UNICAMP – Seminários de Sistemas Estocásticos e Dinâmicos

Stochastic resonance in stochastic PDEs

Nils Berglund

Institut Denis Poisson, University of Orléans, France

INSTITUT DENIS POISSON

24 September 2021 (video talk)

Based on joint works with Rita Nader (Orléans) and Barbara Gentz (Bielefeld)

project PERISTOCH

Stochastic resonance in an SDE

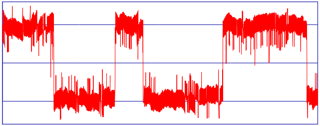
(Link to simulation)

Stochastic resonance in an SDE

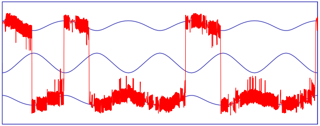
$$dx_t = \underbrace{\left[-x_t^3 + x_t + A\cos(\varepsilon t)\right]}_{=-\frac{\partial}{\partial x}\left[\frac{1}{4}x^4 - \frac{1}{2}x^2 - Ax\cos(\varepsilon t)\right]\Big|_{x_t}} dt + \sigma\, dW_t$$

- ▷ deterministically bistable climate [Croll, Milankovitch]
- ▷ random perturbations due to weather [Benzi-Sutera-Vulpiani, Nicolis-Nicolis]

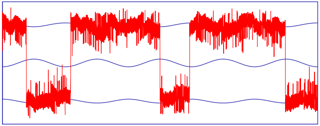
Sample paths $\{x_t\}_t$ for $\varepsilon = 0.001$:



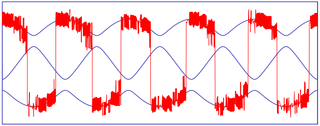
$A = 0, \ \sigma = 0.3$



$A = 0.24, \ \sigma = 0.2$



$A = 0.1, \ \sigma = 0.27$

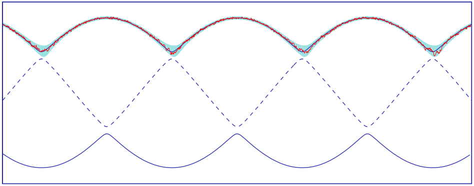


$A = 0.35, \ \sigma = 0.2$

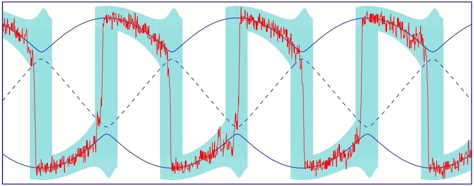
Stochastic resonance in an SDE

Critical noise intensity: $\sigma_{\mathrm{c}} = \max\{\delta, \varepsilon\}^{3/4}$, $\delta = A_{\mathrm{c}} - A$, $A_{\mathrm{c}} = \frac{2}{3\sqrt{3}}$

$\sigma \ll \sigma_{\mathrm{c}}$:
transitions unlikely



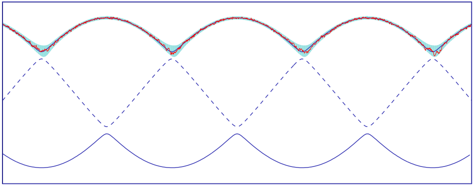
$\sigma \gg \sigma_{\mathrm{c}}$:
synchronisation



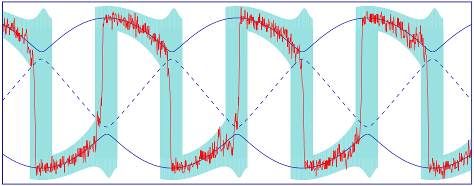
Stochastic resonance in an SDE

Critical noise intensity: $\sigma_c = \max\{\delta, \varepsilon\}^{3/4}$, $\delta = A_c - A$, $A_c = \frac{2}{3\sqrt{3}}$

$\sigma \ll \sigma_c$:
transitions unlikely



$\sigma \gg \sigma_c$:
synchronisation



Theorem [B & Gentz, Annals App. Proba 2002]

- $\sigma < \sigma_c$: transition probability per period $\leqslant e^{-\sigma_c^2/\sigma^2}$
- $\sigma > \sigma_c$: transition probability per period $\geqslant 1 - e^{-c\sigma^{4/3}/(\varepsilon|\log\sigma|)}$

Stochastic resonance in SPDEs

$$\mathrm{d}\phi(t,x) = \left[\Delta\phi(t,x) + f(\varepsilon t, \phi(t,x))\right]\mathrm{d}t + \sigma\,\mathrm{d}W(t,x)$$

- $\phi = \phi(t,x) \in \mathbb{R}$, $\varepsilon t \in [0,T]$ or f is T-periodic, $x \in \mathbb{T} = \mathbb{R}/L\mathbb{Z}$, $L > 0$
- $\phi \mapsto f(s,\phi)$ bistable, e.g. $f(s,\phi) = \phi - \phi^3 + A\cos(s)$
- $\mathrm{d}W(t,x)$ space-time white noise on $\mathbb{R}_+ \times \mathbb{T}$
- $0 < \varepsilon, \sigma \ll 1$
- δ measures closeness to bifurcation (e.g. $A_{\mathrm{c}} - A$)

Stochastic resonance in SPDEs

$$\mathrm{d}\phi(t,x) = \Big[\Delta\phi(t,x) + f(\varepsilon t, \phi(t,x))\Big]\,\mathrm{d}t + \sigma\,\mathrm{d}W(t,x)$$

- $\phi = \phi(t,x) \in \mathbb{R}$, $\varepsilon t \in [0,T]$ or f is T-periodic, $x \in \mathbb{T} = \mathbb{R}/L\mathbb{Z}$, $L > 0$
- $\phi \mapsto f(s,\phi)$ bistable, e.g. $f(s,\phi) = \phi - \phi^3 + A\cos(s)$
- $\mathrm{d}W(t,x)$ space-time white noise on $\mathbb{R}_+ \times \mathbb{T}$
- $0 < \varepsilon, \sigma \ll 1$
- δ measures closeness to bifurcation (e.g. $A_{\mathrm{c}} - A$)

Theorem [B & Nader, arXiv/2107.07292]

- Away from bifurcations, solutions are concentrated around deterministic solutions in Sobolev H^s-norm for any $s < \frac{1}{2}$
- $\sigma < \sigma_{\mathrm{c}} = (\delta \vee \varepsilon)^{3/4}$: transition probability per period $\leqslant \mathrm{e}^{-\sigma_{\mathrm{c}}^2/\sigma^2}$
- $\sigma > \sigma_{\mathrm{c}}$: transition probability per period $\geqslant 1 - \mathrm{e}^{-c\sigma^{4/3}/(\varepsilon|\log\sigma|)}$

Proof ideas, 1D SDE below threshold

On slow time scale $\varepsilon t \to t$:

$$\mathrm{d}x_t = \frac{1}{\varepsilon} f(t, x_t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

$\bar{x}(t)$ deterministic solution tracking stable equilibrium $x^*(t)$.
Write $x_t = \bar{x}(t) + \xi_t$ and Taylor-expand:

$$\mathrm{d}\xi_t = \frac{1}{\varepsilon}\Big[\bar{a}(t)\xi_t + \underbrace{b(t,\xi_t)}_{=\mathcal{O}(\xi_t^2)}\Big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

where $\bar{a}(t) = \partial_x f(t, \bar{x}(t)) = a^\star(t) + \mathcal{O}(\varepsilon) < 0$

Proof ideas, 1D SDE below threshold

On slow time scale $\varepsilon t \to t$:

$$\mathrm{d}x_t = \frac{1}{\varepsilon} f(t, x_t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

$\bar{x}(t)$ deterministic solution tracking stable equilibrium $x^*(t)$.
Write $x_t = \bar{x}(t) + \xi_t$ and Taylor-expand:

$$\mathrm{d}\xi_t = \frac{1}{\varepsilon}\Big[\bar{a}(t)\xi_t + \underbrace{b(t,\xi_t)}_{=\mathcal{O}(\xi_t^2)}\Big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

where $\bar{a}(t) = \partial_x f(t, \bar{x}(t)) = a^\star(t) + \mathcal{O}(\varepsilon) < 0$

Variations of constants (Duhamel formula), if $\xi_0 = 0$:

$$\xi_t = \underbrace{\frac{\sigma}{\sqrt{\varepsilon}} \int_0^t \mathrm{e}^{\bar{\alpha}(t,s)/\varepsilon}\,\mathrm{d}W_s}_{\xi_t^0:\ \text{sol of linearised system}} + \underbrace{\frac{1}{\varepsilon}\int_0^t \mathrm{e}^{\bar{\alpha}(t,s)/\varepsilon}\, b(s,\xi_s)\,\mathrm{d}s}_{\text{treat as a perturbation}}$$

where $\bar{\alpha}(t,s) = \int_s^t \bar{a}(u)\,\mathrm{d}u$

Proof ideas, 1D SDE below threshold

Properties of $\quad \xi_t^0 = \dfrac{\sigma}{\sqrt{\varepsilon}} \displaystyle\int_0^t \mathrm{e}^{\bar\alpha(t,s)/\varepsilon} \,\mathrm{d}W_s$:

- Gaussian process, $\quad \mathbb{E}[\xi_t^0] = 0, \quad \operatorname{Var}(\xi_t^0) = \frac{\sigma^2}{\varepsilon} \int_0^t \mathrm{e}^{2\bar\alpha(t,s)/\varepsilon} \,\mathrm{d}s$
- Confidence interval: $\quad \mathbb{P}\Bigl\{|\xi_t^0| > \frac{h}{\sigma}\sqrt{\operatorname{Var}(\xi_t^0)}\Bigr\} = \mathcal{O}(\mathrm{e}^{-h^2/2\sigma^2})$
- $\sigma^{-2}\operatorname{Var}(\xi_t^0)$ satisfies ODE $\quad \varepsilon\dot v = 2\bar a(t) v + 1$

Proof ideas, 1D SDE below threshold

Properties of $\quad \xi^0_t = \dfrac{\sigma}{\sqrt{\varepsilon}} \displaystyle\int_0^t \mathrm{e}^{\bar\alpha(t,s)/\varepsilon}\,\mathrm{d}W_s$:

- Gaussian process, $\quad \mathbb{E}[\xi^0_t] = 0, \quad \operatorname{Var}(\xi^0_t) = \frac{\sigma^2}{\varepsilon}\int_0^t \mathrm{e}^{2\bar\alpha(t,s)/\varepsilon}\,\mathrm{d}s$
- Confidence interval: $\quad \mathbb{P}\Big\{|\xi^0_t| > \frac{h}{\sigma}\sqrt{\operatorname{Var}(\xi^0_t)}\Big\} = \mathcal{O}(\mathrm{e}^{-h^2/2\sigma^2})$
- $\sigma^{-2}\operatorname{Var}(\xi^0_t)$ satisfies ODE $\quad \varepsilon\dot v = 2\bar a(t)v + 1$

Lemma [B & Gentz, PTRF 2002]

$\bar v(t)$ solution of ODE bounded away from 0: $\quad \bar v(t) = \frac{1}{-2\bar a(t)} + \mathcal{O}(\varepsilon)$

$$\mathbb{P}\bigg\{\sup_{0\leqslant s\leqslant t} \frac{|\xi^0_s|}{\sqrt{\bar v(s)}} > h\bigg\} = C_0(t,\varepsilon)\,\mathrm{e}^{-h^2/2\sigma^2}$$

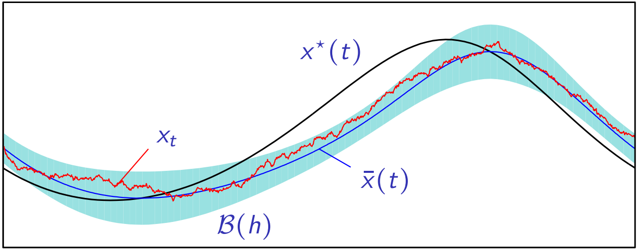
where $C_0(t,\varepsilon) = \sqrt{\frac{2}{\pi}}\,\frac{1}{\varepsilon}\Big|\int_0^t \bar a(s)\,\mathrm{d}s\Big|\frac{h}{\sigma}\Big[1 + \mathcal{O}\big(\varepsilon + \frac{t}{\varepsilon}\,\mathrm{e}^{-h^2/\sigma^2}\big)\Big]$

Proof based on Doob's submartingale inequality and partition of $[0,t]$

Proof ideas, 1D SDE below threshold

Nonlinear equation: $\mathrm{d}\xi_t = \frac{1}{\varepsilon}\big[\bar{a}(t)\xi_t + b(t,\xi_t)\big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$

Confidence strip: $\mathcal{B}(h) = \Big\{|\xi| \leqslant h\sqrt{\bar{v}(t)}\ \forall t\Big\} = \Big\{|x - \bar{x}(t)| \leqslant h\sqrt{\bar{v}(t)}\ \forall t\Big\}$



Theorem B & Gentz, PTRF 2002

$$C(t,\varepsilon)\,\mathrm{e}^{-\kappa_- h^2/2\sigma^2} \leqslant \mathbb{P}\Big\{\text{leaving } \mathcal{B}(h) \text{ before time } t\Big\} \leqslant C(t,\varepsilon)\,\mathrm{e}^{-\kappa_+ h^2/2\sigma^2}$$

where $\kappa_\pm = 1 \mp \mathcal{O}(h)$ and $C(t,\varepsilon) = C_0(t,\varepsilon)\big[1 + \mathcal{O}(h)\big]$ (requires $h \leqslant h_0$)

Notes

Avoided transcritical bifurcation

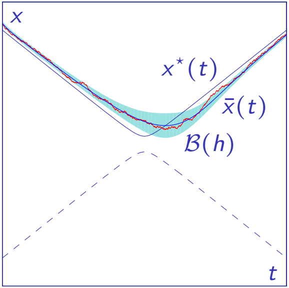
$$\mathrm{d}x_t = \frac{1}{\varepsilon}\Big[t^2 + \delta - x_t^2 + \dots\Big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

Equil. curve: $x^\star(t) \simeq \sqrt{t^2+\delta}$

Slow sol.: $\bar{x}(t) = x^\star(t) + \mathcal{O}(\min\{\frac{\varepsilon}{|t|}, \frac{\varepsilon}{\sqrt{\delta+\varepsilon}}\})$

$$\bar{a}(t) = \partial_x f(t, \bar{x}(t)) \asymp \begin{cases} -|t| & |t| \geqslant \sqrt{\delta+\varepsilon} \\ -\sqrt{\delta+\varepsilon} & |t| \leqslant \sqrt{\delta+\varepsilon} \end{cases}$$

Confidence strip $\mathcal{B}(h)$: width $\asymp h/\sqrt{|\bar{a}(t)|}$

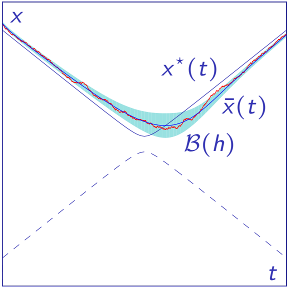


Avoided transcritical bifurcation

$$dx_t = \frac{1}{\varepsilon}\Big[t^2 + \delta - x_t^2 + \dots\Big]\,dt + \frac{\sigma}{\sqrt{\varepsilon}}\,dW_t$$

Equil. curve: $x^\star(t) \simeq \sqrt{t^2+\delta}$

Slow sol.: $\bar{x}(t) = x^\star(t) + \mathcal{O}(\min\{\frac{\varepsilon}{|t|}, \frac{\varepsilon}{\sqrt{\delta+\varepsilon}}\})$



$$\bar{a}(t) = \partial_x f(t, \bar{x}(t)) \asymp \begin{cases} -|t| & |t| \geqslant \sqrt{\delta+\varepsilon} \\ -\sqrt{\delta+\varepsilon} & |t| \leqslant \sqrt{\delta+\varepsilon} \end{cases}$$

Confidence strip $\mathcal{B}(h)$: width $\asymp h/\sqrt{|\bar{a}(t)|}$

Theorem [B & Gentz, AAP 2002]

$$\mathbb{P}\Big\{\text{leaving } \mathcal{B}(h) \text{ before time } t\Big\} \leqslant C(t,\varepsilon)\,e^{-\kappa h^2/2\sigma^2}$$

where $\kappa = 1 - \mathcal{O}(\sup_{s\leqslant t} h|\bar{a}(s)|^{-3/2}) - \mathcal{O}(\varepsilon)$ requires $h < h_0 \inf_{s\leqslant t}|\bar{a}(s)|^{3/2}$

- $\sigma < \sigma_c = \max\{\delta,\varepsilon\}^{3/4}$: result applies $\forall\, t$, $\mathbb{P}\{\text{trans}\} = \mathcal{O}(e^{-\kappa\sigma_c^2/\sigma^2})$
- $\sigma > \sigma_c = \max\{\delta,\varepsilon\}^{3/4}$: result applies up to $t \asymp -\sigma^{2/3}$

Above threshold

What happens for $\sigma > \sigma_c$ and $t > -\sigma^{2/3}$?

General principle: partition $t_0 = s_0 < s_1 < s_2 < \dots < s_n = t$ of $[t_0, t]$

Lemma Let $P_k = \mathbb{P}\{\text{making no transition during } (s_{k-1}, s_k]\}$. Then

$$\mathbb{P}\{\text{making no transition during } [t_0, t]\} \leqslant \prod_{k=1}^{n} P_k$$

Choose partition s.t. each $P_k \leqslant q < 1 \Rightarrow \mathbb{P}\{\text{no transition}\} \leqslant e^{-n \log q}$

Above threshold

What happens for $\sigma > \sigma_c$ and $t > -\sigma^{2/3}$?

General principle: partition $t_0 = s_0 < s_1 < s_2 < \dots < s_n = t$ of $[t_0, t]$

Lemma Let $P_k = \mathbb{P}\{\text{making no transition during } (s_{k-1}, s_k]\}$. Then

$$\mathbb{P}\{\text{making no transition during } [t_0, t]\} \leqslant \prod_{k=1}^{n} P_k$$

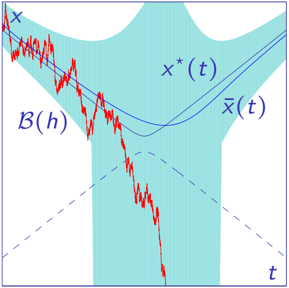
Choose partition s.t. each $P_k \leqslant q < 1 \Rightarrow \mathbb{P}\{\text{no transition}\} \leqslant \mathrm{e}^{-n \log q}$

Define partition such that

$$\int_{s_{k-1}}^{s_k} |\bar{a}(s)| \, \mathrm{d}s = c\varepsilon |\log \sigma| \quad \Rightarrow \quad P_k \leqslant \frac{2}{3}$$

Thm [B & Gentz, AAP 2002]

Transition probability $\geqslant 1 - \mathrm{e}^{-\kappa\sigma^{4/3}/(\varepsilon|\log\sigma|)}$



Notes

SPDE: stable case

$$d\phi(t,x) = \frac{1}{\varepsilon}\big[\Delta\phi(t,x) + f(t,\phi(t,x))\big]\,dt + \frac{\sigma}{\sqrt{\varepsilon}}\,dW(t,x)$$

- $f(t,\phi^*(t)) = 0$ for all $t \in I = [0,T]$
- $a(t) = \partial_\phi f(t,\phi^*(t)) \leqslant -a_- < 0$ for all $t \in I$

In deterministic case $\sigma = 0$: $\exists$ particular solution $\bar\phi(t,x)$ such that

$$\big\|\bar\phi(t,\cdot) - \phi^*(t)e_0\big\|_{H^1} \leqslant C\varepsilon \qquad \forall t \in I$$

SPDE: stable case

$$\mathrm{d}\phi(t,x) = \frac{1}{\varepsilon}\big[\Delta\phi(t,x) + f(t,\phi(t,x))\big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W(t,x)$$

- $f(t,\phi^*(t)) = 0$ for all $t \in I = [0,T]$
- $a(t) = \partial_\phi f(t,\phi^*(t)) \leqslant -a_- < 0$ for all $t \in I$

In deterministic case $\sigma = 0$: $\exists$ particular solution $\bar\phi(t,x)$ such that

$$\big\|\bar\phi(t,\cdot) - \phi^*(t)e_0\big\|_{H^1} \leqslant C\varepsilon \qquad \forall t \in I$$

Theorem [B & Nader 2021]

Fix $s < \frac{1}{2}$, and let $\mathcal{B}(h) = \big\{(t,\phi) \colon t \in I, \|\phi - \bar\phi(t,\cdot)\|_{H^s} < h\big\}$
For any $\nu > 0$

$$\mathbb{P}\Big\{\text{leaving } \mathcal{B}(h) \text{ before time } t\Big\} \leqslant C(t,\varepsilon,s)\exp\Big\{-\kappa\frac{h^2}{\sigma^2}\Big\}\Big[1 - \mathcal{O}\Big(\frac{h}{\varepsilon^\nu}\Big)\Big]$$

holds for some $\kappa > 0$, $h = \mathcal{O}(\varepsilon^\nu)$ and $C(t,\varepsilon,s) = \mathcal{O}(t/\varepsilon)$.

Ideas of proof

Ideas of proof

SPDE near a bifurcation point

$$\mathrm{d}\phi = \frac{1}{\varepsilon}\Big[\Delta\phi + g(t) - \phi^2 - b(t,\phi)\Big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W(t,x)$$

with $g(t) = \delta + t^2 + \mathcal{O}(t^3)$ and $b = \mathcal{O}(\phi^3 + t\phi^2 + t^2\phi)$

- Decompose $\phi(t,x) = \phi_0(t)e_0(x) + \phi_\perp(t,x)$ where e_0 constant fct
- $\phi_\perp$ satisfies similar concentration result as ϕ in stable case
- ϕ_0 satisfies similar equation as in 1D, with error term of order $\|\phi_\perp\|^2_{H^s}$

Notes

Open questions

- Case $x \in \mathbb{T}^2$? Renormalisation needed
- Case $x \in \mathbb{T}^3$? Regularity structures or similar needed . . .

Open questions

- Case $x \in \mathbb{T}^2$? Renormalisation needed
- Case $x \in \mathbb{T}^3$? Regularity structures or similar needed ...

References

- N. B. & Barbara Gentz, *A sample-paths approach to noise-induced synchronization: Stochastic resonance in a double-well potential*, Ann. Appl. Probab., **12**(1):1419–1470, 2002
- N. B. & Barbara Gentz, *Pathwise description of dynamic pitchfork bifurcations with additive noise*, Probab. Theory Related Fields, **122**:341–388, 2002
- N. B. & Barbara Gentz, *Noise-Induced Phenomena in Slow-Fast Dynamical Systems. A Sample-Paths Approach*, Springer, Probability and its Applications (2005)
- N. B. & Rita Nader, *Stochastic resonance in stochastic PDEs*, Preprint, July 2021, arXiv:2107.07292

Thanks for your attention!

Slides available at https://www.idpoisson.fr/berglund/Campinas21.pdf