Equadiff 2015 — MS21, Stochastic Dynamics

On FitzHugh–Nagumo SDEs and SPDEs

Nils Berglund

MAPMO, Université d'Orléans

Lyon, July 8, 2015

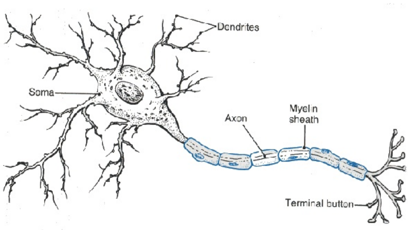
With Christian Kuehn (Vienna) and Damien Landon (Le Mans)

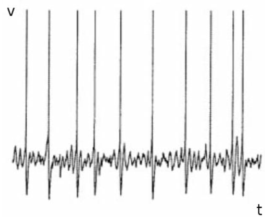
Plan

- FitzHugh–Nagumo SDE
  Dynamics of the membrane potential of a single neuron
  Results on Poissonian vs. non-Poissonian spike statistics
  [N. B. & Damien Landon, Nonlinearity **25**, 2303–2335 (2012)]

- FitzHugh–Nagumo SPDE
  Dynamics of a large ensemble of neurons
  Local existence result for renormalised equation via Martin Hairer's regularity structures
  [N. B. & Christian Kuehn, preprint `arXiv/1504.02953` (2015)]

Neurons and action potentials

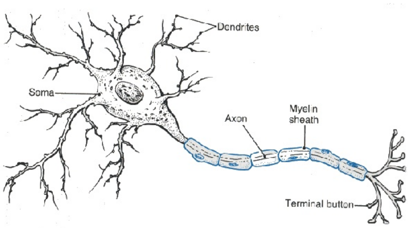


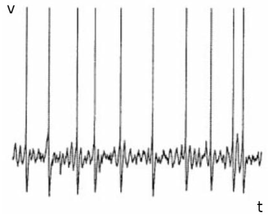


Action potential [Dickson 00]

- Neurons communicate via patterns of spikes in action potentials

Neurons and action potentials





Action potential [Dickson 00]

- Neurons communicate via patterns of spikes in action potentials
- Question: effect of noise on interspike interval statistics?
- Poisson hypothesis: Exponential distribution
  $\Rightarrow$ Markov property

Conduction-based models for action potential

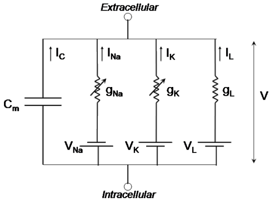
- Hodgkin–Huxley model (1952)

$$C\frac{\mathrm{d}V}{\mathrm{d}t} = -g_{\mathrm{K}} n^4 (V - V_{\mathrm{K}}) - g_{\mathrm{Na}} m^3 h (V - V_{\mathrm{Na}}) - g_{\mathrm{L}}(V - V_{\mathrm{L}}) + I$$

$$\frac{\mathrm{d}n}{\mathrm{d}t} = \alpha_n(V)(1-n) - \beta_n(V)n$$

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \alpha_m(V)(1-m) - \beta_m(V)m$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \alpha_h(V)(1-h) - \beta_h(V)h$$



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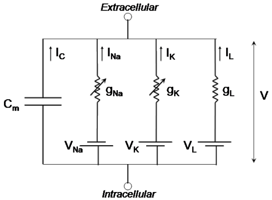
\end{aligned}
$$



- FitzHugh–Nagumo model (1962)

$$
\begin{aligned}
\frac{C}{g}\frac{\mathrm{d}V}{\mathrm{d}t} &= V - V^3 + w \\
\tau\frac{\mathrm{d}w}{\mathrm{d}t} &= \alpha - \beta V - \gamma w
\end{aligned}
$$

- Morris–Lecar model (1982) $2d$, more realistic eq for $\frac{\mathrm{d}V}{\mathrm{d}t}$
- Koper model (1995) $3d$, generalizes FitzHugh–Nagumo

Deterministic FitzHugh–Nagumo (FHN) model

$$\varepsilon\dot{x} = x - x^3 + y$$
$$\dot{y} = a - x - by$$

$b = 0$: fixed pt $P = (a, a^3 - a)$

bifurcation parameter $\delta = \frac{3a^2-1}{2}$

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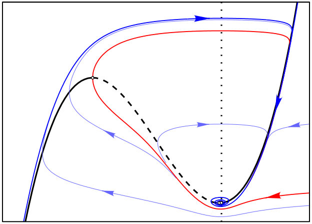
$\delta > 0$:

- P is asymptotically stable
- the system is excitable
- one can define a separatrix



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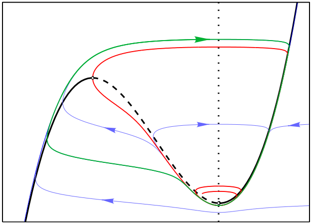
$\exists$ asympt. stable periodic orbit

sensitive dependence on δ:

canard (duck) phenomenon

[Callot, Diener, Diener '78, Benoît '81, ...]



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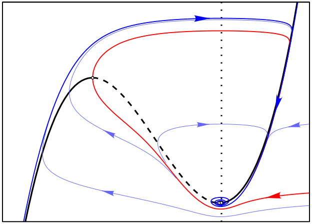

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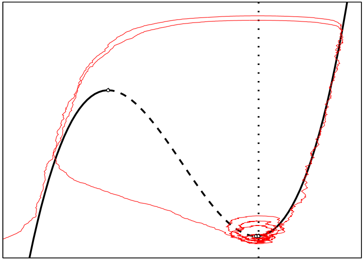
Stochastic FHN equation

$$\mathrm{d}x_t = \frac{1}{\varepsilon}[x_t - x_t^3 + y_t]\,\mathrm{d}t + \frac{\sigma_1}{\sqrt{\varepsilon}}\,\mathrm{d}W_t^{(1)}$$

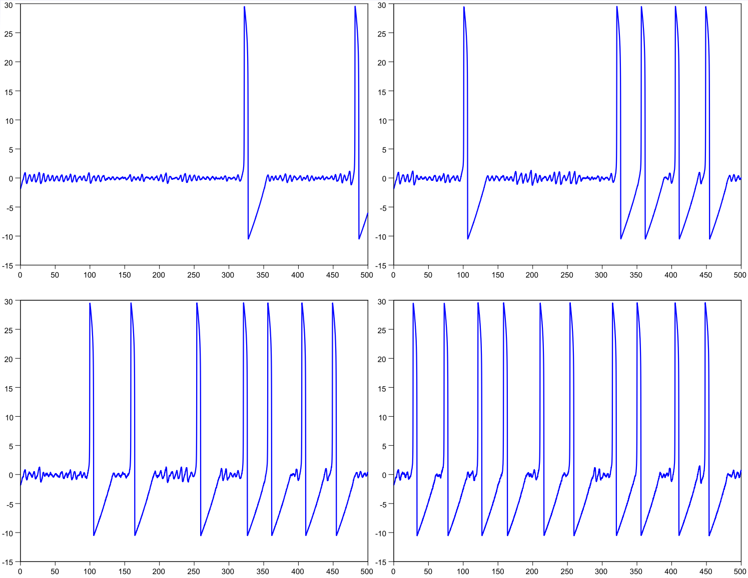
$$\mathrm{d}y_t = [a - x_t - by_t]\,\mathrm{d}t + \sigma_2\,\mathrm{d}W_t^{(2)}$$

- Again $b = 0$ for simplicity in this talk
- $W_t^{(1)}, W_t^{(2)}$: independent Wiener processes (white noise)
- $0 < \sigma_1, \sigma_2 \ll 1$, $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$

$\varepsilon = 0.1$
$\delta = \frac{3a^2-1}{2} = 0.02$
$\sigma_1 = \sigma_2 = 0.03$

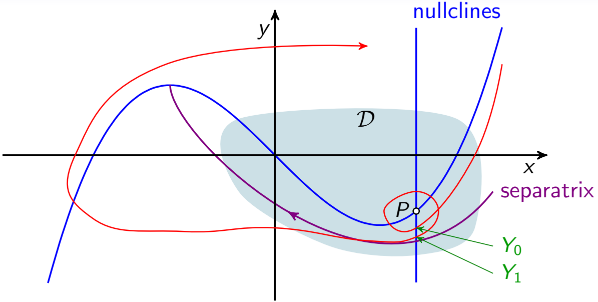


Mixed-mode oscillations (MMOs)



Time series $t \mapsto -x_t$ for $\varepsilon = 0.01,\ \delta = 3 \cdot 10^{-3},\ \sigma = 1.46 \cdot 10^{-4}, \dots, 3.65 \cdot 10^{-4}$

Random Poincaré map



$Y_0, Y_1, \dots$ substochastic Markov chain describing process killed on $\partial\mathcal{D}$
Number of small oscillations N = survival time of Markov chain

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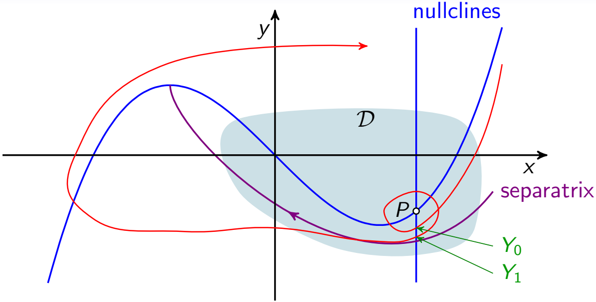

$Y_0, Y_1, \dots$ substochastic Markov chain describing process killed on $\partial\mathcal{D}$
Number of small oscillations N = survival time of Markov chain

Theorem 1 [B & Landon, Nonlinearity 2012]

N is asymptotically geometric: $\lim_{n\to\infty} \mathbb{P}\{N = n+1 | N > n\} = 1 - \lambda_0$
where $\lambda_0 \in \mathbb{R}_+$: principal eigenvalue of the chain, $\lambda_0 < 1$ if $\sigma > 0$ ▸▸ Proof

Transition from weak to strong noise

Theorem 2 [B & Landon, Nonlinearity 2012]

For ε and $\delta/\sqrt{\varepsilon}$ suff. small, $\exists\ \kappa > 0$ s.t. for $\sigma^2 \leqslant (\varepsilon^{1/4}\delta)^2/\log(\sqrt{\varepsilon}/\delta)$

- Principal eigenvalue: $1 - \lambda_0 \leqslant \exp\Bigl\{-\kappa\frac{(\varepsilon^{1/4}\delta)^2}{\sigma^2}\Bigr\}$
- Expected number of small osc.: $\mathbb{E}^{\mu_0}[N] \geqslant C(\mu_0)\exp\Bigl\{\kappa\frac{(\varepsilon^{1/4}\delta)^2}{\sigma^2}\Bigr\}$

Proof: based on construction of set A s.t. $\sup_{y\in A}\mathbb{P}^y\{Y_1 \notin A\}$ exp. small

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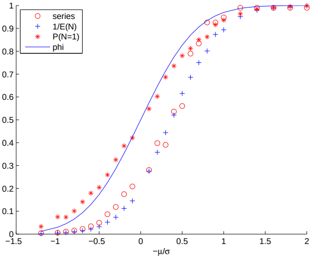
Linearisation around separatrix $\Rightarrow$

$$\mathbb{P}\{N=1\} \simeq \Phi\biggl(-\frac{(\pi\varepsilon)^{1/4}(\delta-\sigma_1^2/\varepsilon)}{\sigma}\biggr)$$

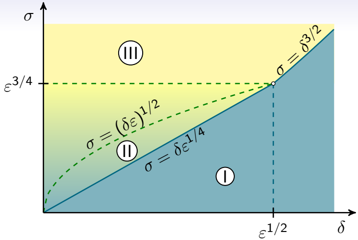
where $\Phi(x) = \int_{-\infty}^x \frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy$

$*$: $\mathbb{P}\{N=1\}$ $\circ$: $1-\lambda_0$

curve: $x \mapsto \Phi(\pi^{1/4}x)$ $+$: $1/\mathbb{E}[N]$



Summary: Parameter regimes

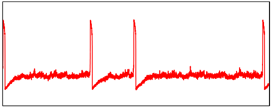


$\sigma_1 = \sigma_2$:
$\mathbb{P}\{N = 1\} \simeq \Phi\Big(-\frac{(\pi\varepsilon)^{1/4}(\delta - \sigma^2/\varepsilon)}{\sigma}\Big)$

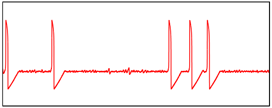
see also

[Muratov & Vanden Eijnden '08]

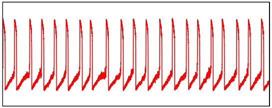
Regime I: rare isolated spikes
Theorem 2 applies ($\delta \ll \varepsilon^{1/2}$)
Interspike interval $\simeq$ exponential



Regime II: clusters of spikes
\# interspike osc asympt geometric
$\sigma = (\delta\varepsilon)^{1/2}$: geom(1/2)



Regime III: repeated spikes
$\mathbb{P}\{N = 1\} \simeq 1$
Interspike interval $\simeq$ constant



FitzHugh–Nagumo SPDE

$$\partial_t u = \Delta u + u - u^3 + v + \xi$$
$$\partial_t v = a_1 u + a_2 v$$

- $u = u(t,x) \in \mathbb{R}$, $v = v(t,x) \in \mathbb{R}^n$, $(t,x) \in D = \mathbb{R}_+ \times \mathbb{T}^d$, $d = 2,3$
- $\xi(t,x)$ Gaussian space-time white noise: $\mathbb{E}\big[\xi(t,x)\xi(s,y)\big] = \delta(t-s)\delta(x-y)$

  ξ: distribution defined by $\langle \xi, \varphi \rangle = W_\varphi$, $\{W_h\}_{h \in L^2(D)}$, $\mathbb{E}[W_h W_{h'}] = \langle h, h' \rangle$

(Link to simulation)

Main result

Mollified noise: $\xi^\varepsilon = \varrho_\varepsilon * \xi$
where $\varrho_\varepsilon(t,x) = \frac{1}{\varepsilon^{d+2}}\varrho\bigl(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\bigr)$ with ρ compactly supported, integral 1

Theorem [B & Kuehn, preprint 2015, arXiv/1504.02953]

There exists a choice of renormalisation constant $C(\varepsilon)$, $\lim_{\varepsilon\to 0} C(\varepsilon) = \infty$, such that

$$\partial_t u^\varepsilon = \Delta u^\varepsilon + [1 + C(\varepsilon)]u^\varepsilon - (u^\varepsilon)^3 + v^\varepsilon + \xi^\varepsilon$$
$$\partial_t v^\varepsilon = a_1 u^\varepsilon + a_2 v^\varepsilon$$

admits a sequence of local solutions $(u^\varepsilon, v^\varepsilon)$, converging in probability to a limit (u, v) as $\varepsilon \to 0$.

- Local solution means up to a random possible explosion time
- Initial conditions should be in appropriate Hölder spaces
- $C(\varepsilon) \asymp \log(\varepsilon^{-1})$ for $d = 2$ and $C(\varepsilon) \asymp \varepsilon^{-1}$ for $d = 3$
- Similar results for general cubic nonlinearity and $v \in \mathbb{R}^n$

Mild solutions of SPDE

$$\partial_t u = \Delta u + F(u) + \xi$$

Construction of mild solution via Duhamel formula:

- $\partial_t u = \Delta u \quad \Rightarrow \quad u(t,x) = \int G(t,x-y)u_0(y)\,\mathrm{d}y =: (\mathrm{e}^{\Delta t}\,u_0)(x)$

  where $G(t,x)$: heat kernel (compatible with bc)

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- $\partial_t u = \Delta u + f \quad \Rightarrow \quad u(t,x) = (\mathrm{e}^{\Delta t} u_0)(x) + \int_0^t \mathrm{e}^{\Delta(t-s)} f(s,\cdot)(x)\, \mathrm{d}s$

  Notation: $u = Gu_0 + G * f$

- $\partial_t u = \Delta u + \xi \quad \Rightarrow \quad u = Gu_0 + G * \xi \quad$ (stochastic convolution)

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- $\partial_t u = \Delta u + \xi + F(u) \quad \Rightarrow \quad u = Gu_0 + G * [\xi + F(u)]$

  Aim: use Banach's fixed-point theorem — but which function space?

Schauder estimates and fixed-point equation

$\mathcal{C}^{\alpha}_{\mathfrak{s}}$: Hölder space for parabolic scaling $\|(t,x)\|_{\mathfrak{s}} = |t|^{1/2} + \sum_{i=1}^{d} |x_i|$

If $\alpha < 0$, $f \in \mathcal{C}^{\alpha}_{\mathfrak{s}} \quad \Leftrightarrow \quad |\langle f, \eta^{\delta}_{t,x} \rangle| \leqslant C\delta^{\alpha}$ where $\eta^{\delta}_{t,x}(s,y) = \frac{1}{\delta^{d+2}}\eta(\frac{s-t}{\delta^2}, \frac{y-x}{\delta})$

Schauder estimate

$$f \in \mathcal{C}^{\alpha}_{\mathfrak{s}} \quad \Rightarrow \quad G * f \in \mathcal{C}^{\alpha+2}_{\mathfrak{s}}$$

Fact: in dimension d, space-time white noise $\xi \in \mathcal{C}^{\alpha}_{\mathfrak{s}}$ a.s. $\forall \alpha < -\frac{d+2}{2}$

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Fixed-point equation: $u = Gu_0 + G * [\xi + F(u)]$

- $d = 1$: $\xi \in \mathcal{C}^{-3/2^-}_{\mathfrak{s}} \Rightarrow G * \xi \in \mathcal{C}^{1/2^-}_{\mathfrak{s}} \Rightarrow F(u)$ defined
- $d = 3$: $\xi \in \mathcal{C}^{-5/2^-}_{\mathfrak{s}} \Rightarrow G * \xi \in \mathcal{C}^{-1/2^-}_{\mathfrak{s}} \Rightarrow F(u)$ not defined
- $d = 2$: $\xi \in \mathcal{C}^{-2^-}_{\mathfrak{s}} \Rightarrow G * \xi \in \mathcal{C}^{0^-}_{\mathfrak{s}} \Rightarrow F(u)$ not defined

  Boundary case, can be treated with Besov spaces [Da Prato and Debussche 2003]

Regularity structures

Basic idea of Martin Hairer [Inventiones Mathematicae, 2014]:

Lift mollified fixed-point equation

$$u = Gu_0 + G * [\xi^\varepsilon + F(u)]$$

to a larger space called a Regularity structure

$$\begin{array}{ccc} (u_0, Z^\varepsilon) & \xmapsto{\;\mathcal{S}\;} & U \\ \Psi \uparrow & & \downarrow \mathcal{R} \\ (u_0, \xi^\varepsilon) & \xmapsto[\;\bar{\mathcal{S}}\;]{} & u^\varepsilon \end{array}$$

- $u^\varepsilon = \bar{\mathcal{S}}(u_0, \xi^\varepsilon)$: classical solution of mollified equation
- $U = \mathcal{S}(u_0, Z^\varepsilon)$: solution map in regularity structure
- $\mathcal{S}$ and $\mathcal{R}$ are continuous (in suitable topology)

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(u_0, MZ^\varepsilon) & \xrightarrow{\mathcal{S}_M} & U_M \\
\uparrow{\scriptstyle M\Psi} & & \downarrow{\scriptstyle \mathcal{R}^M} \\
(u_0, \xi^\varepsilon) & \xrightarrow[\bar{\mathcal{S}}_M]{} & \hat{u}^\varepsilon
\end{array}$$

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- $U = \mathcal{S}(u_0, Z^\varepsilon)$: solution map in regularity structure
- $\mathcal{S}$ and $\mathcal{R}$ are continuous (in suitable topology)
- Renormalisation: modification of the lift Ψ

Regularity structure for $\partial_t u = \Delta u - u^3 + \xi$

New symbols: Ξ, representing ξ, Hölder exponent $|\Xi|_{\mathfrak{s}} = \alpha_0 = -\frac{d+2}{2} - \kappa$
$\mathcal{I}(\tau)$, representing $G * f$, Hölder exponent $|\mathcal{I}(\tau)|_{\mathfrak{s}} = |\tau|_{\mathfrak{s}} + 2$
$\tau\sigma$, Hölder exponent $|\tau\sigma|_{\mathfrak{s}} = |\tau|_{\mathfrak{s}} + |\sigma|_{\mathfrak{s}}$

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$\tau\sigma$, Hölder exponent $|\tau\sigma|_{\mathfrak{s}} = |\tau|_{\mathfrak{s}} + |\sigma|_{\mathfrak{s}}$

τ	Symbol	$\lvert\tau\rvert_{\mathfrak{s}}$	$d = 3$	$d = 2$
Ξ	Ξ	α_0	$-\frac{5}{2} - \kappa$	$-2 - \kappa$
$\mathcal{I}(\Xi)^3$		$3\alpha_0 + 6$	$-\frac{3}{2} - 3\kappa$	$0 - 3\kappa$
$\mathcal{I}(\Xi)^2$		$2\alpha_0 + 4$	$-1 - 2\kappa$	$0 - 2\kappa$
$\mathcal{I}(\mathcal{I}(\Xi)^3)\mathcal{I}(\Xi)^2$		$5\alpha_0 + 12$	$-\frac{1}{2} - 5\kappa$	$2 - 5\kappa$
$\mathcal{I}(\Xi)$		$\alpha_0 + 2$	$-\frac{1}{2} - \kappa$	$0 - \kappa$
$\mathcal{I}(\mathcal{I}(\Xi)^3)\mathcal{I}(\Xi)$		$4\alpha_0 + 10$	$0 - 4\kappa$	$2 - 4\kappa$
$\mathcal{I}(\mathcal{I}(\Xi)^2)\mathcal{I}(\Xi)^2$		$4\alpha_0 + 10$	$0 - 4\kappa$	$2 - 4\kappa$
$\mathcal{I}(\Xi)^2 X_i$	 X_i	$2\alpha_0 + 5$	$0 - 2\kappa$	$1 - 2\kappa$
$\mathbf{1}$	$\mathbf{1}$	0	0	0
$\mathcal{I}(\mathcal{I}(\Xi)^3)$		$3\alpha_0 + 8$	$\frac{1}{2} - 3\kappa$	$2 - 3\kappa$
...	...	...	...	...

The case of the FitzHugh–Nagumo equations

Fixed-point equation

$$u(t,x) = G * [\xi^\varepsilon + u - u^3 + v](t,x) + Gu_0(t,x)$$
$$v(t,x) = \int_0^t u(s,x)\, \mathrm{e}^{(t-s)a_2}\, a_1 \,\mathrm{d}s + \mathrm{e}^{ta_2}\, v_0$$

Lifted version

$$U = \mathcal{I}[\Xi + U - U^3 + V] + Gu_0$$
$$V = \mathcal{E}\, U + Qv_0$$

where $\mathcal{E}$ is an integration map which is not regularising in space
New symbols $\mathcal{E}(\mathcal{I}(\Xi)) =$ , etc. . .

We expect U, and thus also V to be α-Hölder for $\alpha < -\frac{1}{2}$
Thus $\mathcal{I}(U - U^3 + V)$ should be well-defined

The standard theory has to be extended, because $\mathcal{E}$ does not correspond to a smooth kernel

Why do we need to renormalise?

Model (Π, Γ): $\Pi_z \tau$ distribution describing τ near $z \in \mathbb{R}^{d+1}$, $\Pi_{\bar z} = \Pi_z \Gamma_{z\bar z}$

Let $G_\varepsilon = G * \varrho_\varepsilon$ where ϱ_ε is the mollifier

$$(\Pi_{\bar z} \mathbin{\dot{|}})(z) = (G * \xi^\varepsilon)(z) = (G_\varepsilon * \xi)(z) = \int G_\varepsilon(z - z_1)\xi(z_1)\,\mathrm{d}z_1$$

belongs to first Wiener chaos, limit $\varepsilon \to 0$ well-defined

Why do we need to renormalise?

Model (Π, Γ): $\Pi_z \tau$ distribution describing τ near $z \in \mathbb{R}^{d+1}$, $\Pi_{\bar z} = \Pi_z \Gamma_{z \bar z}$

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$$(\Pi_{\bar z} \overset{\bullet}{|})(z) = (G * \xi^\varepsilon)(z) = (G_\varepsilon * \xi)(z) = \int G_\varepsilon(z - z_1)\xi(z_1)\,\mathrm{d}z_1$$

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$$(\Pi_{\bar z} {}^{\bullet}\!\vee\!{}^{\bullet})(z) = (G * \xi^\varepsilon)(z)^2 = \iint G_\varepsilon(z - z_1) G_\varepsilon(z - z_2)\xi(z_1)\xi(z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2$$

diverges as $\varepsilon \to 0$

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Wick product: $\xi(z_1) \diamond \xi(z_2) = \xi(z_1)\xi(z_2) - \delta(z_1 - z_2)$

$$(\Pi_{\bar z} \overset{\bullet\,\bullet}{\vee})(z) = \underbrace{\iint G_\varepsilon(z - z_1) G_\varepsilon(z - z_2) \xi(z_1) \diamond \xi(z_2) \, \mathrm{d}z_1 \, \mathrm{d}z_2}_{\text{in 2nd Wiener chaos, bdd}} + \underbrace{\int G_\varepsilon(z - z_1)^2 \, \mathrm{d}z_1}_{C_1(\varepsilon) \to \infty}$$

Renormalised model: $(\widehat{\Pi}_{\bar z} \overset{\bullet\,\bullet}{\vee})(z) = (\Pi_{\bar z} \overset{\bullet\,\bullet}{\vee})(z) - C_1(\varepsilon)$

Concluding remarks

- Noise can induce spikes that may have non-Poisson interval statistics
- Important tools: random Poincaré maps and quasistationary distributions
- Local existence result for FitzHugh–Nagumo SPDE
  Global existence: proved for Allen–Cahn in 2D [Mourrat and Weber]
- More quantitative results?

Some references

- N. B. & Damien Landon, *Mixed-mode oscillations and interspike interval statistics in the stochastic FitzHugh–Nagumo model*, Nonlinearity **25**, 2303–2335 (2012)
- N. B., Barbara Gentz & Christian Kuehn, *From random Poincaré maps to stochastic mixed-mode-oscillation patterns*, JDDE **27**, 83–136 (2015)
- N. B. & Christian Kuehn, *Regularity structures and renormalisation of FitzHugh–Nagumo SPDEs in three space dimensions*, preprint `arXiv/1504.02953`
- Martin Hairer, *A theory of regularity structures*, Invent. Math. **198** (2), 269–504 (2014)
- Martin Hairer, *Introduction to Regularity Structures*, lecture notes (2013)

Proof of asymptotically geometric distribution

Theorem 1 [B & Landon, Nonlinearity 2012]

N is asymptotically geometric: $\lim_{n\to\infty} \mathbb{P}\{N = n+1 | N > n\} = 1 - \lambda_0$

where $\lambda_0 \in (0,1)$ if $\sigma > 0$ is principal eigenvalue of the chain

Proof:

Markov chain on E, kernel K with density k [Ben Arous, Kusuoka, Stroock '84]

- $\lambda_0 \leqslant \sup_{x\in E} K(x,E) < 1$ by ellipticity (k bounded below)
- $$\begin{aligned}\mathbb{P}^{\mu_0}\{N > n\} = \mathbb{P}^{\mu_0}\{X_n \in E\} &= \int_E \mu_0(\mathrm{d}x) K^n(x,E) \\ &= \int_E \mu_0(\mathrm{d}x) \lambda_0^n h_0(x) \|h_0^*\|_1 [1 + \mathcal{O}((|\lambda_1|/\lambda_0)^n)] \\ &= \lambda_0^n \langle \mu_0, h_0 \rangle \|h_0^*\|_1 [1 + \mathcal{O}((|\lambda_1|/\lambda_0)^n)]\end{aligned}$$
- $$\begin{aligned}\mathbb{P}^{\mu_0}\{N = n+1\} &= \int_E \int_E \mu_0(\mathrm{d}x) K^n(x,\mathrm{d}y)[1 - K(y,E)] \\ &= \lambda_0^n (1-\lambda_0) \langle \mu_0, h_0 \rangle \|h_0^*\|_1 [1 + \mathcal{O}((|\lambda_1|/\lambda_0)^n)]\end{aligned}$$
- Existence of spectral gap follows from positivity condition [Birkhoff '57]

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