Séminaire du LJLL, Paris

Metastability of Allen-Cahn equations with weak space-time white noise

Nils Berglund

Institut Denis Poisson, University of Orléans, France

9 January 2026

Joint works with Barbara Gentz (Bielefeld), Giacomo Di Gesù (Rome), Hendrik Weber (Münster), Tom Klose (Oxford) and Nikolas Tapia (Berlin)





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Plan

1. Metastability in Allen–Cahn equations on $\mathbb{T}^1$

   Joint work with Barbara Gentz

   Electronic J. Probability **18**, (24):1–58 (2013)

2. Metastability in Allen–Cahn equations on $\mathbb{T}^2$

   Joint work with Giacomo Di Gesù & Hendrik Weber

   Electronic J. Probability **22**, (41):1–27 (2017)

3. Current work, motivated by the case of $\mathbb{T}^3$

   Joint works with Tom Klose and Nikolas Tapia

   `arXiv/2207.08555` (2022) (Proceedings, Vienna trimester)

   `arXiv/2507.03820` (2025)

Allen–Cahn equation on $\mathbb{T}^2$

$\partial_t \phi(t,x) = \nu \Delta \phi(t,x) + \phi(t,x) - \phi(t,x)^3 \qquad (\nu = \nu(\varepsilon t))$

(Online: https://youtu.be/yX0EAxZHNCQ)

1. Deterministic Allen–Cahn PDE in $d = 1$

[Chafee & Infante 74, Allen & Cahn 75]

$$\partial_t \phi(t,x) = \Delta\phi(t,x) + f(\phi(t,x))$$

- $x \in [0, L]$, L: bifurcation parameter ($x \mapsto L^{-1}x \Rightarrow \Delta \mapsto \nu\Delta$, $\nu = L^{-2}$)
- $\phi(t,x) \in \mathbb{R}$
- Either periodic or zero-flux Neumann boundary conditions
- In this talk: $f(\phi) = \phi - \phi^3$ (results more general)

Energy function:

$$V[\phi] = \int_0^L \Big[\frac{1}{2}\phi'(x)^2 - \frac{1}{2}\phi(x)^2 + \frac{1}{4}\phi(x)^4\Big] \mathrm{d}x \quad \Rightarrow \quad \partial_t\langle\phi,\psi\rangle = -\nabla_\psi V(\phi)$$

Stationary solutions: $\phi_0''(x) = -\phi_0(x) + \phi_0(x)^3$ critical points of V

Stability: Sturm–Liouville problem $\partial_t\psi_t(x) = \psi_t''(x) + [1 - 3\phi_0(x)^2]\psi_t(x)$

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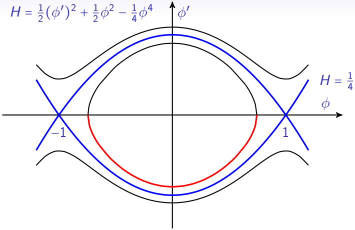
- $\phi_\pm(x) \equiv \pm 1$: stable
- $\phi_0(x) \equiv 0$: unstable
- Nonconstant solutions satisfying b.c.
  (expressible in terms of Jacobi elliptic fcts)



- Neumann b.c: $2k$ nonconstant solutions when $L > k\pi$

Number of positive eigenvalues (= unstable directions)
Transition state

- Periodic b.c: k families when $L > 2k\pi$

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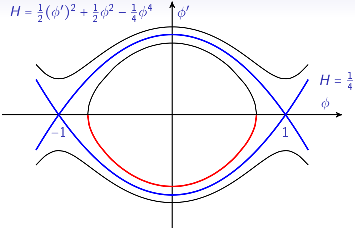

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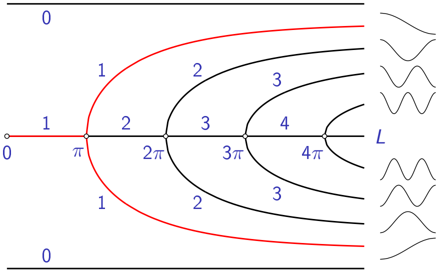

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Allen–Cahn SPDE with space-time white noise

$$\partial_t \phi(t,x) = \Delta\phi(t,x) + f(\phi(t,x)) \;\; + \sqrt{2\varepsilon}\,\xi(t,x) \qquad (f(\phi) = \phi - \phi^3)$$

- Space-time white noise: ξ Gaussian random distribution,
  Formal def: $\mathbb{E}[\xi(t,x)] = 0$, $\mathbb{E}[\xi(t,x)\xi(s,y)] = \delta(t-s)\delta(x-y)$

  Rigorous def: $\mathbb{E}[\langle\xi,\varphi_1\rangle\langle\xi,\varphi_2\rangle] = \langle\varphi_1,\varphi_2\rangle_{L^2}$ $\forall$ test functions φ_1, φ_2

  In Fourier basis: $\xi(t,x) = \sum_{k\in\mathbb{Z}} \dot{W}_k(t) e_k(x)$, $W_k(t)$ independent BM

- Local solution theory: use Duhamel principle
  $\phi = P\phi_0 + (P * F(\phi)) + \sqrt{2\varepsilon}(P * \xi)$ where

  $(P\phi_0)(t,x) = \int_{\mathbb{T}} P_t(x-y)\phi_0(y)\,dy$, P_t heat kernel

  $(P * F)(t,x) = \int_0^t \int_{\mathbb{T}} P_{t-s}(x-y) F(y,s)\,dy\,ds$

  Facts: $(P * \xi)(t,\cdot) \in H^s(\mathbb{T})$ $\forall s < \frac{1}{2}$ and $P * \xi \in \mathcal{C}^{\alpha}(\mathbb{R}_+ \times \mathbb{T})$ $\forall \alpha < \frac{1}{2}$

- Global solution theory: use energy estimates

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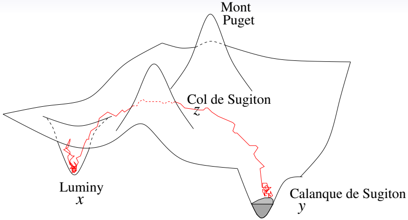
Reversible diffusion in a double-well

$$\mathrm{d}x_t = -\nabla V(x_t)\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t$$

$V : \mathbb{R}^d \to \mathbb{R}$ confining potential

$\tau_y^x = \inf\{t > 0 \colon x_t \in \mathcal{B}_\varepsilon(y)\}$
first-hitting time of small ball $\mathcal{B}_\varepsilon(y)$,
when starting in x



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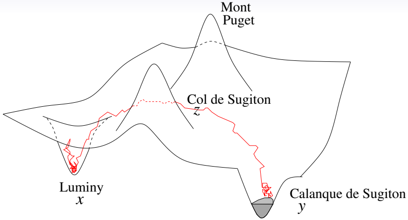

$V : \mathbb{R}^d \to \mathbb{R}$ confining potential

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Arrhenius' law (1889): $\mathbb{E}[\tau_y^x] \simeq e^{[V(z)-V(x)]/\varepsilon}$

Eyring–Kramers law (1935, 1940):
Eigenvalues of Hessian of V at minimum x: $0 < \nu_1 \leqslant \nu_2 \leqslant \dots \leqslant \nu_d$
Eigenvalues of Hessian of V at saddle z: $\lambda_1 < 0 < \lambda_2 \leqslant \dots \leqslant \lambda_d$

$$\mathbb{E}[\tau_y^x] = 2\pi \sqrt{\frac{\lambda_2 \dots \lambda_d}{|\lambda_1| \nu_1 \dots \nu_d}}\, e^{[V(z)-V(x)]/\varepsilon} \bigl[1 + \mathcal{O}_\varepsilon(1)\bigr]$$

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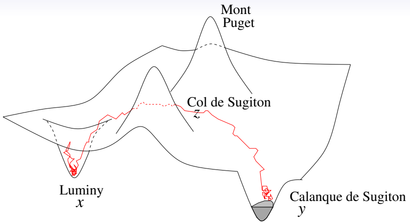

$V : \mathbb{R}^d \to \mathbb{R}$ confining potential

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Arrhenius' law: proved by [Freidlin, Wentzell, 1979] using large deviations

Eyring–Kramers law: [Bovier, Eckhoff, Gayrard, Klein, 2004] using potential theory, [Helffer, Klein, Nier, 2004] using Witten Laplacian, . . .

Potential-theoretic proof of Eyring–Kramers law

- $w_A(x) = \mathbb{E}^x[\tau_A]$ satisfies $\begin{cases} (\mathscr{L} w_A)(x) = -1 & x \in A^c \\ w_A(x) = 0 & x \in A \end{cases}$

  $\mathscr{L} := \varepsilon\Delta - \nabla V \cdot \nabla$

- $h_{AB}(x) = \mathbb{P}^x\{\tau_A < \tau_B\}$ satisfies $\begin{cases} (\mathscr{L} h_{AB})(x) = 0 & x \in (A \cup B)^c \\ h_{AB}(x) = 1 & x \in A \\ h_{AB}(x) = 0 & x \in B \end{cases}$

Theorem: $A, B \subset \mathbb{R}^d$ disjoint. $\exists$ proba measure ν_{AB} on ∂A s.t.

$$\int_{\partial A} \mathbb{E}^x[\tau_B]\nu_{AB}(\mathrm{d}x) = \frac{1}{\operatorname{cap}(A,B)} \int_{B^c} \mathrm{e}^{-V(y)/\varepsilon} h_{AB}(y)\,\mathrm{d}y$$

Apply to A, B neighbourhoods of x, y

- Laplace asymptotics: $\displaystyle\int_{B^c} h_{A,B}(y)\,\mathrm{e}^{-V(y)/\varepsilon}\,\mathrm{d}y \simeq \sqrt{\frac{(2\pi\varepsilon)^d}{\nu_1 \dots \nu_d}}\,\mathrm{e}^{-V(x)/\varepsilon}$
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Potential-theoretic proof of Eyring–Kramers law

Theorem: Dirichlet principle

Let $\mathscr{H}_{AB} = \{h : \mathbb{R}^d \to [0,1] : h|_A = 1, h|_B = 0\}$. Then

$$\operatorname{cap}(A,B) = \inf_{h \in \mathcal{H}_{AB}} \mathscr{E}(h) = \mathscr{E}(h_{AB})$$

Appropriate test function yields $\operatorname{cap}(A,B) \lesssim \varepsilon \sqrt{\frac{|\lambda_1|}{2\pi\varepsilon}} \sqrt{\frac{(2\pi\varepsilon)^{d-1}}{\lambda_2 \dots \lambda_d}} \, \mathrm{e}^{-V(z)/\varepsilon}$

Theorem: Thomson principle [Landim, Mariani, Seo 2018]

Let $\mathcal{U}_{AB} = \{f : \nabla \cdot f|_{(A\cup B)^c} = 0, \int_{\partial A} f(x) \cdot n_A(x) \sigma(\mathrm{d}x) = 1\}$. Then

$$\operatorname{cap}(A,B) = \sup_{f \in \mathcal{U}_{AB}} \frac{1}{\mathscr{D}(f)} = \frac{1}{\mathscr{D}(f_{AB})} \qquad \mathscr{D}(f) = \frac{1}{\varepsilon} \int \mathrm{e}^{V(x)/\varepsilon} |f(x)|^2 \, \mathrm{d}x$$

Appropriate test flow yields $\operatorname{cap}(A,B) \gtrsim \varepsilon \sqrt{\frac{|\lambda_1|}{2\pi\varepsilon}} \sqrt{\frac{(2\pi\varepsilon)^{d-1}}{\lambda_2 \dots \lambda_d}} \, \mathrm{e}^{-V(z)/\varepsilon}$

$$\Rightarrow \quad \mathbb{E}^x[\tau_B] \simeq \int_{\partial A} \mathbb{E}^x[\tau_B] \nu_{AB}(\mathrm{d}x) \simeq 2\pi \sqrt{\frac{\lambda_2 \dots \lambda_d}{|\lambda_1| \nu_1 \dots \nu_d}} \, \mathrm{e}^{-[V(z)-V(x)]/\varepsilon}$$

Potential-theoretic proof of Eyring–Kramers law

Theorem: Dirichlet principle

Let $\mathscr{H}_{AB} = \{h : \mathbb{R}^d \to [0,1] : h|_A = 1, h|_B = 0\}$. Then

$$\operatorname{cap}(A,B) = \inf_{h \in \mathcal{H}_{AB}} \mathscr{E}(h) = \mathscr{E}(h_{AB})$$

Appropriate test function yields $\operatorname{cap}(A,B) \lesssim \varepsilon \sqrt{\frac{|\lambda_1|}{2\pi\varepsilon}} \sqrt{\frac{(2\pi\varepsilon)^{d-1}}{\lambda_2 \dots \lambda_d}} \, \mathrm{e}^{-V(z)/\varepsilon}$

Theorem: Thomson principle [Landim, Mariani, Seo 2018]

Let $\mathcal{U}_{AB} = \{f : \nabla \cdot f|_{(A \cup B)^c} = 0, \int_{\partial A} f(x) \cdot n_A(x) \sigma(\mathrm{d}x) = 1\}$. Then

$$\operatorname{cap}(A,B) = \sup_{f \in \mathcal{U}_{AB}} \frac{1}{\mathscr{D}(f)} = \frac{1}{\mathscr{D}(f_{AB})} \qquad \mathscr{D}(f) = \frac{1}{\varepsilon} \int \mathrm{e}^{V(x)/\varepsilon} |f(x)|^2 \, \mathrm{d}x$$

Appropriate test flow yields $\operatorname{cap}(A,B) \gtrsim \varepsilon \sqrt{\frac{|\lambda_1|}{2\pi\varepsilon}} \sqrt{\frac{(2\pi\varepsilon)^{d-1}}{\lambda_2 \dots \lambda_d}} \, \mathrm{e}^{-V(z)/\varepsilon}$

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Eyring–Kramers law for 1D SPDEs: heuristics

$$\partial_t \phi(t,x) = \Delta\phi(t,x) + f(\phi(t,x)) \quad +\sqrt{2\varepsilon}\,\xi(t,x) \qquad (f(\phi) = \phi - \phi^3)$$

Initial condition: ϕ_{in} near $\phi_- \equiv -1$ with eigenvalues $\nu_k = (\frac{\beta k\pi}{L})^2 + 2$

Target: $\phi_+ \equiv 1$, $\tau_+ = \inf\{t > 0 \colon \|\phi(t,\cdot) - \phi_+\|_{L^\infty} < \rho\}$

Transition state: ($\beta = 1$ for Neumann b.c., $\beta = 2$ for periodic b.c.)

$$\phi_{\text{ts}}(x) = \begin{cases} \phi_0(x) \equiv 0 & \text{if } L \leqslant \beta\pi \quad \text{with ev } \lambda_k = (\frac{\beta k\pi}{L})^2 - 1 \\ \phi_1(x)\ \beta\text{-kink stationary sol.} & \text{if } L > \beta\pi \quad \text{with ev } \lambda'_k \end{cases}$$

[Faris & Jona-Lasinio 82]: large-deviation principle

$\Rightarrow$ Arrhenius law: $\mathbb{E}^{\phi_{\text{in}}}[\tau_+] \simeq e^{(V[\phi_{\text{ts}}] - V[\phi_-])/\varepsilon}$

[Maier & Stein 01]: formal computation; for Neumann b.c. and $L < \pi$

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Eyring–Kramers law for 1D SPDEs: main result

Theorem: Neumann b.c. [B & Gentz, 2013]

- If $L < \pi - c$ with $c > 0$, then

$$\mathbb{E}^{\phi_{\text{in}}}[\tau_+] = 2\pi \sqrt{\frac{1}{|\lambda_0|\nu_0} \prod_{k=1}^{\infty} \frac{\lambda_k}{\nu_k}}\, e^{(V[\phi_{\text{ts}}]-V[\phi_-])/\varepsilon}\big[1 + \mathcal{O}(\varepsilon)\big]$$

- If $L > \pi + c$, then same formula with extra factor $\frac{1}{2}$ (since 2 saddles) and λ'_k instead of λ_k

- Results also for L near π and periodic b.c.
- Proof uses spectral Galerkin approximation & passage to the limit
- Prefactor involves a Fredholm determinant:
  $\Delta_\perp$ Laplacian acting on mean zero functions

$$\prod_{k=1}^{\infty} \frac{\lambda_k}{\nu_k} = \det\big[(-\Delta_\perp - 1)(-\Delta_\perp + 2)^{-1}\big] = \det\big[\mathbb{1} - 3(-\Delta_\perp + 2)^{-1}\big]$$

  converges because $\log\det = \operatorname{Tr}\log$ and $(-\Delta_\perp + 2)^{-1}$ is trace class
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2. The two-dimensional case

- Large-deviation principle: [Hairer & Weber, 2015]
- Naive computation of prefactor fails:

$$\log \prod_{k\in(\mathbb{N}^2)^*} \frac{1-\left(\frac{L}{|k|\pi}\right)^2}{1+2\left(\frac{L}{|k|\pi}\right)^2} \simeq \sum_{k\in(\mathbb{N}^2)^*} \log\left(1-\frac{3L^2}{|k|^2\pi^2}\right)$$

$$\simeq -\sum_{k\in(\mathbb{N}^2)^*} \frac{3L^2}{|k|^2\pi^2} \simeq -\frac{3L^2}{\pi^2}\int_1^\infty \frac{r\,\mathrm{d}r}{r^2} = -\infty$$

- In fact, the equation needs to be renormalised ($P * \xi$ is distribution!)

Theorem: [Da Prato & Debussche 2003]
Let ξ^δ be a mollification on scale δ of white noise. Then

$$\partial_t\phi = \Delta\phi + \phi - \left[\phi^3 - 3\varepsilon C(\delta)\phi\right] + \sqrt{2\varepsilon}\xi^\delta$$

with $C(\delta) \simeq \log(\delta^{-1})$ admits local solution converging as $\delta \to 0$
Moreover, $\phi - \sqrt{2\varepsilon}(P * \xi^\delta) \in \mathscr{C}([0,T], \mathscr{B}^\alpha_{p,r}) \cap L^p([0,T], \mathscr{B}^s_{p,r})$
where $\mathscr{B}^\bullet_{\bullet,\bullet}$ Besov spaces with some $\alpha < 0 < s$, $p > 3$, $r \geqslant 1$

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Computation of the prefactor

- Consider for simplicity $L < \beta\pi \Rightarrow$ transition state in 0
- Galerkin-truncated renormalised potential

$$V_N = \frac{1}{2}\int_{\mathbb{T}^2}\left[\|\nabla\phi_N(x)\|^2 - \phi_N(x)^2\right]\mathrm{d}x + \frac{1}{4}\int_{\mathbb{T}^2} :\phi_N(x)^4:\mathrm{d}x$$

where $:\phi_N^4: = \phi_N^4 - 6\varepsilon C_N\phi_N^2 + 3\varepsilon^2 C_N^2, \qquad C_N = \sum_{|k|\leqslant N}\frac{1}{\lambda_k+1} \asymp \log(N)$

- Using variational principles: $\operatorname{cap}(A,B) \simeq \sqrt{\frac{|\lambda_0|\varepsilon}{2\pi}} \prod_{0<|k|\leqslant N}\sqrt{\frac{2\pi\varepsilon}{\lambda_k}}$
- Symmetry argument:

$$\int_{B^c} h_{A,B}(z)\,\mathrm{e}^{-V_N(z)/\varepsilon}\,\mathrm{d}z = \frac{1}{2}\int \mathrm{e}^{-V_N(z)/\varepsilon}\,\mathrm{d}z = \frac{1}{2}\mathcal{Z}_N(\varepsilon)$$

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- Prefactor proportional to (since $\nu_k = \lambda_k + 3$)

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Main result in dimension 2

Theorem: [B, Di Gesù, Weber, 2017]

For $L < 2\pi$, $A \ni \phi_-$, $B \ni \phi_+$ balls in $\|\cdot\|_{H^s}$, $s < 0$, $\exists \mu_N$ prob measures on ∂A:

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$$\det{}_2(\mathbb{1} + T) = \det(\mathbb{1} + T)\mathrm{e}^{-\operatorname{Tr} T}$$

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  $\det_2$ defined whenever T is only Hilbert–Schmidt (true for $d \leqslant 3$)

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3. Allen–Cahn equation in dimension 3

- Equation needs two counterterms:

$$\partial_t \phi = \Delta\phi + \phi - \left[\phi^3 - \left(3\varepsilon C_1(\delta) - 9\varepsilon^2 C_2(\delta)\right)\phi\right] + \sqrt{2\varepsilon}\xi^\delta$$

with $C_1(\delta) \sim \delta^{-1}$, $C_2(\delta) \sim \log(\delta^{-1})$

- Solution theories:
  - Regularity structures [Hairer 2014]
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- Solution theories:
  - Regularity structures [Hairer 2014]
  - Paracontrolled distributions [Gubinelli–Imkeller-Perkowski 2015, Catellier & Chouk 2018]
  - Wilsonian RG [Kupiainen 2016]

- Metastability: one expects equivalent result (T is still Hilbert–Schmidt)

- Difficulty: lower bound on capacity – finding a good test flow for Thomson principle
  $\Rightarrow$ better understanding of expectations of polynomials under invariant measure

Φ_3^4 measure

- Potential (before renormalisation):

$$V_\alpha(\phi) = \int_\Lambda \Big(\tfrac{1}{2} \|\nabla\phi(x)\|^2 + \tfrac{1}{2}\phi(x)^2 + \alpha\phi(x)^4 \Big) \, dx$$

where $\Lambda = (\mathbb{R}/\mathbb{Z})^3 =: \mathbb{T}^3$, $\alpha \geqslant 0$

- Definition of Gibbs measure?

$$\nu_\alpha(d\phi) \text{ “=” } \frac{1}{\mathcal{Z}_\alpha} e^{-V_\alpha(\phi)} \text{ “}d\phi\text{”}$$

- ν_0 is Gaussian (it's a GFF). For random variable F

$$\mathbb{E}^{\nu_\alpha}[F] = \frac{\mathcal{Z}_0}{\mathcal{Z}_\alpha} \mathbb{E}^{\nu_0}\big[F \, e^{-\alpha \int_\Lambda \phi^4(x) \, dx}\big]$$

In particular, $\dfrac{\mathcal{Z}_\alpha}{\mathcal{Z}_0} = \mathbb{E}^{\nu_0}\big[e^{-\alpha \int_\Lambda \phi^4(x) \, dx}\big]$

- Link with Allen–Cahn equation: write $\phi(x) = \phi_0 + \sqrt{\varepsilon}\phi_\perp(x)$, where ϕ_0 is average of ϕ (and $\alpha = \frac{\varepsilon}{4}$)

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The static Φ_3^4 model

Theorem: Potential needs exactly 4 counterterms:

$$V_\alpha(\phi) = \int_\Lambda \Big(\tfrac{1}{2}\|\nabla\phi(x)\|^2 + \tfrac{1}{2}\big[1 - \alpha^2 C_N^{(2)}\big]\phi(x)^2 + \alpha :\phi(x)^4:_{C_N^{(1)}} + \alpha^2 C_N^{(3)} - \alpha^3 C_N^{(4)} \Big)\, dx$$

where N ($\sim \delta^{-1}$) UV cut-off and

$$C_N^{(1)} = G_N(0) = \mathrm{Tr}\big((-\Delta_N + 1)^{-1}\big) = \mathcal{O}(N)$$

$$C_N^{(2)} = 4^2 3! \int_\Lambda G_N(x)^3\, dx = \mathcal{O}(\log N)$$

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and $G_N(x) = \sum_{|k|\leqslant N} \frac{1}{\lambda_k + 1} e_k(x)$ is the Green function of Δ_N

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Perturbative computation of partition function

- Wick powers: $X =$  $= \int_\Lambda :\phi(x)^4: \mathrm{d}x$, $Y =$  $= \int_\Lambda :\phi(x)^2: \mathrm{d}x$
- Potential: $V_\alpha = V_0 + \alpha X + \beta Y + \gamma$, $\beta = \frac{1}{2}\alpha^2 C_N^{(2)}$, $\gamma = \alpha^2 C_N^{(3)} - \alpha^3 C_N^{(4)}$

  Then $\dfrac{\mathcal{Z}_{N,\alpha}}{\mathcal{Z}_{N,0}} = \mathbb{E}^{\nu_0}\left[\mathrm{e}^{-\alpha X - \beta Y - \gamma}\right] = \mathrm{e}^{-\gamma}\,\mathbb{E}^{\nu_0}\left[\mathrm{e}^{-\alpha X - \beta Y}\right]$
- Let $\Gamma = (\mathscr{V}, \mathscr{E})$ be a multigraph. Its valuation is

$$\Pi_N(\Gamma) = \int_{\Lambda^{\mathscr{V}}} \prod_{e \in \mathscr{E}} G_N(x_{e_+} - x_{e_-})\, \mathrm{d}x$$

  For instance

$C_N^{(1)} = \Pi_N$ "tadpole"

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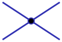
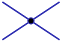
Cumulant expansion

▷ $\mathbb{E}^{\nu_0}\left[\mathrm{e}^{-\alpha X-\beta Y}\right] = \sum_{n\geqslant 0} \frac{1}{n!}\mu_n$

$\mu_n = (-1)^n \mathbb{E}^{\nu_0}\Big[\Big(\alpha$  $+\beta$  $\Big)^n\Big] = (-1)^n \sum_{m=0}^{n} \binom{n}{m} \alpha^m \beta^{n-m} A_{nm}$

where $A_{nm} = \mathbb{E}^{\nu_0}\Big[$  ^{m}  $^{n-m}\Big]$ computed using Isserlis' thm

Examples: $\mu_2 = \alpha^2 4!\Pi_N$ $+ \beta^2 2!\Pi_N$

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$- 3\alpha\beta^2 4!\,\Pi_N$ $- 8\beta^3\Pi_N$

▷ Cumulant expansion: (Leonov & Shiraev)

$-\log \mathbb{E}[\mathrm{e}^{-\alpha X-\beta Y-\gamma}] = \gamma - \sum_{n=2}^{\infty} \frac{\kappa_n}{n!}$ $\qquad \kappa_n = \mu_n - \sum_{m=2}^{n-2} \binom{n-1}{m} \kappa_m \mu_{n-m}$

▷ Linked Cluster Theorem: κ_n projection of μ_n on connected graphs

Proof: for instance Peccati & Taqqu (2011)

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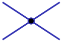
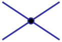
  $\mu_n = (-1)^n \mathbb{E}^{\nu_0}\Big[\Big(\alpha$  $+ \beta$  $\Big)^n\Big] = (-1)^n \sum_{m=0}^{n} \binom{n}{m} \alpha^m \beta^{n-m} A_{nm}$

  where $A_{nm} = \mathbb{E}^{\nu_0}\Big[$  ^{m}  $^{n-m}\Big]$ computed using Isserlis' thm

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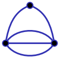
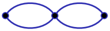
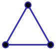
  $\mu_3 = -\alpha^3 \binom{4}{2}^3 2^3 \Pi_N$  $- 3\alpha^2\beta(4^2 \cdot 2 \cdot 3!)\Pi_N$ 

  $- 3\alpha\beta^2 4! \, \Pi_N$  $- 8\beta^3 \Pi_N$ 

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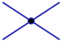
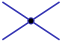
  $\mu_n = (-1)^n \mathbb{E}^{\nu_0}\Big[\Big(\alpha$  $+\beta$  $\Big)^n\Big] = (-1)^n \sum_{m=0}^{n} \binom{n}{m} \alpha^m \beta^{n-m} A_{nm}$

  where $A_{nm} = \mathbb{E}^{\nu_0}\Big[$  ^{m}  $^{n-m}\Big]$ computed using Isserlis' thm

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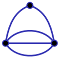
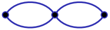
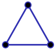
  $\mu_3 = -\alpha^3 \binom{4}{2}^3 2^3 \Pi_N$  $- 3\alpha^2\beta(4^2\cdot 2\cdot 3!)\Pi_N$ 

  $- 3\alpha\beta^2 4!\,\Pi_N$  $- 8\beta^3\Pi_N$ 

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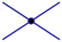
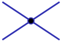
  $\mu_n = (-1)^n\mathbb{E}^{\nu_0}\Big[\Big(\alpha$  $+ \beta$  $\Big)^n\Big] = (-1)^n \sum_{m=0}^{n} \binom{n}{m}\alpha^m\beta^{n-m}A_{nm}$

  where $A_{nm} = \mathbb{E}^{\nu_0}\Big[$  ^{m}  $^{n-m}\Big]$ computed using Isserlis' thm

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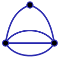
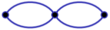
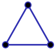
  $\mu_3 = -\alpha^3\binom{4}{2}^3 2^3\Pi_N$  $- 3\alpha^2\beta(4^2\cdot 2\cdot 3!)\Pi_N$ 

  $- 3\alpha\beta^2 4!\,\Pi_N$  $- 8\beta^3\Pi_N$ 

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Divergences and subdivergences

- Degree of Γ: $\deg(\Gamma) = 3(|\mathscr{V}| - 1) - |\mathscr{E}|$. Γ divergent if $\deg(\Gamma) \leqslant 0$.
- Examples:

$\deg($ $) = 0$ $\qquad \Pi_N($ $) = \mathcal{O}(\log N)$

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- It looks like $\Pi_N(\Gamma) = \begin{cases} \mathcal{O}(N^{-\deg(\Gamma)}) & \text{if } \deg(\Gamma) < 0 \\ \mathcal{O}(\log N) & \text{if } \deg(\Gamma) = 0 \end{cases}$

However, $\deg($ $) = 1$, while $\Pi_N($ $) = \mathcal{O}(\log N)$ because it contains a subdivergence

Theorem: [Bogolyubov, Parasiuk, Hepp, Zimmermann]

$\Pi_N^{\mathrm{BPHZ}}(\Gamma) := -\Pi_N \mathscr{A}(\Gamma) 1_{\deg \Gamma > 0}$ is bdd uniformly in N for any Γ where $\mathscr{A}$ defined recursively by $\mathscr{A}(\Gamma) = -\Gamma - \sum_{1 \neq \bar\Gamma \subsetneq \Gamma, \deg(\bar\Gamma) \leqslant 0} \mathscr{A}(\bar\Gamma) \cdot (\Gamma/\bar\Gamma)$

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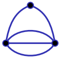
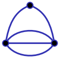
However, deg() = 1, while $\Pi_N($  $) = \mathcal{O}(\log N)$
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Divergences and subdivergences

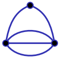
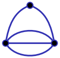
- Degree of Γ: $\deg(\Gamma) = 3(|\mathscr{V}| - 1) - |\mathscr{E}|$. Γ divergent if $\deg(\Gamma) \leqslant 0$.
- Examples:

deg() $= 0$ $\qquad$ $\Pi_N($  $) = \mathcal{O}(\log N)$

deg() $= -1$ $\qquad$ $\Pi_N($  $) = \mathcal{O}(N)$

deg() $= 0$ $\qquad$ $\Pi_N($  $) = \mathcal{O}(\log N)$

- It looks like $\Pi_N(\Gamma) = \begin{cases} \mathcal{O}(N^{-\deg(\Gamma)}) & \text{if } \deg(\Gamma) < 0 \\ \mathcal{O}(\log N) & \text{if } \deg(\Gamma) = 0 \end{cases}$

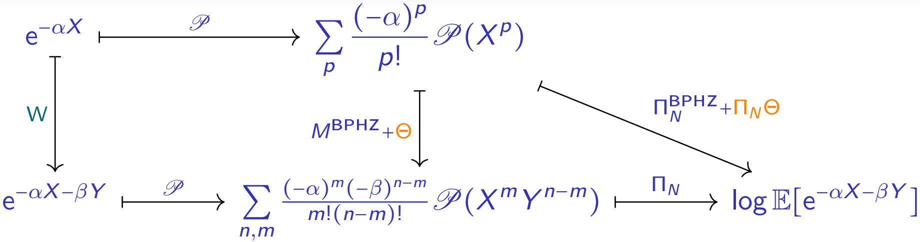
However, deg() $= 1$, while $\Pi_N($  $) = \mathcal{O}(\log N)$ because it contains a subdivergence 

Theorem: [Bogolyubov, Parasiuk, Hepp, Zimmermann]

$\Pi_N^{\text{BPHZ}}(\Gamma) := -\Pi_N \mathscr{A}(\Gamma) 1_{\deg \Gamma > 0}$ is bdd uniformly in N for any Γ

where $\mathscr{A}$ defined recursively by $\mathscr{A}(\Gamma) = -\Gamma - \sum_{1 \neq \bar{\Gamma} \subsetneq \Gamma, \deg(\bar{\Gamma}) \leqslant 0} \mathscr{A}(\bar{\Gamma}) \cdot (\Gamma / \bar{\Gamma})$

Main result: Commutative diagram



- $\mathscr{P} = \Pi_{\text{connected}}\big(\sum_{\text{pairings}}\big)$, satisfies $\deg \mathscr{P}(X^n) = n - 3$
- W: Wick map, $\mathrm{W}(X^n) = H_n(X; \beta Y)$ is Hermite polynomial (for more general models: Bell polynomial)
- Θ associated with energy renormalisation, $\Pi_N \Theta(e^{-\alpha X}) = \gamma$

Theorem: [B, Klose, Tapia 25]

$$\log \frac{\mathcal{Z}_{N,\alpha}}{\mathcal{Z}_{N,0}} = \log \mathbb{E}[e^{-\alpha X - \beta Y}] - \gamma \asymp - \sum_{n \geqslant 4} \frac{(-\alpha)^n}{n!} \Pi_N \mathscr{A}(\mathscr{P}(X^n))$$

as asymptotic expansion with terms uniformly bounded in N

Main result: Commutative diagram

$$
\begin{array}{ccccc}
e^{-\alpha X} & \xrightarrow{\mathscr{P}} & \displaystyle\sum_p \frac{(-\alpha)^p}{p!}\mathscr{P}(X^p) & & \\
\downarrow \mathrm{W} & & \downarrow M^{\mathrm{BPHZ}} + \Theta & \searrow \Pi_N^{\mathrm{BPHZ}} + \Pi_N \Theta & \\
e^{-\alpha X - \beta Y} & \xrightarrow{\mathscr{P}} & \displaystyle\sum_{n,m} \frac{(-\alpha)^m(-\beta)^{n-m}}{m!(n-m)!}\mathscr{P}(X^m Y^{n-m}) & \xrightarrow{\Pi_N} & \log \mathbb{E}[e^{-\alpha X - \beta Y}]
\end{array}
$$

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Main result: Commutative diagram

$$
\begin{array}{ccccc}
e^{-\alpha X} & \xmapsto{\mathscr{P}} & \displaystyle\sum_p \frac{(-\alpha)^p}{p!}\mathscr{P}(X^p) & & \\
\Big\downarrow{\scriptstyle \mathsf{W}} & & \Big\downarrow{\scriptstyle M^{\mathrm{BPHZ}}+\Theta} & \searrow{\scriptstyle \Pi_N^{\mathrm{BPHZ}}+\Pi_N\Theta} & \\
e^{-\alpha X-\beta Y} & \xmapsto{\mathscr{P}} & \displaystyle\sum_{n,m} \frac{(-\alpha)^m(-\beta)^{n-m}}{m!(n-m)!}\mathscr{P}(X^m Y^{n-m}) & \xmapsto{\Pi_N} & \log\mathbb{E}[e^{-\alpha X-\beta Y}]
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Theorem: [B, Klose, Tapia 25]

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as asymptotic expansion with terms uniformly bounded in N

References

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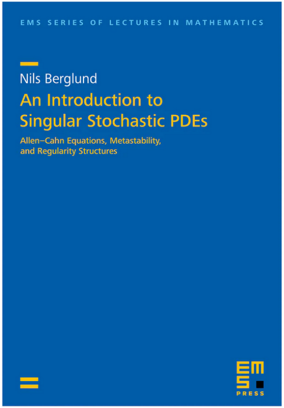
- N. B., Giacomo Di Gesù & Hendrik Weber, *An Eyring–Kramers law for the stoch. Allen–Cahn equation in dimension two*, Electronic J. Probability **22**, 1–27 (2017)
- N. B. & Tom Klose, *Perturbation theory for the* Φ^4_3 *measure, revisited with Hopf algebras*, arXiv/2207.08555 (2022)
- N. B., Tom Klose & Nikolas Tapia, *Perturbative renormalisation of the* $\Phi^4_{4-\varepsilon}$ *model via generalized Wick maps*, arXiv/2507.03820 (2025)
- N. B, *An Introduction to Singular Stochastic PDEs*, EMS Press (2022)



Thanks for your attention!

Slides available at https://www.idpoisson.fr/berglund/LJLL26.pdf

Some literature on the static Φ^4_3 model

- Glimm & Jaffe (1968, 1973), Feldman (1974):
  Combinatorics of Feynman diagrams
- Benfatto, Cassandro, Gallavotti, Nicolò & Olivieri (1978, 1980):
  Renormalisation group (integrating out scales)
- Brydges, Fröhlich & Sokal (1983):
  Generating function and skeleton inequalities
- Brydges, Dimock & Hurd (1995):
  Polymer expansions
- Connes & Kreimer (2000, 2001):
  Hopf algebras
- ...
- Barashkov & Gubinelli (2020):
  Boué–Dupuis formula