Bochum, 16.4.14

Synchronization, noise-induced phase slips and extreme-value theory

with Barbara Gentz

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arXiv / 1208.2557
and arXiv / 1403.7393

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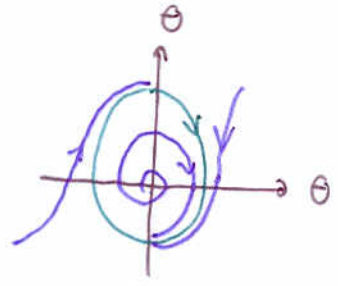
1. Synchronization

A. Deterministic

[Pikovsky, Rosenblum, Kurths 2001]

2 coupled oscillators: $\begin{cases} \dot{x}_1 = f_1(x_1) + \varepsilon g_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_2) + \varepsilon g_2(x_1, x_2) \end{cases}$



vdPol: $\ddot\theta - \gamma(1-\theta^2)\dot\theta + \theta = 0$

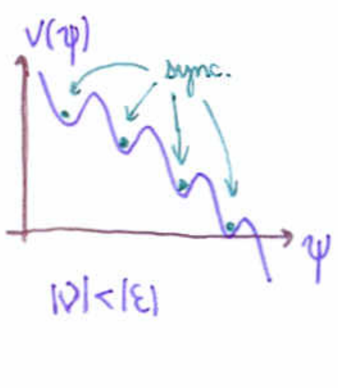
Phase dynamics $\begin{cases} \dot\phi_1 = \omega_1 + \varepsilon Q_1(\phi_1, \phi_2) \\ \dot\phi_2 = \omega_2 + \varepsilon Q_2(\phi_1, \phi_2) \end{cases}$

$\begin{cases} \psi = \phi_1 - \phi_2 \\ \varphi = \frac{1}{2}(\phi_1 + \phi_2) \end{cases} \Rightarrow \begin{cases} \dot\psi = -\nu + \varepsilon q(\psi, \varphi) \\ \dot\varphi = \omega + O(\varepsilon) \end{cases}$ $\quad \nu = \omega_2 - \omega_1$: detuning, $\omega = \frac{1}{2}(\omega_1 + \omega_2)$

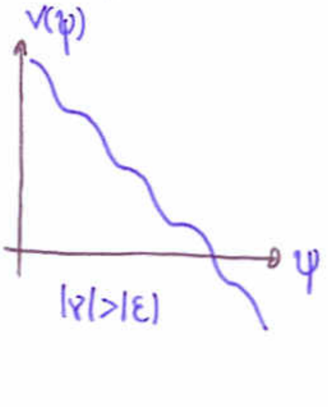
ν, ε small $\Rightarrow \dot\psi \ll \dot\varphi \Rightarrow$ averaging: $\omega \dfrac{d\psi}{d\varphi} = -\nu + \varepsilon \bar{q}(\psi) = -\dfrac{\partial}{\partial\psi} V(\psi)$

$\quad \mathrm{L} = \int_0^1 q(\psi, \varphi)\, d\varphi$

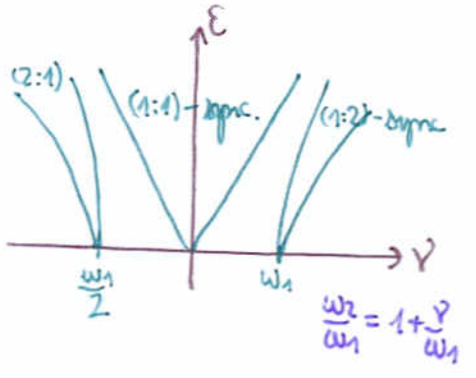
Ex: $\bar{q}(\psi) = \sin(2\pi\psi)$ $\quad$ 2 stat. pts $\Leftrightarrow |\nu| < |\varepsilon|$



$|\nu| < |\varepsilon|$



$|\nu| > |\varepsilon|$

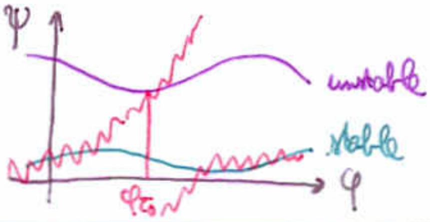


$\frac{\omega_2}{\omega_1} = 1 + \frac{\nu}{\omega_1}$

B. Stochastic

Averaged: $\quad d\psi_t = [-\nu + \varepsilon \bar{q}(\psi_t)]\, dt + \sigma g(\psi_t)\, dW_t$

Non-averaged: $\begin{cases} d\psi_t = [-\nu + \varepsilon q(\psi_t, \varphi_t)]\, dt + \sigma g_\psi(\psi_t, \varphi_t)\, dW_t \\ d\varphi_t = [\omega + \varepsilon \dots]\, dt + \sigma g_\varphi(\dots)\, dW_t \end{cases}$

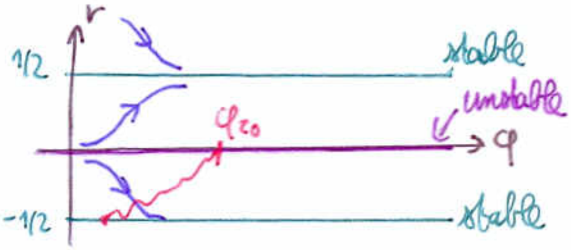


1 $\quad$ Q: Law of φ_{τ_0}?

2. Stochastic exit problem

$x = (r, \varphi)$

$$\begin{cases} dr_t = f_r(r_t, \varphi_t)\,dt + \sigma g_r(r_t, \varphi_t)\,dW_t \\ d\varphi_t = f_\varphi(r_t, \varphi_t)\,dt + \sigma g_\varphi(r_t, \varphi_t)\,dW_t \end{cases}$$



W_t: k-dim BM, $k \geq 2$

$\begin{pmatrix} g_r \\ g_\varphi \end{pmatrix}$: elliptic $\qquad \lambda_+ = \int_0^1 \partial_r f_r(0,\varphi)\,d\varphi \qquad T_+$: period

A. Harmonic measure

$\tau_0 = \inf\{t>0 : r_t = 0\}$

$h(x) = \mathbb{E}^x[\,b(\varphi_{\tau_0})\,] \iff \begin{cases} Lh(x) = 0 & x \in D = \{r<0\} \\ h(x) = b(x) & x \in \partial D = \{r=0\} \end{cases}$

$\mu_x(B) = \mathbb{P}^x\{\varphi_\tau \in B\}$

B. Large deviations

$D = gg^T$

Wentzell–Freidlin: LDP with rate fct $I_{[0,T]}(\gamma) = \frac{1}{2}\int_0^T (\dot\gamma_s - f(\gamma_s))^T D(\gamma_s)^{-1}(\dot\gamma_s - f(\gamma_s))\,ds$

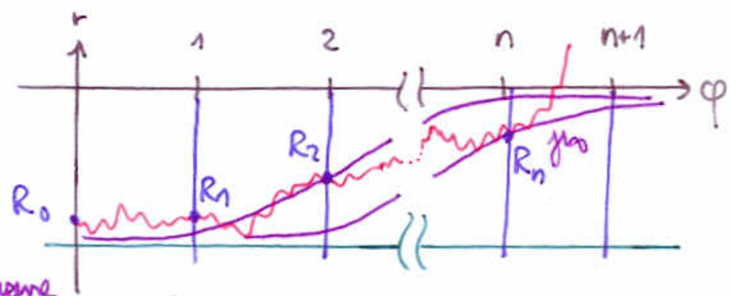
Quasipotential: $\underset{\substack{\in \\ \partial D}}{V(y)} = \inf_{T>0}\ \inf_{\gamma:\ \text{stable orbit} \to \partial D} I_{[0,T]}(\gamma)$

$$\Rightarrow \begin{cases} \lim_{\sigma\to 0} \sigma^2 \log \mathbb{E}^{x_0}[\tau_0] = \inf_{y\in\partial D} V(y) \\ \lim_{\sigma\to 0} \mathbb{P}^{x_0}\{\|x_{\tau_0} - y^*\| > \delta\} = 0 \quad \text{if inf reached at isolated } y^* \end{cases}$$

Problem: V is constant on ∂D ...

Nevertheless: generically, $\exists$ unique (up to translations) minimiser γ_∞

C. Random Poincaré maps



$\tau_n = \inf\{t>0 : \varphi_t = n\}$

$\mathbb{P}\{R_{n+1} \in B \mid r_{\tau_n} = R_n\} = \int_B k(R_n, y)\,dy$

λ_0: Perron–Frobenius $\qquad$ k: harmonic measure

$\varphi = n+s$: $\qquad$ ~Jentzsch $\qquad$ [Ben Arous, Kusuoka, Stroock]

$$\mathbb{P}^{0,R_0}\{\varphi_{\tau_0} \in n+ds\} = \int_E K^n(R_0, dy)\,\mathbb{P}^{0,y}\{\varphi_{\tau_0} \in ds\}$$

Spectral gap $\Rightarrow K^n(R_0, B) = \lambda_0^n\, \pi_0(B)\,[1 + O(\rho^n)]$ $\qquad$ $\mathbb{P}\{R_0 \in B \mid R_n \in E\} = \frac{K^n(R_0,B)}{K^n(R_0,E)} = \underset{[1+O(\rho^n)]}{\pi_0(B)}$ QSD

$$\mathbb{P}^{0,R_0}\{\varphi_{\tau_0} \in n+ds\} = \lambda_0^n \int_E \pi_0(dy)\,\mathbb{P}^{0,y}\{\varphi_{\tau_0} \in ds\}\,[1 + O(\rho^n)]$$

3. Main result - log-periodic oscillations

<u>Theorem:</u> $\exists \beta, c > 0 \quad \forall \delta, \Delta > 0 \quad \exists \sigma_0 > 0 \quad \forall \sigma < \sigma_0$:

Cycling [Day]

$$\boxed{\mathbb{P}\left\{ \frac{\theta_\delta(\varphi_{\tau_0})}{\lambda_+ T_+} \in [t, t+\Delta] \right\} = \Delta \left[1 - \lambda_0(\sigma)\right] \lambda_0(\sigma)^t \, Q_{\lambda_+ T_+}\left(\frac{|\log \sigma|}{\lambda_+ T_+} - t + O(\delta) \right) \times \left[1 + O(e^{-c\varphi/|\log \sigma|}) + O(\delta |\log \delta|) + O(\Delta^\beta) \right]}$$

* $\theta_\delta(\varphi) = \theta(\varphi) - \log \delta$: explicit, $\theta'(\varphi) = \theta'(\varphi + 1) > 0$
* $Q_{\lambda_+ T_+}(x) = \sum_{n=-\infty}^{\infty} G(\lambda_+ T_+ (n - x))$

  $G(x) = \exp\left\{ -2x - \frac{1}{2} e^{-2x} \right\}$ density of $\frac{Z - \log 2}{2}$

  Z standard Gumbel r.v: $\mathbb{P}\{Z \leq t\} = e^{-e^{-t}}$

<u>Corollary:</u> $\exists \ (Y_m^\sigma)_{m \in \mathbb{N}, \sigma > 0}$ asympt. geometric r.v:

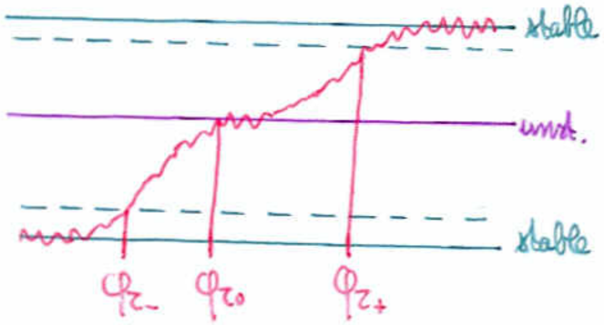
$$\lim_{n \to \infty} \mathbb{P}\left\{ Y_m^\sigma = n+1 \mid Y_m^\sigma > n \right\} = e^{-I_m/\sigma^2}$$

$$I_m = I_\infty + O(e^{-2m\lambda_+ T_+})$$

$$\lim_{m \to \infty} \left[\lim_{\sigma \to 0} \text{Law}\left(\theta(\varphi_{\tau_0}) - |\log \sigma| - \lambda_+ T_+ Y_m^\sigma \right) \right] = \text{Law}\left(\frac{Z}{2} - \frac{\log 2}{2} \right)$$

Y_m^σ: index of chosen translate of γ_∞

<u>More results:</u>



* $\lim_{\sigma \to 0} \text{Law}\left(\theta(\varphi_{\tau_0}) - \theta(\varphi_{\tau_-}) - |\log \sigma| \right) = \text{Law}\left(\frac{Z}{2} - \frac{\log 2}{2} + c \right)$
* $\lim_{\sigma \to 0} \text{Law}\left(\theta(\varphi_{\tau_+}) - \theta(\varphi_{\tau_0}) - |\log \sigma| \right) = \text{Law}\left(\Theta + c \right)$

  $\Theta = -\log |N|$
* $\lim_{\sigma \to 0} \text{Law}\left(\theta(\varphi_{\tau_+}) - \theta(\varphi_{\tau_-}) - 2|\log \sigma| \right) = \text{Law}\left(Z + 2c \right)$
* $\lim_{\sigma \to 0} \text{Law}\left(\theta(\varphi_{\tau_0^{n+1}}) - \theta(\varphi_{\tau_0^n}) \right) = \text{Law}\left(\text{Logistic} \right)$ density $\sim \frac{1}{\cosh^2(\theta)}$

4. Connections to extreme-value theory [Bakhtin, arXiv 2013, Stoch Dyn 14]

$X_1, X_2, \dots$ iid r.v. $\quad M_n = \max\{X_1, \dots X_n\}$

$F(x) = \mathbb{P}\{X_1 \leqslant x\} \Rightarrow \mathbb{P}\{M_n \leqslant t\} = F(t)^n$

Def: $F \in D(\Phi)$ if $\exists\, \Phi$ distr. fct s.t. $\lim_{n\to\infty} F(a_n x + b_n)^n = \Phi(x)$

Thm: [Fréchet, Fisher, Tippett, Gnedenko]

$$F \neq 1_{\{x \geqslant c\}} \Rightarrow \Phi \in \{\text{Gumbel}, \text{Fréchet}, \text{Weibull}\}$$

$$\Lambda(x) = e^{-e^{-x}} \qquad e^{-x^{-\alpha}} 1_{\{x>0\}} \qquad e^{-(-x)^\alpha} 1_{\{x \leqslant 0\}} + 1_{\{x>0\}}$$

Thm: [G] $\quad x_0 = \inf\{x : F(x) = 1\} \in \mathbb{R} \cup \{\infty\}$

$$F \in D(\Lambda) \Leftrightarrow \exists A(z), \lim_{z \to x_0-} A(z) = 0$$

$$\lim_{z \to x_0^-} \underbrace{\mathbb{P}\{X_1 > z[1 + A(z)x] \mid X_1 > z\}}_{\text{residual lifetime}} = -\log \Lambda(x) = e^{-x}$$

[Bakhtin]: $\quad dx_t = \lambda x_t\, dt + \sigma\, dW_t$

$$x_t = e^{\lambda t} \tilde{x}_t \qquad \tilde{x}_t = x_0 + \sigma \int_0^t e^{-\lambda s}\, dW_s$$

$$= x_0 + \tilde{W}_{\sigma^2(1 - e^{-2\lambda t})/(2\lambda)}$$

reflection principle $\Rightarrow \mathbb{P}\{\tau_0 < t \mid \tau_0 < \infty\} = \dfrac{\mathbb{P}\{\tau_0 < t\}}{\mathbb{P}\{\tau_0 < \infty\}}$

$$= \frac{2\, \mathbb{P}\{\tilde{x}_t > 0\}}{2\, \mathbb{P}\{\tilde{x}_\infty > 0\}} = \mathbb{P}\{\tilde{x}_t > 0 \mid \tilde{x}_\infty > 0\}$$

$$\mathbb{P}\{\tau_0 < t + \tfrac{1}{\lambda}|\log\sigma| \mid \tau_0 < \infty\} = \mathbb{P}\{\tilde{x}_{t + \frac{1}{\lambda}|\log\sigma|} > 0 \mid \tilde{x}_\infty > 0\}$$

$$= \mathbb{P}\left\{N > \frac{|x_0|}{\sigma}\sqrt{\frac{2\lambda}{1 - \sigma^2 e^{-2\lambda t}}} \;\middle|\; N > \frac{|x_0|}{\sigma}\sqrt{2\lambda}\right\}$$

uses $\log \Lambda(e^{-x}) = \Lambda(x) \dots$ $\quad \xrightarrow[\sigma \to 0]{} \exp\{-x_0^2 \lambda e^{-2\lambda t}\} = \mathbb{P}\left\{\dfrac{Z + \log(x_0^2\lambda)}{2\lambda} < t\right\}$