Singular SPDEs and Related Topics

Hausdorff Center, Bonn

BPHZ renormalisation and vanishing subcriticality limit of the fractional Φ^3_d model

Nils Berglund

Institut Denis Poisson, Université d'Orléans, France

October 24, 2019

joint works with Christian Kuehn (TU Munich) and Yvain Bruned (Edinburgh)

The fractional Φ^3_d model

$$\partial_t u - \Delta^{\rho/2} u = u^2 + \xi$$

- $u = u(t,x)$, $t \geqslant 0$, $x \in \mathbb{T}^d$
- $\Delta^{\rho/2} := -(-\Delta)^{\rho/2}$ fractional Laplacian, $\rho \in (0,2]$
- ξ space-time white noise

Ill-posed in general, need to consider renormalised equation

$$\partial_t u - \Delta^{\rho/2} u = u^2 + C(\varepsilon, \rho, u) + \xi^\varepsilon$$

where $\xi^\varepsilon = \varrho^\varepsilon * \xi$ mollified noise

Motivations:

- simple yet interesting application of general theory of BPHZ renormalisation
- limit of vanishing local subcriticality as $\rho \searrow \rho_c(d)$
- coupled SPDE–ODE systems, simplification of Fisher–KPP equation

Some recent progress on singular SPDEs

- Martin Hairer, *A theory of regularity structures*, Invent. Math. **198**:269–504, 2014.
  - General theory of function spaces allowing to solve (subcritical) singular SPDEs
  - Ad hoc renormalisation of some particular SPDEs (PAM, Φ^4_3)

- Yvain Bruned, Martin Hairer, and Lorenzo Zambotti, *Algebraic renormalisation of regularity structures*, Invent. Math., **215**:1039–1156, 2019.
- Ajay Chandra and Martin Hairer, *An analytic BPHZ theorem for regularity structures*, `arXiv:1612.08138`, 113 pages, 2016.
- Yvain Bruned, Ajay Chandra, Ilya Chevyrev, and Martin Hairer, *Renormalising SPDEs in regularity structures*, `arXiv:1711.10239`, 85 pages, 2017. To appear in J. Eur. Math. Soc.
  - Systematic way of renormalising subcritical singular SPDEs

Local subcriticality

$$\partial_t u - \Delta^{\rho/2} u = u^2 + \xi \qquad \Rightarrow \qquad u = K_\rho * [u^2 + \xi]$$

Definition 1: The equation is locally subcritical iff the nonlinear term u^2 disappears when zooming in on small scales

$\mathcal{C}^\alpha_\mathfrak{s}(\mathbb{T}^d)$ Besov–Hölder space for scaling $\mathfrak{s} = (\rho, 1, \dots, 1)$

- $\xi \in \mathcal{C}^\alpha_\mathfrak{s}(\mathbb{T}^d)$ for all $\alpha < -\frac{\rho+d}{2}$
- Schauder estimate: $f \in \mathcal{C}^\alpha_\mathfrak{s}(\mathbb{T}^d)$, $\alpha + \rho \notin \mathbb{N} \Rightarrow K_\rho * f \in \mathcal{C}^{\alpha+\rho}_\mathfrak{s}(\mathbb{T}^d)$

Definition 2: The equation is locally subcritical iff when iterating the fixed-point equation, "Hölder regularity stays bounded below"

Proposition: [B & Kuehn, J Stat Phys **168** (2017)]

The equation is locally subcritical iff $\rho > \rho_c = \frac{d}{3}$

$\xi \in \mathcal{C}_\mathfrak{s}^{-\frac{\rho+d}{2}-} \Rightarrow K_\rho * \xi \in \mathcal{C}_\mathfrak{s}^{\frac{\rho-d}{2}-} \Rightarrow$ "$(K_\rho * \xi)^2$ has regularity $\rho - d-$"

$\rho - d > -\frac{\rho+d}{2} \Leftrightarrow \rho > \frac{d}{3}$

Main result

Theorem: [B & Bruned, '19] If $\xi^\varepsilon = \varrho^\varepsilon * \xi$, $\varrho^\varepsilon(t,x) = \frac{1}{\varepsilon^{\rho+d}}\varrho(\frac{t}{\varepsilon^\rho}, \frac{x}{\varepsilon})$,

$$\partial_t u - \Delta^{\rho/2} u = u^2 + C_0(\varepsilon,\rho) + C_1(\varepsilon,\rho)u + \xi^\varepsilon$$

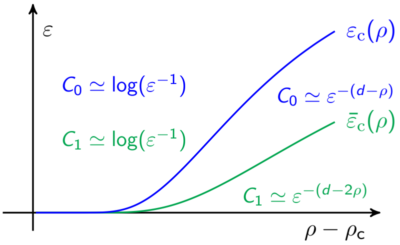
has local solutions admitting limit as $\varepsilon \searrow 0$ for C_0, C_1 s.t.

$$C_0(\varepsilon,\rho) \simeq \begin{cases} \frac{\log(\varepsilon^{-1})}{\varepsilon_c^{d-\rho}} & \varepsilon \geqslant \varepsilon_c \\ \frac{A_0}{\varepsilon^{d-\rho}} & \varepsilon < \varepsilon_c \end{cases} \qquad C_1(\varepsilon,\rho) \simeq \begin{cases} \frac{\log(\varepsilon^{-1})}{\bar\varepsilon_c^{d-2\rho}} & \varepsilon \geqslant \bar\varepsilon_c \\ \frac{\bar A_0}{\varepsilon^{d-2\rho}} & \varepsilon < \bar\varepsilon_c \end{cases}$$

where $\bar\varepsilon_c(\rho) < \varepsilon_c(\rho)$ both of order

$$\exp\left\{-\frac{1}{\rho-\rho_c}\left[\log\left(\frac{const}{\rho-\rho_c}\right) + \mathcal{O}(1)\right]\right\}$$

and $A_0, \bar A_0$ explicit constants

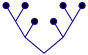


Model space

T_0 set of symbols containing

- $\mathbf{X}^k = X_0^{k_0} \dots X_d^{k_d}$, degree $|\mathbf{X}^k|_\mathfrak{s} = |k|_\mathfrak{s} = \rho k_0 + k_1 + \dots + k_d$
- Ξ representing ξ, degree $|\Xi|_\mathfrak{s} = -\frac{\rho+d}{2} - \kappa$
- $\tau_1, \tau_2 \in T_0 \Rightarrow \tau_1\tau_2 \in T_0$, degree $|\tau_1\tau_2|_\mathfrak{s} = |\tau_1|_\mathfrak{s} + |\tau_2|_\mathfrak{s}$
- $\tau \in T_0$, $\tau \neq \mathbf{X}^k \Rightarrow \mathcal{I}_\rho(\tau) \in T_0$ repres. $K_\rho * u$, $|\mathcal{I}_\rho(\tau)|_\mathfrak{s} = |\tau|_\mathfrak{s} + \rho$
- In some cases, need symbols $\partial^\ell \mathcal{I}_\rho(\tau)$, $|\partial^\ell \mathcal{I}_\rho(\tau)|_\mathfrak{s} = |\tau|_\mathfrak{s} + \rho - |\ell|_\mathfrak{s}$

Convenient graphical notation:

 $= \mathcal{I}_\rho(\Xi)^2$  $= \Big[\mathcal{I}_\rho\Big(\mathcal{I}_\rho(\mathcal{I}_\rho(\Xi)^2)\mathcal{I}_\rho(\Xi)\Big)\Big]^2$

$= \mathcal{I}_\rho(\mathbf{X}^k \partial^\ell \mathcal{I}_\rho(\Xi))$

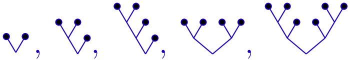
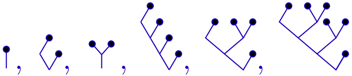
Model space: graded vector space $\mathcal{T}$ spanned by minimal $T \subset T_0$ allowing to represent $U = \mathcal{I}_\rho(\Xi + U^2) + P$ where $P = \sum_k c_k \mathbf{X}^k$ polynomial

Remark: $\rho > \rho_c \Rightarrow$ degrees of $\tau \in T$ bdd below

Model space

Proposition: [B & Kuehn '17]

Symbols $\tau \in T$ of negative degree are

- either complete binary trees, e.g. $\tau =$ 
  $|\tau|_{\mathfrak{s}} = -\frac{2}{3}d + \frac{3m-1}{2}(\rho - \rho_{\mathrm{c}})-$ if τ has $2m$ edges
- or incomplete binary trees, e.g. $\tau =$ 
  $|\tau|_{\mathfrak{s}} = -\frac{1}{3}d + \frac{3\bar{m}+1}{2}(\rho - \rho_{\mathrm{c}})-$ if τ has $2\bar{m} + 1$ edges
- or incomplete trees with one node decoration X_i, $1 \leqslant i \leqslant d$
  (complete trees with decorations don't matter for symmetry reasons)

Proposition: [B & Kuehn '17]

Number of symbols of negative degree is of order $(\rho - \rho_{\mathrm{c}})^{3/2}\,\mathrm{e}^{\beta d/(\rho-\rho_{\mathrm{c}})}$

Proof uses Wedderburn–Etherington numbers (rather than Catalan nbrs)

General formula for the counterterms

Theorem: [Bruned, Hairer, Zambotti; Bruned, Chandra, Chevyrev, Hairer '19]

Counterterms given by

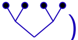
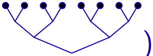
$$C(\varepsilon,\rho,u) = \sum_{\tau\in T\,:\,|\tau|_{\mathfrak{s}}<0} c_\varepsilon(\tau)\frac{\Upsilon^F(\tau)(u)}{S(\tau)}$$

- $\Upsilon^F(\tau)(u)$ given by inductive relation with $\Upsilon^F(\Xi)(u) = 1$; here

$$\Upsilon^F(\tau)(u) = \begin{cases} 2^{n_{\mathrm{inner}}(\tau)} & \text{if } \tau \text{ complete} \\ 2^{n_{\mathrm{inner}}(\tau)} u & \text{if } \tau \text{ incomplete without } X_i \\ 2^{n_{\mathrm{inner}}(\tau)} \partial_{x_i} u & \text{if } \tau \text{ incomplete with } X_i \end{cases}$$

  where $n_{\mathrm{inner}}(\tau)$ number of nodes of τ that are not leaves

- $S(\tau)$ symmetry factor; here $S(\tau) = 2^{n_{\mathrm{sym}}(\tau)}$ where $n_{\mathrm{sym}}(\tau)$ nb of inner nodes with 2 identical lines of offspring, e.g.

$S($$) = S($$) = 2$ $S($$) = 2^3$ $S($$) = 2^7$

Model expectations

- $c_\varepsilon(\tau) = \mathbb{E}[(\boldsymbol{\Pi}^\varepsilon \tilde{\mathcal{A}}_- \tau)(0)] =: E(\tilde{\mathcal{A}}_- \tau)$ where $\tilde{\mathcal{A}}_-$ described below and $\boldsymbol{\Pi}^\varepsilon$ canonical model defined by (writing $z = (t, x)$)

$$(\boldsymbol{\Pi}^\varepsilon \mathbf{1})(z) = 1 \qquad (\boldsymbol{\Pi}^\varepsilon X_i)(z) = z_i \qquad (\boldsymbol{\Pi}^\varepsilon \Xi)(z) = \xi^\varepsilon(z)$$

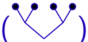
$$(\boldsymbol{\Pi}^\varepsilon \tau\bar{\tau})(z) = (\boldsymbol{\Pi}^\varepsilon \tau)(z)(\boldsymbol{\Pi}^\varepsilon \bar{\tau})(z)$$

$$(\boldsymbol{\Pi}^\varepsilon \partial^k \mathcal{I}_\rho \tau)(z) = \int \partial^k K_\rho(z - \bar{z})(\boldsymbol{\Pi}^\varepsilon \tau)(\bar{z})\,\mathrm{d}\bar{z}$$

Remark: $E(\tau) = 0$ for trees with odd # of leaves, for planted trees $\mathcal{I}_\rho(\tau)$, and for trees with one X_i decoration (and no edge decoration)

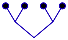
$E($  $) = \mathbb{E}\int K_\rho(-z)\xi^\varepsilon(z)\,\mathrm{d}z = \int K_\rho^\varepsilon(-z)\mathbb{E}[\xi(\mathrm{d}z)] = 0 \qquad K_\rho^\varepsilon = K_\rho * \varrho^\varepsilon$

$E($  $) = \int K_\rho^\varepsilon(-z_1)K_\rho^\varepsilon(-z_2)\mathbb{E}\big[\xi(\mathrm{d}z_1)\xi(\mathrm{d}z_2)\big] = \int K_\rho^\varepsilon(-z_1)^2\,\mathrm{d}z_1$

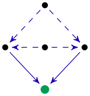
$E($  $) = \mathbb{E}\bigg[\Big(\int K_\rho(-z)K_\rho^\varepsilon(z - z_1)K_\rho^\varepsilon(z - z_2)\xi(\mathrm{d}z_1)\xi(\mathrm{d}z_2)\,\mathrm{d}z\Big)^2\bigg]$

Feynman diagrams

Isserlis–Wick thm: $X \sim \mathcal{N}(0,\Sigma) \Rightarrow \mathbb{E}[X_1 \dots X_{2m}] = \sum_{\text{pairings}} \prod \mathbb{E}[X_i X_j]$

$E($$) = \mathbb{E}\Big[\Big(\int K_\rho(-z) K_\rho^\varepsilon(z-z_1) K_\rho^\varepsilon(z-z_2)\xi(\mathrm{d}z_1)\xi(\mathrm{d}z_2)\,\mathrm{d}z\Big)^2\Big]$

$$= 0 + 2\int K_\rho(-z)K_\rho^\varepsilon(z-z_1)K_\rho^\varepsilon(\bar z-z_1)K_\rho(-\bar z)K_\rho^\varepsilon(z-z_2)K_\rho^\varepsilon(\bar z-z_2)\,\mathrm{d}z\,\mathrm{d}\bar z\,\mathrm{d}z_1\,\mathrm{d}z_2$$

$= \quad 2$ 

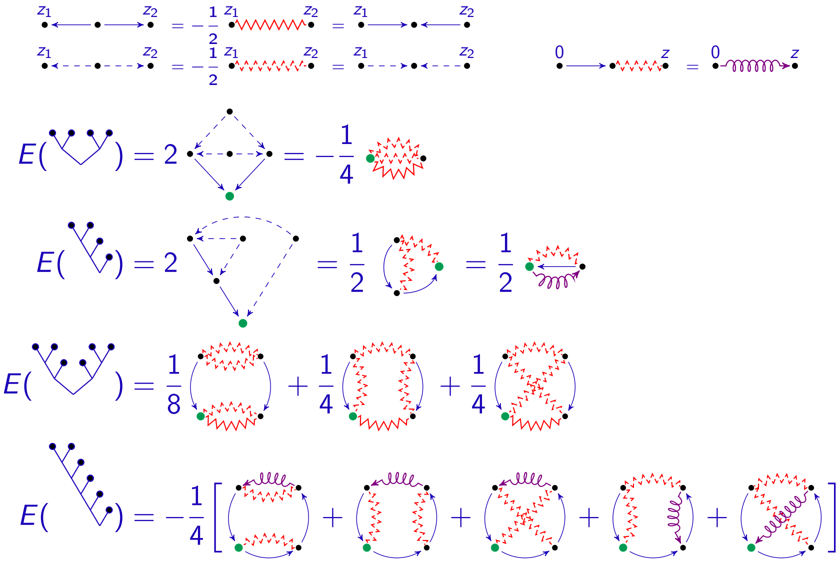
Definition: Feynman (vacuum) diagram

Given by $\Gamma = (\mathscr{V}, \mathscr{E}, v^*)$ directed (multi)graph, v^* distinguished node, $\mathfrak{L}$ finite set of types, a map $\mathfrak{t} : \mathscr{E} \to \mathfrak{L}, e \mapsto \mathfrak{t}(e)$, kernels $K_\mathfrak{t} : (\mathbb{R}^{d+1})^* \to \mathbb{R}$

$$E(\Gamma) = \int_{(\mathbb{R}^{d+1})^{\mathscr{V}\setminus v^\star}} \prod_{e\in\mathscr{E}} K_{\mathfrak{t}(e)}(z_{e_+} - z_{e_-})\,\mathrm{d}z \qquad e = (e_-, e_+),\ z_{v^*} = 0$$

Simplification of Feynman diagrams

v^* can be moved, and vertices of degree 2 can be integrated out:



Degree of Feynman diagrams

Define

$$\deg(\Gamma) = (\rho + d)(|\mathscr{V}| - 1) + \sum_{e \in \mathscr{E}} \deg(\mathfrak{t}(e))$$

where

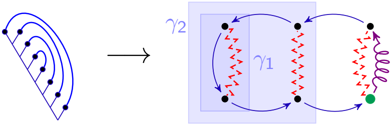
$$\deg(\longrightarrow) = \deg(\dashrightarrow) = -d$$

deg() = deg() $= \rho - d$ deg() $= 2\rho - d$

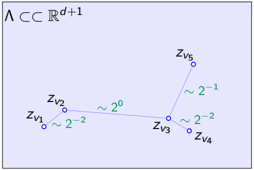
Then for any pairing P, one has $\deg(\Gamma(\tau, P)) = |\tau|_{\mathfrak{s}}\big|_{\kappa=0}$

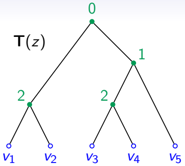
Simple examples suggest that $|E(\Gamma)| \asymp \begin{cases} \varepsilon^{\deg \Gamma} & \text{if } \deg \Gamma < 0 \\ \log(\varepsilon^{-1}) & \text{if } \deg \Gamma = 0 \\ 1 & \text{if } \deg \Gamma > 0 \end{cases}$

This is however not the case in general, because of subdivergences: there can be subgraphs $\gamma \subset \Gamma$ with $\deg \gamma < \deg \Gamma \leqslant 0$



Hepp sectors





$\mathbf{T} = (T, \mathbf{n})$: T binary tree with $|\mathscr{V}|$ leaves, $\mathbf{n}$ increasing node decoration

Hepp sector: $D_{\mathbf{T}} = \{z \in \Lambda^{|\mathscr{V}|} \colon C^{-1}2^{-\mathbf{n}_{i\wedge j}} \leqslant \|x_i - x_j\|_{\mathfrak{s}} \leqslant C2^{-\mathbf{n}_{i\wedge j}}\}$

where $i \wedge j$ last common ancestor in T $\quad\Rightarrow\quad$ $\Lambda^{|\mathscr{V}|} \subset \bigcup_{\mathbf{T}} D_{\mathbf{T}}$

Theorem [Weinberg '66, Hairer '18]

Assume $|K_{\mathfrak{t}}(z)| \lesssim \|z\|_{\mathfrak{s}}^{\deg \mathfrak{t}}$. If $\deg \gamma > 0$ for all $\gamma \subset \Gamma$ then $|E(\Gamma)| < \infty$

Proof idea: $z \in D_{\mathbf{T}} \Rightarrow \prod_{e\in\mathscr{E}} |K(z_{e_+} - z_{e_-})| \lesssim \prod_{e\in\mathscr{E}} 2^{-\mathbf{n}_{e^\uparrow} \deg(e)}$, $e^\uparrow = e_+ \wedge e_-$

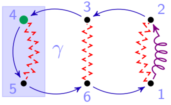
$\mathrm{Vol}(D_{\mathbf{T}}) \lesssim \prod_{v\in T} 2^{-(\rho+d)\mathbf{n}_v}$

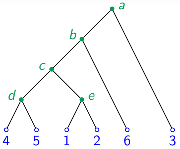
Thus $|E(\Gamma)| \lesssim \sum_{T,\mathbf{n}} \prod_{v\in T} 2^{-\eta_v \mathbf{n}_v}$ where $\eta_v = \rho + d + \sum_{e\in\mathscr{E}} \deg(e) 1_{e^\uparrow}(v)$

$\forall v, \sum_{w\geqslant v} \eta_w = \deg \gamma(v) > 0$; use induction starting from leaves $\square$

Subdivergences

Example:

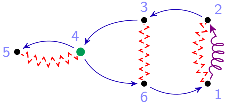
$\Gamma =$ 

$T =$ 

$\deg\Gamma = 10\rho - 4d$, $\deg\gamma = 2\rho - d$ $\quad\Rightarrow\quad$ $\deg\gamma < \deg\Gamma < 0$ if $\frac{3}{8}d < \rho < \frac{2}{5}d$

γ is called unsafe if $\mathbf{n}_d > \mathbf{n}_c$ (it is small and far from its parents)

Define

$\hat{\mathscr{C}}_\gamma\Gamma =$ 

Then $E(\Gamma) - E(\hat{\mathscr{C}}_\gamma\Gamma)$ contains a factor

$$|K_\rho(z_6 - z_5) - K_\rho(z_6 - z_4)| \lesssim |(z_5 - z_4)\cdot\nabla K_\rho(z_6 - z_4)| \lesssim \frac{\|z_5 - z_4\|_\mathfrak{s}}{\|z_6 - z_4\|_\mathfrak{s}^{d+1}}$$

This is smaller than $|K_\rho(z_6 - z_5)|$ by a factor $2^{-(\mathbf{n}_d - \mathbf{n}_b)}$

$\Rightarrow$ if $\deg\gamma > -1$, setting $\tilde{\mathscr{A}}_-\Gamma = -\Gamma + \hat{\mathscr{C}}_\gamma\Gamma$ one has $|E(\tilde{\mathscr{A}}_-\Gamma)| \lesssim \varepsilon^{\deg\Gamma}$

If $\deg\gamma \leqslant -1$, has to push further the Taylor expansion ($\hat{\mathscr{C}}_\gamma\Gamma = \sum_k \dots$)

Zimmermann's forest formula

Inductive def of twisted antipode: $\tilde{A}_-\Gamma = -\Gamma - \sum_{\gamma \subsetneq \Gamma : \deg \gamma < 0} \tilde{A}_-\gamma \cdot \underbrace{\Gamma/\gamma}_{\text{contraction}}$

Definition:

A forest is a collection $\mathscr{F}$ of $\gamma \subset \Gamma$, $\deg \gamma \leqslant 0$, which are pairwise either vertex disjoint, or included one in the other. If $\varrho(\mathscr{F})$ set of roots of $\mathscr{F}$, let $\mathscr{C}_\varnothing \Gamma = \Gamma$ and inductively define $\mathscr{C}_\mathscr{F}\Gamma = \mathscr{C}_{\mathscr{F}\setminus\varrho(\mathscr{F})} \prod_{\gamma\in\varrho(\mathscr{F})} \mathscr{C}_\gamma \Gamma$

Theorem: Zimmermann's forest formula [Zimmermann '66, Hairer '18]

$$\tilde{A}_-\Gamma = -\sum_{\text{forests } \mathscr{F}} (-1)^{|\mathscr{F}|} \mathscr{C}_\mathscr{F}\Gamma$$

- Given a Hepp sector $\mathbf{T} = (T, \mathbf{n})$, a forest is safe if all its γ are safe
- Any forest $\mathscr{F} = \mathscr{F}_\mathrm{s} \sqcup \mathscr{F}_\mathrm{u}$ with $\mathscr{F}_\mathrm{s}$ safe, and all $\gamma \in \mathscr{F}_\mathrm{u}$ unsafe for $\mathscr{F}_\mathrm{s}$
- Then $\tilde{A}_-\Gamma = \sum_{\mathscr{F}_\mathrm{s} \text{ safe}} \prod_{\gamma\in\mathscr{F}_\mathrm{s}} (-\mathscr{C}_\gamma) \prod_{\bar{\gamma} \text{ unsafe for } \mathscr{F}_\mathrm{s}} (\mathrm{id} - \mathscr{C}_{\bar{\gamma}})\Gamma$

Main estimate

$$|c_\varepsilon(\tau)| \leqslant \sum_P \sum_T \sum_{\mathscr{F}_s} \sum_{\mathbf{n}} \int_{D_{T,\mathbf{n}}} \prod_{e\in\mathscr{E}(\tilde{\mathcal{A}}_-\Gamma(\tau,P))} |K_{\mathfrak{t}(e)}(z_{e_+} - z_{e_-})| \, \mathrm{d}z$$

Proposition: [B & Bruned '19]

$$\sum_{\mathbf{n}} \sup_{z\in D_{\mathbf{T}}} \prod_e |K_{\mathfrak{t}(e)}(\dots)| \operatorname{Vol}(D_{\mathbf{T}}) \leqslant \begin{cases} K_1^{|\mathscr{E}|} \varepsilon^{\deg\Gamma} \log(\varepsilon^{-1})^\zeta & \text{if } \deg\Gamma < 0 \\ K_1^{|\mathscr{E}|} \log(\varepsilon^{-1})^{1+\zeta} & \text{if } \deg\Gamma = 0 \end{cases}$$

where K_1 depends only on $K_\mathfrak{t}$ and $\zeta \in \{0,1\}$ # of $\gamma \subset \Gamma$ with $\deg\gamma = 0$

Proof uses lower bound on $\sum_{w\geqslant v} \eta_w$ in terms of $\deg(\gamma(v))$ as in Weinberg's thm

For τ complete with $2k+2$ leaves, $k \leqslant k_{\max} = \frac{d-\rho}{3(\rho-\rho_c)}$:

- # of pairings $P = (2k+1)!! = \prod_{i=1}^k (2i+1)$
- # of Hepp trees $T \leqslant (2k-1)!$
- # of safe forests $\mathscr{F}_s \leqslant 2^k$
- % of pairings yielding $\zeta = 1$ bdd by $\frac{k_{\max}!!(2k-k_{\max})!!}{(2k+1)!!} 1_{k_{\max} \text{ odd} \leqslant 2k+1}$

Main result (precise version)

Theorem: [B & Bruned, arXiv/1907.13028]

$\exists M > 0$ s.t. counterterm $C_0(\varepsilon,\rho) + C_1(\varepsilon,\rho)u$ satisfies

$$\big|C_0(\varepsilon,\rho)\big| \leqslant M\varepsilon_{\rm c}^{-(d-\rho)}\Big[\log(\varepsilon^{-1}) + \frac{1}{\rho-\rho_{\rm c}}\Big(\frac{\varepsilon_{\rm c}}{\varepsilon}\Big)^{3(\rho-\rho_{\rm c})}\Big] \qquad \varepsilon \geqslant \varepsilon_{\rm c}$$

$$\Big|\frac{C_0(\varepsilon,\rho)}{A_0\varepsilon^{-(d-\rho)}} - 1\Big| \leqslant \frac{M}{\rho-\rho_{\rm c}}\Big(\frac{\varepsilon}{\varepsilon_{\rm c}}\Big)^{3(\rho-\rho_{\rm c})} \qquad \varepsilon < \varepsilon_{\rm c}$$

$$\big|C_1(\varepsilon,\rho)\big| \leqslant M\bar\varepsilon_{\rm c}^{\;-(d-2\rho)}\Big[\log(\varepsilon^{-1}) + \frac{1}{\rho-\rho_{\rm c}}\Big(\frac{\bar\varepsilon_{\rm c}}{\varepsilon}\Big)^{3(\rho-\rho_{\rm c})}\Big] \qquad \varepsilon \geqslant \bar\varepsilon_{\rm c}$$

$$\Big|\frac{C_0(\varepsilon,\rho)}{\bar A_0\varepsilon^{-(d-2\rho)}} - 1\Big| \leqslant \frac{M}{\rho-\rho_{\rm c}}\Big(\frac{\varepsilon}{\bar\varepsilon_{\rm c}}\Big)^{3(\rho-\rho_{\rm c})} \qquad \varepsilon < \bar\varepsilon_{\rm c}$$

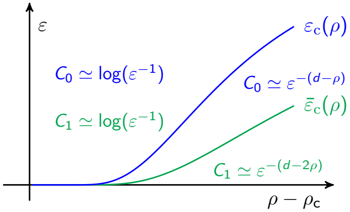
where $\varepsilon_{\rm c} = f(k_{\max})$, $\bar\varepsilon_{\rm c} = f(\bar k_{\max})$,

$$f(k) = \exp\Big\{-\frac{\log k + a - \frac{\log k}{2k})}{\rho-\rho_{\rm c}}\Big\}$$

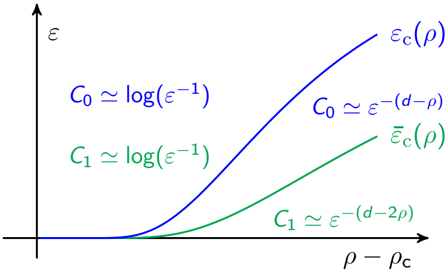
$$k_{\max} = \frac{d-\rho}{3(\rho-\rho_{\rm c})} \qquad \bar k_{\max} = \frac{d-2\rho}{3(\rho-\rho_{\rm c})}$$

$$A_0 = -\lim_{\varepsilon\to0}\varepsilon^{d-\rho}E(\vee)$$

$$\bar A_0 = -4\lim_{\varepsilon\to0}\varepsilon^{d-2\rho}E(\cdot)$$



Thanks for your attention



arXiv/1907.13028

www.idpoisson.fr/berglund/Bonn19_handout.pdf