Pathwise Stochastic Analysis and Applications

CIRM, Marseille (Virtual conference)

Renormalisation when approaching the subcriticality threshold: A simple example

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joint works with Christian Kuehn (TU Munich) and Yvain Bruned (Edinburgh)

The fractional Φ^3_d model

$$\partial_t u + (-\Delta)^{\rho/2} u = u^2 + \xi$$

- $u = u(t, x)$, $t \geqslant 0$, $x \in \mathbb{T}^d$
- $-(-\Delta)^{\rho/2}$ fractional Laplacian, $\rho \in (0, 2]$
- ξ space-time white noise

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Ill-posed in general, need to consider renormalised equation

$$\partial_t u + (-\Delta)^{\rho/2} u = u^2 + C(\varepsilon, \rho, u) + \xi^\varepsilon$$

where $\xi^\varepsilon = \varrho^\varepsilon * \xi$ mollified noise

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Motivations:

- simple yet interesting application of general theory of BPHZ renormalisation (after Bogoliubow, Parasiuk, Hepp & Zimmermann)
- asymptotics of vanishing local subcriticality as $\rho \searrow \rho_c(d)$
- coupled SPDE–ODE systems, simplification of Fisher–KPP equation

Some recent progress on singular SPDEs

- Martin Hairer, *A theory of regularity structures*, Invent. Math. **198**:269–504, 2014.
  - General theory of function spaces allowing to solve (subcritical) singular SPDEs
  - Ad hoc renormalisation of some particular SPDEs (PAM, Φ^4_3)

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- Yvain Bruned, Martin Hairer, and Lorenzo Zambotti, *Algebraic renormalisation of regularity structures*, Invent. Math., **215**:1039–1156, 2019.
- Ajay Chandra and Martin Hairer, *An analytic BPHZ theorem for regularity structures*, `arXiv:1612.08138`, 113 pages, 2016.
- Yvain Bruned, Ajay Chandra, Ilya Chevyrev, and Martin Hairer, *Renormalising SPDEs in regularity structures*, J. European Mathematical Society, **23**:869–947, 2019.
  - Systematic way of renormalising subcritical singular SPDEs

Solving (non-singular) SPDEs

$$\partial_t u + (-\Delta)^{\rho/2} u = u^2 + \xi$$

Duhamel formula: $u = P_\rho u_0 + P_\rho * [u^2 + \xi], \qquad P_\rho = [\partial_t + (-\Delta)^{\rho/2}]^{-1}$

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Scaled Hölder–Besov spaces $\mathcal{C}^\alpha_\mathfrak{s}$:

- $0 < \alpha < 1$: $u \in \mathcal{C}^\alpha_\mathfrak{s} \iff |u(\bar{z}) - u(z)| \lesssim |\bar{z} - z|^\alpha_\mathfrak{s}$,
  where $|z|_\mathfrak{s} := |z_0|^{1/\rho} + \sum_i |z_i|$
- $\alpha > 1$: $u \in \mathcal{C}^\alpha_\mathfrak{s} \iff \mathrm{D}^k u \in \mathcal{C}^{\alpha - |k|_\mathfrak{s}}_\mathfrak{s}$ for $0 < |k|_\mathfrak{s} := \rho k_0 + \sum_i |k_i| < \alpha$
- $\alpha < 0$: $u \in \mathcal{C}^\alpha_\mathfrak{s} \iff |\langle u, \mathscr{S}^\lambda_z \varphi \rangle| \lesssim \lambda^\alpha$
  where $(\mathscr{S}^\lambda_z \varphi)(\bar{z}) = \frac{1}{\lambda^{\rho+d}} \varphi\Big(\frac{\bar{z}_0 - z_0}{\lambda^\rho}, \frac{\bar{z}_i - z_i}{\lambda}\Big)$

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Space-time white noise: $\xi \in \mathcal{C}^\alpha_\mathfrak{s} \quad \forall \alpha < -\frac{\rho+d}{2}$

Schauder estimate: $u \in \mathcal{C}^\alpha_\mathfrak{s}$, $\alpha + \rho \notin \mathbb{Z} \quad\Longrightarrow\quad P_\rho * u \in \mathcal{C}^{\alpha+\rho}_\mathfrak{s}$

Consequence: $P_\rho * \xi \in \mathcal{C}^\alpha_\mathfrak{s} \quad \forall \alpha < \frac{\rho - d}{2}$

Local solutions in the "classical sense" exist iff $\rho > d$

Local subcriticality

$$\partial_t u + (-\Delta)^{\rho/2} u = u^2 + \xi$$

Scaling: $\bar{u}(t,x) = \lambda^\alpha u(\lambda^\beta t, \lambda x)$

$$\Longrightarrow \qquad \partial_t \bar{u} + \lambda^{\beta-\rho}(-\Delta)^{\rho/2}\bar{u} = \lambda^{\beta-\alpha}\bar{u}^2 + \lambda^{\alpha+\frac{\beta}{2}-\frac{d}{2}}\xi$$

$\beta = \rho$, $\alpha = \frac{d-\rho}{2}$ $\qquad \Longrightarrow \qquad \partial_t \bar{u} + (-\Delta)^{\rho/2}\bar{u} = \lambda^{\frac{3}{2}(\rho-\frac{d}{3})}\bar{u}^2 + \xi$

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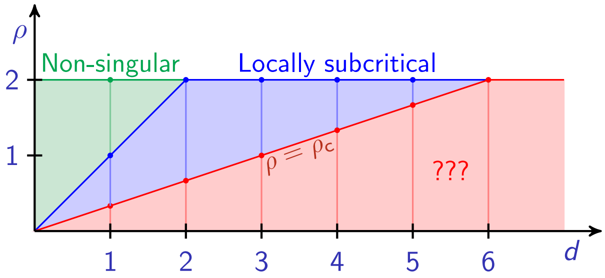
Definition: The equation is locally subcritical iff $\rho > \rho_c = \frac{d}{3}$



Main result

Theorem: [B & Bruned, '19] If $\xi^\varepsilon = \varrho^\varepsilon * \xi$, $\varrho^\varepsilon(t,x) = \frac{1}{\varepsilon^{\rho+d}}\varrho(\frac{t}{\varepsilon^\rho}, \frac{x}{\varepsilon})$,

$$\partial_t u + (-\Delta)^{\rho/2} u = u^2 + C_0(\varepsilon,\rho) + C_1(\varepsilon,\rho) u + \xi^\varepsilon$$

has for $\rho > \rho_c$ local solutions admitting limit as $\varepsilon \searrow 0$

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$$C_0(\varepsilon,\rho) \simeq \begin{cases} \frac{\log(\varepsilon^{-1})}{\varepsilon_c^{d-\rho}} & \varepsilon \geqslant \varepsilon_c \\ \frac{A_0}{\varepsilon^{d-\rho}} & \varepsilon < \varepsilon_c \end{cases} \qquad C_1(\varepsilon,\rho) \simeq \begin{cases} \frac{\log(\varepsilon^{-1})}{\bar{\varepsilon}_c^{\,d-2\rho}} & \varepsilon \geqslant \bar{\varepsilon}_c \\ \frac{\bar{A}_0}{\varepsilon^{d-2\rho}} & \varepsilon < \bar{\varepsilon}_c \end{cases}$$

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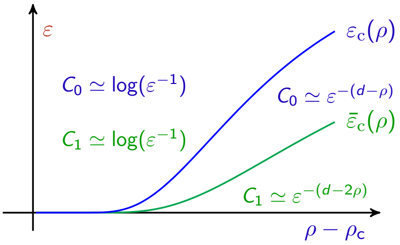
where $\bar\varepsilon_c(\rho) < \varepsilon_c(\rho)$ both of order

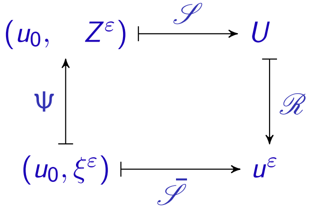
$$\exp\Big\{-\frac{1}{\rho-\rho_c}\Big[\log\Big(\frac{const}{\rho-\rho_c}\Big) + \mathcal{O}(1)\Big]\Big\}$$

and $A_0, \bar A_0$ explicit constants



Regularity structures

Mollified equation: $\partial_t u^\varepsilon + (-\Delta)^{\rho/2} u^\varepsilon = (u^\varepsilon)^2 + \xi^\varepsilon$



- $u^\varepsilon = \bar{\mathscr{S}}(u_0, \xi^\varepsilon)$: fixed point of $u^\varepsilon = P_\rho u_0 + P_\rho * [(u^\varepsilon)^2 + \xi^\varepsilon]$
- $U = \mathscr{S}(u_0, Z^\varepsilon)$: fixed pnt of $U = P_\rho u_0 + \mathcal{I}_\rho[U^2 + \Xi] + \underbrace{p(U)}_{\text{polynomial}}$

  $U \in \mathcal{D}^\gamma$ space of modelled distributions

Regularity structures

Mollified equation: $\partial_t u^\varepsilon + (-\Delta)^{\rho/2} u^\varepsilon = (u^\varepsilon)^2 + \xi^\varepsilon + C(\varepsilon, \rho, u)$

$$\begin{array}{ccc} (u_0, MZ^\varepsilon) & \xrightarrow{\ \mathscr{S}\ } & U_M \\ \uparrow M\Psi & & \downarrow \mathscr{R}^M \\ (u_0, \xi^\varepsilon) & \xrightarrow[\ \bar{\mathscr{S}}_M\ ]{} & u_M^\varepsilon \end{array}$$

- $u_M^\varepsilon = \bar{\mathscr{S}}_M(u_0, \xi^\varepsilon)$: fixed point of $u_M^\varepsilon = P_\rho u_0 + P_\rho * [(u_M^\varepsilon)^2 + \xi^\varepsilon + C]$
- $U_M = \mathscr{S}(u_0, MZ^\varepsilon)$: fixed pnt of $U_M = P_\rho u_0 + \mathcal{I}_\rho[U_M^2 + \Xi] + \underbrace{p(U_M)}_{\text{polynomial}}$

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Model space

T_0 set of symbols containing

- $\mathbf{X}^k = X_0^{k_0} \dots X_d^{k_d}$, degree $|\mathbf{X}^k|_\mathfrak{s} = |k|_\mathfrak{s}$
- Ξ representing ξ, degree $|\Xi|_\mathfrak{s} = -\frac{\rho+d}{2} - \kappa$
- $\tau_1, \tau_2 \in T_0 \Rightarrow \tau_1\tau_2 \in T_0$, degree $|\tau_1\tau_2|_\mathfrak{s} = |\tau_1|_\mathfrak{s} + |\tau_2|_\mathfrak{s}$
- $\tau \in T_0$, $\tau \neq \mathbf{X}^k \Rightarrow \mathcal{I}_\rho(\tau) \in T_0$ repres. $P_\rho * u$, $|\mathcal{I}_\rho(\tau)|_\mathfrak{s} = |\tau|_\mathfrak{s} + \rho$
- In some cases, need symbols $\partial^\ell \mathcal{I}_\rho(\tau)$, $|\partial^\ell \mathcal{I}_\rho(\tau)|_\mathfrak{s} = |\tau|_\mathfrak{s} + \rho - |\ell|_\mathfrak{s}$

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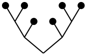
Convenient graphical notation:

$= \mathcal{I}_\rho(\Xi)^2$



$= \Big[\mathcal{I}_\rho\Big(\mathcal{I}_\rho(\mathcal{I}_\rho(\Xi)^2)\mathcal{I}_\rho(\Xi)\Big)\Big]^2$

$= \mathcal{I}_\rho(\mathbf{X}^k \partial^\ell \mathcal{I}_\rho(\Xi))$

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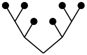
 $= \mathcal{I}_\rho(\Xi)^2$  $= \Big[\mathcal{I}_\rho\Big(\mathcal{I}_\rho\big(\mathcal{I}_\rho(\Xi)^2\big)\mathcal{I}_\rho(\Xi)\Big)\Big]^2$

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Model space: graded vector space $\mathcal{T}$ spanned by minimal $T \subset T_0$ allowing to represent $U = \mathcal{I}_\rho(\Xi + U^2) + p$ where $p = \sum_k c_k \mathbf{X}^k$ polynomial

Remark: $\rho > \rho_c \Leftrightarrow$ degrees of $\tau \in T$ bdd below

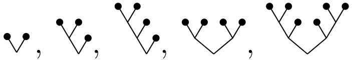
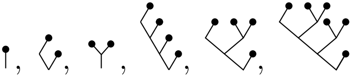
Iterations of the fixed-point equation

$$U = \mathcal{I}_\rho(\Xi + U^2) + c_{\mathbf{1}}(t,x)\mathbf{1} + \sum_{i=0}^{d} c_{\mathbf{X}_i}(t,x)\mathbf{X}_i + \dots$$

Model space

Proposition: [B & Kuehn '17]

Symbols $\tau \in T$ of negative degree are

- either full binary trees, e.g. $\tau =$ 
  $|\tau|_{\mathfrak{s}} = -\frac{2}{3}d + \frac{3m-1}{2}(\rho - \rho_c)-$ if τ has $2m$ edges
- or almost full binary trees, e.g. $\tau =$ 
  $|\tau|_{\mathfrak{s}} = -\frac{1}{3}d + \frac{3\bar{m}+1}{2}(\rho - \rho_c)-$ if τ has $2\bar{m}+1$ edges
- or almost full trees with one node decoration $\mathbf{X}_i$, $1 \leqslant i \leqslant d$
  (complete trees with decorations don't matter for symmetry reasons)

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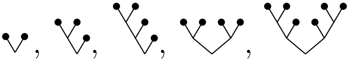
- either full binary trees, e.g. $\tau =$ 
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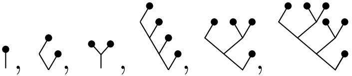
- or almost full binary trees, e.g. $\tau =$ 
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Proposition: [B & Kuehn '17]

Number of symbols of negative degree is of order $(\rho - \rho_c)^{3/2}\, e^{\beta d/(\rho-\rho_c)}$

Proof uses Wedderburn–Etherington numbers (rather than Catalan nbrs)

Model expectations

$E(\tau) := \mathbb{E}[(\mathbf{\Pi}^\varepsilon \tau)(0)]$ where $\mathbf{\Pi}^\varepsilon$ canonical model defined by

$$(\mathbf{\Pi}^\varepsilon \mathbf{1})(z) = 1 \qquad (\mathbf{\Pi}^\varepsilon \mathbf{X}_i)(z) = z_i \qquad (\mathbf{\Pi}^\varepsilon \Xi)(z) = \xi^\varepsilon(z)$$

$$(\mathbf{\Pi}^\varepsilon \tau\bar{\tau})(z) = (\mathbf{\Pi}^\varepsilon \tau)(z)(\mathbf{\Pi}^\varepsilon \bar{\tau})(z)$$

$$(\mathbf{\Pi}^\varepsilon \partial^k \mathcal{I}_\rho \tau)(z) = \int \partial^k K_\rho(z-\bar{z})(\mathbf{\Pi}^\varepsilon \tau)(\bar{z})\, \mathrm{d}\bar{z} \qquad P_\rho = K_\rho + R_\rho$$

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$E($  $) = \mathbb{E}\int K_\rho(-z)\xi^\varepsilon(z)\,\mathrm{d}z = \int K^\varepsilon_\rho(-z)\mathbb{E}[\xi(\mathrm{d}z)] = 0 \qquad K^\varepsilon_\rho = K_\rho * \varrho^\varepsilon$

$E($  $) = \int K^\varepsilon_\rho(-z_1)K^\varepsilon_\rho(-z_2)\mathbb{E}\big[\xi(\mathrm{d}z_1)\xi(\mathrm{d}z_2)\big] = \int K^\varepsilon_\rho(-z_1)^2\,\mathrm{d}z_1$

$E($  $) = \mathbb{E}\left[\left(\int K_\rho(-z)K^\varepsilon_\rho(z-z_1)K^\varepsilon_\rho(z-z_2)\xi(\mathrm{d}z_1)\xi(\mathrm{d}z_2)\,\mathrm{d}z\right)^2\right]$

Isserlis–Wick theorem: $\mathbb{E}[X_1\dots X_{2m}] = \sum_{\text{pairings}}\prod\mathbb{E}[X_iX_j]$

Feynman diagrams

$E($$) = \mathbb{E}\left[\left(\int K_\rho(-z)K_\rho^\varepsilon(z-z_1)K_\rho^\varepsilon(z-z_2)\xi(\mathrm{d}z_1)\xi(\mathrm{d}z_2)\,\mathrm{d}z\right)^2\right]$

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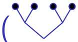
$E($$) = \mathbb{E}\Big[\Big(\int K_\rho(-z)K_\rho^\varepsilon(z-z_1)K_\rho^\varepsilon(z-z_2)\xi(\mathrm{d}z_1)\xi(\mathrm{d}z_2)\,\mathrm{d}z\Big)^2\Big]$

$$= 0 + 2\int K_\rho(-z)K_\rho^\varepsilon(z-z_1)K_\rho^\varepsilon(\bar z-z_1)K_\rho(-\bar z)K_\rho^\varepsilon(z-z_2)K_\rho^\varepsilon(\bar z-z_2)\,\mathrm{d}z\,\mathrm{d}\bar z\,\mathrm{d}z_1\,\mathrm{d}z_2$$

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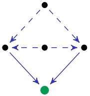
$= \quad 2$ 

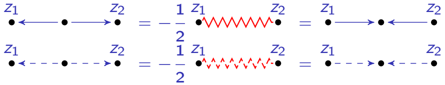
Definition: Feynman (vacuum) diagram

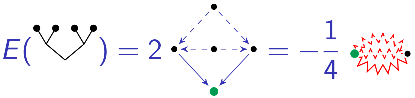
Given by $\Gamma = (\mathscr{V}, \mathscr{E}, v^*)$ directed (multi)graph, v^* distinguished node, $\mathfrak{L}$ finite set of types, a map $\mathfrak{t} : \mathscr{E} \to \mathfrak{L}$, $e \mapsto \mathfrak{t}(e)$, kernels $K_\mathfrak{t} : (\mathbb{R}^{d+1})^* \to \mathbb{R}$

$$E(\Gamma) = \int_{(\mathbb{R}^{d+1})^{\mathscr{V}\setminus v^\star}} \prod_{e\in\mathscr{E}} K_{\mathfrak{t}(e)}(z_{e_+} - z_{e_-})\,\mathrm{d}z \qquad e = (e_-, e_+),\ z_{v^*} = 0$$

Simplification of Feynman diagrams

v^* can be moved, and vertices of degree 2 can be integrated out:





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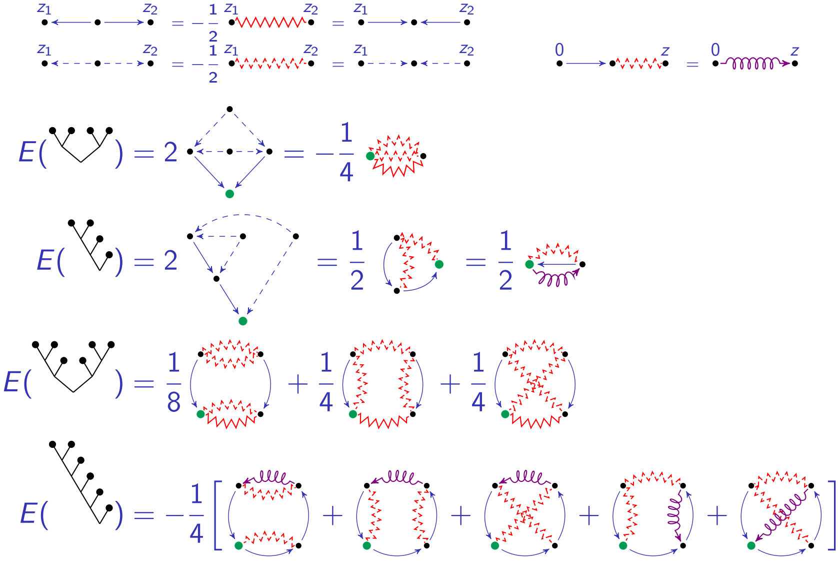

Degree of Feynman diagrams

Define

$$\deg(\Gamma) = (\rho + d)(|\mathscr{V}| - 1) + \sum_{e \in \mathscr{E}} \deg(\mathfrak{t}(e))$$

where

$$\deg(\longrightarrow) = \deg(\dashrightarrow) = -d$$

$$\deg(\leadsto) = \deg(\rightsquigarrow) = \rho - d \qquad \deg(\looparrowright) = 2\rho - d$$

Then for any pairing P, one has $\deg(\Gamma(\tau, P)) = |\tau|_{\mathfrak{s}}\big|_{\kappa=0}$

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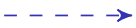
$\deg($  $) = \deg($  $) = -d$

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Simple examples suggest that $|E(\Gamma)| \asymp \begin{cases} \varepsilon^{\deg \Gamma} & \text{if } \deg \Gamma < 0 \\ \log(\varepsilon^{-1}) & \text{if } \deg \Gamma = 0 \\ 1 & \text{if } \deg \Gamma > 0 \end{cases}$

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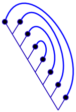
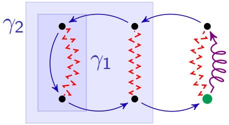
This is however not the case in general, because of subdivergences: there can be subgraphs $\gamma \subset \Gamma$ with $\deg\gamma < \deg\Gamma \leqslant 0$



$\longrightarrow$



Key estimate

Inductive def of twisted antipode: $\tilde{\mathcal{A}}_-\Gamma = -\Gamma - \sum_{\gamma \subsetneq \Gamma : \deg \gamma < 0} \tilde{\mathcal{A}}_-\gamma \cdot \underbrace{\Gamma/\gamma}_{\text{contraction}}$

Proposition: [B & Bruned '19] If τ has p leaves,

$$|E(\tilde{\mathcal{A}}_-(\Gamma))| \leqslant \begin{cases} K_1^p (p-3)!\, \varepsilon^{\deg \Gamma} \log(\varepsilon^{-1})^{\zeta} & \text{if } \deg \Gamma < 0 \\ K_1^p (p-3)!\, \log(\varepsilon^{-1})^{1+\zeta} & \text{if } \deg \Gamma = 0 \end{cases}$$

where K_1 depends only on $K_{\mathfrak{t}}$ and $\zeta \in \{0, 1\}$: # of $\gamma \subset \Gamma$ with $\deg \gamma = 0$

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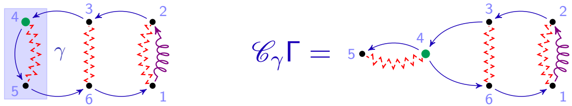
▹ Extracting subdivergences (cf. [Connes & Kreimer]):

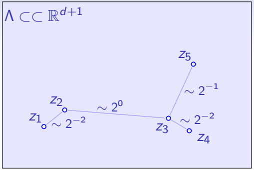
$\Gamma =$ $\qquad\qquad$ $\mathscr{C}_\gamma \Gamma =$

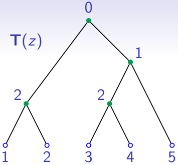


Then $E(\Gamma) - E(\mathscr{C}_\gamma\Gamma)$ contains a factor

$$|K_\rho(z_6 - z_5) - K_\rho(z_6 - z_4)| \lesssim |(z_5 - z_4) \cdot \nabla K_\rho(z_6 - z_4)| \lesssim \frac{\|z_5 - z_4\|_{\mathfrak{s}}}{\|z_6 - z_4\|_{\mathfrak{s}}^{d+1}}$$

▷ Hepp sector:





$\mathbf{T} = (T, \mathbf{n})$: T binary tree, $|\mathscr{V}|$ leaves, $\mathbf{n}$ increasing node decoration

Hepp sector: $D_{\mathbf{T}} = \{z \in \Lambda^{|\mathscr{V}|} \colon C^{-1} 2^{-\mathbf{n}_{i \wedge j}} \leqslant \|z_i - z_j\|_{\mathfrak{s}} \leqslant C 2^{-\mathbf{n}_{i \wedge j}}\}$

where $i \wedge j$ last common ancestor in T $\quad \Rightarrow \quad$ $\Lambda^{|\mathscr{V}|} \subset \bigcup_{\mathbf{T}} D_{\mathbf{T}}$

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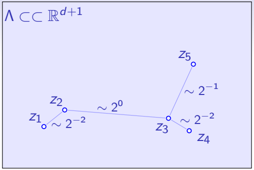

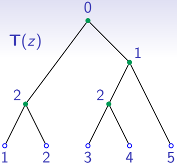

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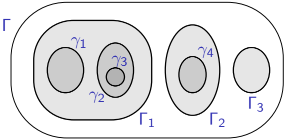
▷ Zimmermann's forest formula:

$$\tilde{\mathcal{A}}\_\Gamma = - \sum_{\text{forests } \mathscr{F}} (-1)^{|\mathscr{F}|} \mathscr{C}_{\mathscr{F}} \Gamma$$



$$\tilde{\mathcal{A}}\_\Gamma = - \sum_{\mathscr{F}_{\mathrm{s}} \text{ safe}} \prod_{\gamma \in \mathscr{F}_{\mathrm{s}}} (-\mathscr{C}_\gamma) \prod_{\bar\gamma \text{ unsafe for } \mathscr{F}_{\mathrm{s}}} (\mathrm{id} - \mathscr{C}_{\bar\gamma}) \Gamma$$

$\bar\gamma$ is unsafe for $\mathbf{T}$ if it is small and far from its parents

General formula for the counterterms

Theorem: [Bruned, Hairer, Zambotti; Bruned, Chandra, Chevyrev, Hairer '19]

Counterterms given by

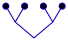
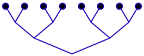
$$C(\varepsilon,\rho,u) = \sum_{\tau\in T:\, |\tau|_{\mathfrak{s}}<0} E(\tilde{\mathcal{A}}_-(\tau))\frac{\Upsilon^F(\tau)(u)}{S(\tau)}$$

- $\tilde{\mathcal{A}}_-(\tau)$ twisted antipode acting on trees
- $\Upsilon^F(\tau)(u)$ given by inductive relation with $\Upsilon^F(\Xi)(u) = 1$; here

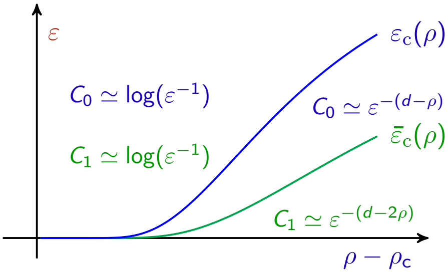
$$\Upsilon^F(\tau)(u) = \begin{cases} 2^{n_{\text{inner}}(\tau)} & \text{if } \tau \text{ full} \\ 2^{n_{\text{inner}}(\tau)}u & \text{if } \tau \text{ almost full without } \mathbf{X}_i \end{cases}$$

  where $n_{\text{inner}}(\tau)$ # of nodes of τ that are not leaves

- $S(\tau)$ symmetry factor; here $S(\tau) = 2^{n_{\text{sym}}(\tau)}$ where $n_{\text{sym}}(\tau)$ # of inner nodes with 2 identical lines of offspring, e.g.

$S($  $) = S($  $) = 2$ $S($  $) = 2^3$ $S($  $) = 2^7$

Thanks for your attention



arXiv/1907.13028

www.idpoisson.fr/berglund/CIRM21.pdf

Main result (precise version)

Theorem: [B & Bruned, arXiv/1907.13028]

$\exists M > 0$ s.t. counterterm $C_0(\varepsilon,\rho) + C_1(\varepsilon,\rho)u$ satisfies

$$\left|C_0(\varepsilon,\rho)\right| \leqslant M\varepsilon_{\rm c}^{-(d-\rho)}\left[\log(\varepsilon^{-1}) + \frac{1}{\rho-\rho_{\rm c}}\Big(\frac{\varepsilon_{\rm c}}{\varepsilon}\Big)^{3(\rho-\rho_{\rm c})}\right] \qquad \varepsilon \geqslant \varepsilon_{\rm c}$$

$$\left|\frac{C_0(\varepsilon,\rho)}{A_0\varepsilon^{-(d-\rho)}} - 1\right| \leqslant \frac{M}{\rho-\rho_{\rm c}}\Big(\frac{\varepsilon}{\varepsilon_{\rm c}}\Big)^{3(\rho-\rho_{\rm c})} \qquad \varepsilon < \varepsilon_{\rm c}$$

$$\left|C_1(\varepsilon,\rho)\right| \leqslant M\bar\varepsilon_{\rm c}^{\;-(d-2\rho)}\left[\log(\varepsilon^{-1}) + \frac{1}{\rho-\rho_{\rm c}}\Big(\frac{\bar\varepsilon_{\rm c}}{\varepsilon}\Big)^{3(\rho-\rho_{\rm c})}\right] \qquad \varepsilon \geqslant \bar\varepsilon_{\rm c}$$

$$\left|\frac{C_0(\varepsilon,\rho)}{\bar A_0\varepsilon^{-(d-2\rho)}} - 1\right| \leqslant \frac{M}{\rho-\rho_{\rm c}}\Big(\frac{\varepsilon}{\bar\varepsilon_{\rm c}}\Big)^{3(\rho-\rho_{\rm c})} \qquad \varepsilon < \bar\varepsilon_{\rm c}$$

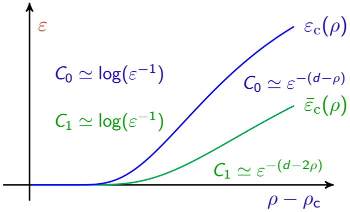
$$\varepsilon_{\rm c} = f(k_{\max}) \quad \bar\varepsilon_{\rm c} = f(\bar k_{\max})$$

$$f(k) = \exp\left\{-\frac{\log k + a - \frac{\log k}{2k})}{\rho-\rho_{\rm c}}\right\}$$

$$k_{\max} = \frac{d-\rho}{3(\rho-\rho_{\rm c})} \qquad \bar k_{\max} = \frac{d-2\rho}{3(\rho-\rho_{\rm c})}$$

$$A_0 = -\lim_{\varepsilon\to0}\varepsilon^{d-\rho}E(\;\;)$$ 

$$\bar A_0 = -4\lim_{\varepsilon\to0}\varepsilon^{d-2\rho}E(\;\;)$$ 



Main estimate

$$|E(\tilde{\mathcal{A}}_-(\tau))| \leqslant \sum_P \sum_T \sum_{\mathscr{F}_s} \sum_{\mathbf{n}} \int_{D_{T,\mathbf{n}}} \prod_{e \in \mathscr{E}(\tilde{\mathcal{A}}_- \Gamma(\tau,P))} |K_{\mathfrak{t}(e)}(z_{e_+} - z_{e_-})| \, \mathrm{d}z$$

Proposition: [B & Bruned '19]

$$\sum_{\mathbf{n}} \sup_{z \in D_{\mathbf{T}}} \prod_e |K_{\mathfrak{t}(e)}(\dots)| \, \mathrm{Vol}(D_{\mathbf{T}}) \leqslant \begin{cases} K_1^{|\mathscr{E}|} \varepsilon^{\deg \Gamma} \log(\varepsilon^{-1})^{\zeta} & \text{if } \deg \Gamma < 0 \\ K_1^{|\mathscr{E}|} \log(\varepsilon^{-1})^{1+\zeta} & \text{if } \deg \Gamma = 0 \end{cases}$$

where K_1 depends only on $K_{\mathfrak{t}}$ and $\zeta \in \{0,1\}$ # of $\gamma \subset \Gamma$ with $\deg \gamma = 0$

For τ complete with $2k+2$ leaves, $k \leqslant k_{\max} = \frac{d-\rho}{3(\rho-\rho_c)}$:

- # of pairings $P = (2k+1)!! = \prod_{i=1}^{k}(2i+1)$
- # of Hepp trees $T \leqslant (2k-1)!$
- # of safe forests $\mathscr{F}_s \leqslant 2^k$
- % of pairings yielding $\zeta = 1$ bdd by $2^{-(2k-k_{\max})}$