Konstanz, 8th International Conference on Random Dynamical Systems

Metastability in a Kuramoto model with nearest neighbour interaction

Nils Berglund

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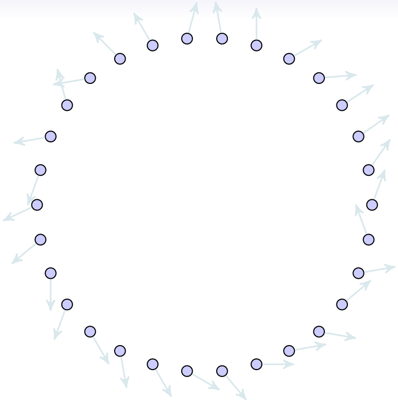
22 July 2025

Joint work with Georgi Medvedev and Gideon Simpson (Drexel Univ.)

The model



- ▷ Lattice $\Lambda = \mathbb{Z}/n\mathbb{Z}$, $n \geqslant 1$
- ▷ Configuration: $u : \Lambda \to \mathbb{S} = \mathbb{R}/\mathbb{Z}$ (config. space: $\mathbb{S}^\Lambda = (\mathbb{R}/\mathbb{Z})^{\mathbb{Z}/n\mathbb{Z}}$)
- ▷ Energy of configuration:

$$V(u) = -\frac{K}{2\pi} \sum_{i \in \Lambda} \cos\big(2\pi(u_{i+1} - u_i)\big) \qquad K > 0$$

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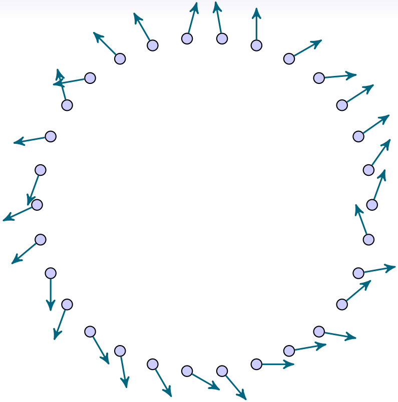

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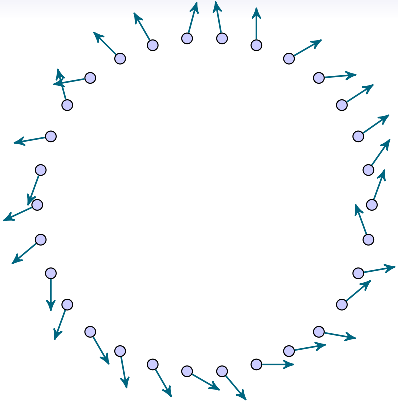

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Dynamics

- Gradient dynamics with noise ($0 < \varepsilon \ll 1$):

$$\mathrm{d}u(t) = -\nabla V(u(t))\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t$$

- In components:

$$\mathrm{d}u_i = K\big[\sin(2\pi(u_{i+1} - u_i)) + \sin(2\pi(u_{i-1} - u_i))\big]\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t^{(i)}$$

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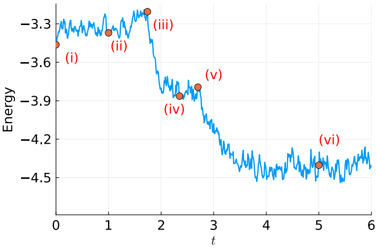

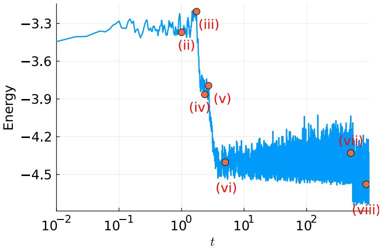

Reversible diffusions

$$\mathrm{d}x_t = -\nabla V(x_t)\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t$$

$V : \mathbb{R}^n \to \mathbb{R}$ confining potential, bounded below

- Invariant Gibbs measure: $\mu(\mathrm{d}x) = \frac{1}{Z}\,\mathrm{e}^{-V(x)/\varepsilon}\,\mathrm{d}x$
- Dynamics is reversible wrt μ (detailed balance)
- $\varepsilon = 0$: stationary states x^* satisfy $\nabla V(x^*) = 0$
- Assume V is Morse function: Hessian of V non-degenerate at any x^*
- Morse index of x^*: number of negative eigenvalues of Hessian at x^*
  - index 0: local minima, stable
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Metastability in a double-well potential

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first-hitting time of small ball $\mathcal{B}_\varepsilon(y)$,
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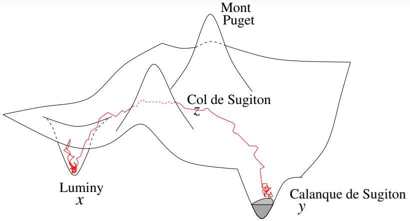

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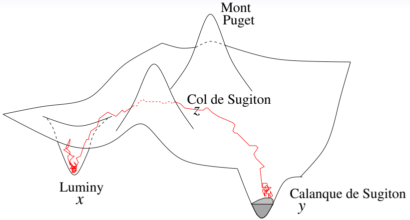

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Eyring–Kramers law (1935, 1940):
Eigenvalues of Hessian of V at minimum x: $0 < \nu_1 \leqslant \nu_2 \leqslant \dots \leqslant \nu_d$
Eigenvalues of Hessian of V at saddle z: $\lambda_1 < 0 < \lambda_2 \leqslant \dots \leqslant \lambda_d$

$$\mathbb{E}[\tau_y^x] = 2\pi\sqrt{\frac{\lambda_2\dots\lambda_d}{|\lambda_1|\nu_1\dots\nu_d}}\,\mathrm{e}^{[V(z)-V(x)]/\varepsilon}\big[1 + \mathcal{O}_\varepsilon(1)\big]$$

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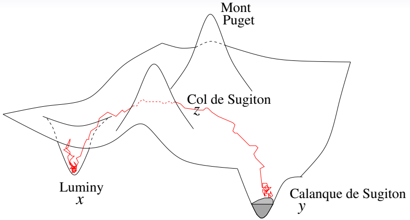

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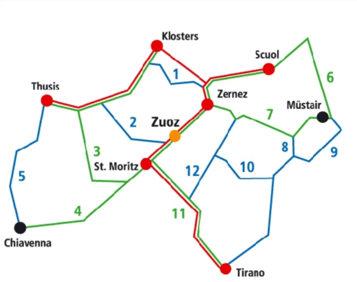
Arrhenius' law: proved by [Freidlin, Wentzell, 1979] using large deviations

Eyring–Kramers law: [Bovier, Eckhoff, Gayrard, Klein, 2004] using potential theory, [Helffer, Klein, Nier, 2004] using Witten Laplacian, ...

Multiwell landscapes



Rot Rhätische Bahn
Grün ganzjährig offen
Blau Wintersperre

Nr.	Pass	Land	Passhöhe (m.ü.M.)
1	Flüela	CH	2383
2	Albula	CH	2312
3	Julier	CH	2284
4	Maloja	CH	1815
5	Splügen	I - CH	2115
6	Reschen	A - I	1507
7	Ofen	CH	2149
8	Umbrail	CH - I	2502
9	Stilfserjoch	I	2757
10	Foscagno	I	2291
11	Bernina	CH - I	2323
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- $G = (\mathcal{V}, \mathcal{E})$: graph where $\mathcal{V}$ is set of local minima of V
  $x^*, y^* \in \mathcal{V}$ connected by an edge $\Leftrightarrow \exists$ 1-saddle whose unstable manifold reaches x^* and y^*
- Dynamics resembles markovian jump process on $\mathcal{V}$ with transition rates given by inverses of EK law [Landim, Seo, ..., B, ...]

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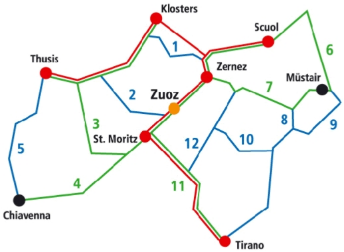

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- Problem: V is degenerate in $(1, 1, \dots, 1)^\top$ direction $\Rightarrow$ not a Morse function
- Solution: rotation, $u = \bar{u} q_0 + Qv$, $\bar{u} \in \mathbb{R}$, $v \in \mathbb{R}^{n-1}$ $(q_0 | Q)$ orthogonal matrix, $q_0 = \frac{1}{\sqrt{n}}(1, 1, \dots, 1)^\top$

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with $V_\perp(v) = V(Qv)$, $(\widehat{W}_t^0)_t$ and $(\widehat{W}_t^\perp)_t$ indep. B.M.

$\Rightarrow$ $\bar{u}_t$ performs B.M. independent of v_t, $V_\perp$ Morse function

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Kuramoto model: q-twisted states

- q-twisted state: $u_i^{(q)} = q\frac{i}{n}$, for $-\frac{n}{2} < q \leqslant \frac{n}{2}$, $q \in \mathbb{Z}$

$q = 0$ $q = 1$ $q = 2$ $q = -1$

- stable for $|q| < \frac{n}{4}$ (unstable with Morse index $n-1$ if $|q| > \frac{n}{4}$)
- 1-saddles: $u_i^{(r)} = \hat{q}\frac{i-i_0}{n}$, with $\hat{q} = r\frac{n}{n-2} = r(1+\mathcal{O}(n^{-1}))$, $r \in \mathbb{Z} + \frac{1}{2}$

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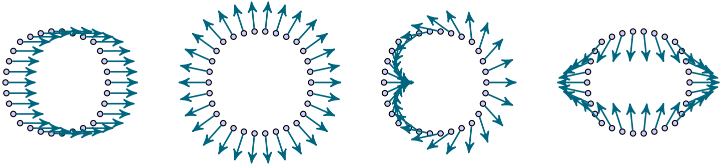

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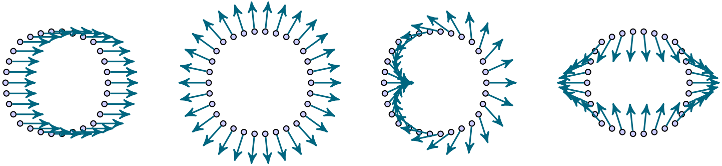

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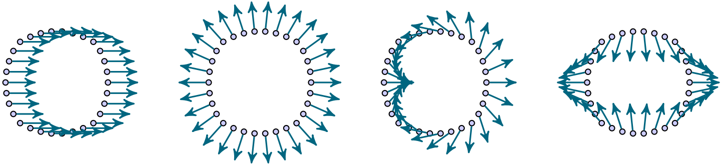

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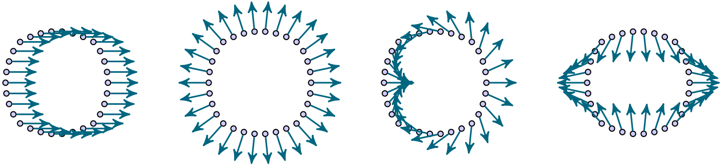

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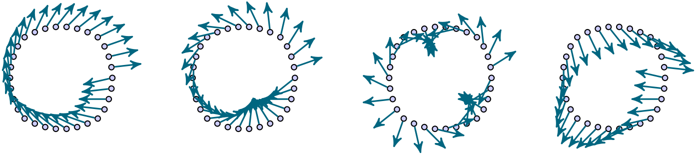

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Classification of equilibria

$$\nabla V(u) = 0 \quad \Leftrightarrow \quad \sin(2\pi(\underbrace{u_{i+1} - u_i}_{a_i})) = \sin(2\pi(\underbrace{u_i - u_{i-1}}_{a_{i-1}})) \quad \forall i \in \Lambda$$

$\Rightarrow$ the $2\pi a_i$ can only take two supplementary values

Proposition

Assume $n \geqslant 5$.

- If all a_i equal and in $(-\frac{1}{4}, \frac{1}{4})$: q-twisted state $u^{(q)}$ with $|q| < \frac{n}{4}$, stable
- If all but one a_i equal and in $(-\frac{1}{4}, \frac{1}{4})$: 1-saddle $u^{(r)}$ with $|r| < \frac{n}{4} - \frac{1}{2}$
- All other cases: saddle of index $\geqslant 2$

- q-twisted state: Hess V is circulant matrix, use discrete Fourier transf.
- $u^{(r)}$: Hess V is rank 1 perturbation of circulant matrix
- Other cases: find 2-dim subspace on which $\langle v, (\text{Hess } V)v\rangle < 0$ if $v \neq 0$

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- If all but one a_i equal and in $(-\frac{1}{4}, \frac{1}{4})$: 1-saddle $u^{(r)}$ with $|r| < \frac{n}{4} - \frac{1}{2}$
- All other cases: saddle of index $\geqslant 2$

- q-twisted state: Hess V is circulant matrix, use discrete Fourier transf.
- $u^{(r)}$: Hess V is rank 1 perturbation of circulant matrix
- Other cases: find 2-dim subspace on which $\langle v, (\text{Hess}\, V)v \rangle < 0$ if $v \neq 0$

Symmetries

The potential V has the following symmetries ($V(g(u)) = V(u) \forall u$):

- Integer translations: $T_k(u) = (u_0 + k_0, \ldots, u_{n-1} + k_{n-1})$, $k \in \mathbb{Z}^n$
- Global phase shift: $S_\varphi(u) = (u_0 + \varphi, \ldots, u_{n-1} + \varphi)$, $\varphi \in \mathbb{R}$
- Cyclic permutation: $C_p(u) = (u_p, \ldots, u_{n-1+p})$, $p \in \Lambda$
- Inversion: $I(u) = -u$

Fundamental domains:

- For integer translations: $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$
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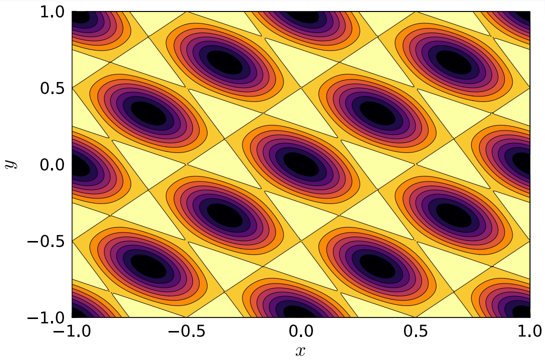
Case $N = 3$



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- $u^{(1)}, u^{(-1)}$: 2-saddles
- $u^{(1/2)}, C_1 u^{(1/2)}, C_2 u^{(1/2)}$: 1-saddles

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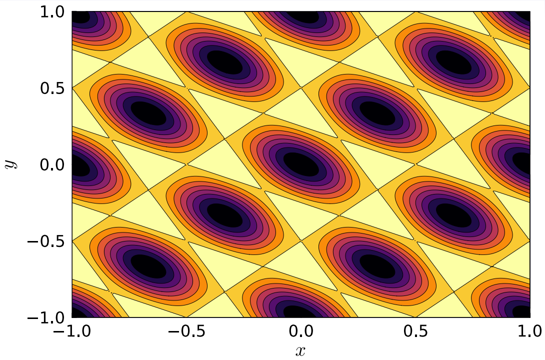
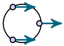
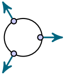
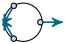
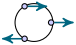
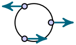


- $u^{(0)}$: stable 
- $u^{(1)}, u^{(-1)}$: 2-saddles  
- $u^{(1/2)}, C_1 u^{(1/2)}, C_2 u^{(1/2)}$: 1-saddles   

Potential landscape

$$V(u) = -\frac{K}{2\pi}\sum_{i\in\Lambda}\cos\big(2\pi(u_{i+1}-u_i)\big) \qquad\qquad K>0$$

$$V(u^{(q)}) = -n\tfrac{K}{2\pi}\cos\Big(\tfrac{2\pi q}{n}\Big)$$

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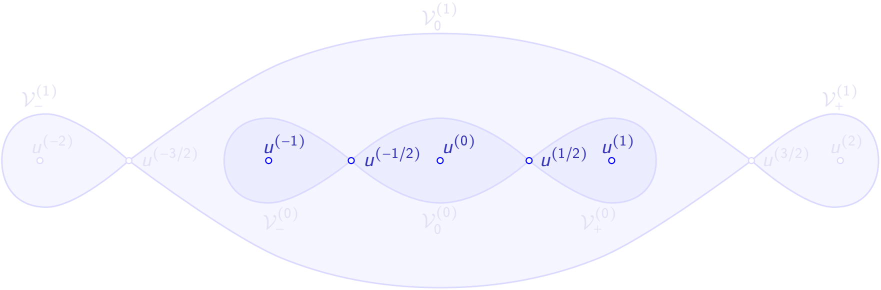

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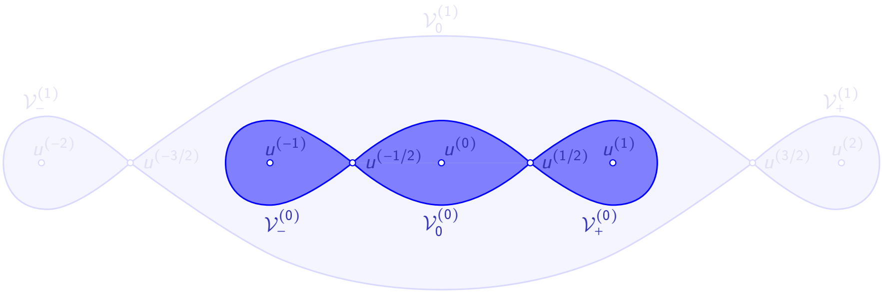

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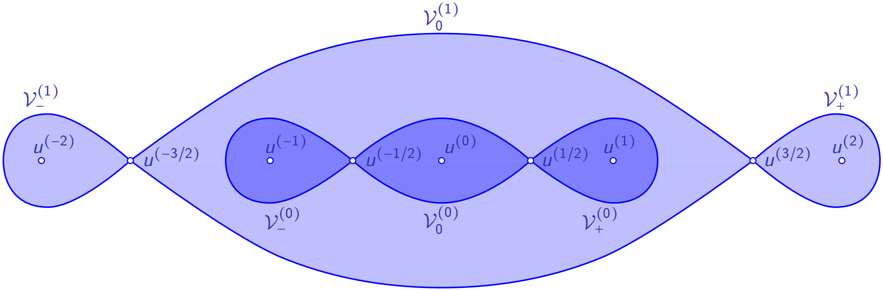

Potential landscape

$$H_q = V(u^{(q-1/2)}) - V(u^{(q)}) = \frac{K}{\pi} - \left(q - \frac{1}{4}\right)\frac{K\pi}{n} + \mathcal{O}(n^{-2})$$

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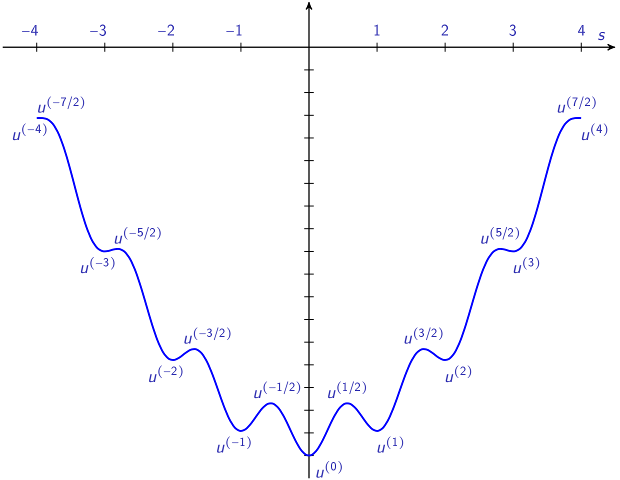

Mean transition times

Eyring–Kramers law for minimum $u^{(q)}$ and saddle $u^{(q-1/2)}$:

$$\mathbb{E}^{u^{(q)}}[\tau_{q-1}] = \frac{2\pi}{|\mu_1|}\sqrt{\frac{|\det M_\perp|}{\det L_\perp}}\,\mathrm{e}^{H_q/\varepsilon}(1+\mathcal{O}_\varepsilon(1))$$

where $L_\perp = \operatorname{Hess} V_\perp(u^{(q)})$, $M_\perp = \operatorname{Hess} V_\perp(u^{(q-1/2)})$, μ_1 negative ev of $M_\perp$

Proposition

$$\frac{\det M_\perp}{\det L_\perp} = -1 + \frac{2}{n}$$

Proof: Take limit as $\varepsilon \to 0$ of regularization

$$\frac{\det(\varepsilon\mathbb{1}+M)}{\det(\varepsilon\mathbb{1}+L)} = \det\left[(\varepsilon\mathbb{1}+M)(\varepsilon\mathbb{1}+L)^{-1}\right] = \det\left[\mathbb{1}+(M-L)(\varepsilon\mathbb{1}+L)^{-1}\right]$$

where $M - L = \psi\psi^\top$ rank 1, $(\varepsilon\mathbb{1}+L_\perp)^{-1} = \sum_k(\varepsilon+\lambda_k)^{-1}\Pi_k$ explicit. □

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Eyring–Kramers law

Theorem

For $0 \leqslant q < \frac{n}{4}$, $\delta > 0$, first-hitting time
$\tau_q = \inf\left\{t > 0 \colon \operatorname{dist}(u_t, \{u^{(0)}, \dots, u^{(q)}\}) < \delta\right\}$ satisfies

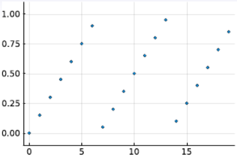
$$\mathbb{E}^{u^{(q+1)}}\left[\tau_q\right] = C(q,n)\, \mathrm{e}^{H_{q+1}/\varepsilon}\left[1 + \mathcal{O}_\varepsilon(1)\right]$$

where

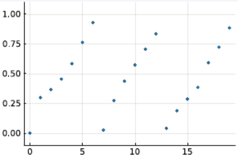
$$C(q,n) = \frac{3}{4Kn}\left(1 + \frac{\pi^2(4q+3) - 4}{4n}\right) + \mathcal{O}(n^{-3})$$

$$H_{q+1} = \frac{K}{\pi} - \left(q + \frac{3}{4}\right)\frac{K\pi}{n} + \mathcal{O}(n^{-2})$$

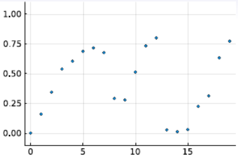
Simulation



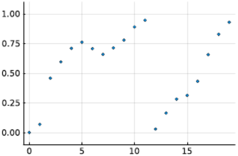
(a) Point (i), $t = 0$



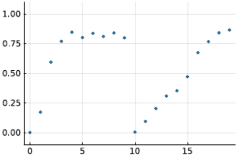
(b) Point (ii), $t = 1$



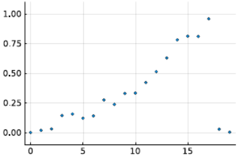
(c) Point (iii), $t = 1.74$



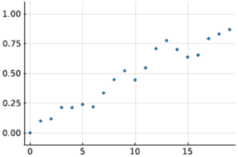
(d) Point (iv), $t = 2.35$



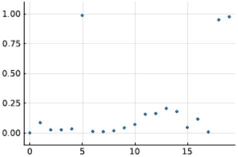
(e) Point (v), $t = 2.7$



(f) Point (vi), $t = 5$

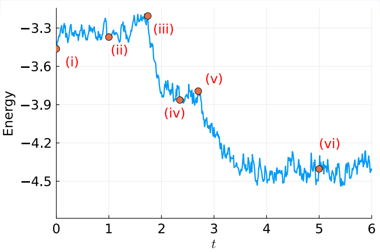


(g) Point (vii), $t = 510.65$

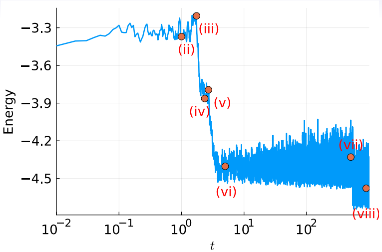


(h) Point (viii), $t = 900$

Simulation



Energy
−3.3
−3.6
−3.9
−4.2
−4.5
0
1
2
3
4
5
6
t
(i)
(ii)
(iii)
(iv)
(v)
(vi)



Energy
−3.3
−3.6
−3.9
−4.2
−4.5
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
t
(ii)
(iii)
(iv)
(v)
(vi)
(vii)
(viii)

Outlook

- ▷ Limit $n \to \infty$? For what scaling?
- ▷ Beyond nearest-neighbour coupling?
- ▷ Λ of higher dimension? Effect of topology?

References

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- ▷ N. B., *Reducing metastable continuous-space Markov chains to Markov chains on a finite set*, 44p (2023), to appear in Annales de l'Institut Henri Poincaré
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Model on $\mathbb{T}^2$

(Online: https://youtube.com/shorts/xKe9_RCcRho)

Model on $\mathbb{S}^2$

(Online: https://youtube.com/shorts/FQYA5udyY_E)