$

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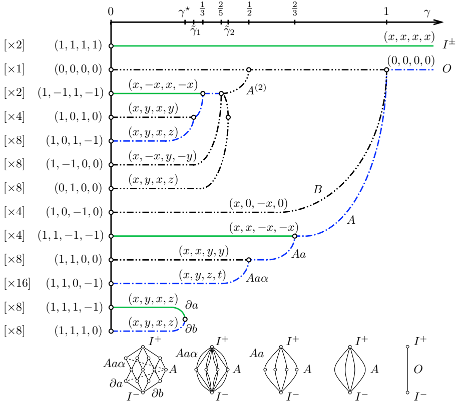

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Problem: no metastable hierarchy $\Rightarrow$ Usual Eyring–Kramers law invalid

Limitations of the standard Eyring–Kramers law

- **Question 1:**
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- **Question 3:**
  What happens when $\gamma \sim N^2$ and $N \to \infty$ in example 1?
  One expects convergence to Allen–Cahn SPDE

  $$\partial_t u(t,x) = \frac{\gamma}{N^2} \Delta u(t,x) + u(t,x) - u(t,x)^3 + \sqrt{2\varepsilon}\xi(t,x)$$

  where ξ is space-time white noise

  Is there an Eyring–Kramers law for such SPDEs?

Q1: Markovian jump processes with symmetries

Generator L: transition rates $L_{ij} = \frac{c_{ij}}{m_i} e^{-h_{ij}/\varepsilon}$, $\quad c_{ij} = c_{ji} \quad \forall i, j \in \mathcal{X}_0$

Assumptions

- Reversibility: $m_i e^{-V_i/\varepsilon} L_{ij} = m_j e^{-V_j/\varepsilon} L_{ji} \qquad \forall i, j \in \mathcal{X}_0$
- Symmetry: $L_{ij} = L_{g(i)g(j)} \quad \forall g \in G$, $(G, *)$ a finite group
  i.e. $\pi(g)L = L\pi(g)$ where $\pi(g)_{ab} = 1_{\{g(a)=b\}}$ permutation matrix
- Metastable order on the set of G-orbits $A_1 \prec \cdots \prec A_m$
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Main observation: $\quad \pi$ is a representation: $\pi(g * h) = \pi(g)\pi(h) \; \forall g,h \in G$

Representation theory of finite groups: $\pi = \bigoplus_{p=0}^{r-1} \alpha^{(p)} \pi^{(p)}$

where $\pi^{(p)}$: irreducible representations of G

$$P^{(p)} L = L P^{(p)} \qquad p = 0, \dots, r-1$$

where $P^{(p)}$: projector on $\operatorname{im} \pi^{(p)} \Rightarrow$ each subspace $P^{(p)}\mathbb{C}^n$ invariant for L

Each restricted generator satisfies an asymmetric Eyring–Kramers law

Q1: Modified Eyring–Kramers law

Trivial representation: $\pi^{(0)}(g) = 1 \ \forall g \quad \Rightarrow \quad m$ ev ($m = \#$ orbits)

Theorem [B, Dutercq, J Theor Proba 2015]

$k \leqslant m$, initial distribution μ uniform on each A_i, $i \geqslant k$

τ_{k-1} = first-hitting time of $A_1 \cup A_2 \cup \dots \cup A_{k-1}$ $\qquad G_a := \{g : g(a) = a\}$

$$\mathbb{E}^{\mu}[\tau_{k-1}] = \frac{|G_{a_i} \cap G_{a_j}|}{|G_{a_k}|} \frac{m_{a_k}}{c_{a_i a_j}} \mathrm{e}^{H_k/\varepsilon} \big[1 + \mathcal{O}(\mathrm{e}^{-\theta/\varepsilon})\big] = \frac{1 + \mathcal{O}(\mathrm{e}^{-\theta/\varepsilon})}{\lambda_k^{(0)}}$$

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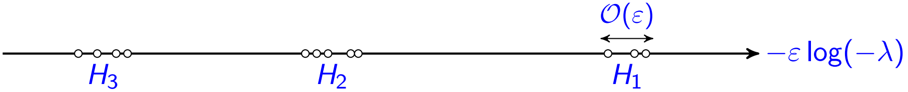
Other representations: Similar result for process on set of active orbits

$\Rightarrow$ clustering of eigenvalues: $\lambda_k^{(p)} = C_k^{(p)} \, \mathrm{e}^{-H_k/\varepsilon}$



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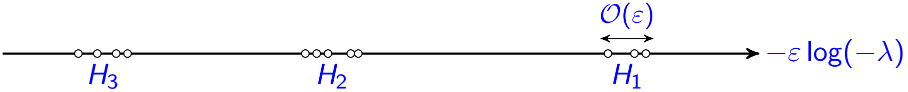

Case of diffusions: similar results [S. Dutercq, PhD thesis, 2015]

Q2: Eyring–Kramers law for nonquadratic saddles

Facts from potential theory: $A, B \subset \mathbb{R}^d$, $\tau_A = \inf\{t > 0\colon x_t \in A\}$

Committor function: $h_{A,B}(x) = \mathbb{P}^x\{\tau_A < \tau_B\}$

Capacity: $\operatorname{cap}(A,B) = \displaystyle\int_{(A\cup B)^c} \|\nabla h_{A,B}(x)\|^2 \, \mathrm{e}^{-V(x)/\varepsilon} \, \mathrm{d}x$

$$\frac{\displaystyle\int_{A^c} h_{B,A}(y)\, \mathrm{e}^{-V(y)/\varepsilon}\,\mathrm{d}y}{\operatorname{cap}(B,A)} = \mathbb{E}^\mu[\tau_A] \overset{B=\mathcal{B}_\varepsilon(x))}{\simeq} \mathbb{E}^x[\tau_A] \qquad (\operatorname{supp}\mu \subset \partial B)$$

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Committor function: $h_{A,B}(x) = \mathbb{P}^x\{\tau_A < \tau_B\}$

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$$\frac{\displaystyle\int_{A^c} h_{B,A}(y)\, \mathrm{e}^{-V(y)/\varepsilon} \, \mathrm{d}y}{\operatorname{cap}(B,A)} = \mathbb{E}^{\mu}[\tau_A] \overset{B=\mathcal{B}_\varepsilon(x))}{\simeq} \mathbb{E}^{x}[\tau_A] \qquad (\operatorname{supp}\mu \subset \partial B)$$

Theorem: [B & Gentz, MPRF 2010]

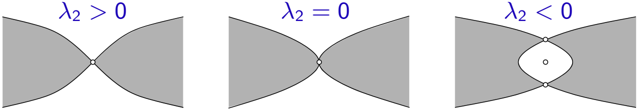
- Saddle in 0, separating A and B
- $V(x) = -u_1(x_1) + u_2(x_2,\dots,x_q) + \frac{1}{2}\sum_{j=q+1}^d \lambda_j x_j^2 + \cdots, \qquad \lambda_j > 0$

$$\operatorname{cap}(A,B) = \varepsilon \frac{\displaystyle\int \mathrm{e}^{-u_2(x_2,\dots,x_q)/\varepsilon} \, \mathrm{d}x_2 \dots \mathrm{d}x_q}{\displaystyle\int \mathrm{e}^{-u_1(x_1)/\varepsilon} \, \mathrm{d}x_1} \prod_{j=q+1}^d \sqrt{\frac{2\pi\varepsilon}{\lambda_j}} \bigl[1 + \mathcal{O}((\varepsilon|\log\varepsilon|)^\alpha)\bigr]$$

with α related to growth of u_1 and u_2

Q2: Example – Transverse pitchfork bifurcation

$$V(x) = -\frac{1}{2}|\lambda_1|x_1^2 + \frac{1}{2}\lambda_2 x_2^2 + C_4 x_2^4 + \frac{1}{2}\sum_{j=3}^{d} \lambda_j x_j^2 + \cdots$$



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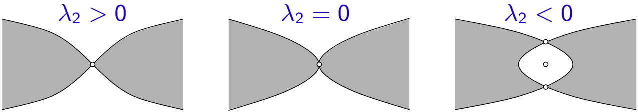


$$\mathbb{E}^x[\tau_A] = 2\pi\sqrt{\frac{(\lambda_2 + \sqrt{2\varepsilon C_4})\lambda_3 \dots \lambda_d}{|\lambda_1| \det(\nabla^2 V(x))}}\,\frac{e^{\overline{V}(x,A)/\varepsilon}}{\Psi_+(\lambda_2/\sqrt{2\varepsilon C_4})}\left[1 + R(\varepsilon)\right]$$

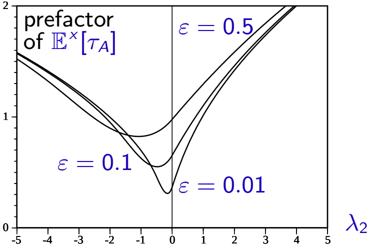
for $\lambda_2 > 0$ where

$$\Psi_+(\alpha) = \sqrt{\frac{\alpha(1+\alpha)}{8\pi}}\, e^{\alpha^2/16}\, K_{1/4}\left(\frac{\alpha^2}{16}\right)$$

$\lim_{\alpha\to+\infty} \Psi_+(\alpha) = 1$

$\lim_{\alpha\to 0} \Psi_+(\alpha) = \frac{\Gamma(1/4)}{2^{5/4}\pi^{1/2}} \simeq 0.860$

Similar expression for $\lambda_2 < 0$ with $\Psi_-(\alpha)$ involving $I_{\pm 1/4}$



Q3: Eyring–Kramers law for parabolic SPDEs

$$\partial_t u_t(x) = \partial_{xx} u_t(x) + f(u_t(x)) + \sqrt{2\varepsilon}\,\xi(t,x) \qquad \text{e.g. } f(u) = u - u^3$$

$x \in [0, L]$ with periodic or Neumann b.c.

$$u_t(x) = \frac{1}{\sqrt{L}} \sum_{k\in\mathbb{Z}} z_k(t)\,\mathrm{e}^{\mathrm{i}\,\pi k x/L} \quad \Rightarrow \quad \mathrm{d}z_t = -\nabla V(z_t)\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t$$

$$V = \int_0^L \left[\tfrac{1}{2}u'^2 - \tfrac{1}{2}u^2 + \tfrac{1}{4}u^4\right]\mathrm{d}x = \frac{1}{2}\sum_{k\in\mathbb{Z}} \lambda_k |z_k|^2 + \frac{1}{4L}\sum_{\sum k_i = 0} z_{k_1} z_{k_2} z_{k_3} z_{k_4}$$

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$$V = \int_0^L \Big[\tfrac12 u'^2 - \tfrac12 u^2 + \tfrac14 u^4\Big]\,\mathrm{d}x = \tfrac12 \sum_{k\in\mathbb{Z}} \lambda_k |z_k|^2 + \frac{1}{4L} \sum_{\sum k_i = 0} z_{k_1} z_{k_2} z_{k_3} z_{k_4}$$

Initial cond $u_{\text{in}} \simeq -1$. Target $u_+ \equiv 1$, $\tau_+ = \inf\{t > 0 \colon \|u_t - u_+\|_\infty\} < \rho$

Transition state: ($\beta = 1$ for Neumann b.c. $\beta = 2$ for periodic b.c.)

$$u_{\text{ts}}(x) = \begin{cases} u_0(x) \equiv 0 & \text{if } L \leqslant \beta\pi \\ u_1(x)\ \beta\text{-kink stationary sol.} & \text{if } L > \beta\pi \end{cases} \qquad \text{with ev } \lambda_k = \left(\tfrac{\beta k\pi}{L}\right)^2 - 1$$

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$$\partial_t u_t(x) = \partial_{xx} u_t(x) + f(u_t(x)) + \sqrt{2\varepsilon}\,\xi(t,x) \qquad \text{e.g. } f(u) = u - u^3$$

$x \in [0, L]$ with periodic or Neumann b.c.

$$u_t(x) = \frac{1}{\sqrt{L}} \sum_{k\in\mathbb{Z}} z_k(t)\, e^{i\pi kx/L} \quad \Rightarrow \quad dz_t = -\nabla V(z_t)\, dt + \sqrt{2\varepsilon}\, dW_t$$

$$V = \int_0^L \left[\tfrac{1}{2}u'^2 - \tfrac{1}{2}u^2 + \tfrac{1}{4}u^4\right] dx = \tfrac{1}{2}\sum_{k\in\mathbb{Z}} \lambda_k |z_k|^2 + \frac{1}{4L}\sum_{\sum k_i = 0} z_{k_1} z_{k_2} z_{k_3} z_{k_4}$$

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[Faris & Jona-Lasinio 82]: LDP $\Rightarrow \mathbb{E}^{u_{\rm in}}[\tau_+] \simeq e^{(V[u_{\rm ts}] - V[u_-])/\varepsilon}$

[Maier & Stein 01]: formal computation; for Neumann b.c.

$$\Rightarrow \mathbb{E}^{u_{\rm in}}[\tau_+] \simeq 2\pi \sqrt{\frac{1}{|\lambda_0|\nu_0} \prod_{k=1}^{\infty} \frac{\lambda_k}{\nu_k}}\, e^{(V[u_{\rm ts}] - V[u_-])/\varepsilon} \qquad (\nu_k = \text{ev at } u_-)$$

Q3: Eyring–Kramers law for parabolic SPDEs

Theorem: [B & Gentz, Elec J Proba 2013]

Neumann b.c:

- If $L < \pi - c$, then

$$\mathbb{E}^{u_{\rm in}}[\tau_+] = 2\pi \sqrt{\frac{1}{|\lambda_0|\nu_0} \prod_{k=1}^{\infty} \frac{\lambda_k}{\nu_k}}\, \mathrm{e}^{(V[u_{\rm ts}]-V[u_-])/\varepsilon} \bigl[1 + \mathcal{O}(\varepsilon^{1/2}|\log\varepsilon|^{3/2})\bigr]$$

- If $L > \pi + c$, then same formula with extra factor $\frac{1}{2}$ (since 2 saddles)

Q3: Eyring–Kramers law for parabolic SPDEs

Theorem: [B & Gentz, Elec J Proba 2013]

Neumann b.c:

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- If $L > \pi + c$, then same formula with extra factor $\frac{1}{2}$ (since 2 saddles)
- If $\pi - c \leqslant L \leqslant \pi$, then

$$\mathbb{E}^{u_{\rm in}}[\tau_+] = 2\pi \sqrt{\frac{\lambda_1 + \sqrt{3\varepsilon/2L}}{|\lambda_0|\nu_0\nu_1} \prod_{k=2}^{\infty} \frac{\lambda_k}{\nu_k}}\, \frac{\mathrm{e}^{(V[u_{\rm ts}]-V[u_-])/\varepsilon}}{\Psi_+(\lambda_1/\sqrt{3\varepsilon/2L})} \bigl[1 + R(\varepsilon)\bigr]$$

  with Ψ_+ as before
- If $\pi \leqslant L \leqslant \pi + c$, similar formula with Ψ_-

Periodic b.c: Similar expressions with different $\Psi_\pm$ and extra factor $\varepsilon^{1/2}$

Concluding remarks

- Irreversible systems:
  Cycling: $\partial\mathcal{D}$ periodic orbit, WKB doesn't work [Day '92, B & Gentz '14]
  Transition-path theory [Vanden-Eijnden & E '06, Lu & Nolen '15]
- SPDEs in higher space dim: Regularity structures [Hairer & Weber '14]

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