Imperial College London – DynamIC Seminar

Precise estimates on noise-induced transitions in oscillating double-well potentials

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INSTITUT ORLÉANS · TOURS Mathématiques & Physique Théorique
DENIS POISSON

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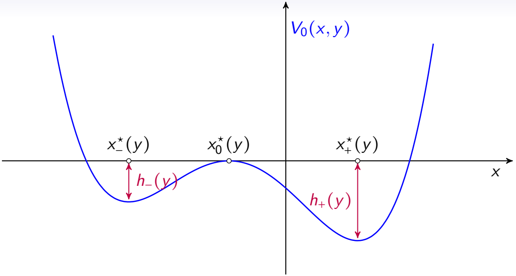
partly based on joint work with Barbara Gentz (Bielefeld)

PROJET FINANCÉ PAR L'ANR · PROJECT FUNDED BY THE ANR · ANR

project PERISTOCH

The problem



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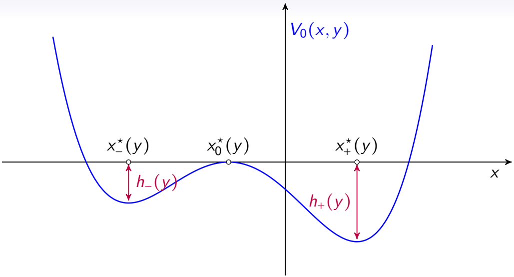

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  $V_0(x, y+1) = V_0(x,y)$
- $0 \leqslant \varepsilon, \sigma \ll 1$, $\varrho > 0$
- W_t^x, W_t^y independent standard Wiener processes

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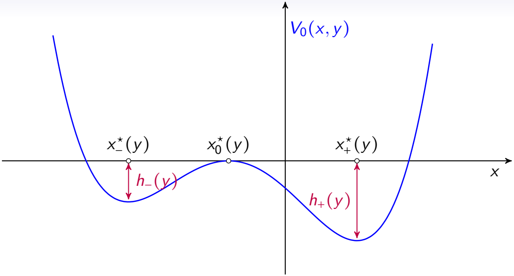

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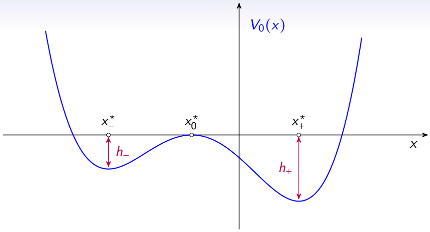
Question: describe law of $\tau_+ = \inf\{t > 0 \colon x_t = x_+^*(y_t) | (x_0 = x_-^*(y_0), y_0)\}$

Static case

$$dx_t = -V_0'(x_t)\,dt + \sigma\,dW_t$$

$$\omega_\pm = \sqrt{V_0''(x_\pm^*)} \quad \omega_0 = \sqrt{-V_0''(x_0^*)}$$



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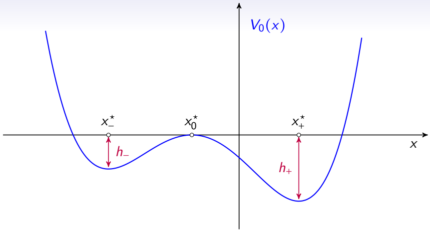

- By Dynkin's equation, $\forall x < x_+^*$,

$$\mathbb{E}^x[\tau_+] = \frac{2}{\sigma^2}\int_x^{x_+^*}\int_{-\infty}^{x_2} \mathrm{e}^{2[V_0(x_2)-V_0(x_1)]/\sigma^2}\,\mathrm{d}x_1\,\mathrm{d}x_2$$

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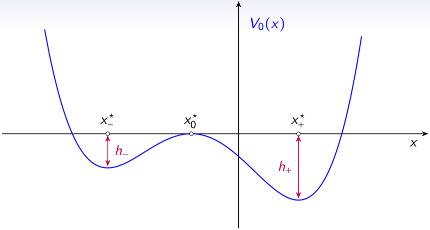

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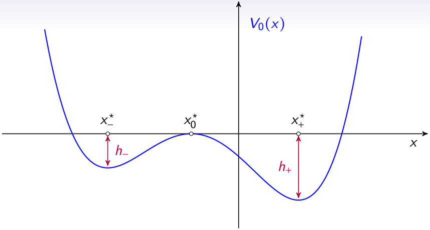

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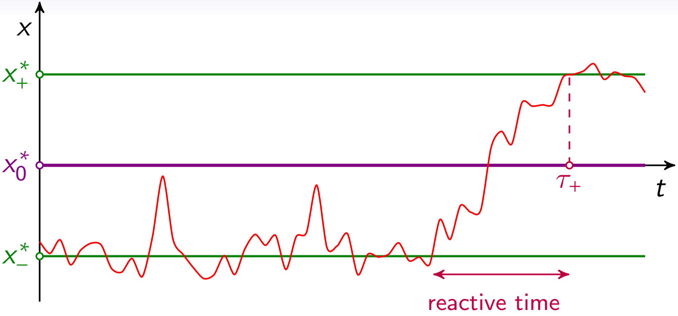
- [Day '83]: $\forall s \geqslant 0$, $\displaystyle\lim_{\sigma\to 0} \mathbb{P}^{x_-^*}\Big\{\tau_+ > s\,\mathbb{E}^{x_-^*}[\tau_+]\Big\} = \mathrm{e}^{-s}$

(Convergence to exponential law $\mathscr{E}(1)$)

Static case: reactive time



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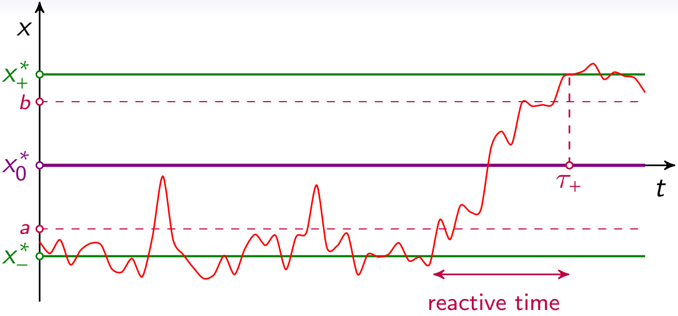

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$$\lim_{\sigma\to 0} \mathrm{Law}\Big(\omega_0^2\tau_b - 2\log(\sigma^{-1}) \,\Big|\, \tau_b < \tau_a\Big) = \mathrm{Law}\Big(\underbrace{\mathcal{G}}_{\text{Gumbel}} + \underbrace{T(x_0,b)}_{\text{deterministic}}\Big)$$

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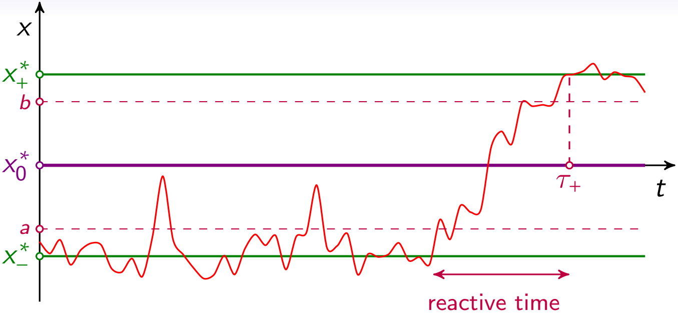

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Gumbel law: $\mathbb{P}\{\mathcal{G} < t\} = \mathrm{e}^{-\mathrm{e}^{-t}} \quad \forall t \in \mathbb{R}$

(max-stable distribution from extreme value theory, cf. [Bakhtin '15])

$\Rightarrow$ reactive time $\simeq \omega_0^{-2}[2\log(\sigma^{-1}) + \mathcal{G} + T(x_0,b)]$

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  - [Bouchet & Reygner 2016]: Formal computations → Eyring–Kramers law in bistable situations
  - [Landim, Mariani & Seo 2019]: Non-reversible potential theory
    Confirms result by [B & R 2016] for some systems with known π
  - [Le Peutrec & Michel 2019]: Semiclassical analysis for systems with known π

Back to the problem

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Theorem: [B & Gentz, SIAM J Math Analysis 2014]

$$\lim_{\sigma\to 0} \mathrm{Law}\Big(\theta(y_{\tau_0}) - \log(\sigma^{-1}) - \frac{\lambda_+}{\varepsilon} Y^\sigma\Big) = \mathrm{Law}\Big(\frac{\mathcal{G}}{2} - \frac{\log 2}{2}\Big)$$

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- $\theta(y)$: explicit parametrisation of $\bar{x}_0(y)$, $\theta(y+1) = \theta(y) + \frac{\lambda_+}{\varepsilon}$
- λ_+: Lyapunov exponent of $\bar{x}_0(y)$ ($\lambda_+ = \int_0^1 \omega_0(y)^2\,\mathrm{d}y + \mathcal{O}(\varepsilon)$)
- $Y^\sigma \in \mathbb{N}$: asymptotically geometric $\mathbb{N}$-valued r.v:

$$\lim_{n\to\infty} \mathbb{P}\{Y^\sigma = n+1 | Y^\sigma > n\} = p(\sigma)$$

$p(\sigma) \simeq \mathrm{e}^{-\mathcal{I}/\sigma^2}$, $\mathcal{I}$ Freidlin–Wentzell quasipotential, $\mathbb{E}[\tau_0] \simeq p(\sigma)^{-1}$

Sketch of proof

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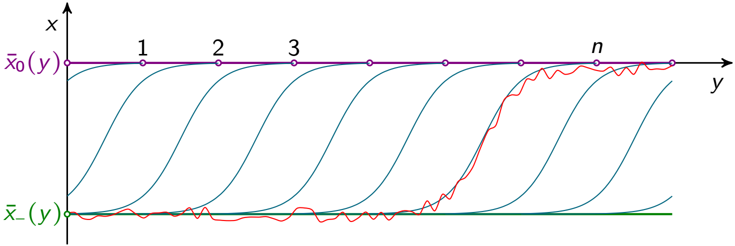

Instantons: minimize Freidlin–Wentzell large-deviation rate function

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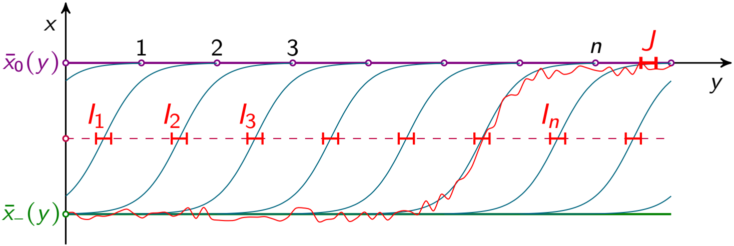

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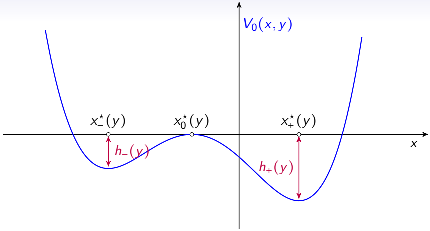
$$\mathbb{P}\{y_{\tau_0} \in J\} \simeq \sum_k \underbrace{\mathbb{P}\{y_{\tau_-} \in I_k\}}_{\simeq \mathbb{P}\{Y^\sigma = k\}} \; \underbrace{\mathbb{P}^{I_k}\{y_{\tau_0} \in J\}}_{\simeq \mathbb{P}\{\frac{\mathcal{G}}{2} + const \,\in J - k\}}$$

Eyring–Kramers-type law for $\mathbb{E}[\tau_+]$

$$\omega_\pm(y) = \sqrt{\partial_{xx} V_0(x_\pm^*(y), y)}$$

$$\omega_0(y) = \sqrt{-\partial_{xx}(x_0^*(y), y)}$$

$$r_\pm(y) = \frac{\omega_\pm(y)\omega_0(y)}{2\pi}\,\mathrm{e}^{-2h_\pm(y)/\sigma^2}$$



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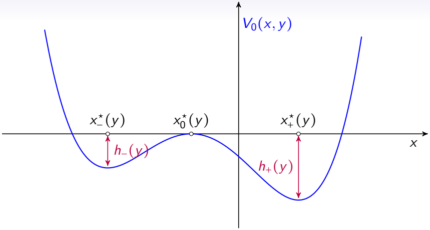

▷ Leading eigenvalue of $-\mathscr{L}_x = -\frac{\sigma^2}{2}\partial_{xx} + \partial_x V_0 \partial_x$:

$$\lambda_1(y) = [r_+(y) + r_-(y)][1 + \mathcal{O}(\sigma^2)]$$

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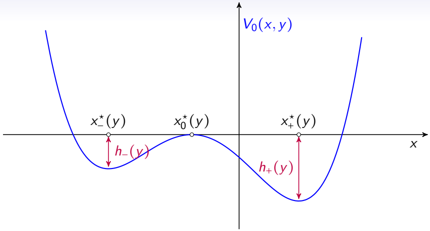

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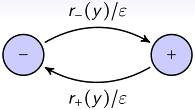
Theorem: [B 2020, `arXiv:2007.08443`]

$$\mathbb{E}^{(x^*_-(y_0), y_0)}[\tau_+] = \frac{2\pi\varepsilon[1 + R(\varepsilon,\sigma)]}{\displaystyle\int_0^1 \omega_0(y)\omega_-(y)\,\mathrm{e}^{-2h_-(y)/\sigma^2}\,\mathrm{d}y}$$

where $R(\varepsilon,\sigma)$ complicated but small if $\langle\lambda_1\rangle \ll \varepsilon \ll \langle\lambda_1\rangle^{1/4}$

Heuristics: two-state jump process



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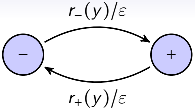

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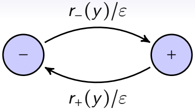

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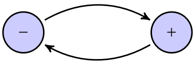

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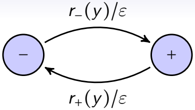

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$$= \frac{1}{1-\mathrm{e}^{-R_-(1,0)/\varepsilon}} \int_0^1 \mathrm{e}^{-R_-(y_0+y,y_0)/\varepsilon}\,\mathrm{d}y \qquad \text{(by periodicity of } r_-\text{)}$$

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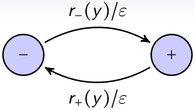

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&= \frac{1}{1-\mathrm{e}^{-R_-(1,0)/\varepsilon}}\int_0^1 \mathrm{e}^{-R_-(y_0+y,y_0)/\varepsilon}\,\mathrm{d}y \qquad \text{(by periodicity of } r_-\text{)} \\
&\simeq \begin{cases}
\dfrac{\varepsilon}{R_-(1,0)} = \dfrac{2\pi\varepsilon}{\int_0^1 \omega_0(y)\omega_-(y)\,\mathrm{e}^{-2h_-(y)/\sigma^2}\,\mathrm{d}y} & \text{if } \varepsilon \gg \max\limits_{y\in[0,1]} r_-(y) \\
\dfrac{\varepsilon}{r_-(y_0)} & \text{if } \varepsilon \ll \min\limits_{y\in[0,1]} r_-(y) \\
\text{In between: Stochastic resonance}
\end{cases}
\end{aligned}$$

Potential theory for non-reversible SDEs

[Landim, Mariani & Seo 2019]:

- Generator $\mathscr{L} = \frac{1}{\varepsilon}\mathscr{L}_x + \mathscr{L}_y, \quad \mathscr{L}_y = \frac{\varrho^2\sigma^2}{2}\partial_{yy} + \partial_y$

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- Decompose $\mathscr{L} = \mathscr{L}_\mathrm{s} + \mathscr{L}_\mathrm{a}$ where
  - $\mathscr{L}_\mathrm{s} = \frac{\sigma^2}{2\varepsilon}\mathrm{e}^{2V/\sigma^2}\,\nabla\cdot D\,\mathrm{e}^{-2V/\sigma^2}\,\nabla$, where $D = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon\varrho^2 \end{pmatrix}$, is self-adjoint wrt π
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Theorem: [LMS 2019] For any $A, B \subset \mathbb{R}^2$, $A \cap B = \varnothing$

$$\int_{\partial A} \mathbb{E}^{(x,y)}\big[\tau_B\big]\,d\nu_{AB} = \frac{1}{\operatorname{cap}(A,B)} \int_{B^c} h^*_{AB}(x,y)\,d\pi$$

- $d\nu_{AB}$ probability measure on ∂A
- $\operatorname{cap}(A,B)$: capacity, satisfies variational principles
- $h^*_{AB}(x,y) = \mathbb{P}^{*,(x,y)}\{\tau_A < \tau_B\}$ committor for adjoint dynamics

Estimating the capacity

- ▷ For $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$, define $\mathscr{D}(\varphi) = \frac{2\varepsilon}{\sigma^2} \int_{(A\cup B)^c} \varphi \cdot (D^{-1}\varphi) \frac{\mathrm{d}x\,\mathrm{d}y}{\pi(x,y)}$

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Proposition: [LMS 2019] Dirichlet principle

$$\operatorname{cap}(A,B) = \inf_{f\in\mathscr{H}_{AB}^{1,0}} \inf_{\varphi\in\mathscr{F}_{AB}^{0}} \mathscr{D}(\Phi_f - \varphi)$$

Infimum reached for $f = \frac{1}{2}(h_{AB} + h_{AB}^*)$ and $\varphi = \Phi_f - \Psi_{h_{AB}}$

Proposition: [LMS 2019] Thomson principle

$$\operatorname{cap}(A,B) = \sup_{f\in\mathscr{H}_{AB}^{0,0}} \sup_{\varphi\in\mathscr{F}_{AB}^{1}} \frac{1}{\mathscr{D}(\Phi_f - \varphi)}$$

Supremum reached for $f = \frac{1}{2\operatorname{cap}(A,B)}(h_{AB} - h_{AB}^*)$ and $\varphi = \Phi_f - \frac{1}{\operatorname{cap}(A,B)}\Psi_{h_{AB}}$

Main difficulty: estimating $\pi(x, y)$

Static eigenfunctions: $\mathscr{L}_x \phi_n(x|y) = -\lambda_n(y)\phi_n(x|y)$, $\phi_0(x|y) = 1$

Proposition:

$$\pi(x, y) = \frac{\mathrm{e}^{-2V_0(x,y)/\sigma^2}}{Z_0(y)}\Big[1 + \alpha_1(y)\phi_1(x|y) + \Phi_\perp(x, y)\Big]$$

- $\alpha_1(y)$ well-approximated in terms of jump process
- $\Phi_\perp(x, y) \perp \operatorname{span}\{\phi_0, \phi_1\}$, satisfies $\langle \pi_0, \Phi_\perp \rangle^{1/2} \lesssim \frac{\varepsilon}{\sigma^2} \cosh\Big(\frac{h_+(y) - h_-(y)}{\sigma^2}\Big)$

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Case $\varrho = 0$: $\pi = \frac{e^{-2V_0/\sigma^2}}{Z_0}[1 + \sum_{n\geqslant 1}\alpha_n\phi_n]$. $\quad \mathscr{L}^\dagger \pi = 0 \Leftrightarrow \alpha_n$ satisfy ODE

$$\varepsilon\alpha_n' = -\lambda_n(y)\alpha_n - \frac{\varepsilon}{\sigma^2}f_{n0}(y) - \frac{\varepsilon}{\sigma^2}\sum_{m\geqslant 1} f_{nm}(y)\alpha_m$$

with $f_{nm}(y) = -\sigma^2\langle \pi_0\phi_m, \partial_y\phi_n\rangle$

- if $\varepsilon \gg \langle\lambda_1\rangle$, then (an affine function of) α_1 is slow variable
- if $\varepsilon \ll 1$, then all α_n for $n \geqslant 2$ are fast variables

Open questions

- ▷ Larger values of ε?
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References

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Thanks for your attention!

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